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TEMPORAL GROWTH AND EVENTUAL PERIODICITY FOR DISPERSIVE WAVE EQUATIONS IN A QUARTER PLANE

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Abstract. Studied here is the large-time behavior and eventual periodicity of solutions of initial-boundary-value problems for the BBM equation and the KdV equation, with and without a Burgers-type dissipation appended. It is shown that the total energy of a solution of these problems grows at an algebraic rate which is in fact sharp for solutions of the associated linear equations. We also establish that solutions of the linear problems are eventually periodic if the boundary data are periodic.

1. **Introduction.** Initial-boundary-value problems for the KdV equation or the BBM equation arise naturally in modeling small-amplitude long waves in a channel with a wavemaker mounted at one end, or in modeling coastal zone motions generated by long-crested waves propagating shoreward from deep water (see, for example [9], [3], [4]). Such mathematical formulations have received considerable attention in the past, and a satisfactory theory of global well-posedness is in place for initial and boundary conditions satisfying physically relevant smoothness and consistency assumptions (see e.g. [5], [10], [12], [15] and the references contained therein).

Our major interest here is certain qualitative aspects of solutions, suggested by experiments, that are connected with their large-time behavior. The issues are of obvious importance for understanding the kind of time-dependent equilibrium that is reached in a laboratory channel under constant periodic forcing or in a near-shore zone approached by a regular wavetrain with a long fetch. In particular, we will address the question of energy growth, both locally and globally and the issue of what we term *eventual periodicity* which is exhibited by solutions of initial-boundary-value problems (IBVP henceforth) for the generalized BBM equation and the generalized KdV equation

$$u_t + u_x + u^p u_x - u_{xxt} = 0 \quad \text{for } x, t \geq 0,$$

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$$u_t + u_x + u^p u_x + u_{xxx} = 0 \quad \text{for } x, t \geq 0,$$

respectively, and their dissipative counterparts which includes a Burgers-type term, namely

$$u_t + u_x + u^p u_x - u_{xxt} - \nu u_{xx} = 0 \quad \text{for } x, t \geq 0, \tag{1}$$

$$u_t + u_x + u^p u_x + u_{xxx} - \nu u_{xx} = 0 \quad \text{for } x, t \geq 0, \tag{2}$$

where $\nu > 0$. All these equations will be supplemented with the initial and boundary conditions

$$u(x, 0) = f(x) \quad \text{for } x \geq 0 \quad \text{and} \quad u(0, t) = g(t) \quad \text{for } t \geq 0. \tag{3}$$

As is now well known, if $\nu > 0$, order-one solutions of the pure initial-value problems posed on the whole real line $\mathbb{R}$ for either (1) or (2) decay to zero as t approaches infinity (see e.g. [2], [6]). In contrast, the total energy of solutions of the IBVP for the above equations may grow as t increases. Intuitively, the growth is due to energy that is transmitted into the system at the boundary. One issue that will occupy us here is to determine the rate at which the total energy in the system, as measured by a natural norm, increases. Indeed, it will be shown that if the boundary data g satisfies the growth conditions

$$\int_0^t g(\tau)^2 d\tau = O(t) \quad \text{and} \quad \int_0^t g'(\tau)^2 d\tau = O(t)$$

as $t \to \infty$, then the L_2-norms of u and u_x grow at most algebraically with t. For solutions of the associated linear problems, the sharp growth rate $t^{1/2}$ is established. It is interesting to note that the presence of a Burgers-type dissipation is not strong enough to offset the influx of energy from the boundary. This is in contrast to the situation that obtains when damping of the form μu, $\mu > 0$, is introduced. In this case, the total energy of solutions of the initial-boundary-value problems

$$u_t + u_x + u^p u_x + \mu u - u_{xxt} = 0 \tag{4}$$

and

$$u_t + u_x + u^p u_x + \mu u + u_{xxx} = 0 \tag{5}$$

posed for $x, t \geq 0$, and with boundary conditions as in (3), are uniformly bounded in time (c.f. [11]). For the nonlinear problems (1) and (2), it may be the case that the nonlinear term dominates and that solutions grow faster than $t^{1/2}$. At least our energy-type arguments do not suffice to deny this possibility. If it is true that the sharp rates of growth for the nonlinear problem are not the same as for the linear boundary-value problems, it would be in contrast with the situation that obtains for the pure initial-value problems, in which the asymptotic rates of boundedness, or decay in case $\nu > 0$, of the energy is the same for solutions of the linear and the corresponding nonlinear equations. Finally, it is also worth remark that the present results represent a significant improvement over the current exponential growth rates in the literature (see e.g. [7]).

A second, related point will then occupy attention. Laboratory experiments in a channel with a flap-type or piston-type wavemaker mounted at one end show an interesting phenomenon. If the wavemaker is oscillated periodically, say with a period T, and the wave elevation $g_{x_0}(t) = \eta(x_0, t)$ is observed as a function of time at some station x_0 down the channel, then it appears that in due course, g_{x_0} becomes periodic of period T. Of course, at different stations x_0, the associated function g_{x_0} varies in its structure. Our second overall goal is to establish this observation as a mathematically exact fact about solutions of the aforementioned model equations. To this end, the boundary data $g = g_0$ will be assumed periodic of period $T > 0$, say,

$$g(t+T) = g(t) \quad \text{for all } t \geq 0.$$

In case $\nu > 0$, it is then shown that solutions u of the associated linear problems are eventually periodic of period T, which is to say

$$\|u(\cdot, t+T) - u(\cdot, t)\|_{L_\infty} \to 0 \quad \text{as } t \to \infty,$$

where $\|\cdot\|_{L_\infty}$ is the L_∞-norm in the x-variable. Similar, but somewhat weaker results hold for $\nu = 0$. As $t \to \infty$, the solution $u(x,t)$ with periodic forcing converges to a function $u^*(x,t)$ which is a solution of the boundary-value problem that is exactly periodic in t.

These developments complement the work in [11] where the issue of asymptotic periodicity was initiated in the context of (5) with $\mu > 0$. In [11], results are established for the nonlinear problem that are similar to those derived here for the linear problem corresponding to (1) and (2) with $\nu > 0$. The analysis in the linear case that appears here, in particular the integral representations, will very likely find use in a theory of eventual periodicity for the nonlinear problems (see, again, [11]). It should be remarked that for the pure initial-value problems posed on all of $\mathbb{R}$, the temporal decay rates of (2) with $\nu > 0$ and (5) with $\mu > 0$ are quite different, the former being generically of order $t^{-\frac{1}{2}}$ whilst the latter throws up exponential decay. This disparity is a hint as to why asymptotic periodicity is a little more subtle with the Burgers-type dissipation than it is for the dissipation featured in (4) and (5).

Considerable effort has gone into the well-posedness theory for initial-boundary-value problems of the form depicted in (2)-(3) (see [5], [8], [10], [12], [13], [14] and [15]). Part of the purpose of the more recent of these works was to extend the range of the earlier theory as exemplified in [7], [8], [12] and [13] to include much weaker regularity assumptions on the auxiliary data f and g. Our goals here are different, and consequently we will not struggle to state our conclusions under the most general hypotheses for which they might hold. Rather, we will be content to establish our theory under assumptions on f and g strong enough to directly justify the intermediate calculations that come to the fore in our analysis. Appropriate continuous dependence results may then be invoked to obtain sharper conclusions, but on the whole, we eschew this exercise.

Notation and Outline

The notation in force throughout is standard. The symbol $L_2(\mathbb{R}^+)$ signifies the usual Hilbert space of measurable, square-integrable functions on $\mathbb{R}^+$ and the norm of any $f \in L_2(\mathbb{R}^+)$ is written unadorned as $\|f\|$. If $f \in H^s(\mathbb{R}^+)$, the Sobolev class of $L_2(\mathbb{R}^+)$-functions whose first s derivatives also lie in $L_2(\mathbb{R}^+)$, its norm will be denoted by $\|f\|_s$. The norm of all other Banach spaces X will be denoted $\|\cdot\|_X$.

