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A new solution representation for the BBM equation in a quarter plane and the eventual periodicity

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Abstract

The initial-boundary-value problem for the Benjamin–Bona–Mahony (BBM) equation is studied in this paper. The goal is to understand the periodic behaviour (termed as eventual periodicity) of its solutions corresponding to periodic boundary condition or periodic forcing. Towards this end, we derive a new formula representing solutions of this initial- and boundary-value problem by inverting the operator $\partial_t + \alpha\partial_x - \gamma\partial_{xxt}$ defined in the space–time quarter plane. The eventual periodicity of the linearized BBM equation with periodic boundary data and forcing term is established by combining this new representation formula and the method of stationary phase. The eventual periodicity of the full BBM equation is obtained under a suitable assumption imposed on its solution.

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1. Introduction

This paper is concerned with the initial- and boundary-value problem (IBVP) for the Benjamin–Bona–Mahony (BBM) equation in the quarter plane $\{(x,t), x \geqslant 0, t \geqslant 0\}$,

$$\begin{cases} u_t + \alpha u_x + \beta u u_x - \gamma u_{xxt} = f, & x \geqslant 0,\ t \geqslant 0, \\ u(x,0) = u_0(x), & x \geqslant 0, \\ u(0,t) = g(t), & t \geqslant 0, \end{cases} \tag{1.1}$$

where $\gamma > 0$, α and β are real parameters. Our goal is two-fold: first, to establish a new representation formula for (1.1) by inverting the operator $\partial_t + \alpha\partial_x - \gamma\partial_{xxt}$; second, to understand whether the solution of (1.1) exhibits certain time-periodic behaviour if the boundary data g and the forcing term f are periodic.

The study on the large-time periodic behaviour is partially motivated by a laboratory experiment involving water waves generated by a wavemaker mounted at the end of a water channel. It is observed that if the wavemaker is oscillated periodically, say with a long period T_0, it appears that in due course, at any fixed station down the channel, the wave elevation becomes periodic of period T_0. Professor Jerry L Bona proposed the problem of establishing this observation as a mathematically exact fact about solutions of the suitable model equations for water waves. One goal of this paper is to determine whether the solution u of (1.1) exhibits eventual periodicity. More precisely, we investigate whether the difference

$$u(x, t + T_0) - u(x, t) \tag{1.2}$$

approaches zero as $t \to \infty$ if g and f are periodic of period T_0. Since the solution of (1.1) grows in time (measured in the norm of Sobolev spaces H^k with $k \geqslant 0$)(see [7]), the issue of eventual periodicity appears to be extremely difficult.

The eventual periodicity has previously been studied in several works. In [7], Bona and Wu thoroughly investigated the large-time behaviour of solutions to the BBM equation and the KdV equation including the eventual periodicity. A formula representing the solution of the BBM equation was derived through the Laplace transform with respect to temporal variable t and the eventual periodicity is shown for the linearized BBM equation with zero initial data and no forcing term. It appears that the formula derived there cannot be easily extended to include a nonzero initial data or a forcing term. Bona *et al* in [6] established the eventual periodicity in the context of the damped KdV equation

$$u_t + u_x + uu_x + u_{xxx} + u = 0$$

with small amplitude boundary data $u(0, t) = g(t)$. They were able to obtain time-decaying bounds for solutions of this equation and the eventual periodicity follows as a consequence. Through the Laplace transform with respect to the spatial variable x, Shen *et al* in a recent work [10] obtained a new solution representation formula for the KdV equation and re-established the eventual periodicity of the linearized KdV equation. In addition, the eventual periodicity of the full KdV equation was studied there through extensive numerical experiments. We also mention the work of Usman and Zhang [11], in which the eventual periodicity of the KdV equation in a bounded domain is studied.

In this paper, we first derive a new solution formula for the IBVP

$$\begin{cases} u_t + \alpha u_x - \gamma u_{xxt} = f, & x \geqslant 0,\ t \geqslant 0, \\ u(x, 0) = u_0(x), & x \geqslant 0, \\ u(0, t) = g(t), & t \geqslant 0. \end{cases} \tag{1.3}$$

This explicit formula reads

$$\begin{aligned} u(x, t) = {} & g(t)\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}} + \int_0^\infty \Gamma(x - y, t) u_0(y)\, \mathrm{d}y \\ & + \int_0^t \int_0^\infty \Phi(x - y, t - \tau) \left[f(y, \tau) + \frac{\alpha}{\sqrt{\gamma}}\, g(\tau)\, \mathrm{e}^{-\frac{y}{\sqrt{\gamma}}} \right] \mathrm{d}y\, \mathrm{d}\tau, \end{aligned} \tag{1.4}$$

where Γ and Φ are given by

$$\Gamma(x, t) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\, \mathrm{d}\xi,$$

$$\Phi(x, t) = \int_{-\infty}^\infty \frac{1}{1 + \gamma\xi^2} \mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\, \mathrm{d}\xi.$$

Γ should be understood as a distribution. To obtain (1.4), we consider both the even and odd extensions of (1.3) to the whole spatial line. By taking the Fourier transforms of these extensions and solving the resulting equations simultaneously, we are able to represent

$$\int_0^\infty \sin(x\xi)\, u(x,t)\, \mathrm{d}x \qquad \text{and} \qquad \int_0^\infty \cos(x\xi)\, u(x,t)\, \mathrm{d}x \tag{1.5}$$

in terms of f, u_0 and g. (1.4) is then established by taking the inverse Fourier transform of the quantities in (1.5). Corollaries of (1.4) include explicit solution formulae of (1.3) with $\alpha = 0$ and $\gamma > 0$ and of (1.3) with $\gamma = 0$. More details can be found in the second section.

In [3, 5], the IBVP (1.1) has been recast as an integral equation through the inversion of the operator $\partial_t - \gamma\partial_{xxt}$,

$$u(x,t) = u_0(x) + g(t)\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}} + \int_0^t \int_0^\infty K(x,y)\left(\alpha u + \tfrac{1}{2}\beta u^2\right)(y,\tau)\, \mathrm{d}y\, \mathrm{d}\tau, \tag{1.6}$$

where the kernel function $K(x,y)$ is given by

$$K(x,y) = \frac{1}{2\gamma}\left[\mathrm{e}^{-\frac{x+y}{\sqrt{\gamma}}} + \mathrm{sgn}(x-y)\mathrm{e}^{-\frac{|x-y|}{\sqrt{\gamma}}}\right]. \tag{1.7}$$

While this representation is handy in dealing with the well-posedness issue, we find it inconvenient in studying the eventual periodicity of (1.1) due to the inclusion of the linear term on the right. The new representation (1.4) allows us to show that any solution of (1.3) is eventual periodic if f and g are periodic of the same period. The precise statement and its proof are presented in the third section.

This new representation is also very useful in studying the eventual periodicity of the full BBM equation. Borrowing some ideas from John Albert ([1, 2]), we first obtain suitable large-time decay properties of the kernel functions. By representing the solution of (1.1) in terms of this new formula and taking advantage of these decay results, we are able to establish the eventual periodicity for the IBVP (1.1) of the full BBM equation under a suitable ansatz on its solution. The details of this result and its proof are provided in section 4.

We mention that there is adequate theory of well-posedness and regularity on the IBVP (1.1). The following theorem of Bona and Luo [4] serves our purpose well. In the following theorem, we write $\boldsymbol{R}^+ = [0,\infty)$ and C_b^k is exactly like C^k except that the functions and its first k derivatives are required to be bounded.

