ALGORITHMIC OBSTRUCTIONS AND ORDER-PRESERVING BRAIDS

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Abstract. An n-strand braid is order-preserving if its induced action on the fundamental group G of the disk with n punctures preserves some bi-ordering of G. A braid β is order-preserving if and only if the link L obtained as the union of β and its axis has bi-orderable complement. We describe and implement an algorithm which, given a braid β, terminates if β is *not* order-preserving. Additionally, our algorithm returns a complete proof that β is not order-preserving. The algorithm relies on the fact that the fundamental group of the n-punctured disk is a free group.

1. Introduction

A group G is called *bi-orderable* if there is a strict total ordering $<$ on G that is invariant under both left and right multiplication. Free groups are bi-orderable, and in fact have uncountably many distinct bi-orderings. Given a braid β in the n-strand braid group B_n, there is a specific action of $\beta \in B_n$ on F_n coming the induced action of β on the fundamental group of the n-punctured disk; see Section 2. We aim to classify braids via properties of this action with respect to bi-orders of the free group. In particular, if the action β preserves a given bi-ordering of F_n, the we say that β is *order-preserving*, or that β preserves a bi-order of F_n.

Question 1. Which n-strand braids preserve a bi-order of the free group of rank n?

Kin and Rolfsen resolved this question for periodic braids, and several families of non-periodic braids [KR18]. We produce a new infinte family of 3-braids which are *not* order-preserving.

Theorem 2. *The braids $\sigma_1\sigma_2^{2k+1}$ are not order-preserving for any integer k.*

We discovered this family, and the proof that the family is not order-preserving, with the help of our implementation of a new algorithm.

Algorithm. Given a braid which is *not* order-preserving, our algorithm can determine in finite time that the braid is indeed not order-preserving, and return a proof of this fact.

Our algorithm is inspired by an algorithm of Calegari and Dunfield which can be applied to any finitely presented group; if the group itself is not left-orderable, their algorithm will find an obstruction in finite time [CD03; Dun19].

The braid group B_n is isomorphic to the mapping class group of the punctured disk $\mathrm{Mod}(D_n)$. With this in mind braids can classified by their Nielsen-Thurston type as either periodic, pseudo-Anosov, or reducible. The periodic braids are know to be order-preserving by Kin-Rolfsen. A natural next class of 3-braids to consider are pseudo-Anosov braids.

Murasugi proved that every psuedo-Anosov 3-braid is conjugate to $h^d\sigma_1\sigma_2^{-a_1}\cdots\sigma_1\sigma_2^{-a_n}$ with $a_i \geq 0$ with at least one $a_i \neq 0$ [Mur74] where $h = (\sigma_1\sigma_2)^3$ is the full twist. Considering this classification, the simplest family of 3-braids to consider is $\sigma_1\sigma_2^{-k}$. We note that Kin-Rolsen showed that when $k = 1$ the braid is not order-preserving; our Theorem 2 extends this result to an infinite family when k is odd. One of our goals in creating and implementing the above algorithm is to increase examples of braids known to be not order-preserving, especially among pseudo-Anosov 3-braids. Theorem 2 is a concrete step towards this goal.

2020 *Mathematics Subject Classification.* 57K10, 57K20.

Key words and phrases. ordered groups, 3-manifold groups, braids .

1.1. **Motivation coming from 3-manifold topology.** A 3-manifold is *bi-orderable* when its fundamental group is a bi-orderable group. Orderability has played a significant role in studying when a 3-manifold is an *L-space*, that is, a manifold with the simplest possible Heegaard Floer homology, and when that manifold admits a geometric decomposition called a taut foliation. In particular, the L-space Conjecture posits that a closed irreducible 3-manifold is not an L-space, and admits a taut foliation, if and only if it is left-orderable [BGW13; Juh15]. It is another question altogether how bi-orderability relates to Floer theoretic or topological properties of a 3-manifold.

Question 3. Is there a topological characterization of 3-manifolds with bi-orderable fundamental group?

In the 1960's, Lickorish-Wallace proved that any closed, orientable, connected 3-manifold may be obtained by performing Dehn surgery on a link in S^3 [Lic62; Wal60]. Bi-orderability makes an appearance in this setting; a link, K, is said to be *bi-orderable* if $\pi_1(S^3 - K)$ is bi-orderable. Clay and Rolfsen show that no Dehn surgery on a bi-orderable <u>knot</u> produces an L-space [CR12], however this theorem is not true for <u>links</u> in general.

Problem 4. Classify bi-orderable links in S^3.

A *braided link* is the closure $\widehat{\beta}$ of an n-strand braid β together with the braid axis, as pictured in Figure 1a. Utilizing the structure of the braided link complement, the above classification problem can be reinterpreted as the algebraic Question 1 about the *action the braid group B_n on the free group F_n*. For a braid $\beta \in B_n$, the braided link $\widehat{\beta}$ is bi-orderable if and only if there is a bi-order on F_n that is preserved by the action of β [KR18].

1.2. **Organization of the paper.** In Section 2 we define the explicit action of a braid on the free group that we refer to throughout, and organize some background information about braids and bi-orderings of groups. Finding a bi-ordering of the free group preserved by a braid can be reduced to finding a certain order on zero-exponent sum words in the free group as we show in Section 3. We describe our algorithm and its implementation in Section 4. In Section 5, we prove Theorem 2.