The plan of this paper is as follows. Section 2 is concerned with the large-time bounds on solutions of all four of the initial-boundary-value problems mentioned above. These are obtained by energy-type estimates. They would be standard except for the non-homogeneous boundary conditions at $x = 0$. Attention is then restricted to the linear versions of the IBVP's in Section 3 with the boundary data taken to be periodic. In this context, eventual periodicity of solutions is established. The script concludes with a brief summary and comments about directions that may be worth further development.

2. **Large-time bounds.** In this section, attention is given to the large-time behavior of solutions of the IBVP's for the BBM equation, the KdV equation, the BBM-Burgers equation and the KdV-Burgers equation. The goal is to understand fairly precisely how norms of solutions behave for large t. To begin, consider the first-order linear equation

$$u_t + u_x = 0 \quad \text{for } x, t \geq 0,$$

with $u(x,0) = f(x)$ and $u(0,t) = g(t)$. In the KdV-range of small-amplitude long waves, this equation is the lowest-order approximation for unidirectional motions. One easily verifies that

$$\|u(\cdot,t)\|^2 = \|f\|^2 + \int_0^t g(\tau)^2 d\tau, \qquad \|u_x(\cdot,t)\| = \|f_x\|^2 + \int_0^t g'(\tau)^2 d\tau.$$

Therefore, if $f \in H^1(\mathbb{R}^+)$ and

$$\int_0^t g(\tau)^2 d\tau = O(t) \quad \text{and} \quad \int_0^t g'(\tau)^2 d\tau = O(t) \tag{6}$$

as $t \to \infty$, then

$$\|u(\cdot,t)\|_1 = O(t^{\frac{1}{2}})$$

as $t \to \infty$. This simple calculation indicates the sharp growth of order $t^{\frac{1}{2}}$ in both L_2 and H^1 that one might expect to obtain for solutions of the full nonlinear, dispersive and dissipative IBVP's. The growth assumptions (6) on g are natural. Indeed, many physically relevant boundary data such as periodic functions, or more generally, data that along with its derivatives, is uniformly bounded in time conform to such an assumption.

2.1. **The BBM equation.** In this subsection, interest is focused on the IBVP for the generalized BBM equation

$$u_t + u_x + u^p u_x - u_{xxt} = 0, \qquad \text{for } x, t \geq 0, \tag{7}$$

with

$$u(x,0) = f(x) \qquad \text{for } x \geq 0 \quad \text{and} \quad u(0,t) = g(t) \qquad \text{for } t \geq 0. \tag{8}$$

Assume that u is a solution of (7)-(8) which, along with its first few partial derivatives, lies in $L_2(\mathbb{R}^+)$ for each $t \geq 0$ and is smooth up to the boundary. Such an assumption is certainly justified by the existing theory provided the consistency condition

$$f(0) = g(0) \tag{9}$$

is satisfied (see e.g. [7], [8], [12] and [13]). This condition will be in force throughout for both KdV-type and BBM-type equations.

If (7) is multiplied by $2u(x,t)$, and the result integrated over $\mathbb{R}^+$ and then $[0,t]$, we obtain after appropriate integrations by parts

$$\|u(\cdot,t)\|_1^2 = \|f\|_1^2 + \int_0^t \Big[\frac{2}{p+2} g^{p+2} + g^2 - 2g u_{xt}(0,\tau)\Big] d\tau.$$

Applying the Cauchy-Schwartz inequality gives

$$\begin{aligned} \|u(\cdot,t)\|_1^2 \leq \|f\|_1^2 + \int_0^t \Big[\frac{2}{p+2} g^{p+2} + g^2\Big] d\tau \\ +2\Big(\int_0^t g^2 d\tau\Big)^{\frac{1}{2}} \Big(\int_0^t u_{xt}^2(0,\tau) d\tau\Big)^{\frac{1}{2}}. \end{aligned} \tag{10}$$

To deal with the potentially troublesome term involving $u_{xt}(0,t)$ in the above inequality, multiply (7) by $2u_{xt}(x,t)-\frac{2}{p+1}u^{p+1}$ and integrate over $\mathbb{R}^+\times[0,t]$ to obtain

$$\begin{aligned}\|u_x(\cdot,t)\|^2 + \int_0^t u_{xt}^2(0,\tau)\,d\tau - \frac{2}{(p+1)(p+2)}\int_0^\infty u^{p+2}\,dx \qquad (11)\\ = \|f'\|^2 - \frac{2}{(p+1)(p+2)}\int_0^\infty f^{p+2}\,dx\\ + \int_0^t \Big[(g')^2 + \frac{2}{p+1}g^{p+1}u_{xt}(0,\tau) - \frac{1}{(p+1)^2}g^{2(p+1)} - \frac{2}{(p+1)(p+2)}g^{p+2}\Big]d\tau.\end{aligned}$$

Theorem 2.1. *If $f\in H^1(\mathbb{R}^+)$ and $g\in C^1(\mathbb{R}^+)$ satisfy the conditions*

$$\int_0^t g(\tau)^2 d\tau = O(t),\ \int_0^t g(\tau)^{2p+2}d\tau = O(t),\ \int_0^t g'(\tau)^2 d\tau = O(t), \qquad (12)$$

as $t\to\infty$, then any solution u of the IBVP (7) and (8) with $0\le p<2$ satisfies

$$\|u(\cdot,t)\|_1 \le C(t+1)^{\frac{1}{2-p}},$$

where C is a constant depending only on $\|f\|_1$. In particular, if $p=1$,

$$\|u(\cdot,t)\|_1 = O(t)$$

as $t\to\infty$.

Remarks. If $p=0$, we are considering the linearized BBM-equation. Note in this case that one of the hypotheses is redundant. Note also that, for $p\ge 1$, $\int_0^t |g(\tau)|^r d\tau = O(t)$ for any $r\in[2,2p+2]$ by interpolation. If g and g' are uniformly bounded, then (12) holds. Thus (12) holds for $g(t)=a\sin(bt)$ say, $a,b>0$, or indeed for any smooth periodic function. For non-integer values of p, it is assumed throughout that $p=m/n$ is rational with m,n relatively prime and n odd. In this case, a branch of the mapping $z\to z^{1/n}$ may be chosen so that u^p is real if u is real.

Proof. For $f\in H^1$ and $g\in C^1$ satisfying (12), the inequality (2.1) gives the bound

$$\|u(\cdot,t)\|_1^2 \le C(t+1) + C(t+1)^{\frac12}\Big(\int_0^t u_{xt}^2(0,\tau)\,d\tau\Big)^{\frac12}. \qquad (13)$$

Applying Young's inequality to (11) yields

$$\begin{aligned}\frac12\int_0^t u_{xt}^2(0,\tau)\,d\tau \le \|f'\|^2 + \frac{2}{(p+1)(p+2)}\int_0^\infty \big[u^{p+2}-f^{p+2}\big]\,dx\\ + \int_0^t \Big[(g')^2 + \frac{1}{(p+1)^2}g^{2(p+1)} - \frac{2}{(p+1)(p+2)}g^{p+2}\Big]d\tau. \qquad (14)\end{aligned}$$

Thus, for $f\in H^1(\mathbb{R}^+)$ and g satisfying (12), it follows from (14) that

$$\int_0^t u_{xt}^2(0,\tau)\,d\tau \le C(t+1) + \frac{4}{(p+1)(p+2)}\int_0^\infty u^{p+2}\,dx$$

where C is a constant depending only on $\|f\|_1$ and the constants implied by (12) related to g. Because of the elementary inequality

$$\|h\|_{L_\infty(\mathbb{R}^+)}^2 \le \sqrt{2}\|h\|\,\|h_x\|, \qquad (15)$$

it transpires that

$$\int_0^\infty u^{p+2}\,dx \le \|u(\cdot,t)\|_{L_\infty}^p \|u(\cdot,t)\|^2 \le \|u(\cdot,t)\|^{2+\frac p2}\|u_x(\cdot,t)\|^{\frac p2} \le \|u(\cdot,t)\|_1^{p+2}.$$