Theorem 1.1. *Let $I = [0,T]$ if T is positive or $I = [0,\infty)$ if $T = \infty$. Assume that $g \in C^1(I)$ and $u_0 \in C_b^2(\boldsymbol{R}^+) \cap H^2(\boldsymbol{R}^+)$. Then (1.1) is globally well-posed in the sense that there is a unique classical solution $u \in C^1(I, C_b^\infty(\boldsymbol{R}^+)) \cap C(I, H^2(\boldsymbol{R}^+))$ which depends continuously on $g \in C^1(I)$ and $u_0 \in C_b^2(\boldsymbol{R}^+) \cap H^2(\boldsymbol{R}^+)$.*

A more recent work [3] reduces the regularity assumptions to $g \in C(I)$ and $u_0 \in C_b^1(\boldsymbol{R}^+)$ while (1.1) still has a unique global solution in a slightly weak sense. We shall not attempt to optimize these regularity assumptions in this paper. The rest of this paper is divided into three sections and two appendices.

2. The inversion of the operator $\partial_t + \alpha\partial_x - \gamma\partial_{xxt}$

This section explicitly solves the IBVP of the linearized BBM equation

$$\begin{cases} u_t + \alpha u_x - \gamma u_{xxt} = f, & x \geqslant 0,\ t \geqslant 0, \\ u(x,0) = u_0(x), & x \geqslant 0, \\ u(0,t) = g(t), & t \geqslant 0, \end{cases} \tag{2.1}$$

where $\gamma \geqslant 0$ and α are real parameters. This amounts to inverting the operator $\partial_t + \alpha\partial_x - \gamma\partial_{xxt}$ for the quarter-plane problem. Without loss of generality, we assume $g(0) = 0$.

In the case when $\gamma > 0$, we consider a new dependent variable

$$w(x,t) = u(x,t) - g(t)\,\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}}, \tag{2.2}$$

which satisfies the equations

$$\begin{cases} w_t + \alpha w_x - \gamma w_{xxt} = \tilde{f}, & x \geqslant 0,\ t \geqslant 0 \\ w(x,0) = u_0(x), & x \geqslant 0, \\ w(0,t) = 0, & t \geqslant 0, \end{cases} \tag{2.3}$$

where

$$\tilde{f}(x,t) = f(x,t) + \frac{\alpha}{\sqrt{\gamma}}\,g(t)\,\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}}. \tag{2.4}$$

The IBVP (2.3) has its boundary data equal to zero and is solved through odd and even extensions to the whole spatial line. The solution u of (2.1) is then obtained by (2.2).

Theorem 2.1. *Let $I = [0,T]$ if T is positive or $I = [0,\infty)$ if $T = \infty$. Let $u_0 \in C_b^2(\boldsymbol{R}^+) \cap H^2(\boldsymbol{R}^+)$, $g \in C^1(I)$ and $f \in C(I, L^2(\mathbf{R}^+))$. Without loss of generality, assume $u_0(0) = g(0) = 0$. Then the unique classical solution of (2.1) can be written as*

$$\begin{aligned} u(x,t) = g(t)\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}} &+ \int_0^\infty \Gamma(x-y,t)u_0(y)\,\mathrm{d}y \\ &+ \int_0^t \int_0^\infty \Phi(x-y,t-\tau)\left[f(y,\tau) + \frac{\alpha}{\sqrt{\gamma}}\,g(\tau)\,\mathrm{e}^{-\frac{y}{\sqrt{\gamma}}}\right]\mathrm{d}y\,\mathrm{d}\tau, \end{aligned} \tag{2.5}$$

where Γ and Φ are given by

$$\Gamma(x,t) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi, \tag{2.6}$$

$$\Phi(x,t) = \int_{-\infty}^\infty \frac{1}{1+\gamma\xi^2}\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi. \tag{2.7}$$

Γ in (2.6) should be understood as a distribution.

To gain an initial understanding of the formula in this theorem, we consider two special cases. The first is $\alpha = 0$ and $f \equiv 0$. When $\alpha = 0$,

$$\Gamma(x,t) = \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi = \delta(x),$$

where δ denotes the Dirac delta. Therefore, for $x \geqslant 0$,

$$\int_0^\infty \Gamma(x-y,t)u_0(y)\,\mathrm{d}y = u_0(x).$$

Corollary 2.2. *The solution of (2.1) with $\alpha = 0$ and $f \equiv 0$ is given by*

$$u(x,t) = u_0(x) + g(t)\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}}.$$

The second special case is when $\gamma = 0$ and $f \equiv 0$. Although $\gamma = 0$ is not allowed in theorem 2.1, the solution formula for this case can still be obtained similarly as (2.5). Instead of (2.2), one considers

$$w(x,t) = u(x,t) - g(t)\mathrm{e}^{-x}$$

which solves

$$\begin{cases} w_t + \alpha w_x = (\alpha g(t) - g'(t))\mathrm{e}^{-x}, & x \geqslant 0,\ t \geqslant 0, \\ w(x,0) = u_0(x), & x \geqslant 0, \\ w(0,t) = 0, & t \geqslant 0. \end{cases}$$

Then, as in theorem 2.1, the solution for this special case is

$$u(x,t) = g(t)\mathrm{e}^{-x} + \int_0^\infty \Gamma(x-y,t)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^\infty \Phi(x-y,t-\tau)\left[\alpha g(\tau)\mathrm{e}^{-y} - g'(\tau)\mathrm{e}^{-y}\right]\mathrm{d}y\,\mathrm{d}\tau. \tag{2.8}$$

This representation allows us to extract a simple solution formula for the IBVP (2.1) with $\gamma = 0$ and $f \equiv 0$.

Corollary 2.3. *The solution of the IBVP (2.1) with $\gamma = 0$ and $f \equiv 0$ is given by*

$$u(x,t) = \begin{cases} u_0(x-\alpha t), & \text{if } x \geqslant \alpha t, \\ g\left(t - \dfrac{x}{\alpha}\right), & \text{if } x < \alpha t. \end{cases}$$

Proof of corollary 2.3. When $\gamma = 0$,

$$\Gamma(x,t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x-\alpha t)\xi}\,\mathrm{d}\xi = \delta(x-\alpha t), \qquad \Phi(x,t) = \delta(x-\alpha t).$$

If $x - \alpha t \geqslant 0$, then

$$\int_0^\infty \Gamma(x-y,t)u_0(y)\,\mathrm{d}y = \int_0^\infty \delta(x-\alpha t-y)u_0(y)\,\mathrm{d}y = u_0(x-\alpha t) \tag{2.9}$$

and

$$\begin{aligned} &\int_0^t \int_0^\infty \Phi(x-y,t-\tau)\left[\alpha g(\tau)\mathrm{e}^{-y} - g'(\tau)\mathrm{e}^{-y}\right]\mathrm{d}y\,\mathrm{d}\tau \\ &\quad = \int_0^t \left[\alpha g(\tau) - g'(\tau)\right] \int_0^\infty \delta(x-\alpha t+\alpha\tau-y)\,\mathrm{e}^{-y}\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad = \int_0^t \left[\alpha g(\tau) - g'(\tau)\right]\mathrm{e}^{-(x-\alpha t+\alpha\tau)}\,\mathrm{d}\tau \\ &\quad = \mathrm{e}^{-x+\alpha t}\left[\int_0^t \alpha g(\tau)\,\mathrm{e}^{-\alpha\tau}\,\mathrm{d}\tau - \int_0^t g'(\tau)\,\mathrm{e}^{-\alpha\tau}\,\mathrm{d}\tau\right] \\ &\quad = -g(t)\mathrm{e}^{-x}. \end{aligned} \tag{2.10}$$

Inserting (2.9) and (2.10) in (2.8) yields

$$u(x,t) = u_0(x-\alpha t) \qquad \text{if } x - \alpha t \geqslant 0.$$

If $x - \alpha t < 0$, then

$$\int_0^\infty \delta(x-\alpha t-y)u_0(y)\,\mathrm{d}y = 0$$

and

$$\int_0^\infty \delta(x-\alpha t+\alpha\tau-y)\,\mathrm{e}^{-y}\,\mathrm{d}y = \begin{cases} \mathrm{e}^{-(x-\alpha t+\alpha\tau)}, & \text{if } \tau \geqslant t - \alpha^{-1}x, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\int_0^t \int_0^\infty \Phi(x-y,t-\tau)\left[\alpha g(\tau)\mathrm{e}^{-y} - g'(\tau)\mathrm{e}^{-y}\right] \mathrm{d}y\,\mathrm{d}\tau$$

$$= \int_{t-\alpha^{-1}x}^{t} \mathrm{e}^{-(x-\alpha t+\alpha\tau)} \left[\alpha g(\tau) - g'(\tau)\right] \mathrm{d}\tau$$

$$= -g(t)\mathrm{e}^{-x} + g(t-\alpha^{-1}x).$$

Therefore, for $x < \alpha t$,

$$u(x,t) = g(t-\alpha^{-1}x).$$

This completes the proof of corollary 2.3.

