Heegaard Floer Homology Exercise Set #3

Exercise 1: Show that the following is a short exact sequence of chain complexes.

\[ 0 \longrightarrow \widehat{CF}(\Sigma, \alpha, \beta, z, s) \overset{\iota}{\longrightarrow} CF^+(\Sigma, \alpha, \beta, z, s) \overset{U}{\longrightarrow} CF^+(\Sigma, \beta, \alpha, z, s) \longrightarrow 0 \]

Here \( \iota(x) = [x, 0] \).

Exercise 2: Recall that \( U \) is defined as an isomorphism of \( CF^{\infty} \). This restricts to a map \( U^- \) in \( CF^- \) and induces a map \( U^+ \) on \( CF^+ \). Furthermore, these maps induce \( U_* \), \( U^-_* \), and \( U^+_* \) on their respective homologies. (In practice, abusing notation all of these maps are referred to as \( U \).)

1. Which of these maps are always injective? ...surjective?
2. Recall the long exact sequence

\[ \cdots \longrightarrow HF^-(Y, s) \overset{i_*}{\longrightarrow} HF^{\infty}(Y, s) \overset{\pi_*}{\longrightarrow} HF^+(Y, s) \longrightarrow \cdots \]

Show that \( \ker((U_*^-)^k) = \ker(i_*) \) and that \( \text{im}((U_*^+)^k) = \text{im}(\pi_*) \).
3. Prove that \( HF_{red}^-(Y, s) \cong HF_{red}^+(Y, s) \).

Exercise 3: Prove that \( Y \) is an L-space if and only if \( \text{rk} \, HF^-(Y) = |H_1(Y; \mathbb{Z})| \).

Exercise 4: Consider the \( Y \) from last time (Heegaard diagram pictured below).

Let \( s = s_z(x_1y_1) \). Draw the complexes \( CF^-(\Sigma, \alpha, \beta, z, s) \) and \( CF^+(\Sigma, \alpha, \beta, z, s) \).

![Heegaard diagram example](image.png)