Differential Forms Crash Course

Jiří Lebl

October 17, 2022

The classical theorems we learned this semester can be conveniently stated in a way that gives a vast generalization in one simple statement, and also allows one to more easily remember/derive the statements of the theorems, and simplify computations. We will only scratch the surface (no pun intended) here. What we are aiming at is the so-called *Generalized Stokes' Theorem*:

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega.$$

If rowdy mathematicians write graffiti on bathroom walls, this is a good candidate of what they would write. It says that the integral over an object of the derivative of something is an integral of that something over the boundary. To make all the theorems fit within this, we have to figure out what all the objects mean, what is a boundary, and what is a derivative. An amazing thing is that the "d" operator is the right derivative in the right context. It is the gradient when it needs to be a gradient, it is a curl when it needs to be a curl, it is a divergence when it needs to be the divergence, etc. The differential forms also include all the information needed to compute the integrals, to deal with orientation, or to change coordinates.

We will mostly worry about 3 dimensions and see how the ideas apply in 2 dimensions. But the ideas apply in any number of dimensions with almost no change.

In 3 dimensions, there are 4 different kinds of what are called *differential forms*. There are 0-forms, 1-forms, 2-forms, 3-forms. You have seen 0-forms and 1-forms without knowing about it. Differential forms are things that are "integrated" on the geometric object of the corresponding dimension (point, path, surface, region). In n dimensions there would be $n + 1$ different kinds of differential forms, but let us stick to 3 dimensions for simplicity.

0-forms

In the context of differential forms, functions are called 0-*forms*. These 0-forms are "integrated" on points, as points are the 0-dimensional objects. That is, functions are evaluated at points. If P is point, then let

$$\int_{P} f = f(P).$$

For example, if $f(x, y, z) = x^2 - 1 + z$ and $P = (1, 2, 3)$, then

$$\int_P f = f(1, 2, 3) = 1^2 - 1 + 3 = 3.$$

Points can have orientation, that is, positive or negative. Above, we dealt with a positively oriented P. If Q is negatively oriented, then

$$\int_Q f = -f(Q).$$

If $Q = (2, 1, 0)$ is negatively oriented, then

$$\int_Q f = -f(2, 1, 0) = -(2^2 - 1 + 0) = -3.$$

We can also add and subtract points. So suppose that $P = (1, 2, 3)$ and $R = (0, 0, 2)$ are both positively oriented, we write $-P$ as the negatively oriented P and then we could write $R - P$. Then

$$\int_{R-P} f = f(R) - f(P) = f(0, 0, 2) - f(1, 2, 3) = 1 - 3 = -2.$$

That looks a lot like the "integral" of the "boundary" of a segment of a curve that starts at P and ends at R, and this is exactly where this notation will show up. You then have to be careful not to do arithmetic on the components of $R - P$, despite what it looks like. These are points, not vectors, and when points add or subtract, it is in the sense above.

We haven't really done anything except make up new notation so far, and it may seem like we're making up nonsense, but the notation will be useful for stating the fundamental theorem of calculus as the same theorem as Green's, Stokes', divergence, etc.

1-forms

One-forms are expressions such as

$$f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz.$$

For example,

$$x^2 y\, dx + 3xe^z\, dy + (z + y)\, dz.$$

So a 1-form is a combination of dx, dy, and dz. We cannot just multiply the dx, dy, dz, although more on that later. These objects keep track of how we integrate. In some sense they are the "derivatives" of the coordinate functions x, y, and z.

One-forms are things that are integrated on (oriented) paths, as paths are one-dimensional. If C is a path, then we define

$$\int_C f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz = \int_C \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle \cdot \hat{t}\, ds.$$

And you have seen this expression before. We use the following formula for the actual computation. Suppose the path C is parametrized by t for $a \leq t \leq b$. That is, x, y, z are functions of t. Then

$$\int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$$
$$= \int_a^b \left(f\big(x,y,z\big)\frac{dx}{dt} + g\big(x,y,z\big)\frac{dy}{dt} + h\big(x,y,z\big)\frac{dz}{dt} \right) dt. \quad (1)$$