Proof of theorem 2.1. The major idea is to extend the equation of (2.3) from the half line $\{x : x > 0\}$ to the whole line $x \in \boldsymbol{R}$ so that the method of Fourier transforms can be employed. Both odd and even extensions will be considered. The rest of the proof is divided into four major steps.

Step 1. Odd extension. Denote by W the odd extension of w in x, namely

$$W(x,t) = \begin{cases} w(x,t) & \text{if } x \geqslant 0, \\ -w(-x,t) & \text{if } x < 0. \end{cases}$$

If W_0 and F are the odd extensions of w_0 and $\tilde{f}$, respectively, then W solves the following initial-value problem:

$$\begin{cases} W_t + \alpha\,\mathrm{sgn}(x) W_x - \gamma W_{xxt} = F, & x \in \boldsymbol{R},\ t > 0 \\ W(x,0) = W_0(x), & x \in \boldsymbol{R}. \end{cases} \tag{2.11}$$

Let $\widehat{W}$ denote the Fourier transform of W, namely

$$\widehat{W}(\xi,t) = \mathcal{F}(W)(\xi,t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x} W(x,t)\,\mathrm{d}x.$$

After applying a basic property of the Fourier transform, we obtain

$$(1+\gamma\xi^2)\,\partial_t \widehat{W}(\xi,t) + \alpha\,\mathcal{F}\,(\mathrm{sgn}(x)W_x) = \widehat{F}(\xi,t). \tag{2.12}$$

According to the definition of W, we have

$$\widehat{W}(\xi,t) = \int_0^\infty (\mathrm{e}^{-\mathrm{i}x\xi} - \mathrm{e}^{\mathrm{i}x\xi}) w(x,t)\,\mathrm{d}x = -2\mathrm{i}\int_0^\infty \sin(x\xi) w(x,t)\,\mathrm{d}x. \tag{2.13}$$

In addition,

$$\mathcal{F}\,(\mathrm{sgn}(x)W_x)\,(\xi,t) = \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\xi x}\mathrm{sgn}(x) W_x\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-\mathrm{i}\xi x}\mathrm{sgn}(x) W_x\,\mathrm{d}x$$

$$= \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\xi x}(-w_x(-x,t))\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-\mathrm{i}\xi x} w_x(x,t)\,\mathrm{d}x.$$

Making the substitution $y = -x$ and integrating by parts yield

$$\mathcal{F}\,(\mathrm{sgn}(x)W_x)\,(\xi,t) = -\int_0^\infty \mathrm{e}^{\mathrm{i}y\xi} w_y(y,t)\,\mathrm{d}y + \int_0^\infty \mathrm{e}^{-\mathrm{i}\xi x} w_x(x,t)\,\mathrm{d}x.$$

$$= \mathrm{i}\xi \int_0^\infty (\mathrm{e}^{\mathrm{i}x\xi} + \mathrm{e}^{-\mathrm{i}x\xi})\,w(x,t)\,\mathrm{d}x$$

$$= 2\mathrm{i}\xi \int_0^\infty \cos(x\xi)\,w(x,t)\,\mathrm{d}x. \tag{2.14}$$

We can also write $\widehat{F}$ in terms of $\tilde{f}$ as

$$\widehat{F}(\xi,t) = -2\mathrm{i}\int_0^\infty \sin(x\xi)\,\tilde{f}(x,t)\,\mathrm{d}x. \tag{2.15}$$

Inserting (2.13), (2.14) and (2.15) in (2.12), we obtain

$$\partial_t X(\xi,t) - \beta(\xi)Y(\xi,t) = h_1(\xi,t), \tag{2.16}$$

where

$$X(\xi,t) = \int_0^\infty \sin(x\xi)\,w(x,t)\,\mathrm{d}x, \qquad Y(\xi,t) = \int_0^\infty \cos(x\xi)\,w(x,t)\,\mathrm{d}x,$$

$$\beta(\xi) = \frac{\alpha\xi}{1+\gamma\xi^2} \qquad \text{and} \qquad h_1(\xi,t) = \frac{1}{1+\gamma\xi^2}\int_0^\infty \sin(x\xi)\,\tilde{f}(x,t)\,\mathrm{d}x.$$

Step 2. Even extension. Denote by $V(x,t)$ the even extension of w, namely

$$V(x,t) = \begin{cases} w(x,t) & \text{if } x \geqslant 0, \\ w(-x,t) & \text{if } x < 0. \end{cases}$$

It can be easily verified that V satisfies

$$\begin{cases} V_t + \alpha\,\mathrm{sgn}(x)V_x - \gamma V_{xxt} = H, & x \in \boldsymbol{R},\ t > 0 \\ V(x,0) = V_0(x), & x \in \boldsymbol{R}, \end{cases} \tag{2.17}$$

where H and V_0 are the even extensions of $\tilde{f}$ and w_0, respectively. As in step 1, we have

$$\widehat{V}(\xi,t) = 2\int_0^\infty \cos(x\xi)\,w(x,t)\,\mathrm{d}x, \tag{2.18}$$

and

$$\widehat{H}(\xi,t) = 2\int_0^\infty \cos(x\xi)\,\tilde{f}(x,t)\,\mathrm{d}x. \tag{2.19}$$

Furthermore,

$$\begin{aligned} \mathcal{F}\left(\mathrm{sgn}(x)V_x\right)(\xi,t) &= \int_{-\infty}^0 \mathrm{e}^{-\mathrm{i}\xi x}\mathrm{sgn}(x)V_x\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-\mathrm{i}\xi x}\mathrm{sgn}(x)V_x\,\mathrm{d}x \\ &= \int_{-\infty}^0 \mathrm{e}^{-\mathrm{i}\xi x}w_x(-x,t)\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-\mathrm{i}\xi x}w_x(x,t)\,\mathrm{d}x. \end{aligned}$$

Making the substitution $y = -x$ in the first integral and integrating by parts leads to

$$\begin{aligned} \mathcal{F}\left(\mathrm{sgn}(x)V_x\right)(\xi,t) &= \int_0^\infty \mathrm{e}^{\mathrm{i}\xi y}w_y(y,t)\,\mathrm{d}y + \int_0^\infty \mathrm{e}^{-\mathrm{i}\xi x}w_x(x,t)\,\mathrm{d}x \\ &= -\mathrm{i}\xi\int_0^\infty \mathrm{e}^{\mathrm{i}\xi x}w(x,t)\,\mathrm{d}x + \mathrm{i}\xi\int_0^\infty \mathrm{e}^{-\mathrm{i}\xi x}w(x,t)\,\mathrm{d}x \\ &= 2\xi\int_0^\infty \sin(x\xi)\,w(x,t)\,\mathrm{d}x. \end{aligned} \tag{2.20}$$