Combining the above estimates leads securely to the inequality

$$\|u(\cdot,t)\|_1^2 \le C\,(t+1) + C(t+1)^{\frac{1}{2}}\left[C(t+1) + C\|u(\cdot,t)\|_1^{p+2}\right]^{\frac{1}{2}}. \tag{16}$$

For any $a, b \ge 0$, $(a+b)^{\frac{1}{2}} \le a^{\frac{1}{2}} + b^{\frac{1}{2}}$; thus (16) implies that

$$\|u(\cdot,t)\|_1^2 \le C(t+1) + C(t+1)^{\frac{1}{2}}\|u(\cdot,t)\|_1^{1+\frac{p}{2}}. \tag{17}$$

The advertised inequality now follows from the latter inequality by applying the next, elementary lemma which may be found, for example, in [17]. ☐

Lemma 2.2. *Let P, Q and $\beta < 2$ be positive numbers. If $Y \ge 0$ satisfies*

$$Y^2 \le PY^\beta + Q,$$

then Y is bounded in terms of P, Q and β, and in fact

$$Y \le \max\left\{(2P)^{\frac{1}{2-\beta}}, (2Q)^{\frac{1}{2}}\right\}.$$

2.2. **The KdV equation.** Consideration is now given to determining the growth rates for solutions of the IBVP

$$u_t + u_x + u^p u_x + u_{xxx} = 0 \qquad \text{for } x, t \ge 0, \tag{18}$$

with

$$\begin{aligned} u(x,0) &= f(x) \qquad \text{for } x \ge 0, \\ u(0,t) &= g(t) \qquad \text{for } t \ge 0 \end{aligned} \tag{19}$$

for the generalized KdV equation. Just as for the BBM equation, the consistency condition (9) will be presumed. As needed for regularity, higher-order consistency conditions will also be assumed (see [13]).

Multiplying equation (18) by $2u(x,t)$ and integrating over $\mathbb{R}^+ \times [0,t]$, there obtains

$$\begin{aligned} &\|u(\cdot,t)\|^2 + \int_0^t u_x^2(0,\tau)d\tau \\ &= \|f\|^2 + \int_0^t \left[g^2 + 2gu_{xx}(0,\tau) + \frac{2}{p+2}g^{p+2}\right]d\tau \end{aligned} \tag{20}$$

after integrations by parts. Next, multiply (18) by the combination $2u_{xx} + \frac{2}{p+1}u^{p+1}$ and integrate to arrive at the relation

$$\begin{aligned} &\|u_x(\cdot,t)\|^2 + \int_0^t u_x^2(0,\tau)d\tau + \int_0^t \left[u_{xx}(0,\tau) + \frac{1}{p+1}g^{p+1}\right]^2 d\tau \\ &= \|f'\|^2 + \frac{2}{(p+1)(p+2)}\int_0^\infty u^{p+2}\,dx - \frac{2}{(p+1)(p+2)}\int_0^\infty f^{p+2}\,dx \\ &\quad - \int_0^t \left[\frac{2}{(p+1)(p+2)}g^{p+2} + 2g'(\tau)u_x(0,\tau)\right]d\tau. \end{aligned} \tag{21}$$

Theorem 2.3. *Let $f \in H^1(\mathbb{R}^+)$ and let $g \in C^1(\mathbb{R}^+)$ satisfy*

$$\int_0^t g(\tau)^2 d\tau = O(t) \quad \textit{and} \quad \int_0^t g'(\tau)^2 d\tau = O(t)$$

as $t \to \infty$. Then, for $0 \le p < \frac{4}{3}$, the IBVP (18)-(19) satisfies the global bounds

$$\|u(\cdot,t)\| = O\left(t^{\frac{4-p}{2(4-3p)}}\right) \quad \textit{and} \quad \|u_x(\cdot,t)\| = O\left(t^{\frac{p+4}{2(4-3p)}}\right)$$

as $t \to \infty$, where the constants implied depend on $\|f\|_1$ and on the constants implied in the assumptions on g. In particular these conclusions for $p = 1$ amount to

$$\|u(\cdot,t)\| = O\left(t^{\frac{3}{2}}\right) \quad \textit{and} \quad \|u_x(\cdot,t)\| = O\left(t^{\frac{5}{2}}\right)$$

as $t \to \infty$.

Remark. As for Theorem 2.1, the case $p = 0$ is the linear KdV equation. The hypotheses on g are valid for any smooth periodic or almost periodic boundary forcing.

Proof. Applying the Cauchy-Schwartz inequality to (20) yields

$$\begin{aligned}\|u(\cdot,t)\|^2 &\le \|f\|^2 + \int_0^t \left[g^2 + \frac{2}{p+2} g^{p+2}\right] d\tau + 2\left(\int_0^t g^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t u_{xx}^2(0,\tau)d\tau\right)^{\frac{1}{2}} \\ &\le C_2(t+1) + C_3(t+1)^{\frac{1}{2}} \left(\int_0^t u_{xx}^2(0,\tau)d\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{22}$$

It then follows that for all $t > 0$,

$$\begin{aligned}\left|\int_0^\infty u^{p+2} dx\right| &\le \sqrt{2}\|u(\cdot,t)\|^{2+\frac{p}{2}} \|u_x(\cdot,t)\|^{\frac{p}{2}} \\ &\le C\left[(t+1) + (t+1)^{\frac{1}{2}}\left(\int_0^t u_{xx}^2(0,\tau)d\tau\right)^{\frac{1}{2}}\right]^{1+\frac{p}{4}} \|u_x(\cdot,t)\|^{\frac{p}{2}}.\end{aligned}$$

Inserting this estimate into (21), there appears the inequality

$$\begin{aligned}\|u_x(\cdot,t)\|^2 + \int_0^t u_{xx}^2(0,\tau)d\tau &\le C_4(t+1) \\ &+ C\left[(t+1) + (t+1)^{\frac{1}{2}}\left(\int_0^t u_{xx}^2(0,\tau)d\tau\right)^{\frac{1}{2}}\right]^{1+\frac{p}{4}} \|u_x(\cdot,t)\|^{\frac{p}{2}}\end{aligned} \tag{23}$$

valid for all $t > 0$. It is convenient to write

$$X(t) = \|u_x(\cdot,t)\| \quad \text{and} \quad Y^2(t) = \int_0^t u_{xx}^2(0,\tau)d\tau.$$

With this notation, (23) becomes

$$X^2 + Y^2 \le C(t+1) + C\left[(t+1) + (t+1)^{\frac{1}{2}}\, Y\right]^{1+\frac{p}{4}} X^{\frac{p}{2}}$$

where the constants only depend on $\|f\|_1$ and the constants implied by the assumptions on g. For $\mu \ge 0$, $q > 1$ and $r > 1$, Young's inequality insures that

$$\begin{aligned}X^2 + Y^2 &\le C_4(t+1) + C_8(t+1)^{q\left(1+\frac{p}{4}-\mu\right)} \\ &+ C_6(t+1)^{\mu r} X^{\frac{pr}{2}} + C_9(t+1)^{q\left(\frac{1}{2}+\frac{p}{8}-\mu\right)} Y^{q\left(1+\frac{p}{4}\right)}.\end{aligned} \tag{24}$$

We need the following, more general version of Lemma 2.2

Lemma 2.4. *Let $\alpha < 2, \beta < 2$, P_1, P_2 and Q be nonnegative numbers. If $X \ge 0$ and $Y \ge 0$ satisfy*

$$X^2 + Y^2 \le P_1 X^\alpha + P_2 Y^\beta + Q,$$

then

$$\max\{X, Y\} \le \max\left\{\sqrt{2Q},\, (2P_1 + 2P_2)^{\frac{1}{2-\max\{\alpha,\beta\}}}\right\}.$$