For example, suppose C is the straight line from $(0,0,0)$ to $(1,2,3)$ parametrized by $x(t) = t$, $y(t) = 2t$, $z(t) = 3t$, for $0 \leq t \leq 1$. Then

$$\int_C x^2y\,dx + 3xe^z\,dy + (z+y)\,dz = \int_0^1 \Big((t)^2(2t)\big)(1) + \big(3te^{3t}\big)(2) + \big(3t+2t\big)(3) \Big)\,dt$$
$$= \left[\frac{t^4}{2} + \frac{6t-2}{3}e^{3t} + \frac{15t^2}{2} \right]_{t=0}^1 = \frac{4}{3}e^3 + \frac{26}{3}.$$

We often give a name to the one-form. We say $\omega = f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$. Then

$$\int_C \omega = \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz.$$

One way that one-forms arise is as derivatives of functions. Let f be a function, then what you called *total derivative* in multivariable calculus, is really the "d operator" on 0-forms giving 1-forms. That is,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

For example, if $f(x,y,z) = x^2e^yz$, then

$$df = 2xe^yz\,dx + x^2e^yz\,dy + x^2e^y\,dz.$$

Not every vector field is a gradient vector field, and so similarly, not every 1-form is a derivative of a function.

For example, $\omega = -y\,dx + x\,dy$ is not the total derivative of any function f. If it were, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial(-y)}{\partial y} = -1, \quad \text{but} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(x)}{\partial x} = 1,$$

and that is impossible.

Notice that the dx is the derivative of x. That is, if $f(x,y,z) = x$, then

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 1\,dx + 0\,dy + 0\,dz = dx.$$

Similarly for dy and dz. So the notation keeps track of changes of variables and chain rule, as we saw above. That is, dx becomes $\frac{dx}{dt}\,dt$ when we integrate with respect to t. Similarly, if we parametrize a curve with respect to x, we do not need to change the dx. Consider a curve C given by $y = x^2$, $z = x^3$ for $0 \leq x \leq 1$. Let us compute a simple integral over C:

$$\int_C x\,dx + y\,dy + z\,dz = \int_0^1 x\,dx + y\,\frac{dy}{dx}\,dx + z\,\frac{dz}{dx}\,dx = \int_0^1 \big(x + x^2(2x) + x^3(3x^2)\big)\,dx = \frac{3}{2}.$$

Boundaries of paths and the fundamental theorem

If C is a path from point Q to point P, then we say that the boundary of C is P with positive orientation and Q with negative orientation. This is written as $P - Q$. We write the boundary as $\partial C = P - Q$.

The upshot of all this is the easy statement of the fundamental theorem of calculus that will look like all the other statements of the fundamental theorem. We can simply write it as

$$\int_C df = \int_{\partial C} f$$

Let's interpret this equation. The left-hand side is

$$\int_C df = \int_C \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

While the right-hand side, assuming C goes from Q to P, is

$$\int_{\partial C} f = f(P) - f(Q).$$

If $f(x, y, z) = x^2 e^y z$ as above, and C is the path parametrized by $\gamma(t) = (t, 3t, t + 1)$ for $0 \leq t \leq 1$, so starting at $(0, 0, 1)$ and ending at $(1, 3, 2)$, then

$$\int_C df = \int_{\partial C} f = f(1, 3, 2) - f(0, 0, 1) = 1^2 e^3 2 - 0^2 e^0 1 = 2e^3.$$

Another example of this use is to compute a path integral by computing the antiderivative. For example, suppose C is a straight line from $(0, 0, 0)$ to $(1, 2, 3)$, and we want to compute

$$\int_C y\,dx + x\,dy + 2z\,dz.$$

If we can find an f whose total derivative is the form above, then we are done. If f exists then $\frac{\partial f}{\partial x} = y$, so $f = xy + g(y, z)$ for some function g. Taking the derivative with respect to y gets us $\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}$, so g is independent of y. Taking the derivative with respect to z we find $2z = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z}$, so $g = z^2$ (plus a constant, but we just need one antiderivative). So $f = xy + z^2$. In other words:

$$\int_C y\,dx + x\,dy + 2z\,dz = \int_C df = \int_{\partial C} f = f(1, 2, 3) - f(0, 0, 0) = 1 \cdot 2 + 3^2 = 11.$$