Taking the Fourier transform of (2.17) and applying (2.18), (2.19) and (2.20), we obtain

$$\partial_t Y(\xi,t) + \beta(\xi)X(\xi,t) = h_2(\xi,t), \tag{2.21}$$

where

$$h_2(\xi,t) = \frac{1}{1+\gamma\xi^2}\int_0^\infty \cos(x\xi)\,\tilde{f}(x,t)\,\mathrm{d}x.$$

Step 3. Solving for $X(\xi,t)$ and $Y(\xi,t)$. Solving the system of (2.16) and (2.21), we find

$$X(\xi,t)=\int_0^\infty \sin(x\xi+\beta t)\,u_0(x)\,\mathrm{d}x+\frac{1}{1+\gamma\xi^2}\int_0^t\int_0^\infty \sin(x\xi+\beta(t-\tau))\,\tilde f(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau,$$

$$Y(\xi,t)=\int_0^\infty \cos(x\xi+\beta t)\,u_0(x)\,\mathrm{d}x+\frac{1}{1+\gamma\xi^2}\int_0^t\int_0^\infty \cos(x\xi+\beta(t-\tau))\,\tilde f(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

where $\tilde f$ is defined in (2.4). We give the details of the derivation in appendix A.

Step 4. Finding $w(x,t)$ through the inverse Fourier transform. To find $w(x,t)$, we first note that

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}w(y,t)\,\mathrm{d}y=Y(\xi,t)-\mathrm{i}X(\xi,t).$$

Applying the formulae in the previous step, we have

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}w(y,t)\,\mathrm{d}y=\mathrm{e}^{-\mathrm{i}\beta t}\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}u_0(y)\,\mathrm{d}y+\frac{1}{1+\gamma\xi^2}\int_0^t \mathrm{e}^{-\mathrm{i}\beta(t-\tau)}\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}\,\tilde f(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \tag{2.22}$$

We shall now establish a theorem that allows us to obtain $w(x,t)$ by taking the inverse Fourier transform of (2.22).

Theorem 2.4. *Fix $t>0$. If $u(x,t)\in L^2(\boldsymbol{R}^+)$, then, for any $x\geqslant 0$,*

$$\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x\xi}\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}u(y,t)\,\mathrm{d}y\,\mathrm{d}\xi=u(x,t). \tag{2.23}$$

Proof of theorem 2.4.. Recall that if $g_\epsilon(\xi)=\exp(-\epsilon\pi\xi^2)$, then

$$\widehat{g_\epsilon}(x)=\frac{1}{\epsilon^{1/2}}\exp\left(-\frac{\pi x^2}{\epsilon}\right).$$

In addition, for any $f\in L^p(\boldsymbol{R})$ with $1\leqslant p<\infty$,

$$\widehat{g_\epsilon}*f\to f\qquad \text{in } L^p(\boldsymbol{R}).$$

These basic facts can be found in Lieb and Loss [8]. Now, consider

$$\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x\xi}g_\epsilon(\xi)\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}u(y,t)\,\mathrm{d}y\,\mathrm{d}\xi.$$

According to lemma 2.5, if $u(y,t)\in L^2(\boldsymbol{R}^+)$, then

$$P(u)(\xi,t)\equiv\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}u(y,t)\,\mathrm{d}y\in L^2(\boldsymbol{R}).$$

Since $g_\epsilon(\xi)\to 1$ as $\epsilon\to 0$, the dominated convergence theorem implies

$$g_\epsilon P(u)\to P(u)\qquad \text{in } L^2(\boldsymbol{R}).$$

Then,

$$\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x\xi}\,g_\epsilon(\xi)\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}u(y,t)\,\mathrm{d}y\,\mathrm{d}\xi=\int_0^\infty u(y,t)\int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}(y-x)\xi}g_\epsilon(\xi)\,\mathrm{d}\xi\,\mathrm{d}y=\int_0^\infty u(y,t)\widehat{g_\epsilon}(y-x)\,\mathrm{d}y. \tag{2.24}$$

As in the proof of theorem 2.16 of [8], we can prove

$$\int_0^\infty u(y,t)\widehat{g}_\epsilon(y-x)\,\mathrm{d}y \to u(x,t) \qquad \text{in } L^2(\boldsymbol{R}^+).$$

Letting $\epsilon \to 0$ in (2.24) yields (2.23). This proves theorem 2.4.

Lemma 2.5. *If $u(y,t) \in L^2(\boldsymbol{R}^+)$, then*

$$P(u)(\xi,t) \equiv \int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi} u(y,t)\,\mathrm{d}y \in L^2(\boldsymbol{R}).$$

Proof of lemma 2.5. For any $\epsilon > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} |P(u)|^2(\xi,t)g_\epsilon(\xi)\,\mathrm{d}\xi &= \int_{-\infty}^{\infty} P(u)(\xi,t)\overline{P(u)}(\xi,t)g_\epsilon(\xi)\,\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} g_\epsilon(\xi)\int_0^\infty \mathrm{e}^{-\mathrm{i}x\xi}u(x,t)\,\mathrm{d}x \int_0^\infty \mathrm{e}^{\mathrm{i}y\xi}u(y,t)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \int_0^\infty\int_0^\infty u(x,t)u(y,t)\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(x-y)\xi}g_\epsilon(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^\infty u(x,t)\int_0^\infty \widehat{g}_\epsilon(x-y)u(y,t)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

As $\epsilon \to 0$,

$$\int_0^\infty \widehat{g}_\epsilon(x-y)u(y,t)\,\mathrm{d}y \to u(x,t) \qquad \text{in } L^2(\boldsymbol{R}^+)$$

and $u(x,t) \in L^2(\boldsymbol{R}^+)$ implies $\int_{-\infty}^{\infty} |P(u)|^2(\xi,t)g_\epsilon(\xi)\,\mathrm{d}\xi$ is bounded uniformly. Since

$$g_\epsilon(\xi) = \exp(-\epsilon\pi|\xi|^2) \to 1 \text{ as } \epsilon \to 0,$$

we obtain by applying the monotone convergence theorem that

$$\int_{-\infty}^{\infty} |P(u)|^2(\xi,t)\,\mathrm{d}\xi = \int_0^\infty |u(x,t)|^2\,\mathrm{d}x.$$

This proves lemma 2.5.

Taking the inverse Fourier transform (denoted by $\mathcal{F}^{-1}$) of (2.22) and applying theorem 2.4 and the basic property $\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}(f) * \mathcal{F}^{-1}(g)$, we obtain

$$w(x,t) = \int_0^\infty \Gamma(x-y,t)u_0(y)\,\mathrm{d}y + \int_0^t\int_0^\infty \Phi(x-y,t-\tau)\tilde{f}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau,$$

where, noticing $\beta = \frac{\alpha\xi}{1+\gamma\xi^2}$,

$$\begin{aligned}\Gamma(x,t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi, \\ \Phi(x,t) &= \int_{-\infty}^{\infty} \frac{1}{1+\gamma\xi^2}\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi.\end{aligned}$$

Therefore, by (2.2),

$$\begin{aligned}u(x,t) &= w(x,t) + g(t)\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}} \\ &= g(t)\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}} + \int_0^\infty \Gamma(x-y,t)u_0(y)\,\mathrm{d}y \\ &\quad + \int_0^t\int_0^\infty \Phi(x-y,t-\tau)\left[f(y,\tau) + \frac{\alpha}{\sqrt{\gamma}}g(\tau)\,\mathrm{e}^{-\frac{y}{\sqrt{\gamma}}}\right]\mathrm{d}y\,\mathrm{d}\tau.\end{aligned}$$

This completes the proof of theorem 2.1.

