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Problem: Let D be a nonempty, open, connected and relatively compact set in a metric space X with metric d. Prove that if f is a continuous map from D into D such that $f(D)$ is open, then there exists a point $x_0 \in D$ so that $d(x_0, \partial D) = d(f(x_0), \partial D)$.

We know that $d(x, \partial D)$ is a continuous map (easy to see by triangle inequality). Then $g(x) = d(f(x), \partial D) - d(x, \partial D)$ is a continuous map of D to $\mathbb{R}$. If we can show that $\exists z \in D$ s.t. $g(z) \geq 0$ and $\exists y \in D$ s.t. $g(y) \leq 0$, then $\exists x_0$ s.t. $d(f(x_0), \partial D) = d(x_0, \partial D)$ by the intermediate value theorem.

As $d(x, \partial D)$ is continuous and D is relatively compact which means that $\overline{D}$ is compact, then there exists a $z \in \overline{D}$ such that $d(z, \partial D)$ is maximal. Since D is non-empty and open then any point in D will have some positive distance to the boundary and thus this maximal z is in D. Then since $f(z) \in D$ we have that $d(f(z), \partial D) \leq d(z, \partial D)$ and so we have shown that $\exists z \in D$ s.t. $g(z) \leq 0$.

To prove that $\exists y$ s.t. $g(y) \geq 0$ we consider 2 cases. First let's suppose that $f(D) = D$. In this case we can use a similar argument as above to find y such that $d(f(y), \partial D)$ is maximal and then $g(y) \geq 0$.

Now suppose that $f(D) \subsetneq D$. This means that there exists a point $\alpha \in D$ s.t. $\alpha \in \partial f(D)$, this follows by connectedness of D. So take a sequence of $f(y_i)$ so that they converge to α. Since $\overline{D}$ is compact we can pick y_i such that this is a convergent sequence as well (we could always just pick a convergent subsequence) and we suppose that $y_i \to \beta$. Now β must be on the boundary of D, since if it wasn't, then $f(\beta) = \alpha$ by continuity and so α would be in $f(D)$, but it is also on the boundary of $f(D)$, and this is not possible since $f(D)$ is open.

Since $\alpha \in D$ it is a positive distance from the boundary since D is open, so say $d(\alpha, \partial D) = \delta$. So now pick a point y_j from $\{y_i\}$ such that $d(y_j, \partial D) < \frac{\delta}{2}$ and such that $d(f(y_j), \alpha) < \frac{\delta}{2}$. Then by triangle inequality we have $d(f(y_j), \partial D) \geq \frac{\delta}{2}$ and so $g(y_j) \geq 0$. QED!