1.3. **Acknowledgments.** This material is based upon work supported by the National Science Foundation under Grant No. DMS-1929284 while the three authors were in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the 'Braids' program.

We continued the collaboration through the Collaborate@ICERM program. We are very grateful to ICERM for making the collaboration possible and supporting our work. Part of this research was conducted using computational resources and services at the Center for Computation and Visualization, Brown University. The second author was partially supported by an AMS-Simons travel grant. The third author was also partially supported by Grant No. DMS-1745583.

2. Braids and orders

The braid group B_n embeds as a subgroup of the automorphism group $\mathrm{Aut}(F_n)$ of the free group, F_n via the following action where $F_n = \langle x_1, \cdots, x_n \rangle$.

$$\sigma_i \mapsto \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \mapsto x_j \end{cases} \tag{1}$$

This action comes from the identification of B_n with the mapping class group $\mathrm{Mod}(D_n)$ of the n-punctured disk. Thus the braid group acts on $\pi_1(D_n) \cong F_n$.

We interpret an n-strand braid β as an automorphism of F_n, which we read from *right to left* so that braids act on elements of F_n on the left. This also means that we read the action of σ_i on the punctured disk by tracing the paths of the strands as we flow up the braid so that the i^{th} puncture passes in front of the $(i+1)^{th}$ puncture. Our convention for the generators of $\pi_1(D_n)$ is that the basepoint is chosen along the bottom of the disk and the i^{th} generator loops once in a clockwise

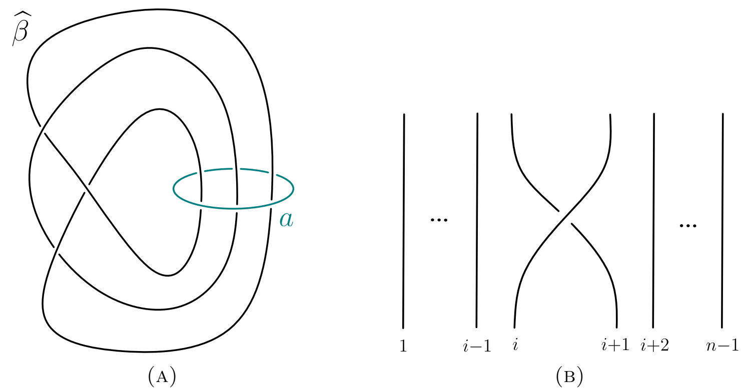

FIGURE 1. (A) The closure $\widehat{\beta}$ of of a 3-braid β together with its axis a forms a braided link. (B) The Artin generator σ_i.

direction around the i^{th} puncture. We note that the action of β on F_n with our convention is the inverse automorphism which Kin-Rolfsen consider for the same β.

Definition 5. P is a *positive cone* for a bi-ordering of F_n if

(1) $P \cdot P \subset P$,
(2) $F_n = P \sqcup P^{-1} \sqcup \{1\}$, and
(3) $gPg^{-1} = P$ for all $g \in F$.

A positive cone P determines a (bi-)order in the following way: say that $f < g$ if and only if $f^{-1}g \in P$. This definition automatically guarantees the order will be left-invariant. Condidition (1) assures transitivity, and (2) gives totality and strictness. Condition (3) is equivalent to $<$ being right-invariant. Thus defining $<$ in this way from P gives a bi-ordering of F_n. One can show that conversely, a bi-order on F_n determines a positive cone of F_n satisfying the three conditions.

Definition 6. An n-braid β is called *order-preserving* if there exists a positive cone P of F_n preserved by β. That is $\beta(P) = P$, set-wise.

Remark 7. Describing a positive cone is equivalent to describing a negative cone. Because of this, you can always assume your favorite element x of F_n is in the positive cone since $x \in P$ or $x \in -P$.

For each braid β we obtain a link in S^3 by taking the union of the closure $\hat{\beta}$ of β with the braid axis a; see Figure 1a.

Proposition 8 ([KR18])**.** The link $\hat{\beta} \cup a$ is bi-orderable if and only if the action β on F_n preserves some bi-ordering on F_n

2.1. **Precones.** Let $k \geq 1$, and let W_k be the set of reduced words in $x_1 \ldots, x_n$ of length less than of equal to k. We define a k-precone of F_n to be the part of a cone of F_n restricted to words of length k, as made precise in the following definition.

Definition 9. A subset P is a *k-precone* of F_n if

(1) $(P \cdot P) \cap W_k \subset P$,
(2) $W_k = P \sqcup P^{-1} \sqcup \{1\}$, and
(3) $(gPg^{-1}) \cap W_k \subset P$ for all $g \in W_k$.

Notice that a k-precone is not necessarily closed under multiplication (or conjugation) since many products of elements in the precone may be too long. Given a subset S of F_n there is an action of the braid on this subset which we denote by $\beta(S)$.

Definition 10. Given $k \in \mathbb{N}$, a k-precone is *preserved by* an automorphsim φ if $\varphi(P) \cap W_k \subset P_k$.