3. Eventual periodicity for the linearized equation

This section studies the eventual periodicity of solutions to the IBVP for the linearized BBM equation

$$\begin{cases} u_t + \alpha u_x - \gamma u_{xxt} = f(x,t), & x \geqslant 0,\ t \geqslant 0, \\ u(x,0) = u_0(x), & x \geqslant 0, \\ u(0,t) = g(t), & t \geqslant 0, \end{cases} \tag{3.1}$$

where g and f are assumed to be periodic in t. We prove the following theorem.

Theorem 3.1. *Let $\gamma > 0$ and α be real parameters. Let $I = [0, T]$ if T is positive or $I = [0, \infty)$ if $T = \infty$. Let $u_0 \in C_b^2(\boldsymbol{R}^+)$ and $u_0' \in L^1(\boldsymbol{R}^+)$. Let $g \in C(I)$ with $g(0) = 0$ and $f \in C(I, L^1(\boldsymbol{R}^+))$. Assume g and f are periodic of period T_0 in t, namely, for all $t \geqslant 0$,*

$$g(t + T_0) = g(t) \qquad \text{and} \qquad f(x, t + T_0) = f(x,t). \tag{3.2}$$

Then, for any fixed $x > 0$, the solution u of (3.1) satisfies

$$\lim_{t\to\infty} (u(x, t + T_0) - u(x,t)) = 0. \tag{3.3}$$

That is, u is eventually periodic of period T_0.

Remark 3.2. When $\gamma = 0$, the eventual periodicity is easily obtained from the explicit formula in corollary 2.3.

Proof of theorem 3.1. Consider the new function

$$v(x,t) = u(x,t) - u_0(x),$$

which satisfies

$$\begin{cases} v_t + \alpha v_x - \gamma v_{xxt} = f(x,t) - \alpha\, u_0'(x), & x \geqslant 0,\ t \geqslant 0, \\ v(x,0) = 0, & x \geqslant 0, \\ v(0,t) = g(t), & t \geqslant 0 \end{cases} \tag{3.4}$$

Applying the explicit formula in theorem 2.1 to (3.4) gives

$$v(x,t) = g(t)\mathrm{e}^{-\frac{x}{\sqrt{\gamma}}} + \int_0^t \int_0^\infty \Phi(x-y, t-\tau)\left[f(y,\tau) - \alpha u_0'(y) + \frac{\alpha}{\sqrt{\gamma}}\, g(\tau)\, \mathrm{e}^{-\frac{y}{\sqrt{\gamma}}}\right] \mathrm{d}y\, \mathrm{d}\tau,$$

where Φ is given by (2.7). Noting the conditions in (3.2), we obtain after making a substitution,

$$\begin{aligned} u(x, t+T_0) - u(x,t) &= v(x, t+T_0) - v(x,t) \\ &= \int_0^{t+T_0} \int_0^\infty \Phi(x-y, t+T_0-\tau)\left[f(y,\tau) - \alpha u_0'(y) + \frac{\alpha}{\sqrt{\gamma}}\, g(\tau)\, \mathrm{e}^{-\frac{y}{\sqrt{\gamma}}}\right] \mathrm{d}y\, \mathrm{d}\tau \\ &\quad - \int_0^t \int_0^\infty \Phi(x-y, t-\tau)\left[f(y,\tau) - \alpha u_0'(y) + \frac{\alpha}{\sqrt{\gamma}}\, g(\tau)\, \mathrm{e}^{-\frac{y}{\sqrt{\gamma}}}\right] \mathrm{d}y\, \mathrm{d}\tau \\ &= \int_{-T_0}^0 \int_0^\infty \Phi(x-y, t-\tau)\left[f(y,\tau) - \alpha u_0'(y) + \frac{\alpha}{\sqrt{\gamma}}\, g(\tau)\, \mathrm{e}^{-\frac{y}{\sqrt{\gamma}}}\right] \mathrm{d}y\, \mathrm{d}\tau. \end{aligned} \tag{3.5}$$

To show (3.3), we first show that, for any $\epsilon > 0$, there is $K > 0$ such that

$$|\Phi(x,t)| = \left| \int_{-\infty}^\infty \frac{1}{1+\gamma\xi^2}\, \mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\, \mathrm{d}\xi \right| < \epsilon \qquad \text{when } t > K.$$

First, we choose $M = M(\epsilon) > 0$ such that

$$\int_{-\infty}^{-M} \frac{1}{1+\gamma\xi^2}\,\mathrm{d}\xi + \int_{M}^{\infty} \frac{1}{1+\gamma\xi^2}\,\mathrm{d}\xi < \frac{\epsilon}{2}.$$

Next, we apply the method of stationary phase to show the following asymptotics.

Proposition 3.3. *For any fixed $M > 0$,*

$$\int_{-M}^{M} \frac{1}{1+\gamma\xi^2}\,\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi = O\left(\frac{1}{\sqrt{t}}\right)$$

as $t \to \infty$. This large-time asymptotics is uniform in $x \in \boldsymbol{R}^{+}$.

We note that the result of this proposition does not necessarily hold for

$$\int_{-\infty}^{\infty} \frac{1}{1+\gamma\xi^2}\,\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi$$

because one of the conditions in the method of stationary phase is violated. More details on this point will be provided in appendix B. The following estimate is a special consequence of this proposition.

Corollary 3.4. *There exists $K = K(M)$ such that*

$$\left|\int_{-M}^{M} \frac{1}{1+\gamma\xi^2}\,\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi\right| < \frac{\epsilon}{2}$$

whenever $t > K$.

We resume the proof of theorem 3.1. It then follows from (3.5) that, for $t > K$,

$$|u(x,t+T) - u(x,t)| \leqslant \epsilon\left[\int_{-T_0}^{0}\int_{0}^{\infty} |f(y,\tau)|\,\mathrm{d}y\,\mathrm{d}\tau + \alpha T\int_{0}^{\infty} |u_0'(y)|\,\mathrm{d}y + \alpha\int_{-T_0}^{0} g(\tau)\,\mathrm{d}\tau\right].$$

This completes the proof of theorem 3.1.

Proof of proposition 3.3. We apply the method of stationary phase. To do so, we split the integral into four parts

$$\int_{-M}^{M} \frac{1}{1+\gamma\xi^2}\,\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi = I_1 + I_2 + I_3 + I_4 \equiv \int_{-M}^{-\frac{1}{\sqrt{\gamma}}} + \int_{-\frac{1}{\sqrt{\gamma}}}^{0} + \int_{0}^{\frac{1}{\sqrt{\gamma}}} + \int_{\frac{1}{\sqrt{\gamma}}}^{M}.$$

Note that $\pm 1/\sqrt{\gamma}$ are the zero points of the derivative of $p(\xi) = \alpha\xi/(1+\gamma\xi^2)$ and $p'(\xi)$ is nonzero for ξ in each of these intervals. It suffices to consider I_2 and I_4. Without loss of generality, we assume $\alpha > 0$. Direct applications of the method of stationary phase concludes that

$$I_2 \sim \frac{\sqrt{\pi}\,\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\,\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{\gamma}}(-x+\frac{\alpha t}{2})}}{2\sqrt{2\alpha}\sqrt[4]{\gamma}\sqrt{t}}, \tag{3.6}$$

$$I_4 \sim \frac{\sqrt{\pi}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\,\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{\gamma}}(x-\frac{\alpha t}{2})}}{2\sqrt{2\alpha}\sqrt[4]{\gamma}\sqrt{t}}. \tag{3.7}$$

For the readers' convenience, this method is recalled in appendix B and the details leading to these estimates can also be found there.