Applying Lemma 2.4 to the inequality (24), it is determined that $X(t)$ and $Y(t)$ are bounded above by

$$\max\left\{\left[(t+1)+(t+1)^{q\left(1+\frac{p}{4}-\mu\right)}\right]^{\frac{1}{2}},\right.$$
$$\left.\left[(t+1)^{\mu r}+(t+1)^{q\left(\frac{1}{2}+\frac{p}{8}-\mu\right)}\right]^{\frac{1}{2-\max\{(1+p/4)q,pr/2\}}}\right\}.$$

If μ, q and r are chosen to be

$$\mu=\frac{p(p+4)}{4(4+3p)},\quad q=\frac{4+3p}{4+p},\quad r=\frac{4+3p}{2p},$$

then for $p<\frac{4}{3}$,

$$X(t)+Y(t)\le C(t+1)^{\frac{p+4}{2(4-3p)}},$$

which is to say that for $p<\frac{4}{3}$,

$$\|u_x(\cdot,t)\|\le C(t+1)^{\frac{p+4}{2(4-3p)}}\quad\text{and}\quad\int_0^t u_{xx}^2(0,\tau)d\tau\le C(t+1)^{\frac{p+4}{2(4-3p)}}. \tag{25}$$

In particular, if $p=1$,

$$\|u_x(\cdot,t)\|\le C(t+1)^{\frac{5}{2}}.$$

Inserting (25) into (22), there obtains

$$\|u(\cdot,t)\|\le C(t+1)^{\frac{4-p}{2(4-3p)}},$$

which specializes to $\|u(\cdot,t)\|\le C(t+1)^{\frac{3}{2}}$ for $p=1$. This completes the proof of Theorem 2.3. □

2.3. **The BBM-Burgers equation.** Attention is now given to situations where a Burgers-type damping is featured in addition to nonlinearity and dispersion. We start with the IBVP

$$u_t+u_x+u^pu_x-\nu u_{xx}-u_{xxt}=0\qquad\text{for } x,t\ge 0, \tag{26}$$

with

$$\begin{aligned} u(x,0)&=f(x)\qquad\text{for } x\ge 0,\\ u(0,t)&=g(t)\qquad\text{for } t\ge 0\end{aligned} \tag{27}$$

for the BBM-Burgers equation.

Energy-type methods are used to derive three equalities. These results are the key to our further ruminations. Multiplying (26) by $2u$, u_{xx} and $2\left(\frac{u^{p+1}}{p+1}-u_{xt}\right)$, respectively, and integrating over $\mathbb{R}^+\times[0,t]$, we obtain after integrations by parts that

$$\begin{aligned}\int_0^\infty(u^2+u_x^2)dx+2\nu\int_0^t\int_0^\infty u_x^2\,dxd\tau&=\int_0^\infty(f^2+f_x^2)dx-2g(t)u_x(0,t)+2g(0)f'(0)\\&\quad+\int_0^t\left[\frac{2}{p+2}g^{p+2}+g^2-2\nu gu_x(0,\tau)+2g'u_x(0,\tau)\right]d\tau,\end{aligned} \tag{28}$$

$$\begin{aligned}\|u_x(\cdot,t)\|^2+\|u_{xx}(\cdot,t)\|^2+2\nu\int_0^t\|u_{xx}(\cdot,\tau)\|^2d\tau+2\int_0^t u_x^2(0,\tau)d\tau&\\=\|f_x\|^2+\|f_{xx}\|^2&\\-\int_0^t\int_0^\infty pu^{p-1}u_x^3\,dxd\tau-\int_0^t[2g'u_x(0,\tau)-g^pu_x^2(0,\tau)]d\tau&\end{aligned} \tag{29}$$

and

$$(1+\nu)\|u_x(\cdot,t)\|^2+\nu u_x^2(0,t)+\int_0^t\left(\frac{1}{p+1}g^{p+1}-u_{xt}(0,\tau)\right)^2d\tau$$
$$=(1+\nu)\|f_x\|^2+\nu u_x^2(0,0)+2\nu\int_0^t\int_0^\infty u^p u_x^2\,dxd\tau+\frac{2}{(p+1)(p+2)}\int_0^\infty(u^{p+2}-f^{p+2})dx$$
$$+\int_0^t\left[(g')^2-\frac{2}{(p+1)(p+2)}g^{p+2}+\frac{2\nu}{p+1}g^{p+1}u_x(0,\tau)\right]d\tau. \qquad (30)$$

Prior to this work, estimates obtained indicated exponential growth rates (see e.g. [7]); these are here considerably improved, though they are not in line with the associated linear theory.

Theorem 2.5. *Assume that $f\in H^2(\mathbb{R}^+)$ and $g\in C^1(\mathbb{R}^+)$ and suppose $|g(t)|\le 1$ for all $t\ge 0$. If there exists a constant C_0 such that g satisfies*

$$\int_0^t g'(\tau)^2d\tau<C_0t$$

for $t\ge 0$, then, for $0\le p<\frac{3}{2}$, the solution u of the IBVP (26)-(27) with auxiliary data f and g has the properties

$$\|u(\cdot,t)\|\le C_1(t+1)^{\frac{3}{2(3-2p)}},\ \|u_x(\cdot,t)\|\le C_1(t+1)^{\frac{2p+3}{2(3-2p)}},\ \|u_{xx}(\cdot,t)\|\le C_1(t+1)^{\frac{2p+3}{2(3-2p)}},$$

where C_1 is a constant depending only on $\|f\|_2$ and C_0. In particular, for $p=1$,

$$\|u(\cdot,t)\|\le C_1(t+1)^{\frac{3}{2}},\quad \|u_x(\cdot,t)\|\le C_1(t+1)^{\frac{5}{2}},\quad \|u_{xx}(\cdot,t)\|\le C_1(t+1)^{\frac{5}{2}}.$$

Remark. For the linearized BBM-Burgers equation, the growth rate of the $H^2(\mathbb{R}^+)$-norm is optimal. That is, when f and g satisfy the conditions of Theorem 2.5, the corresponding solution u of the linear BBM-Burgers equation satisfies

$$\|u(\cdot,t)\|\le C(t+1)^{\frac{1}{2}},\quad \|u_x(\cdot,t)\|\le C(t+1)^{\frac{1}{2}},\quad \|u_{xx}(\cdot,t)\|\le C(t+1)^{\frac{1}{2}}.$$

Proof of Theorem 2.5. First, note that since $|g(t)|\le 1$ for all $t\ge 0$, it follows that

$$\int_0^t g^2(t)dt\le t\quad\text{and}\quad\int_0^t g^{p+2}(t)dt\le t$$

for all $t\ge 0$. Inserting the simple estimates

$$-2g(t)u_x(0,t)\le 2|g(t)|\|u_x(\cdot,t)\|_{L^\infty}\le 2^{\frac{3}{2}}|g(t)|\|u_x(\cdot,t)\|^{\frac{1}{2}}\|u_{xx}(\cdot,t)\|^{\frac{1}{2}}$$

and

$$\int_0^t\left[\frac{2}{p+2}g^{p+2}+g^2-2\nu g u_x(0,\tau)+2g'u_x(0,\tau)\right]d\tau$$
$$\le\frac{p+4}{p+2}t+2(\nu+C_0)t^{\frac{1}{2}}\left(\int_0^t u_x^2(0,\tau)d\tau\right)^{\frac{1}{2}}$$

into (28), there obtains

$$\|u(\cdot,t)\|^2+\|u_x(\cdot,t)\|^2+2\nu\int_0^t\int_0^\infty u_x^2(x,\tau)dxd\tau \qquad (31)$$
$$\le C(t+1)+2^{\frac{3}{2}}\|u_x(\cdot,t)\|^{\frac{1}{2}}\|u_{xx}(\cdot,t)\|^{\frac{1}{2}}+C't^{\frac{1}{2}}\left(\int_0^t u_x^2(0,\tau)d\tau\right)^{\frac{1}{2}}.$$