We point out that if P is a cone of F_n preserved by β, then $P \cap W_k$ for any positive integer k is a k-precone preserved by β by checking the definitions. There can be many different cones that have the same k-precone for a given k. The following proposition also asserts the converse statement.

Proposition 11. β preserves a positive cone of F if and only if β preserves a k-precone of F for every $k \in \mathbb{N}$.

Proposition 11 is the crucial result used in our algorithm. To show a braid β does not preserve a positive cone of F_n, it suffices to show that for some k, β does not preserve any k-precones of F. For any fixed k, there are a finite number of k-precones of F, each with finite cardinality. So Proposition 11 reduces the infinite problem to a finite problem – assuming the braid does not preserve any order. The following Lemma is needed to prove Proposition 11.

Lemma 12. *Suppose that for each positive integer k, we have a k-precone P_k of F_n preserved by β. If $P_k \subset P_l$ for all $k \leq l$, then $P = \bigcup P_k$ is a positive cone of F preserved by β.*

Proof. To show that P is positive cone preserved by β we need to check the three conditions of Definition 5, and finally to check that β preserves P.

To do these checks, it will be convenient first to show that $P \cap W_k = P_k$. For each k we certainly have that $P_k \subset P \cap W_k$. Now suppose $x \in P \cap W_k$ for some k. Since $x \in P$, $x \in P_l$ for some l. If $l \leq k$ then $x \in P_l \subset P_k$. Suppose $l > k$. Since $x \in W_k$, either $x \in P_k$, $x \in P_k^{-1}$, or $x = 1$. Since $x \in P_l$, $x \neq 1$. Also, since $P_k^{-1} \subset P_l^{-1}$ and $x \in P_l$, x cannot be in P_l^{-1}, nor can x be in P_k^{-1}. Thus, we must have $x \in P_k$ so $P \cap W_k \subset P_k$.

Condition (1): Suppose $a, b \in P$. For some large enough k, we must have a, b, and $ab \in W_k$. Since $P_k = P \cap W_k$, $a, b \in P_k$. Since P_k is a precone, $ab \in P_k \subset P$ as desired.

Condition (2): Since $P = \cup P_k$ we also have that $P^{-1} = \cup P_k^{-1}$; we claim that $P \sqcup P^{-1} \sqcup \{1\} = F_n$. Suppose $g \in F_n$ so $g \in W_k$ for some k. Thus, either we have that $g \in P_k \subset P$, or $g \in P_k^{-1} \subset P^{-1}$, or $g = 1$. In any case, g is in $P \sqcup P^{-1} \sqcup \{1\}$ and hence we have that $F_n = P \sqcup P^{-1} \sqcup \{1\}$.

Condition (3): Suppose $g \in F_n$ and $x \in P$. For some k,we have that $x, gxg^{-1} \in W_k$. Since $P_k = P \cap W_k$ and is a precone, we also have that $gxg^{-1} \in P_k \subset P$.

Preserved by β: Suppose $x \in P$. For some k, we have that $x, \beta(x) \in W_k$,. Since $P_k P \cap W_k$ and is a precone, we have that $\beta(x) \in P_k \subset P$. □

Lemma 13. *Suppose X is a compact space with a countable family $\mathcal{C}$ of closed nested subsets. If each $C \in \mathcal{C}$ is nonempty then the intersection of all $C \in \mathcal{C}$ is also nonempty.*

Proof. This follows from Theorem 26.9 of Munkres. □

Proof of Proposition 11. As noted in the discussion before Proposition 11, if β preserves a positive cone P, then $P \cap W_k$ is a k-precone preserved by β.

Now suppose that for each positive k, the braid β preserves k-precone for each k which are nested. Consider 2^F, the powerset of F. Each $A \in 2^F$ is identified with an indicator function $f_A : F \to \{0, 1\}$ defined as follows.

$$f_A(g) = \begin{cases} 1 & g \in A \\ 0 & g \notin A \end{cases}$$

2^F can be given a topology by identifying it with the product topology on $\{0, 1\}^F$. Here we use the discrete topology on $\{0, 1\}$.

Given an element $g \in F$, define U_g to be the collection of subsets of F which contain g so $U_g^c = 2^F - U_g$ is the collection of sets which do not contain g. For each g, U_g and U_g^c are open in 2^F.

For each k, define $\mathcal{S}_k \subset 2^F$ be the collection of all subsets $A \subset F$ such that $A \cap W_k$ is a k-precone of F preserved by β. This is a nested family as follows.

$$\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \cdots$$

Since β preserves a k-precone for each k, each $\mathcal{S}_k$ is nonempty.

Claim: Each $\mathcal{S}_k$ is a closed subset of 2^K.

Consider a "point" S in $\mathcal{S}_k^c = 2^F - \mathcal{S}_k$; then S is a subset of F such that $S \cap W_k$ is not a k-precone. Since W_k is finite, the set

$$U_S = \Big(\bigcap_{g \in S \cap W_k} U_g \Big) \cap \Big(\bigcap_{g \notin S^c \cap W_k} U_g^c \Big)$$

is open in 2^F. Note that a set A is in the collection U if and only if $A \cap W_k = S \cap W_k$. It follows that $S \in U_S \subset \mathcal{S}_k^c$ for each set $S \in \mathcal{S}_k^c$. Thus, $\mathcal{S}_k^c$ is $\bigcup_{S \in \mathcal{S}_k^c} U_S$. Since each U_S is open, $\mathcal{S}_k^c$ is open. Therefore, $\mathcal{S}_k$ is closed.