2-forms

OK, so far we've only justified notation you have seen before. Let us now move to the surface integral (the flux integral) and how to frame it in terms of differential forms. For 2-forms we need to be a bit more careful with orientation, and we need to keep track of it on the form side of things. For this purpose, we introduce a new object, the so-called *wedge* or *wedge product*. It is a way to put together forms. The wedge product takes two 1-forms ω and η and gets a 2-form $\omega \wedge \eta$. Let us start with wedging together dx, dy, and dz. We write

$$dx \wedge dy, \qquad dy \wedge dz, \qquad dz \wedge dx.$$

We define that

$$dx \wedge dy = -dy \wedge dx, \qquad dy \wedge dz = -dz \wedge dy, \qquad dz \wedge dx = -dx \wedge dz.$$

Finally, a wedge of something with itself is just zero:

$$dx \wedge dx = 0, \qquad dy \wedge dy = 0, \qquad dz \wedge dz = 0.$$

This is true for any 1-form: $\omega \wedge \omega = 0$.

An arbitrary 2-form is an expression of the form

$$\omega = f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy.$$

If any other wedges appear, we can (if we really want to) use the rules above to convert them to this form. For example,

$$x^2\, dy \wedge dz + y\, dx \wedge dz + z^2\, dx \wedge dx = x^2\, dy \wedge dz - y\, dz \wedge dx.$$

We also impose some further algebra rules on this product. Anything we would call a "product" had better be what we call bilinear: If ω, η, and γ are one-forms, then

$$\begin{aligned}(\omega + \eta) \wedge \gamma &= \omega \wedge \gamma + \eta \wedge \gamma, \\ \omega \wedge (\eta + \gamma) &= \omega \wedge \eta + \omega \wedge \gamma.\end{aligned}$$

If f is a function, then

$$(f\omega) \wedge \eta = f(\omega \wedge \eta) = \omega \wedge (f\eta).$$

Let's see these rules on an example:

$$\begin{aligned}(x^2y\, dx + z^2\, dz) \wedge (e^z\, dy + 8\, dz) &= x^2y\, dx \wedge (e^z\, dy + 8\, dz) + z^2\, dz \wedge (e^z\, dy + 8\, dz) \\ &= x^2ye^z\, dx \wedge dy + 8x^2y\, dx \wedge dz + z^2e^z\, dz \wedge dy + 8z^2\, dz \wedge dz \\ &= -z^2e^z\, dy \wedge dz - 8x^2y\, dz \wedge dx + x^2ye^z\, dx \wedge dy.\end{aligned}$$

In general,

$$\begin{aligned}(f\, dx + g\, dy + h\, dz) \wedge (a\, dx + b\, dy + c\, dz) &= fa\, dx \wedge dx + fb\, dx \wedge dy + fc\, dx \wedge dz \\ &\quad + ga\, dy \wedge dx + gb\, dy \wedge dy + gc\, dy \wedge dz \\ &\quad + ha\, dz \wedge dx + hb\, dz \wedge dy + hc\, dz \wedge dz \\ &= (gc - hb)\, dy \wedge dz + (ha - fc)\, dz \wedge dx + (fb - ga)\, dx \wedge dy.\end{aligned}$$

You should recognize the formula for the cross product. That is, the result is a 2-form whose coefficients are $\langle f, g, h\rangle \times \langle a, b, c\rangle$. The wedge product is always the right product in the right context.