4. Eventual periodicity for the full BBM equation

This section investigates the eventual periodicity of the IBVP for the full BBM equation (1.1). The lack of suitable bounds on its solutions in terms of the time variable makes this problem extremely difficult. By imposing a condition on the solution and making full use of the solution representation formula derived in section 2, we are able to establish the eventual periodicity.

Theorem 4.1. *Consider the IBVP for the full BBM equation (1.1) with real parameters $\gamma > 0$ and $\alpha \neq 0$. Let $I = [0, \infty)$ and assume u_0, g and f satisfy*

$$u_0 \in C_b^2(\boldsymbol{R}^+) \cap H^4(\boldsymbol{R}^+), \qquad g \in C^1(I) \qquad \text{and} \qquad f \in C(I, H^4(\boldsymbol{R}^+)).$$

In addition, g and f are periodic of period T_0 in t.

Let u be the corresponding solution of (1.1). Set

$$u_1(x,t) = u(x,t) + u(x, t+T_0)$$

and

$$\widehat{u}_1(\xi, t) = \int_0^\infty \mathrm{e}^{-\mathrm{i}x\xi}\, u_1(x,t)\,\mathrm{d}x.$$

Suppose that for some index $p > 1$ and a suitable constant $C_0 > 0$

$$A(t) \equiv C_0\,\beta \int_0^t (A_1(\tau) + A_2(\tau))\,\mathrm{d}\tau \leqslant A_0 < 1 \qquad \text{for all } t \geqslant 1,$$

where

$$A_1(\tau) = \frac{1}{\sqrt[3]{1+t-\tau}} \int_0^\infty |u_1(y,\tau)|\,\mathrm{d}y,$$

$$A_2(\tau) = \left[\frac{1}{\sqrt[9]{1+t-\tau}} \int_{\xi > (t-\tau)^{1/9}} |\widehat{u}_1(\xi,\tau)|^p\,\mathrm{d}\xi\right]^{\frac{1}{p}}.$$

Then, for $t \geqslant 1$,

$$\sup_{x \geqslant 0} |u(x, t+T_0) - u(x,t)| \leqslant \frac{C_1}{1 - A(t)}\,(1+t)^{-\frac{1}{3}}, \tag{4.1}$$

where $C_1 = \|u_0\|_{L^1} + \|u_0\|_{H^4} + \|u(\cdot, T_0)\|_{L^1} + \|u(\cdot, T_0)\|_{H^4}$. In particular, u is eventually periodic of period T_0.

To prove this theorem, we need the following decay estimates.

Lemma 4.2. *Let $\gamma > 0$ and $\alpha \neq 0$. Then, for $t \geqslant 1$,*

$$\left| \int_{|\xi| \leqslant t^{1/9}} \mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi \right| \leqslant C(1+t)^{-\frac{1}{3}}, \tag{4.2}$$

$$\left| \int_{|\xi| \leqslant t^{1/9}} \frac{\mathrm{i}\xi}{1+\gamma\xi^2}\,\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi \right| \leqslant C(1+t)^{-\frac{1}{3}}. \tag{4.3}$$

Proof of lemma 4.2. (4.2) is essentially lemma 5 of Albert [2]. To prove (4.3), we note that $\mathrm{i}\xi/(1+\gamma\xi^2)$ is the Fourier transform of

$$E(x) = -\frac{1}{2\gamma}\,(\operatorname{sign} x)\,\mathrm{e}^{-\frac{|x|}{\sqrt{\gamma}}}$$

and $E \in L^1$. Therefore, by Young's inequality,

$$\begin{aligned}\left|\int_{|\xi|\leqslant t^{1/9}} \frac{\mathrm{i}\xi}{1+\gamma\xi^2}\, \mathrm{e}^{\mathrm{i}x\xi-\mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi\right| &= \left|E * \int_{|\xi|\leqslant t^{1/9}} \mathrm{e}^{\mathrm{i}x\xi-\mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi\right| \\ &\leqslant \|E\|_{L^1}\left|\int_{|\xi|\leqslant t^{1/9}} \mathrm{e}^{\mathrm{i}x\xi-\mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi\right| \\ &\leqslant C\|E\|_{L^1}(1+t)^{-\frac{1}{3}}.\end{aligned}$$

Proof of theorem 4.1. Consider the difference

$$w(x,t) = u(x,t+T_0) - u(x,t),$$

which satisfies

$$\begin{aligned}&w_t + \alpha w_x - \gamma w_{xxt} = -\beta u(x,t+T_0)u_x(x,t+T_0) + \beta u(x,t)u_x(x,t),\\ &w(x,0) = w_0(x) \equiv u(x,T_0) - u_0(x), \qquad (4.4)\\ &w(0,t) = 0.\end{aligned}$$

According to (2.5), w can be represented by

$$w(x,t) = \int_0^\infty \Gamma(x-y,t)\, w_0(y)\,\mathrm{d}y - N(x,t), \qquad (4.5)$$

where

$$N(x,t) = \frac{\beta}{2}\int_0^t\int_0^\infty \Phi(x-y,t-\tau)\,\left[\partial_y u^2(y,T_0+\tau) - \partial_y u^2(y,\tau)\right]\mathrm{d}y\,\mathrm{d}\tau.$$

Through an integration by parts, N can be rewritten as

$$N(x,t) = \frac{\beta}{2}\int_0^t\int_0^\infty \Psi(x-y,t-\tau)\, w(y,\tau)\, u_1(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \qquad (4.6)$$

with

$$\Psi(x,t) = \int_{-\infty}^\infty \frac{\mathrm{i}\,\xi}{1+\gamma\xi^2}\,\mathrm{e}^{\mathrm{i}x\xi-\mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi. \qquad (4.7)$$

To estimate the first term in (4.5), we recall the formula for Γ (2.6) and write

$$\Gamma(x,t) = \int_{|\xi|\leqslant t^{1/9}} \mathrm{e}^{\mathrm{i}x\xi-\mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi + \int_{|\xi|> t^{1/9}} \mathrm{e}^{\mathrm{i}x\xi-\mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi.$$

Applying (4.2) in lemma 4.2, we have

$$\begin{aligned}\left|\int_0^\infty \Gamma(x-y,t)\, w_0(y)\,\mathrm{d}y\right| &\leqslant C(1+t)^{-\frac{1}{3}}\int_0^\infty |w_0(y)|\,\mathrm{d}y \\ &\quad + \int_{|\xi|>t^{1/9}}\left|\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}\, w_0(y)\,\mathrm{d}y\right|\mathrm{d}\xi. \qquad (4.8)\end{aligned}$$

To further bound (4.8), it suffices to consider the integral over $[t^{1/9},\infty)$. By Hölder's inequality,

$$\begin{aligned}\int_{t^{1/9}}^\infty \left|\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}\, w_0(y)\,\mathrm{d}y\right|\mathrm{d}\xi &\leqslant \int_{t^{1/9}}^\infty \xi^{-4}\left|\xi^4\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi}\, w_0(y)\,\mathrm{d}y\right|\mathrm{d}\xi \\ &\leqslant C\,t^{-\frac{7}{18}}\,\|\xi^4\widehat{w_0}\|_{L^2} \leqslant C\,t^{-\frac{1}{3}}\,\|w_0\|_{H^4}. \qquad (4.9)\end{aligned}$$

Combining (4.8) and (4.9) and noticing that $w_0 = u(x, T_0) - u_0(x)$, we obtain

$$\left| \int_0^\infty \Gamma(x-y,t)\, w_0(y)\, \mathrm{d}y \right| \leqslant C_1(1+t)^{-\frac{1}{3}}. \tag{4.10}$$