Since the discrete topology on $\{0,1\}$ is compact, 2^F is compact by the Tychonoff theorem. Since each $\mathcal{S}_k$ is closed, $\bigcap \mathcal{S}_k$ is also nonempty by Lemma 13.

Let $P \in \bigcap \mathcal{S}_k$, and let $P_k = P \cap W_k$ for each $k \in \mathbb{Z}_+$. Thus, $P = \bigcup P_k$, and when $k \leq l$, $P_k \subset P_l$. Since $P \in \mathcal{S}_k$ for each k, each P_k is a k-precone preserved by β. Therefore P is a positive cone of F preserved by β by Lemma 12. □

3. Words with zero exponent sum

In order to improve the efficiency of our algorithm, we would like to minimize the number of k-precones we need to consider. Towards this end, we focus on words in the free group with zero exponent sum, a subgroup which we call K_0. We prove the following general proposition which will imply that certain bi-orders of K_0 preserved by β are related to bi-orders on the free group preserved by β.

Proposition 14. Suppose we have the short exact sequence where A is an abelian group.

$$1 \to K \to F_n \overset{\phi}{\to} A \to 1$$

Let β be a braid (or more generally any automorphism of F_n) with the property $\beta(K) = K$. A positive cone P_K of K that is preserved by the action of β and invariant under conjugation by elements of F_n exists if and only if a conjugate invariant positive cone P_F of F_n preserved by β exists.

Proof. ($\Rightarrow$) Given P_K, the cone P_F we seek is

$$P_F := \{x \in F_n \,|\, \phi(x) \in P_A, \text{ or } \phi(x) = 0 \text{ with } x \in P_K\}$$

where P_A is the conjugate invariant positive cone of A that is preserved by the ϕ-induced action of β.

P_F **is invariant under action of** β: Let $x \in P_F$. If $x \notin K$, then $\phi(x) \in P_A$. $\phi(\beta(x)) = \beta_\phi(x) \in P_A$ since P_A is closed under the action of β, so $\beta(x) \in P_F$. If $x \in K$, then $x \in P_K$. Since P_K is closed under the action of β, then $\beta(x) \in P_K$ and $\beta(x) \in P_F$.

P_F **is conjugation invariant**: Let $g \in F_n$ and supposed $x \in P_F$. If $x \in K$, or $x \in P_K$, then by assumption $gxg^{-1} \in P_K$ (note that $gxg^{-1} \in K$ as kernels are normal). If $x \notin K$, then $\phi(x) \neq 0 \in P_A$ and since A is abelian $\phi(gxg^{-1}) = \phi(x) \in P_A$.

($\Leftarrow$) Suppose that we have a a conjugate invariant positive cone P_F of F_n preserved by β. Then we define $P_K := P_F \cap K$. Since K and P are both conjugate invariant in F_n, P_K is a positive cone of K that is closed under conjugation by F_n. Since K is preserved by β by assumption, then P_K is preserved by β. □

Let $t : F_n \to \mathbb{Z}$ be the exponent sum map. Taking $K = K_0$ and ϕ to be t, the following Corollary and Theorem follow directly from Proposition 14 and Lemma 15.

Lemma 15. For any braid β, we have that $\beta(K_0) = K_0$.

Proof. We see from equation (1) that the action of a generator σ_i preserves the exponent sum of a word in F_n. It follows that for any braid β, we have that $t(\beta(w)) = t(w)$ for $w \in F_n$. Since $K_0 = \ker t$, we have $\beta(K_0) = K_0$. □

Corollary 16. *A braid β preserves a bi-order of F_n if and only if it preserves a bi-order on K_0, the subgroup of zero exponent sum elements, which is conjugation invariant under elements of F_n.*

Theorem 17. *An n-braid β preserves a positive cone of F_n if and only if β preserves a positive cone P' where any word in F_n with positive exponent sum is in P'.*

In principle, this means that we can seed our k-precones immediately with all words of positive exponent sum. In reality, our algorithm instead searches for intersections of k-precones with K_0 which are preserved by β and are not only conjugate invariant in K_0, but in the larger group F_n, as in the following definition.

Definition 18. A subset Q of K_0 is a *k-zerocone* of F_n if

(1) $(Q \cdot Q) \cap W_k \subset Q$,
(2) $Q \sqcup Q^{-1} \sqcup \{1\} = W_k \cap K_0$, and
(3) $(gQg^{-1}) \cap W_k \subset Q$ for all $g \in W_k$.

Notice that the conjugation in condition (3) is by all elements of W_k, not just the ones in K_0. Now, to relate k-zerocones to k-precones of F_n, we define the set $\mathrm{Pos}_k(Q)$ obtained by adding to Q all positive exponent sum words of length at most k as follows,

$$\mathrm{Pos}_k(Q) := Q \cup \Big(W_k \cap t^{-1}(\mathbb{Z}^+)\Big),$$

where $t : F_n \to \mathbb{Z}$ is the exponent sum map.