OK, now that we know what 2-forms are, what do we do with them. First, let's see how to differentiate 1-forms to get 2-forms, with the d operator. We want the derivative to be linear, so that in particular $d(\omega + \eta) = d\omega + d\eta$. When we have an expression such as $f\,dx$, we define

$$d(f\,dx) = df \wedge dx.$$

Similarly for dy and dz. Let's compute the derivative of any 1-form:

$$\begin{aligned}
d(f\,dx + g\,dy + h\,dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz \\
&= \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\right) \wedge dx \\
&\qquad + \left(\frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy + \frac{\partial g}{\partial z}\,dz\right) \wedge dy \\
&\qquad + \left(\frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy + \frac{\partial h}{\partial z}\,dz\right) \wedge dz \\
&= \frac{\partial f}{\partial x}\,dx \wedge dx + \frac{\partial f}{\partial y}\,dy \wedge dx + \frac{\partial f}{\partial z}\,dz \wedge dx \\
&\qquad + \frac{\partial g}{\partial x}\,dx \wedge dy + \frac{\partial g}{\partial y}\,dy \wedge dy + \frac{\partial g}{\partial z}\,dz \wedge dy \\
&\qquad + \frac{\partial h}{\partial x}\,dx \wedge dz + \frac{\partial h}{\partial y}\,dy \wedge dz + \frac{\partial h}{\partial z}\,dz \wedge dz \\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dy \wedge dx.
\end{aligned}$$

You should recognize the formula for the curl. That is, if the functions f, g, h are coefficients of a vector field, then the coefficients of the derivative of the one-form are the coefficients of the curl of the vector field. Again the wedge product and d gets us the right thing in the right context. And to do computations with d and the wedge is much easier to do because one only needs to follow a couple of simple rules.

For example,

$$\begin{aligned}
d(x\,dx + y^2\,dz) &= 1\,dx \wedge dx + 0\,dy \wedge dx + 0\,dz \wedge dz + 0\,dx \wedge dz + 2y\,dy \wedge dz + 0\,dz \wedge dz \\
&= 2y\,dy \wedge dz.
\end{aligned}$$

Of course, we do not need to do this in excruciating detail; we know which derivatives will end up zero, and which wedge products will end up zero. We need to only look at those. So perhaps,

$$\begin{aligned}
d(xy\,dx + z^2\,dy + y^2\,dz) &= x\,dy \wedge dx + 2z\,dz \wedge dy + 2y\,dy \wedge dz \\
&= (2y - 2z)\,dy \wedge dz - x\,dx \wedge dy.
\end{aligned}$$

The formula $\nabla \times \nabla f = \vec{0}$ appears in the fact that

$$d(df) = 0.$$

This is a general feature of the d operator, and it is sometimes written as $d^2 = 0$.

OK, now that we have the derivative, we also want to integrate 2-forms. 2-forms are integrated over surfaces. Let S be an oriented surface, where $\hat{n}$ is the unit normal that gives the orientation. Suppose S is a graph of $z = \varphi(x, y)$ and $\hat{n}$ is the upward unit normal. We define

$$\int_S f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy = \iint_S \langle f, g, h \rangle \cdot \hat{n}\, dS.$$

We use only one integral sign for integrals of forms by convention, even though it is a surface integral. The definition works for any surface integral, not just a graph, if you figure out the correct orientation.

Again, we have only defined a new notation for something we knew how to compute already, the flux integral. But using this notation, a way to compute surface integrals is suggested by the change of variables formula from multivariable calculus. And in this way we can compute the integral for any parametrized surface easily. Denote

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

This expression is the determinant of the derivative from the change of variables formula for 2 dimensional integrals. This formula is called the Jacobian determinant. Let S be parametrized by (u, v) ranging over a domain D, where the ordering u and then v gives the orientation of S via the right-hand rule. That is, if we curl the fingers on our right hand, first in the u direction, then in the v direction, then our thumb would be the unit normal giving the orientation. So x, y, and z are functions of (u, v). Then

$$\int_S f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy = \iint_D \left(f\frac{\partial(y, z)}{\partial(u, v)} + g\frac{\partial(z, x)}{\partial(u, v)} + h\frac{\partial(x, y)}{\partial(u, v)} \right) du\, dv.$$

Compare this to how we computed 1-form integrals above in equation (1), and it will feel very familiar.