To bound N in (4.6), we split the integral in (4.7) into two parts: $|\xi| \leqslant t^{1/9}$ and $|\xi| > t^{1/9}$ and bound the first part through lemma 4.2. Therefore, for $\tau \leqslant t$,

$$\int_0^\infty \Psi(x-y, t-\tau)\, w(y,\tau)\, u_1(y,\tau)\, \mathrm{d}y \leqslant Q_1 + Q_2 \tag{4.11}$$

with

$$Q_1 = C(1+t-\tau)^{-\frac{1}{3}} \int_0^\infty |w(y,\tau)u_1(y,\tau)|\, \mathrm{d}y,$$

$$Q_2 = \int_0^\infty \int_{|\xi|>(t-\tau)^{1/9}} \frac{\mathrm{i}\xi}{1+\gamma\xi^2}\, \mathrm{e}^{\mathrm{i}(x-y)\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}(t-\tau)}\, \mathrm{d}\xi\, w(y,\tau)u_1(y,\tau)\, \mathrm{d}y.$$

Obviously,

$$Q_1 \leqslant C \sup_{y\geqslant 0} |w(y,\tau)|\, \frac{1}{\sqrt[3]{1+t-\tau}} \int_0^\infty |u_1(y,\tau)|\, \mathrm{d}y. \tag{4.12}$$

Exchanging the order of the integrals in Q_2, we find

$$Q_2 \leqslant \int_{|\xi|>(t-\tau)^{1/9}} \frac{\xi}{1+\gamma\xi^2} \left| \int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi} w(y,\tau)\, u_1(y,\tau)\, \mathrm{d}y \right| \mathrm{d}\xi.$$

By symmetry, it suffices to consider the interval $\xi > (t-\tau)^{1/9}$. Realizing that

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi} w(y,\tau)\, u_1(y,\tau)\, \mathrm{d}y = \widehat{w}(\cdot,\tau) * \widehat{u}_1(\cdot,\tau),$$

we apply Hölder's and Young's inequalities to obtain

$$Q_2 \leqslant \left[\int_{\xi>(t-\tau)^{1/9}} \left(\frac{\xi}{1+\gamma\xi^2} \right)^q \mathrm{d}\xi \right]^{\frac{1}{q}} \int_{\xi>(t-\tau)^{1/9}} |\widehat{w}(\xi,\tau)|\, \mathrm{d}\xi \left[\int_{\xi>(t-\tau)^{1/9}} |\widehat{u}_1(\cdot,\tau)|^p\, \mathrm{d}\xi \right]^{\frac{1}{p}},$$

where $1/q + 1/p = 1$. These integrals can be further bounded as follows.

$$\left[\int_{\xi>(t-\tau)^{1/9}} \left(\frac{\xi}{1+\gamma\xi^2} \right)^q \mathrm{d}\xi \right]^{\frac{1}{q}} \leqslant C\,(1+t-\tau)^{\frac{1}{9}(-1+\frac{1}{q})} = C\,(1+t-\tau)^{-\frac{1}{9p}},$$

$$\begin{aligned} \int_{\xi>(t-\tau)^{1/9}} |\widehat{w}(\xi,\tau)|\, \mathrm{d}\xi &\leqslant \int_0^\infty \left| \int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi} w(y,\tau)\, \mathrm{d}y \right| \mathrm{d}\xi \\ &= \sup_{x\geqslant 0} \left| \int_0^\infty \mathrm{e}^{\mathrm{i}x\xi} \int_0^\infty \mathrm{e}^{-\mathrm{i}y\xi} w(y,\tau)\, \mathrm{d}y\, \mathrm{d}\xi \right| \\ &\leqslant \sup_{x\geqslant 0} |w(x,\tau)| \end{aligned}$$

Therefore,

$$Q_2 \leqslant C \sup_{x\geqslant 0} |w(x,\tau)| \left[\frac{1}{\sqrt[9]{1+t-\tau}} \int_{\xi>(t-\tau)^{1/9}} |\widehat{u}_1(\xi,\tau)|^p\, \mathrm{d}\xi \right]^{\frac{1}{p}}. \tag{4.13}$$

Combining (4.6), (4.11), (4.12) and (4.13), we obtain

$$N \leqslant A(t) \sup_{\tau\in[0,t]} \sup_{x\geqslant 0} |w(x,\tau)|. \tag{4.14}$$

Finally, we obtain (4.1) by putting together (4.5), (4.10) and (4.14). This concludes the proof of theorem 4.1.

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Appendix A

This appendix provides the details of solving the system of ODEs (2.16) and (2.21). Consider the general nonhomogeneous linear systems

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = P(t)\boldsymbol{x}(t) + \boldsymbol{g}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$, $P \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{g} \in \boldsymbol{R}^n$. By variation of parameters, its solution can be written as

$$\boldsymbol{x}(t) = \Psi(t)\Psi^{-1}(t_0)\boldsymbol{x}_0 + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\boldsymbol{g}(s)\,\mathrm{d}s, \tag{A.1}$$

where Ψ denotes a fundamental matrix of the homogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = P(t)\boldsymbol{x}(t).$$

Since the system of equations we are solving can be written as

$$\partial_t \begin{bmatrix} X(\xi,t) \\ Y(\xi,t) \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} X(\xi,t) \\ Y(\xi,t) \end{bmatrix} + \begin{bmatrix} h_1(\xi,t) \\ h_2(\xi,t) \end{bmatrix}$$

and a fundamental matrix of the corresponding homogenous system is given by

$$\Psi(t) = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\beta t} & \mathrm{e}^{-\mathrm{i}\beta t} \\ \mathrm{i}\,\mathrm{e}^{\mathrm{i}\beta t} & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\beta t} \end{bmatrix},$$

we apply (A.1) to obtain that

$$\begin{bmatrix} X(\xi,t) \\ Y(\xi,t) \end{bmatrix} = \Psi(t)\Psi^{-1}(0) \begin{bmatrix} X(\xi,0) \\ Y(\xi,0) \end{bmatrix} + \Psi(t)\int_0^t \Psi^{-1}(\tau) \begin{bmatrix} h_1(\xi,\tau) \\ h_2(\xi,\tau) \end{bmatrix} \mathrm{d}\tau. \tag{A.2}$$

Inserting

$$\Psi^{-1}(t) = \frac{1}{2}\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\beta t} & -\mathrm{i}\mathrm{e}^{-\mathrm{i}\beta t} \\ \mathrm{e}^{\mathrm{i}\beta t} & \mathrm{i}\mathrm{e}^{\mathrm{i}\beta t} \end{bmatrix}, \qquad \begin{bmatrix} X(\xi,0) \\ Y(\xi,0) \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^\infty \sin(x\xi)u_0(x)\,\mathrm{d}x \\ \displaystyle\int_0^\infty \cos(x\xi)u_0(x)\,\mathrm{d}x \end{bmatrix}$$

and

$$\begin{bmatrix} h_1(\xi,t) \\ h_2(\xi,t) \end{bmatrix} = \begin{bmatrix} \displaystyle\frac{1}{1+\gamma\xi^2}\int_0^\infty \sin(x\xi)\tilde{f}(x,t)\,\mathrm{d}x \\ \displaystyle\frac{1}{1+\gamma\xi^2}\int_0^\infty \cos(x\xi)\tilde{f}(x,t)\,\mathrm{d}x \end{bmatrix}$$

in (A.2), we find after some simplification

$$X(\xi,t)=\int_0^\infty \sin(x\xi+\beta t)\,u_0(x)\,\mathrm{d}x+\frac{1}{1+\gamma\xi^2}\int_0^t\int_0^\infty \sin(x\xi+\beta(t-\tau))\,\tilde{f}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau,$$

$$Y(\xi,t)=\int_0^\infty \cos(x\xi+\beta t)\,u_0(x)\,\mathrm{d}x+\frac{1}{1+\gamma\xi^2}\int_0^t\int_0^\infty \cos(x\xi+\beta(t-\tau))\,\tilde{f}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Appendix B

This appendix offers an expanded commentary on the asymptotic analysis of the oscillatory integrals discussed in section 3. This analysis relies upon standard results in the theory of stationary phase, e.g. theorem 13.1 in Olver's book [9]. For readers' convenience, we first recall this theory here.