Lemma 19. (a) If P is a k-precone, then the intersection $P \cap K_0$ is a k-zerocone.
(b) Suppose Q is a k-zerocone. The set $\mathrm{Pos}_k(Q)$ is a k-precone.
(c) In particular, the set of k-zerocones of K_0 are precisely the set of intersections of k-precones of F_n and K_0.

Proof. (a) Suppose P is a k-precone of F_n. We show that $Q = P \cap K_0$ is a k-zerocone.

Condition (1): Suppose $a, b \in Q$. Then when $ab \in W_k$, we have that $ab \in P$. Thus since K_0 is a subgroup of F_n, $ab \in P \cap K_0 = Q$.

Condition (2):

$$\begin{aligned} W_k \cap K_0 =& \Big(P \sqcup P^{-1} \sqcup \{1\}\Big) \cap K_0 \\ =& \Big(P \cap K_0\Big) \sqcup \Big(P^{-1} \cap K_0\Big) \sqcup \{1\} \\ =& Q \sqcup Q^{-1} \sqcup \{1\} \end{aligned}$$

Condition (3): Suppose $g \in W_k$ and $x \in Q$. When $gxg^{-1} \in W_k$, we have that $gxg^{-1} \in P$. Thus since K_0 is normal, $gxg^{-1} \in P \cap K_0 = Q$.

(b) For the second statement, we show that $\mathrm{Pos}_k(Q)$ is a k-precone.

Condition (1): Suppose $a, b \in \mathrm{Pos}_k(Q)$. If a and b are both in Q then by definition of a k-zerocone, when $ab \in W_k$, we have that $ab \in Q \subset \mathrm{Pos}_k(Q)$. If either a or b is not in Q, then since $Q \subset K_0$, and $K_0 = \ker t$, we have $t(ab) > 0$. Thus, when $ab \in W_k$, we have that $ab \in W_k \cap t^{-1}(\mathbb{Z}^+) \subset \mathrm{Pos}_k(Q)$.

Condition (2): First, we note that

$$\mathrm{Pos}_k(Q)^{-1} = Q^{-1} \cup \Big(W_k \cap t^{-1}(\mathbb{Z}^-)\Big).$$

For every element x in W_k, exactly one of $t(x) > 0$, $t(x) < 0$, or $x \in K_0$ is true. Thus, we have that

$$\begin{aligned} W_k =& \Big(W_k \cap t^{-1}(\mathbb{Z}^+)\Big) \sqcup \Big(W_k \cap K_0\Big) \sqcup \Big(W_k \cap t^{-1}(\mathbb{Z}^-)\Big) \\ =& \Big(W_k \cap t^{-1}(\mathbb{Z}^+)\Big) \sqcup Q \sqcup \{1\} \sqcup Q^{-1} \sqcup \Big(W_k \cap t^{-1}(\mathbb{Z}^-)\Big) \\ =& \mathrm{Pos}_k(Q) \sqcup \{1\} \sqcup \mathrm{Pos}_k(Q)^{-1} \end{aligned}$$

Condition (3): Suppose $g \in W_k$ and $x \in \mathrm{Pos}_k(Q)$. If $x \in Q$ then when by definition of a k-zerocone, when $gxg^{-1} \in W_k$, we have that $gxg^{-1} \in Q \subset \mathrm{Pos}_k(Q)$. If $x \in W_k \cap t^{-1}(\mathbb{Z}^+)$ then $t(gxg^{-1}) = t(x) > 0$. Thus, when $gxg^{-1} \in W_k$, we have that $gxg^{-1} \in W_k \cap t^{-1}(\mathbb{Z}^+) \subset \mathrm{Pos}_k(Q)$.

(c) For the final statement, we have already shown that an intersection of a k-precone and K_0 is a k-zerocone by Part (a). For the other inclusion, we have that any k-zerocone Q is the intersection $\mathrm{Pos}_k(Q) \cap K_0$ by definition of $\mathrm{Pos}_k(Q)$. □

Definition 20. We say a k-zerocone Q_k of K_0 is *preserved by* an automorphsim φ of F_n if $\varphi(Q_k) \cap W_k \subset Q_k$.

Lemma 21. For each $k \in \mathbb{N}$, the braid β preserves a k-precone of F_n if and only if β preserves a k-zerocone of K_0.

Proof. Suppose β preserves a k-precone P_k of F_n. Then, $Q_k = P_k \cap K_0$ is a k-zerocone by Lemma 19. Since β is a bijection, then $\beta(Q_k) = \beta(P_k \cap K_0) = \beta(P_k) \cap \beta(K_0) = \beta(P-k) \cap K_0$. Using the fact that β preserves P_k, we have $\beta(P_k) \cap W_k \subset P_k$, and so

$$\beta(Q_k) \cap W_k = W_k \cap \beta(P_k) \cap K_0 \subset P_k \cap K_0 = Q_k.$$

Conversely, suppose β preserves a k-zerocone Q_k of K_0 so $\beta(Q_k) \cap W_k \subset Q_k$. Define the k-precone P_k as in Lemma 19 as follows.