The constant C depends on $\|f\|_1$ while C' depends on ν and C_0. Since $|g(t)| \le 1$, it follows from (29) that

$$\|u_x(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 + 2\,\nu \int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau + \frac{1}{2}\int_0^t u_x^2(0,\tau)d\tau \le C''(t+1)$$
$$+\sqrt{2}p \sup_{0\le s\le t}\left\{\|u(\cdot,s)\|^{\frac{p-1}{2}}\,\|u_x(\cdot,s)\|^{\frac{p}{2}}\,\|u_{xx}(\cdot,s)\|^{\frac{1}{2}}\right\}\int_0^t\int_0^\infty u_x^2(x,\tau)dxd\tau$$

where C depends on $\|f\|_2$ and C_0. Using the bounds in (31), the latter inequality can be extended to

$$\|u_x(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 + 2\,\nu \int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau + \frac{1}{2}\int_0^t u_x^2(0,\tau)d\tau$$
$$\le C''(t+1) + \frac{1}{2\nu}\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^{\frac{1}{2}}$$
$$\times\left[C(t+1) + 2^{\frac{3}{2}}\sup_{0\le s\le t}\|u_x(\cdot,s)\|^{\frac{1}{2}}\|u_{xx}(\cdot,s)\|^{\frac{1}{2}} + C't^{\frac{1}{2}}\left(\int_0^t u_x^2(0,\tau)d\tau\right)^{\frac{1}{2}}\right]^{\frac{2p+3}{4}}.$$

The simple inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$, which holds for $a\ge 0, b\ge 0$ and $p \ge 1$, allows us to infer further that

$$\|u_x(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 + 2\,\nu \int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau + \frac{1}{2}\int_0^t u_x^2(0,\tau)d\tau$$
$$\le C''(t+1) + C'''(t+1)^{\frac{2p+3}{4}}\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^{\frac{1}{2}} + C_1\sup_{0\le s\le t}\|u_x(\cdot,s)\|^{\frac{2p+3}{8}}\|u_{xx}(\cdot,s)\|^{\frac{2p+7}{8}}$$
$$+C^{iv}t^{\frac{2p+3}{8}}\left(\int_0^t u_x^2(0,\tau)d\tau\right)^{\frac{2p+3}{8}}\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^{\frac{1}{2}}.$$

Here, C'', C''' and C^{iv} depend upon C_0 and $\|f\|_2$, but C_1 is a constant depending only on ν^{-1}. By Young's inequality, for $p < \frac{3}{2}$,

$$\|u_x(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 + 2\,\nu \int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau + \frac{1}{2}\int_0^t u_x^2(0,\tau)d\tau$$
$$\le \left[C''(t+1) + C^v(t+1)^{\frac{2p+3}{3}} + \frac{1}{4}\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^2\right]$$
$$+\left[C_2 + \frac{1}{2}\sup_{0\le s\le t}\|u_x(\cdot,s)\|^2 + \frac{1}{4}C^v\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^2\right]$$
$$+\left[C^{vi}(t+1)^{\frac{2p+3}{3-2p}} + \frac{1}{4}\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^2 + \frac{1}{4}\int_0^t u_x^2(0,\tau)d\tau\right].$$

The constants C^v and C^{vi} depend on C_0 and on $\|f\|_2$, while C_2 depends only on ν^{-1} and p. If we let

$$M(t) = \sup_{0\le s\le t}\|u_x(\cdot,s)\|^2 + \sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^2,$$

then since the right-hand side of the last inequality is increasing with t, it follows that

$$M(t) + 2\nu\int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau + \frac{1}{2}\int_0^t u_x^2(0,\tau)d\tau$$
$$\le C^{vi}(t+1)^{\frac{2p+3}{3}} + \frac{1}{2}M(t) + \frac{1}{4}\int_0^t u_x^2(0,\tau)d\tau,$$

from which it is deduced that, for all $t \geq 0$,

$$\|u_x(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 + 4\nu \int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau + \frac{1}{2}\int_0^t u_x^2(0,\tau)d\tau \leq C(t+1)^{\frac{2p+3}{3-2p}},$$

which is to say,

$$\|u_x(\cdot,t)\| \leq C(t+1)^{\frac{2p+3}{2(3-2p)}}, \quad \|u_{xx}(\cdot,t)\| \leq C(t+1)^{\frac{2p+3}{2(3-2p)}},$$

$$\int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau \leq C(t+1)^{\frac{2p+3}{3-2p}}, \qquad \int_0^t u_x^2(0,\tau)d\tau \leq C(t+1)^{\frac{2p+3}{3-2p}}.$$

Using the inequality $2\|u_x\|^{\frac{1}{2}}\|u_{xx}\|^{\frac{1}{2}} \leq \|u_x\| + \|u_{xx}\|$ in (31), there obtains

$$\|u(\cdot,t)\|^2 \leq C(t+1) + \sqrt{2}\|u_x(\cdot,t)\| + \sqrt{2}\|u_{xx}(\cdot,t)\| + C' t^{\frac{1}{2}} \left(\int_0^t u_x^2(0,\tau)d\tau\right)^{\frac{1}{2}}$$

which certainly implies that

$$\|u(\cdot,t)\| \leq C(t+1)^{\frac{3}{2(3-2p)}}.$$

Combining these inequalities gives immediately that

$$\|u(\cdot,t)\|_{L_\infty} \leq C(t+1)^{\frac{p+3}{2(3-2p)}}.$$

□

2.4. **The KdV-Burgers equation.** Attention is turned to the IBVP for the generalized KdV-Burgers equation

$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0 \quad \text{for } x, t \geq 0, \tag{32}$$

with

$$\begin{aligned} u(x,0) &= f(x) \qquad && \text{for } x \geq 0, \\ u(0,t) &= g(t) && \text{for } t \geq 0. \end{aligned} \tag{33}$$

Multiplying the equation by $2u$ and by $2u_{xx} + \frac{2}{p+1}u^{p+1}$, respectively, and integrating over $\mathbb{R}^+ \times [0,t]$ leads to

$$\|u(\cdot,t)\|^2 + 2\nu \int_0^t \int_0^\infty u_x^2\, dx d\tau + \int_0^t u_x^2(0,\tau)d\tau \tag{34}$$

$$= \|f\|^2 + \int_0^t \left[\frac{2}{p+2}g^{p+2} + g^2 - 2\nu g u_x(0,\tau) + 2g u_{xx}(0,\tau)\right] d\tau$$

and

$$\|u_x(\cdot,t)\|^2 + 2\nu \int_0^t \int_0^\infty u_{xx}^2\, dx d\tau + \int_0^t u_x^2(0,\tau)d\tau + \int_0^t \left[u_{xx}(0,\tau) + \frac{1}{p+1}g^{p+1}\right]^2 d\tau$$

$$= \|f'\|^2 + \frac{2}{(p+1)(p+2)}\int_0^\infty u^{p+2}\, dx - \frac{2}{(p+1)(p+2)}\int_0^\infty f^{p+2}\, dx$$

$$+ 2\nu \int_0^t \int_0^\infty u^p u_x^2\, dx d\tau$$

$$- \int_0^t \Big[\frac{2}{(p+1)(p+2)}g^{p+2} - \frac{2\nu}{p+1}g^{p+1}u_x(0,\tau) + 2g'(\tau)u_x(0,\tau)\Big] d\tau. \tag{35}$$

It is found that solutions of the generalized KdV-Burgers equation possess algebraic growth bounds.

Theorem 2.6. *Let $f \in H^1(\mathbb{R}^+)$ and $g \in C^1(\mathbb{R}^+)$. Suppose that $0 \le p < \frac{4}{3}$ and that there exists a constant C_0 such that*

$$\int_0^t g(\tau)^2 d\tau \le C_0 t, \quad \int_0^t g'(\tau)^2 d\tau \le C_0 t, \quad \int_0^t g(\tau)^{2p+2} d\tau \le C_0 t$$

for $t \ge 0$. Then the associated solution u of the IBVP (32)-(33) has temporal growth bounded above as follows:

$$\|u(\cdot,t)\| \le C(t+1)^{\frac{4-p}{8-6p}}, \quad \|u_x(\cdot,t)\| \le C(t+1)^{\frac{p+4}{4-3p}}. \tag{36}$$

In particular, if $p = 1$,

$$\|u(\cdot,t)\| \le C(t+1)^{\frac{3}{2}}, \quad \|u_x(\cdot,t)\| \le C(t+1)^{\frac{5}{2}}.$$

The constant C depends only on C_0 and $\|f\|_1$.

