Exercise Set #3

Exercise 1: Suppose $Q$ is an integral form. Show that the following are equivalent:

(i) $Q$ is even.

(ii) Every matrix representation of $Q$ has diagonal with all even entries.

(iii) At least one matrix representation of $Q$ has diagonal with all even entries.

Exercise 2: Compute $\text{sign}(E_8)$.

$$E_8 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

Exercise 3: Prove that any closed oriented simply-connected 4-manifold with even intersection form and vanishing signature is homeomorphic to $S^4$ or a connected sum of some number of $S^2 \times S^2$.

Exercise 4:

a) Given a bilinear integral form $Q : L \times L \to \mathbb{Z}$, define $F : L \to L^*$ by $F(x) = F_x$ where $F_x(y) = Q(x, y)$. Show that $Q$ is unimodular if and only if $F$ is an isomorphism.

b) Show that if $M$ is a closed simply-connected orientable 4-manifold, then the intersection form $Q_M$ is unimodular. (Hint: Use Poincaré duality.)

Exercise 5: Let $U_1$ and $U_2$ be disjoint (not necessarily unlinked) framed unknots in the boundary of a 0-handle, $B$, with framings $n_1$ and $n_2$. Let $M$ be the 4-manifold obtained by attaching 2-handles to $B$ along $U_1$ and $U_2$. For $i = 1, 2$, let $F_i$ be the sphere obtained by pushing the interior of a disk in $\partial B$ bound by $U_i$ into the interior of $B$ and capping of the disk with the core of the attached 2-handle. Compute $F_1 \cdot F_2$ and $F_1 \cdot F_1$.

Exercise 6: Let $M$ be a 4-dimensional manifold constructed by attaching 1- and 2-handles to a 0-handle.

a) Let $M'$ be the manifold obtained by attaching a zero framed 2-handle to the belt sphere of one the 2-handles of $M$. Show that $\pi_1(M) \cong \pi_1(M')$.

b) Find a handle decomposition of the double of $M$ which is $DM := M \cup_{id_M} \overline{M}$.

c) Let $G$ be a finitely presented group. Find a closed orientable 4-manifold $M$ with $\pi_1(M) \cong G$. 