$$P_k = Q_k \cup \Big(W_k \cap t^{-1}(\mathbb{Z}^+)\Big)$$

Since β is bijective and doesn't affect exponent sum,

$$\begin{aligned}
\beta(P_k) \cap W_k =& \Big[\beta(Q_k) \cup \Big(\beta(W_k) \cap t^{-1}(\mathbb{Z}^+)\Big)\Big] \cap W_k \\
=& \Big(\beta(Q_k) \cap W_k\Big) \cup \Big(\beta(W_k) \cap t^{-1}(\mathbb{Z}^+) \cap W_k\Big) \\
\subset& Q_k \cup \Big(t^{-1}(\mathbb{Z}^+) \cap W_k\Big) = P_k.
\end{aligned}$$

□

Combining Proposition 11 and Lemma 21, we can detect non-order-preserving braids by obstructing k-zerocones of K_0 by the following proposition.

Proposition 22. The braid β is not order-preserving if and only if for some integer k we have that β does not preserve any k-zerocone of K_0.

4. Algorithms

Calegari and Dunfield described a theoretical algorithm for deciding the left-orderability of a group [CD03, Section 8]. For a finitely presented group G with a solution to the word problem, when G is not left-orderable group, their algorithm produces an obstruction to left-orderability in finite time. When G is left-orderable, their algorithm does not halt. Dunfield implemented this algorithm to obstruct left-orderability for many hyperbolic 3-manifold groups [Dun19]. His implementation uses a $PSL(2, \mathbb{C})$ representation to solve the word problem.

Taking inspiration from Calegari and Dunfield's work, we describe and implement an algorithm to answer the following question.

Question 23. Suppose β is an n-strand braid, and let k be a positive integer. Does β preserve a k-precone of F_n in the sense of Definition 10?

By Proposition 11, a braid β is order-preserving if and only if the answer to Question 23 is "yes" for every positive integer k. The recursive algorithm PreservePreCone(P) defined in Algorithm 24 returns True or False when the answer to Question 23 is "yes" or "no" respectively. Note that since we are working with the free group, we can solve the word problem by greedy reduction.

To understand Algorithm 24, recall that W_k is the set of words in $F_n = \langle x_1, \dots, x_n \rangle$ with word length k or less, as defined in Section 2.1. Given a automorphism $f \in \mathrm{Aut(F_n)}$, and a set $P \subset F_n$, define $S_f(P)$ as follows:

$$S_f(P) := (P \cdot P) \cup \Big(\bigcup_{i=1}^{n} \{x_i P x_i^{-1}\}\Big) \cup f(P) \cup f^{-1}(P).$$

When applied to a set P, the operation S_f returns a set that is not closed, but closer to being closed under multiplication, conjugation, and the actions of f and f^{-1}. The foundation of Algorithm 24 is recursive applications of S_f, when $f = \beta$ is an n-strand braid.

Algorithm 24. PreservePreCone(P)

> **while** $S_\beta(P) \cap W_k \not\subset P$ **do**
> $P := S_\beta(P) \cap W_k$
> **if** $1 \in P$ **then**
>   **return** False
> **if** $P \cup P^{-1} \cup \{1\} = W_k$ **then**
>   **return** True
> $g :=$ shortest word in $W_k - (P \cup P^{-1} \cup \{1\})$
> **return** PreservePreCone($P \cup \{g\}$) **or** PreservePreCone($P \cup \{g^{-1}\}$)

When executing Algorithm 24, every element of W_k must be placed in P at least once either by the function S_β or during the recursive branching step. Since the number of words in F with length k is $6 \cdot 5^{k-1}$, the time complexity of Algorithm 24 is at least exponential in k. When implemented, this algorithm does not complete in a reasonable time for $k > 6$.

When β maps short words to significantly longer words, it is easy to find a k-precone P_k of the free group where $\beta(P_k) \cap W_k$ is small. (For example, the braid $\sigma_1^3\sigma_2^{-3}\sigma_1$ maps x_2 to a word of length 21, and you wouldn't see much of the braid action in W_k until $k = 21$ or higher.) In this case, when β is non-order-preserving, k must be large for the answer to Question 23 to be "no". This means that in practice, Algorithm 24 is not so useful for obstructing order-preservingness of most braids.

In our implementation, we make several modifications to improve the effectiveness of obstructing order-preservingness. First, instead of using the action of β and β^{-1}, we use automorphisms $b = \psi_1 \circ \beta$ and $b' = \psi_2 \circ \beta^{-1}$ where ψ_1 and ψ_2 are inner automorphisms. The automorphisms ψ_1 and ψ_2 are chosen to minimize the longest possible length of the images $b(w)$ and $b'(w)$. Lemma 25 describes that composition with an inner automorphism does not change a preserved positive cone. So the choices of ψ_1 and ψ_2 only help us to find a contradiction sooner (for smaller k) by changing the order in which elements are added to a precone.

Lemma 25. *A positive cone P is preserved by β if and only if P is preserved by $\phi \circ \beta$ for ϕ an inner automorphism of F_n.*

Proof. Suppose P is a positive cone. Since P is conjugate invariant, any inner automorphism will preserve P. Thus, for any braid β and inner automorphism ϕ, we have that $\beta(P) = P$ if and only if $\phi(\beta(P)) = P$. □

Second, in light of Corollary 16, we only add words with exponent sum zero to our prospective k-precone. While this doesn't change the time complexity of the algorithm, it significantly reduce the number of words we need to consider. To do this, instead of seeding our prospective precone with shortest elements in W_k, we seed with words in Z_k, the subset of words in W_k with zero exponent sum.

