**Problem 11002, year 2003, page 240**

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*Problem: Pooh Bear has $2N+1$ honey pots. No matter which one of them he sets aside, he can split the remaining $2N$ pots into two sets of the same total weight each consisting of $N$ pots. Must all $2N+1$ pots weight the same?*

We answer the question in the positive. To prove this we must show that for some $N$, if we take any one of the honey pots away, the remaining cannot be split into two groups of the same weight and same number of pots unless all the pots weight the same. A solution would then have to satisfy $2N+1$ equations (one for each pot taken away). Each of these equations be written in the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{2N+1} x_{2N+1} = 0,$$

where $\alpha$'s are either $1$, $-1$ or $0$. There is only one $\alpha$ which is $0$ and the number of $1$'s and $-1$'s is equal. So what we are looking for is really the nullspace of a matrix which has zeros on the diagonal and $1$'s and $-1$'s elsewhere, provided that the number of $1$'s and $-1$'s in each row is equal. For example when $2N+1=5$ then we may have

$$\begin{bmatrix} 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}.$$

This matrix has nullity 1. What we would need to show is that every such $5 \times 5$ matrix has nullity 1 and that would mean that there is only one solution (all pots of equal weight) no matter how Pooh divides the remaining $2N$ pots. We note that we could always just make the first column all ones (except for the zero), since that would not change the solutions. So we really have matrices of the form

$$\begin{bmatrix} 0 & \beta \\ 1 & A \end{bmatrix},$$

where $A$ is a $2N \times 2N$ matrix and $\beta$ is just a vector of $1$'s and $-1$'s. If we can show that every $2N \times 2N$ matrix with zeros on the diagonal and $1$'s and $-1$'s elsewhere is invertible then we are done.

So we now focus on computing the determinant of this matrix $A$. We look at the algebraic approach for computing determinants, that is

$$det(A) = \sum_{\sigma \in S_{2N}} \epsilon(\sigma) \prod_{i=1}^{2N} a_{i\sigma(i)},$$

where $S_{2N}$ is the group of permutations of $2N$ elements and $\epsilon(\sigma) = 1$ if $\sigma$ is an even permutation and $\epsilon(\sigma) = -1$ if $\sigma$ is odd. Now each of the products above is going to be either $1$, $-1$ or $0$. If we can show that the number of products that is $1$ or $-1$ is odd, then this would mean the determinant will be odd, and thus the matrix will be invertible, as then the determinant can't be zero. So we want to find all the products that do not involve any element of the diagonal (which are zero) and count these. So in the first row we can pick $2N-1$ elements to start with, since we can't pick the diagonal element. So suppose we pick the element $a_{1\,i}$, where $i \neq 1$. Now we look at the row $i$, since that row has a zero in the $i$'th column. So from row $i$ we are still free to pick $2N-1$ elements. We repeat the argument with a row we have not yet picked from (likely row 2, unless $i$ was 2). From this row we can pick $2N-3$ elements, and then using the same argument we can pick $2N-3$ from yet another row. We continue until we get the entire product which is either $1$ or $-1$. So this means that the number of different products which are $1$ or $-1$ is

$$(2N-1)(2N-1)(2N-3)(2N-3)\cdots(3)(3)(1)(1),$$

and this is just an odd number. This means that the determinant of $A$ is odd, and thus not $0$, and this means that

$$nullity \begin{bmatrix} 0 & \beta \\ 1 & A \end{bmatrix} = 1,$$

which means that Pooh's honey pots better all be of the same weight.