Proof. After inserting the inequalities

$$\int_0^\infty \left|u^{p+2}\right| dx \le \sqrt{2}\|u(\cdot,t)\|^{2+\frac{p}{2}} \|u_x(\cdot,t)\|^{\frac{p}{2}}$$

and

$$2\nu \int_0^t \int_0^\infty |u|^p u_x^2 \, dx d\tau \le 2 \max_{0 \le s \le t} \|u(\cdot,s)\|_{L_\infty}^p \left[\nu \int_0^t \int_0^\infty u_x^2 \, dx d\tau\right]$$

in (35) and applying some straightforward estimates, we obtain

$$\|u_x(\cdot,t)\|^2 + \int_0^t u_{xx}^2(0,\tau) d\tau \le C(t+1)$$

$$+C\left[[(t+1) + (t+1)^{\frac{1}{2}} \left(\int_0^t u_{xx}^2(0,\tau)d\tau\right)^{\frac{1}{2}}\right]^{1+\frac{p}{4}} \|u_x(\cdot,t)\|^{\frac{p}{2}}.$$

This estimate is similar to those obtained in (23); the inequality (36) follows immediately. □

3. **Eventual periodicity.** Our goal in this section is to demonstrate that solutions of the linearized versions of the equations under consideration are eventually periodic if the boundary data g is periodic. At the outset, it is interesting to consider again the first-order linear wave equation

$$u_t + u_x = 0, \qquad \text{for } x, t \ge 0,$$

with the initial and boundary conditions

$$u(x,0) = f(x) \quad \text{for } x \ge 0, \qquad u(0,t) = g(t) \quad \text{for } t \ge 0.$$

Its solution is indeed eventually periodic if g is periodic. In fact,

$$u(x,t) = \tilde{g}(t-x),$$

where

$$\tilde{g}(z) = \begin{cases} g(z) & \text{if } z \ge 0, \\ f(-z) & \text{if } z \le 0. \end{cases}$$

The function $\tilde{g}$ is continuous on account of the consistency condition (9). If $x = x_0$ is fixed, then for $t > x_0$, $u(x_0,t) = g(t-x_0)$ is exactly periodic.

In this section, we study in detail the eventual periodicity properties for the more complex problems presented by the linear IBVP for the BBM-Burgers equation, the KdV-Burgers equation, and also for their non-dissipative counterparts.

3.1. **The linear BBM-Burgers equation.** We start with the linear BBM-Burgers equation

$$u_t + u_x - u_{xxt} - \nu u_{xx} = 0 \quad \text{for } x, t \geq 0 \tag{37}$$

with $\nu > 0$ and

$$u(x,0) = f(x) \quad \text{for } x \geq 0, \qquad u(0,t) = g(t) \quad \text{for } t \geq 0. \tag{38}$$

Assume that g is periodic so that for some $T > 0$,

$$g(T+t) = g(t) \qquad \text{for any } t \geq 0.$$

It is our aim to show that any solution of IBVP (37)-(38) is eventually periodic of the same period, which is to say

$$\|u(\cdot, T+t) - u(\cdot, t)\|_{L_\infty} \to 0, \quad \text{as } t \to \infty. \tag{39}$$

To proceed, consider a new function $v(x,t) = u(x, T+t) - u(x,t)$ which satisfies

$$\begin{array}{ll} v_t + v_x - v_{xxt} - \nu v_{xx} = 0 & \text{for } x, t \geq 0, \\ v(x,0) = u(x,T) - f(x) = F(x) & \text{for } x \geq 0, \\ v(0,t) = 0 & \text{for } t \geq 0. \end{array} \tag{40}$$

Then (39) is equivalent to

$$\|v(\cdot, t)\|_{L_\infty} \to 0, \quad \text{as } t \to \infty. \tag{41}$$

Of course, the function F depends both explicitly and implicitly on f and on $g(t)$, $0 \leq t \leq T$. By the well-posedness theory for this problem (see [7]), if $f \in H^2(\mathbb{R}^+)$, say, and $g \in C^1(\mathbb{R}^+)$, then $u(\cdot, T)$ lies in $H^2(\mathbb{R}^+)$ and hence so does F.

Theorem 3.1. *Let v be the solution of (40). Then, it follows that*

- *(a) $\|v(\cdot,t)\|_1 \leq \|F\|_1$, $\quad |v(x,t)| \leq \|F\|_1$ for all $x, t \in \mathbb{R}^+$;*
- *(b) $v_x \in L_2(\mathbb{R}^+ \times \mathbb{R}^+)$, $\quad v_{xx} \in L_2(\mathbb{R}^+ \times \mathbb{R}^+)$;*
- *(c) $\|v_x(\cdot,t)\|$, $\|v_{xx}(\cdot,t)\|$, $\|v(\cdot,t)\|_{L_\infty} \to 0$ as $t \to \infty$.*

In consequence, the solution u of (37)-(38) with g periodic of period T is eventually periodic of period T.

Proof. Multiply the equation by $2v(x,t)$ and integrate over $\mathbb{R}^+ \times [0,t]$. Integration by parts and use of the zero boundary condition lead to

$$\|v(\cdot,t)\|_1^2 + 2\nu \int_0^t \int_0^\infty v_x^2 \, dx \, d\tau = \|F\|_1^2.$$

This time-independent bound implies (a) and the first part of (b). To complete the proof, multiply the equation by $2v_{xx}$ and integrate over $\mathbb{R}^+$ to obtain, after integration by parts, the exact relation

$$\frac{d}{dt} \int_0^\infty \left[v_x^2 + v_{xx}^2 \right] dx = -2\nu \int_0^\infty v_{xx}^2 dx - v_x^2(0,t). \tag{42}$$

Integrate (42) over the temporal interval $[0,t]$, thereby reaching the relation

$$\int_0^\infty \left[v_x^2 + v_{xx}^2 \right] dx + 2\nu \int_0^t \int_0^\infty v_{xx}^2 \, dx d\tau + \int_0^t v_x^2(0,\tau) d\tau = \int_0^\infty \left[F_x^2 + F_{xx}^2 \right] dx,$$

which implies that

$$v_{xx} \in L_2(\mathbb{R}^+ \times \mathbb{R}^+) \quad \text{and} \quad v_x^2(0,t) \in L_1(\mathbb{R}^+).$$

Returning to (42), the quantity

$$V(t) \equiv \int_0^\infty \left[v_x^2 + v_{xx}^2\right] dx \tag{43}$$

is determined to be decreasing and is obviously bounded below, hence it approaches a limit as $t \to \infty$. The limit must be zero since V lies in $L_1(0,\infty)$. That is, $\|v_x(\cdot,t)\|$, $\|v_{xx}(\cdot,t)\| \to 0$ as $t \to \infty$. Indeed, since $V \geq 0$ is a decreasing L_1-function, we must have that

$$V(t) \leq \max\left\{V(0),\, \frac{1}{t}\|V\|_{L_1(\mathbb{R}^+)}\right\} \leq \frac{C}{t+1}$$

for $t \geq 0$, where $C = \max\{V(0), 2\|V\|_{L_1(\mathbb{R}^+)}\}$, say.