Finally, instead of restricting ourselves to working with words at most length k, we allow our algorithm to "remember" words of longer length without using these extra elements in the computation to $S_b(P)$. This means for a given k our algorithm will find contradictions for preserving larger precones without having to perform extra computations.

After these modifications we get Algorithm 26.

Algorithm 26. ModPreservePreCone(P,E)

> **while** $S(P) \cap Z_k \not\subset P$ **do**
> $P_* := S_f(P) \cap Z_k$
> $E := E \cup (S_f(P) - P_*)$ ▷ Tracking extra elements
> **if** $1 \in P_* \cup E$ **then**
>   **return** False
> **if** $P_* \cup P_*^{-1} \cup \{1\} = Z_k$ **then**
>   **return** True
> $g :=$ shortest word in $Z_k - (P_* \cup P_*^{-1} \cup \{1\})$

```
preserve_order_obstruct([1,-2,-2,-2],4,track_extra_elements =True, tree=True)
```

```
0
    parent
        start
    node element
        x1^-1*x2

1
    parent
        0
    node element
        x2^-1*x3
    Proof info
        ['x3^-1*x2^-1*x1*x2=b(conj([x2^-1*x3] by x3^-1))', 'x2^-1*x1^-1*x2*x3=conj([x1^-1*x
2] by x2^-1) * x2^-1*x3']

2
    parent
        0
    node element
        x3^-1*x2
    Proof info
        ['x2^-2*x1*x2*x3*x2^-1=conj([b(conj([conj([x1^-1*x2] by x3) * b(x3^-1*x2)] by x2))]
by x2)', 'x2*x3^-1*x2^-1*x1^-1*x2^2=conj([x3^-1*x2] by x2) * conj([x1^-1*x2] by x2^-1)']

False
```

FIGURE 2. Output of the implementation of Algorithm 26 applied to the braid $\sigma_1\sigma_2^{-3}$ with word length restriction to $k = 4$.

return ModPresPreCone($P \cup \{g\}$,E) **or** ModPresPreCone($P \cup \{g^{-1}\}$,E)

If ModPreservePreCone($\{x_1^{-1}x_2\}$,$\emptyset$) returns True, if the answer to Question 23 is "yes". When ModPreservePreCone($\{x_1^{-1}x_2\}$,$\emptyset$) returns False, we can't conclude that answer to Question 23 is "no" for k, but we can conclude that answer to Question 23 is "no" for some positive integer by Proposition 22. More importantly, the same proposition implies that when the algorithm ModPreservePreCone($\{x_1^{-1}x_2\}, \emptyset$) returns False, the braid β is not order-preserving.

4.1. **Implemented outputs.** Algorithms 24 and 26 are implemented in SageMath and Python which will be available on Github [JST23]. When the implementation of Algorithm 26 returns False, it also returns a proof that the braid is not order-preserving. Figure 2 shows the output the of implemented Algorithm 26 applied to the braid $\sigma_1\sigma_2^{-3}$. The proof output is a binary tree showing attempts to build precones. Each child node is an attempt to add the node element to the precone. If the attempt was successful, there will be no proof information. If the attempt was unsuccessful, the proof info will output two elements of the attempted precone that are inverses, as well as instructions for how the elements were added to the precone. The algorithm proved that $\sigma_1\sigma_2^{-3}$ is not order-preserving, which is a new result.

Proposition 27. The braid $\sigma_1\sigma_2^{-3}$ is not order-preserving.

Proof. Figure 2 shows the output the of implemented Algorithm 26 applied to the braid $\sigma_1\sigma_2^{-3}$ with word length restriction to $k = 4$. Diagrammatically, the binary tree for $\sigma_1\sigma_2^{-3}$ is depicted below.

seeding element: $y = x_1^{-1}x_2$

node element: $\alpha = x_2^{-1}x_3$	node element: $\alpha^{-1} = (x_2^{-1}x_3)^{-1}$
Contradiction elements:	Contradiction elements:
$x_3^{-1}x_2^{-1}x_1x_2 = \beta(x_3^{-1}\alpha x_3)$	$x_2^{-2}x_1x_2x_3x_2^{-1} = x_2\beta(x_2x_3yx_3^{-1}\beta(\alpha^{-1})x_2^{-1})x_2^{-1}$
$(x_3^{-1}x_2^{-1}x_1x_2)^{-1} = \alpha x_3^{-1}yx_3^{-1}$	$(x_2^{-2}x_1x_2x_3x_2^{-1})^{-1} = (x_2\alpha^{-1}x_2^{-1})(x_2^{-1}\alpha x_2)$

This information is interpreted as a proof by supposing $\sigma_1\sigma_2^{-3}$ preserves a precone P. As stated in Remark 7, we may assume $x_1^{-1}x_2$ is an element of P (this is the seeding element). If $\alpha = x_2^{-1}x_3 \in P$ (one node element) then the element $x_3^{-1}x_2^{-1}x_1x_2$ and its inverse (contradiction elements) are both in P, which is a contradiction. If $\alpha^{-1} = (x_2^{-1}x_3)^{-1} \in P$ (the other node element) then the element $x_2^{-2}x_1x_2x_3x_2^{-1}$ and its inverse are both in P, which is a contradiction. Thus no such P can exists and $\sigma_1\sigma_2^{-3}$ is not order-preserving. □

In Section 5, we show how to mimic and generalize this computer generated proof to a proof that all $\sigma_1\sigma_2^{2k+1}$ are not order-preserving.