For example, let $\omega = x\, dy \wedge dz + y\, dz \wedge dz + z\, dx \wedge dy$ be the 2-form, and let S be the surface given by the graph $z = x^2 + y^2$ where x and y lie in the unit square $0 \le x, y \le 1$. We have $x = u$, $y = v$, $z = u^2 + v^2$. The domain D is the unit square $0 \le u, v \le 1$. Then

$$\begin{aligned} \int_S \omega = \int_S x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \\ = \int_0^1 \int_0^1 \left(u(-2u) + v(-2v) + (u^2 + v^2) \right) du\, dv \\ = \int_0^1 \int_0^1 (-u^2 - v^2)\, du\, dv = \frac{-2}{3}. \end{aligned}$$

For another example, suppose $\eta = xz\, dy \wedge dz$, and let the surface S be the cylinder of radius 1 around the z-axis for $0 \le z \le 1$ oriented with the normal outwards (away from the z-axis). Let us compute $\int_S \eta$.

First we parametrize S. Let (u, v) map to $(\cos u, \sin u, v)$ for $0 \le u \le 2\pi$ and $0 \le v \le 1$. We check that the right-hand rule, curling our fingers around the u direction followed by the v direction gets us the outward normal. If it didn't, we could just swap u and v.

So

$$\int_S xz\, dy \wedge dz = \int_0^1 \int_0^{2\pi} \underbrace{(\cos u)}_{x} \underbrace{v}_{z} \underbrace{(\cos u)}_{\frac{\partial(y,z)}{\partial(u,v)}} \, du\, dv$$

$$= \int_0^1 \int_0^{2\pi} (\cos u)^2 v\, du\, dv = \pi.$$

This is a good way to remember how to integrate parametrized surfaces. Another advantage is that you do not have to always put everything into the normal form. Perhaps in the last example you swap dy and dz (and so introduce a negative sign) and write the integral as $\int_S -xz\, dz \wedge dy$. We can just compute the integral that way:

$$\int_S -xz\, dz \wedge dy = \int_0^1 \int_0^{2\pi} -\underbrace{(\cos u)}_{x} \underbrace{v}_{z} \underbrace{(-\cos u)}_{\frac{\partial(z,y)}{\partial(u,v)}} \, du\, dv$$

$$= \int_0^1 \int_0^{2\pi} (\cos u)^2 v\, du\, dv = \pi.$$

We computed $\frac{\partial(z,y)}{\partial(u,v)}$ because we had $dz \wedge dy$. Before we computed $\frac{\partial(y,z)}{\partial(u,v)}$ because we had $dy \wedge dz$. That's what we meant when we said the wedge product keeps track of orientation. It keeps track of how you are supposed to integrate a 2-form, no matter how we write the 2-form.

Stokes' Theorem

The classical Stokes' Theorem can now be stated. Let S be an oriented surface and ∂S be the boundary curve of S oriented according to the right-hand rule as we have for the classical Stokes' Theorem. Let ω be a 1-form. Then Stokes' Theorem in terms of differential forms is

$$\int_S d\omega = \int_{\partial S} \omega.$$

If $\omega = f\, dx + g\, dy + h\, dz$, then $d\omega$, as we saw above, is really the 2-form whose coefficients are the components of $\nabla \times \langle f, g, h \rangle$. So the left-hand side is

$$\int_S d\omega = \iint_S \nabla \times \langle f, g, h \rangle \cdot \hat{n}\, dS.$$

The right-hand side is the integral

$$\int_{\partial S} \omega = \int_{\partial S} \langle f, g, h \rangle \cdot \hat{t}\, ds.$$

That is, we have the classical Stokes'. Notice how the expression

$$\int_S d\omega = \int_{\partial S} \omega$$

is now the same for both the Stokes' Theorem and the Fundamental Theorem of Calculus. The only difference is that S is now a surface and not a curve and ω is a 1-form and not a 0-form (function).