It then follows from part (a) that

$$\|v(\cdot,t)\|_{L_\infty}^2 \leq \|v(\cdot,t)\|\|v_x(\cdot,t)\| \leq \|F\|\|v_x(\cdot,t)\|,$$

whence,

$$\|v(\cdot,t)\|_{L_\infty} \to 0 \quad \text{as } t \to \infty,$$

as advertised. (Indeed, it follows that $\|v(\cdot,t)\|_{L_\infty} \leq C\, t^{-1/4}$ as $t \to \infty$). Thus, the solution u of the linear equation is eventually periodic of period T. $\square$

3.2. **The linear KdV-Burgers equation.** Attention is now given to the linear KdV-Burgers equation

$$u_t + u_x + u_{xxx} - \nu u_{xx} = 0 \tag{44}$$

with

$$u(x,0) = f(x) \quad \text{for } x \geq 0, \qquad u(0,t) = g(t) \quad \text{for } t \geq 0. \tag{45}$$

We show that its solution is eventually periodic if g is periodic. The approach is, as for the BBM-Burgers equation, to consider the function

$$v(x,t) = u(x,t+T) - u(x,t). \tag{46}$$

One easily verifies that v solves

$$v_t + v_x + v_{xxx} - \nu v_{xx} = 0$$

with

$$v(x,0) = u(x,T) - f(x) = F(x) \quad \text{and} \quad v(0,t) = 0.$$

Theorem 3.2. *Suppose to be given an initial datum $f \in H^1(\mathbb{R}^+)$ and boundary data $g \in C^1(\mathbb{R}^+)$ which is periodic of period $T > 0$. Let u be the solution of (44)-(45) and let v be as defined in (46). Then v satisfies*

(a) $\|v(\cdot,t)\|_1 \leq \|F\|$, $\quad v_x,\, v_{xx} \in L_2(\mathbb{R}^+ \times \mathbb{R}^+)$,

(b) $\|v_x(\cdot,t)\|$, $\|v(\cdot,t)\|_{L_\infty} \to 0$ *as* $t \to \infty$.

Thus the solution u is eventually periodic of period T.

This theorem can be proven in a the same fashion as was Theorem 3.1. The details are therefore omitted.

3.3. **The Linear KdV equation.** Attention is now turned to a more complex issue, which is the same problem as considered in the previous subsection, but with $\nu = 0$, thus, the issue is eventual periodicity without the aid of a dissipative term. The simple energy-type analysis effected in the previous two subsections appears to have little chance of success without a sink for energy. Indeed, the solution of the IBVP for the linear KdV-equation

$$u_t + u_x + u_{xxx} = 0 \quad \text{for } x, t \geq 0 \tag{47}$$

with

$$u(x,0) = f(x) \quad \text{for } x \geq 0, \qquad u(0,t) = g(t) \quad \text{for } t \geq 0, \tag{48}$$

is expected to possess infinite energy in the limit as $t \to \infty$. This is because energy is being continually supplied and in the absence of dissipation there is no obvious mechanism for its reduction. (There is, however, a subtle damping effect associated with the implementation of a boundary condition at $x = 0$. This is explained in [10] and [11], for example, and its ramifications appear in the present analysis.) Thus an analysis leading to some kind of eventual periodicity will have to be more involved. It will be shown that in a weaker sense, eventual periodicity still obtains.

We will study the special case wherein the initial configuration f is zero. This corresponds to a semi-infinite stretch of the medium of propagation being at rest, and then disturbed by periodic input at one end. The case where $f \neq 0$ can also be handled, but the ideas are clearer if the medium is initially at rest.

As the solution of (47) is defined in the quarter plane $\mathbb{R}^+ \times \mathbb{R}^+$, it is natural to think of using the Laplace transform as in [10]. In the proof of the following theorem, the linear KdV equation is solved using the Laplace transform with respect to the temporal variable. The resulting representation of the solution allows one to deduce eventual periodicity.

Theorem 3.3. *Suppose $g \in C^1(\mathbb{R}^+)$ to be periodic of period $T > 0$. Then the solution u of the IBVP (47)-(48) is eventually periodic in the sense that*

$$u(\cdot, t+T) - u(\cdot, t) \to 0$$

as $t \to \infty$, uniformly on compact subsets of $(0, \infty)$.

Proof. An explicit representation of solutions of the linear KdV equation (47) is derived via the Laplace transform. Let $v(x,\sigma)$ denote the Laplace transform of $u(x,t)$ in the temporal variable, which is to say,

$$v(x,\sigma) = \mathcal{L}(u)(x,\sigma) = \int_0^\infty e^{-\sigma t} u(x,t)dt.$$

The original function u can be recovered from v through the inverse Laplace transform

$$u(x,t) = \frac{1}{2\pi i} \lim_{L\to\infty} \int_{b-iL}^{b+iL} e^{\sigma t} v(x,\sigma) d\sigma.$$

(This formula holds for any $b > 0$ since v is analytic in the right-half plane.) Taking the Laplace transform with respect to t of (47) and remembering the auxiliary condition (48), the IBVP (47)-(48) is converted into a one-parameter family of third-order boundary value problems, *viz.*

$$\sigma v + v_x + v_{xxx} = 0$$

with boundedness conditions as $x \to \infty$ and with

$$v(0,\sigma) = G(\sigma)$$

where $G(\sigma)$ denotes the Laplace transform of $g(t)$. Solving this equation leads to

$$v(x,\sigma) = e^{R(\sigma)x}G(\sigma)$$

where $R(\sigma)$ is the unique solution of

$$\sigma + R + R^3 = 0$$

whose real-part is non-positive. Through the inverse Laplace transform, u is then given formally by

$$u(x,t) = \frac{1}{2\pi i}\int_{b-\infty\, i}^{b+\infty\, i} e^{\sigma t}\left[e^{R(\sigma)x}G(\sigma)\right] d\sigma \tag{49}$$

for $x, t \geq 0$ and any fixed $b > 0$. In fact, as (49) does not depend on $b > 0$, we can contemplate the limit as $b \to 0$. It is straightforward to see that this latter limit exists as a convergent integral, so we may take $b = 0$, which makes the further calculations easier (see again [10]). Elementary manipulations with these integrals allows one to write this last representation in the alternative form

$$u(x,t) = \int_0^t g(t-s)M(x,s)ds$$

where

$$M(x,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iys + r(y)x} dy \tag{50}$$

with $r(y) = R(iy)$. It will be shown presently that the kernel M has the property that corresponding to any compact set $[\alpha, A]$ where $0 < \alpha < A < \infty$, there is a constant $C = C(\alpha, A)$ such that

$$|M(x,t)| \leq \frac{C}{t^{3/2}}. \tag{51}$$

Similar estimates hold for $\partial_x^j M(x,t)$ for $j = 1, 2, \cdots$.

Consider now the function

$$\begin{aligned} u(x,t+T) - u(x,t) &= \int_0^{t+T} g(t+T-s)M(x,s)ds - \int_0^t g(t-s)M(x,s)ds \\ &= \int_t^{t+T} g(t-s)M(x,s)ds, \end{aligned}$$

for $x > 0$, where we have used the periodicity of g. For fixed $\alpha > 0$ and $A < \infty$, the inequality (51) implies that

$$\sup_{\alpha \leq x \leq A} |u(x,t+T) - u(x,t)| \leq \|g\|_{L^\infty} T \frac{C}{t^{3/2}}$$

from which it is clear that u is eventually periodic, uniformly on compact subsets of $(0,\infty)$. $\square$

Corollary 3.4. *Under the hypotheses of Theorem 3.3, the solution u of (47)-(48) converges as $t \to \infty$ to a function $u_\infty(x,t)$ which is a strictly periodic function of t of period T, and which takes on the boundary value g at $x = 0$.*

Proof. Define, for $n = 0, 1, 2, \cdots$,

$$u_n(x,t) = u(x, t + nT).$$

Fix constants α and A with $0 < \alpha < A < +\infty$. We claim that $\{u_n\}_0^\infty$ is a Cauchy sequence in $X = C_b([\alpha, A] \times [0, \infty))$. In fact, as above, there is a constant C depending on α and A such that for any $s \geq 0$,

$$\sup_{x \in [\alpha, A]} |u_{n+1}(x,s) - u_n(x,s)| \leq \frac{\|g\|_\infty TC}{(nT)^{3/2}} = \frac{C'}{n^{3/2}}.$$