5. A Family of Non-Order-Preserving Braids

We prove that the braids $\sigma_1\sigma_2^{2k+1}$ are not order-preserving. Our proof is based off the computer generated proof resulting from the implemented Algorithm 26 applied to the braid $\sigma_1\sigma_2^{-3}$, as discussed in Section 4.1 and shown in Figure 2.

Theorem 2. *The braids $\sigma_1\sigma_2^{2k+1}$ are not order-preserving for any integer k.*

Proof. Let $\psi \in \mathrm{Inn}(\mathrm{F}_3)$ be conjugation by w^{-k} where $w = x_2^{-1}x_1x_2x_3$, let f be the automorphism $\psi \circ \beta$ in $\mathrm{Aut}(\mathrm{F}_3)$ which is defined by the following action.

$$x_1 \longmapsto w^{-k}x_2w^k \qquad x_2 \longmapsto x_2^{-1}x_1x_2x_3x_2^{-1}x_1^{-1}x_2$$
$$x_3 \longmapsto x_2^{-1}x_1x_2$$

The automorphism f and the braid β preserve the same bi-orders of F_3 by Lemma 25.

Suppose P is a positive cone of F_3 preserved by f. We may assume without loss of generality that $x_1^{-1}x_2 \in P$.

Now, either $x_2^{-1}x_3$ or $x_3^{-1}x_2$ must be in P. Suppose first that $x_2^{-1}x_3 \in P$. Then

$$f(x_2^{-1}x_3) = x_2^{-1}x_1x_2x_3^{-1} \in P.$$

Additionally, we have that $x_3x_2^{-1}$ is in P by conjugating $x_2^{-1}x_3$. However, since $x_1^{-1}x_2$ is also in P, we have that $x_1^{-1}x_2f(x_2^{-1}x_3)x_3x_2^{-1} = 1 \in P$ which is a contradiction.

On the other hand, suppose that that $x_3^{-1}x_2 \in P$. Then, since $x_1^{-1}x_2 \in P$,

$$x_3(x_1^{-1}x_2)x_3^{-1} \cdot f(x_3^{-1}x_2) = x_3x_1^{-2}x_2 \in P.$$

Since,

$$\begin{aligned} f(x_3x_1^{-2}x_2) =& x_2^{-1}x_1x_2w^{-k}x_2^{-2}w^kx_2^{-1}x_1x_2x_3x_2^{-1}x_1^{-1}x_2 \\ =& x_2^{-1}x_1x_2w^{-k}x_2^{-2}w^{k+1}x_2^{-1}x_1^{-1}x_2, \end{aligned}$$

we have that

$$x_2^{-2}w \in P$$

after conjugating by $x_2^{-1}x_1x_2w^{-k}$.

However, since $x_1^{-1}x_2$ and $x_3^{-1}x_2$ are in P, the elements $x_2^{-1}x_1$ and $x_3x_2^{-1}$ are in P^{-1}. Thus,

$$x_2^{-1}\left[x_2^{-1}(x_2^{-1}x_1)x_2 \cdot x_3x_2^{-1}\right]x_2 = x_2^{-2}w \in P^{-1}$$

which is a contradiction. □

Once a braid is known to be order-preserving, or not order-preserving, there are many relations to create new examples from the known examples. Many of of these relations follow from basic order-preserving theory, or are proved in [KR18].

A braid $\beta \in B_n$ is order-preserving if and only if

- any integer power of β is order-preserving, including β^{-1},
- $\beta\Delta_n^{2k}$ is order-preserving for ever $k \in \mathbb{Z}$, where Δ_n is the half twist,
- any conjugate of β is order-preserving (this includes any cyclic re-ordering of the braid word for β),
- the braid tensor $\beta \otimes \alpha$ is order-preserving, where $\alpha \in B_m$ is an order-preserving braid, See Section 4.7 of [KR18],

- the mirror of β is order-preserving.

Applying these relations to $\sigma_1\sigma_2^{2k+1}$ gives several more examples of non-order-preserving braids.

Corollary 28. For all $k, \ell \in \mathbb{Z}$ with $\ell \neq 0$, the following braids are not order-preserving.

a.) $(\sigma_1^{\pm 1}\sigma_2^{2k+1})^\ell$
b.) $\sigma_1^{\pm 1}\sigma_2^{2k+1}\Delta_3^{2\ell}$
c.) $(\sigma_2^{2k+1}\sigma_1^{\pm 1})^\ell$
d.) $(\sigma_1^{\pm 1}\sigma_2^{2k+1}) \otimes \alpha$ for any braid $\alpha \in B_\ell$

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