3-forms and the Divergence Theorem

If we take one more wedge, we find that the only forms that survive our rules, namely that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$, are the ones that look like

$$f\, dx \wedge dy \wedge dz.$$

Notice that

$$\begin{aligned} dx \wedge dy \wedge dz &= dz \wedge dx \wedge dy = dy \wedge dz \wedge dx \\ &= -dy \wedge dx \wedge dz = -dx \wedge dz \wedge dy = -dz \wedge dy \wedge dx. \end{aligned}$$

Integrating 3-forms is easy. Write the 3-form as $f\, dx \wedge dy \wedge dz$ and then, given a region R in 3-space, we have

$$\int_R f\, dx \wedge dy \wedge dz = \iiint_R f\, dV,$$

where dV is the volume measure. We also put orientation on R, and the above is for positive orientation. If orientation is not mentioned, we always mean the positive orientation. If R would be oriented negatively, then we define the integral to be the negative of the integral for positive orientation. Let us not worry about it, and just do positively oriented regions in 3-space.

Example: Let R be the region defined by $-1 < x < 2$, $2 < y < 3$, $0 < z < 1$. Then

$$\begin{aligned} \int_R x^2 y e^z\, dx \wedge dy \wedge dz &= \int_{-1}^{2} \int_{2}^{3} \int_{0}^{1} x^2 y e^z\, dz\, dy\, dx = \int_{-1}^{2} \int_{2}^{3} x^2 y(e-1)\, dy\, dx \\ &= \int_{-1}^{2} x^2 \left(\frac{3^2}{2} - \frac{2^2}{2} \right) (e-1)\, dx = \left(\frac{2^3}{3} - \frac{(-1)^3}{3} \right) \left(\frac{3^2}{2} - \frac{2^2}{2} \right) (e-1). \end{aligned}$$

Next, how do we differentiate 2-forms to get 3-forms? We apply essentially the same formula as before:

$$d(f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy) = df \wedge dy \wedge dz + dg \wedge dz \wedge dx + dh \wedge dx \wedge dy.$$

Let us carry this through. For example, let's start with the first term:

$$
\begin{aligned}
df \wedge dy \wedge dz &= \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\right) \wedge dy \wedge dz \\
&= \frac{\partial f}{\partial x}\,dx \wedge dy \wedge dz + \frac{\partial f}{\partial y}\,dy \wedge dy \wedge dz + \frac{\partial f}{\partial z}\,dz \wedge dy \wedge dz = \frac{\partial f}{\partial x}\,dx \wedge dy \wedge dz.
\end{aligned}
$$

In the second term, it is only the $\frac{\partial g}{\partial y}$ term to survive, and in the third term it is only the $\frac{\partial h}{\partial z}$ term.

All in all we find that for $\omega = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$,

$$
d\omega = d(f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy) = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz.
$$

And again, notice the expression for the divergence pops up. We are then not surprised that the Divergence Theorem

$$
\iiint_R \nabla \cdot \langle f, g, h\rangle\,dV = \iint_{\partial R} \langle f, g, h\rangle \cdot \hat{n}\,dS,
$$

where R is oriented positively and $\hat{n}$ is the outward unit normal on the boundary ∂R, takes the form

$$
\int_R d\omega = \int_{\partial R} \omega.
$$

Generalized Stokes' Theorem

The formula

$$
\int_\Omega d\omega = \int_{\partial\Omega} \omega.
$$

is called the *Generalized Stokes' Theorem*. Here ω is a $(k-1)$-form and Ω is a k-dimensional geometric object over which to integrate. In 3-space, ω is a 0-, 1-, or 2-form, and Ω is a path (1-dimensional), a surface (2-dimensional), or a region (3-dimensional).

Another thing to notice is the following diagram:

$$
\text{0-forms} \xrightarrow{d} \text{1-forms} \xrightarrow{d} \text{2-forms} \xrightarrow{d} \text{3-forms}
$$

corresponds to the diagram

$$
\text{functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\nabla\times} \text{vector fields} \xrightarrow{\nabla\cdot} \text{functions.}
$$

We mentioned above that $\nabla \times \nabla f = \vec{0}$ is the formula $d(df) = 0$ for a function (0-form) f. Similarly, $\nabla \cdot \nabla \times \vec{F} = 0$ is the formula $d(d\omega) = 0$ for a 1-form ω. It is always true that using the d operator on an output of a d operator, that is a d-derivative of a d-derivative, is 0. In other words,

$$
d(d\omega) = 0
$$

for all differential forms ω. It is sometimes shortened to $d^2 = 0$.