Suppose that in the integral

$$I(t)=\int_a^b \mathrm{e}^{\mathrm{i}tp(y)}\,q(y)\,\mathrm{d}y$$

the limits a and b are independent of t, a being finite and $b(>a)$ finite or infinite. The functions $p(y)$ and $q(y)$ are independent of t, $p(y)$ being real and $q(y)$ either real or complex. We also assume that the only point at which $p'(y)$ vanishes is a. Without loss of generality, both t and $p'(y)$ are taken to be positive; cases in which one of them is negative can be handled by changing the sign of i throughout. We require

(i) In (a,b), the functions $p'(y)$ and $q(y)$ are continuous, $p'(y)>0$, and $p''(y)$ and $q'(y)$ have at most a finite number of discontinuities and infinities.

(ii) As $y\to a+$,

$$p(y)-p(a)\sim P(y-a)^{\mu},\qquad q(y)\sim Q(y-a)^{\lambda-1},\tag{B.1}$$

the first of these relations being differentiable. Here P, μ and λ are positive constants, and Q is a real or complex constant.

(iii) For each $\epsilon\in(0,b-a)$,

$$\mathcal{V}_{a+\epsilon,b}\left\{\frac{q(y)}{p'(y)}\right\}\equiv\int_{a+\epsilon}^b\left|\left(\frac{q(y)}{p'(y)}\right)'\right|\,\mathrm{d}y<\infty.$$

(iv) As $t\to b-$, the limit of $q(y)/p'(y)$ is finite, and this limit is zero if $p(b)=\infty$.

With these conditions, the nature of asymptotic approximation to $I(t)$ for large t depends on the sign of $\lambda-\mu$. In the case $\lambda<\mu$, we have the following theorem.

Theorem B.1. *In addition to the above conditions, assume that $\lambda<\mu$, the first of (B.1) is twice differentiable and the second of (B.1) is differentiable, then*

$$I(t)\sim \mathrm{e}^{\lambda\pi\mathrm{i}/(2\mu)}\,\frac{Q}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\frac{\mathrm{e}^{\mathrm{i}tp(a)}}{(Pt)^{\lambda/\mu}}$$

as $t\to\infty$.

We now provide the details leading to (3.6). It suffices to check the conditions of theorem B.1. Setting

$$a=-\frac{1}{\sqrt{\gamma}},\qquad b=0,\qquad p(\xi)=\frac{\alpha\xi}{1+\gamma\xi^2}\qquad\text{and}\qquad q(\xi)=\frac{\mathrm{e}^{\mathrm{i}\xi x}}{1+\gamma\xi^2},$$

we have

(i) p, p', p'', q and q' are all continuous in $(-1/\sqrt{\gamma}, 0)$, and $p'(\xi) > 0$.

(ii) As $\xi \to -\frac{1}{\sqrt{\gamma}}+$,

$$p(\xi) - p\left(-\frac{1}{\sqrt{\gamma}}\right) \sim \frac{\alpha\sqrt{\gamma}}{2}\left(\xi + \frac{1}{\sqrt{\gamma}}\right)^2, \qquad q\left(-\frac{1}{\sqrt{\gamma}}\right) \sim \frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{\gamma}}x}.$$

That is, $P = \frac{\alpha\sqrt{\gamma}}{2}$, $\mu = 2$, $Q = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{1}{\sqrt{\gamma}}x}$ and $\lambda = 1$.

(iii) For any fixed $\epsilon > 0$, $\mathcal{V}_{-\frac{1}{\sqrt{\gamma}}+\epsilon,\,0}(q/p') < \infty$. In fact,

$$\frac{q}{p'} = \frac{(1+\gamma\xi^2)\mathrm{e}^{\mathrm{i}x\xi}}{\alpha(1-\gamma\xi^2)}, \qquad \mathcal{V}_{-\frac{1}{\sqrt{\gamma}}+\epsilon,\,0}\left(\frac{q}{p'}\right) = \int_{-\frac{1}{\sqrt{\gamma}}+\epsilon}^{0} \left|\left(\frac{q}{p'}\right)'\right| \mathrm{d}\xi < \infty.$$

(iv) As $\xi \to 0-$, $q/p' \to 1/\alpha$.

Theorem B.1 then implies

$$\int_{-\frac{1}{\sqrt{\gamma}}}^{0} \frac{1}{1+\gamma\xi^2}\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi \sim \mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\frac{1}{4}\mathrm{e}^{-\mathrm{i}\frac{x}{\sqrt{\gamma}}}\,\Gamma\left(\frac{1}{2}\right)\frac{\mathrm{e}^{\mathrm{i}t\left(\frac{\alpha}{2\sqrt{\gamma}}\right)}}{(\alpha/2\sqrt{\gamma}t)^{1/2}}$$

$$= \frac{\sqrt{\pi}\,\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\,\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{\gamma}}\left(-x+\frac{\alpha t}{2}\right)}}{2\sqrt{2\alpha}\sqrt[4]{\gamma}\sqrt{t}}.$$

Estimate (3.7) also follows from theorem B.1. The conditions can be similarly checked for this integral. In fact, for

$$a = \frac{1}{\sqrt{\gamma}}, \qquad b = M, \qquad p(\xi) = -\frac{\alpha\xi}{1+\gamma\xi^2} \qquad \text{and} \qquad q(\xi) = \frac{\mathrm{e}^{\mathrm{i}\xi x}}{1+\gamma\xi^2},$$

we have

$$p(\xi) - p\left(\frac{1}{\sqrt{\gamma}}\right) \sim \frac{\alpha\sqrt{\gamma}}{2}\left(\xi - \frac{1}{\sqrt{\gamma}}\right)^2, \qquad q\left(\frac{1}{\sqrt{\gamma}}\right) \sim \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{1}{\sqrt{\gamma}}x}.$$

It is also easy to verify that $\mathcal{V}_{\frac{1}{\sqrt{\gamma}}+\epsilon,\,M}(q/p') < \infty$ for any fixed $\epsilon > 0$. In fact,

$$\frac{q}{p'} = -\frac{(1+\gamma\xi^2)\mathrm{e}^{\mathrm{i}x\xi}}{\alpha(1-\gamma\xi^2)}, \qquad \mathcal{V}_{\frac{1}{\sqrt{\gamma}}+\epsilon,\,M}\left(\frac{q}{p'}\right) = \int_{\frac{1}{\sqrt{\gamma}}+\epsilon}^{M} \left|\left(\frac{q}{p'}\right)'\right| \mathrm{d}\xi < \infty. \quad \text{(B.2)}$$

In addition, as $\xi \to M$, q/p' tends to a finite limit. It then follows from theorem B.1 that

$$\int_{\frac{1}{\sqrt{\gamma}}}^{M} \frac{1}{1+\gamma\xi^2}\mathrm{e}^{\mathrm{i}x\xi - \mathrm{i}\frac{\alpha\xi}{1+\gamma\xi^2}t}\,\mathrm{d}\xi \sim \frac{\sqrt{\pi}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\,\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{\gamma}}\left(x-\frac{\alpha t}{2}\right)}}{2\sqrt{2\alpha}\sqrt[4]{\gamma}\sqrt{t}}.$$

When $M = \infty$, (B.2) cannot be verified and theorem B.1 does not apply to the integral on $(-\infty, \infty)$.

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