Applying in the plane

In the plane, think of everything as if it were in three space but with no z dependence, so no dz. So there are only 0-forms, 1-forms and 2-forms. The only 2-form that appears is $f\,dx \wedge dy$, since the other possible wedge product gets you $dy \wedge dx = -dx \wedge dy$. The derivative of a one-form is

$$\begin{aligned}
d(f\,dx + g\,dy) &= df \wedge dx + dg \wedge dy \\
&= \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right) \wedge dx + \left(\frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy\right) \wedge dy \\
&= \frac{\partial f}{\partial y}\,dy \wedge dx + \frac{\partial g}{\partial x}\,dx \wedge dy \\
&= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,dx \wedge dy.
\end{aligned}$$

If R is a region in the plane and ∂R is its boundary, the Generalized Stokes' Theorem says:

$$\int_{\partial R} f\,dx + g\,dy = \int_R d(f\,dx + g\,dy) = \int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,dx \wedge dy.$$

And you will recognize Green's Theorem.

Changing coordinates

Differential forms take care of changing coordinates easily. The trick is to know that dx, dy, dz are the derivatives of the x, y, and z coordinate functions. Suppose we wish to write down everything in terms of dr, $d\theta$, dz of cylindrical coordinates. Consider

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Then

$$\begin{aligned}
dx &= d(r\cos\theta) = \cos\theta\,dr - r\sin\theta\,d\theta, \\
dy &= d(r\sin\theta) = \sin\theta\,dr + r\cos\theta\,d\theta, \\
dz &= d(z) = dz.
\end{aligned}$$

Consider the one-form

$$\omega = (x^2 + y^2)\,dx + z\,dy + dz$$

Let us change this one-form into cylindrical coordinates.

$$\begin{aligned}
\omega = r^2\,dx + z\,dy + dz &= r^2(\cos\theta\,dr - r\sin\theta\,d\theta) + z(\sin\theta\,dr + r\cos\theta\,d\theta) + dz \\
&= (r^2\cos\theta + z\sin\theta)\,dr + (-r^3\sin\theta + z\cos\theta)\,d\theta + dz.
\end{aligned}$$

Now suppose we wish to find

$$\int_C \omega,$$

where C is the spiral given in cylindrical coordinates by $r = 1$, $\theta = t$, $z = t$ for $0 \leq t \leq 2\pi$. So

$$dr = \frac{dr}{dt}\,dt = 0, \qquad d\theta = \frac{d\theta}{dt}\,dt = dt, \qquad dz = \frac{dz}{dt}\,dt = dz.$$

And so plugging it in we compute

$$\begin{aligned}\int_C \omega &= \int_C (r^2 \cos\theta + z\sin\theta)\,dr + (-r^3\sin\theta + z\cos\theta)\,d\theta + dz \\ &= \int_0^{2\pi} (-\sin t + t\cos t + 1)\,dt = 2\pi.\end{aligned}$$

Changing variables for two-forms and three-forms is exactly the same idea since they are constructed out of dx, dy, dz.

For example, what about the area measure on the xy-plane in cylindrical (so in polar). In the plane the area measure dA is $dx \wedge dy$, so

$$\begin{aligned}dx \wedge dy &= (\cos\theta\,dr - r\sin\theta\,d\theta) \wedge (\sin\theta\,dr + r\cos\theta\,d\theta) \\ &= (\cos\theta)(r\cos\theta)\,dr \wedge d\theta + (-r\sin\theta)(\sin\theta)\,d\theta \wedge dr \\ &= (\cos\theta)(r\cos\theta)\,dr \wedge d\theta + (r\sin\theta)(\sin\theta)\,dr \wedge d\theta \\ &= r(\cos^2\theta + \sin^2\theta)\,dr \wedge d\theta = r\,dr \wedge d\theta.\end{aligned}$$

We obtain the familiar $r\,dr\,d\theta$ from calculus.

We get the volume form dV too,

$$dx \wedge dy \wedge dz = r\,dr \wedge d\theta \wedge dz.$$