Thus there is a constant C independent of u such that

$$\|u_{n+1} - u_n\|_X \leq \frac{C}{n^{3/2}}.$$

It follows that

$$\sum_{n=1}^{\infty} \|u_{n+1} - u_n\|_X < \infty,$$

from which one concludes immediately that $\{u_n\}_1^\infty$ is a Cauchy sequence in X since, if $n > m$, then

$$\|u_n - u_m\|_X \leq \sum_{k=m}^{n-1} \|u_{k+1} - u_k\|_X \leq C' \sum_{k=m}^{n-1} \frac{1}{k^{3/2}} \leq \frac{C''}{m^{1/2}}.$$

Let

$$u_\infty = \lim_{n \to \infty} u_n.$$

The function u_∞ is periodic of period T because

$$\begin{aligned} |u_\infty(x, t+T) - u_\infty(x,t)| &\leq |u_\infty(x, t+T) - u_{n+1}(x,t)| + |u_{n+1}(x,t) - u_\infty(x,t)| \\ &= |u_\infty(x, t+T) - u_n(x, t+T)| + |u_{n+1}(x,t) - u_\infty(x,t)| \to 0, \end{aligned}$$

as $n \to \infty$. This function is defined for $[\alpha, A] \times [0, \infty)$ for arbitrary fixed α and A. One easily obtains a globally defined function on $\mathbb{R}^+ \times \mathbb{R}^+$ using these local functions and the uniqueness part of the well-posedness theory.

Finally it is shown that for $\alpha \leq x \leq A$,

$$|u(x,t) - u_\infty(x,t)| \leq \frac{C''}{t^{1/2}}$$

where C'' depends on α, A and T. To see this, argue as follows. As $u = u_0$, we may write

$$u(x,t) - u_k(x,t) = \sum_{n=0}^{k-1} \Big(u_n(x,t) - u_{n+1}(x,t) \Big).$$

It thus transpires that

$$\begin{aligned} |u(x,t) - u_k(x,t)| &\leq \sum_{n=0}^{k-1} |u_n(x,t) - u_{n+1}(x,t)| \\ &\leq \sum_{n=0}^{\infty} \frac{\|g\|_\infty TC}{(t + nT)^{3/2}} \leq \frac{C'}{t^{1/2}}. \end{aligned}$$

Similar arguments applied to

$$\partial_x^j u_* = \int_0^t \partial_x^j M(x,s)\, g(t-s) ds$$

show that the sequence $\{u_n\}_{n=0}^\infty$ is in fact Cauchy in $C_b^k([\alpha, A] \times [0, \infty))$ for any k and that

$$\left|\partial_x^j u(x,t) - \partial_x^j u_\infty(x,t)\right| \le \frac{C_j}{t^{1/2}}$$

for a constant C_j depending on j, α, T, A and $\|g\|_{L^\infty(\mathbb{R}^+)}$. It follows that u and u_∞ are C^∞-functions on $(0, \infty) \times (0, \infty)$ and that u_∞ solves the linear KdV equation. Thus u_∞ is, as advertised, a T-periodic solution of the evolution equation whose boundary trace at $x = 0$ is g. □

Attention is now given over to the verification that the kernel M has the requisite properties used in the above analysis.

Lemma 3.5. *For any fixed $j \in \mathbb{N}$ and $0 < \alpha < A < +\infty$, there is a constant $C_j = C_j(\alpha, A)$ for which the kernel*

$$M(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda t}\, e^{r(\lambda)x} d\lambda$$

has the property

$$|\partial_x^j\, M(x,t)| \le \frac{C_j}{t^{3/2}} \tag{52}$$

for all $t > 0$.

Proof. Consider first the case $j = 0$. Integration by parts once results in

$$M(x,t) = \frac{ix}{2\pi t} \int_{-\infty}^{\infty} e^{iy\,t} e^{r(y)x} r'(y) dy. \tag{53}$$

Here, we have used the fact that $r(y)$ has negative real part for large values of y and that $x \in [\alpha, A]$ to eliminate the boundary terms. It follows that

$$|M(x,t)| \le \frac{C}{t},$$

uniformly for $x \in [\alpha, A]$. Further progress depends on analyzing the integral in (53). Break it into three parts, *viz.*

$$I + II + III =$$
$$\int_{-\infty}^{-\frac{2}{3\sqrt{3}}} e^{iyt} e^{r(y)x}\, r'(y) dy + \int_{-\frac{2}{3\sqrt{3}}}^{\frac{2}{3\sqrt{3}}} e^{iyt} e^{r(y)x}\, r'(y) dy + \int_{\frac{2}{3\sqrt{3}}}^{\infty} e^{iyt} e^{r(y)x}\, r'(y) dy$$

where $r(y) = s(y) + i\,\tau(y)$ has a negative or zero real part. One determines straightforwardly that

$$s = -\sqrt{3\tau^2 - 1}, \quad 8\,\tau^3 - 2\tau + y = 0 \quad \text{if } y \in \left(-\infty, -\tfrac{2}{3\sqrt{3}}\right], \text{where } \tau \ge \tfrac{1}{\sqrt{3}}\ ;$$
$$s = 0, \quad \tau^3 - \tau - y = 0 \quad \text{if } y \in \left[-\tfrac{2}{3\sqrt{3}}, \tfrac{2}{3\sqrt{3}}\right], \text{ where } \tau \in \left[-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right]\ ;$$
$$s = -\sqrt{3\tau^2 - 1}, \quad 8\,\tau^3 - 2\tau + y = 0 \quad \text{if } y \in \left[\tfrac{2}{3\sqrt{3}}, \infty\right), \text{ where } \tau \le -\tfrac{1}{\sqrt{3}}.$$

Consider the second term and make the change of variables

$$y = z^3 - z, \quad z \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right].$$

Noticing that y is a monotonic function of z whenever $-1/\sqrt{3} \le z \le 1/\sqrt{3}$, we have

$$\tau(z) = z, \quad \tau'(y) = \frac{1}{3\tau^2 - 1} = \frac{1}{3z^2 - 1}, \quad dy = (3z^2 - 1)dz,$$

whence, as an integral in the $z-$variable,

$$II = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} e^{i(z^3-z)t} e^{izx} dz.$$

Break up this integral further by integrating over $[-\frac{1}{\sqrt{3}}, 0]$ and $[0, \frac{1}{\sqrt{3}}]$ separately;

$$II = II_1 + II_2 = \int_{-\frac{1}{\sqrt{3}}}^{0} e^{-ip(z)t} q(z) dz + \int_{0}^{\frac{1}{\sqrt{3}}} e^{ip(z)t} q(z) dz$$

with

$$p(z) = -z^3 + z \quad \text{and} \qquad q(z) = e^{izx}.$$

It then follows from the theory of stationary phase (Theorem A.1) that

$$II_1 \sim \frac{\sqrt{\sqrt{3}\pi}\, e^{-i(\frac{\pi}{4} + \frac{x}{\sqrt{3}} - \frac{2t}{3\sqrt{3}})}}{2\sqrt{t}} \tag{54}$$

as $t \to \infty$. Details leading to this asymptotic approximation are provided in the Appendix. Similarly,

$$II_2 \sim O\Big(\frac{1}{\sqrt{t}}\Big)$$

as $t \to \infty$.

Since I and III are virtually the same, it suffices to consider I. Make the change of variable

$$y = -8z^3 + 2z, \quad \text{for } z \in \Big[\frac{1}{\sqrt{3}}, \infty\Big).$$

For z in this range, y is a monotonic function of z. Correspondingly,

$$\tau(z) = z, \qquad r(z) = s(z) + i\tau(z) = -\sqrt{3z^2 - 1} + iz,$$
$$r'(y) = r'(z)\frac{dz}{dy} = \Big(i - \frac{3z}{\sqrt{3z^2-1}}\Big)\frac{dz}{dy}.$$

Therefore,

$$I = \int_{\frac{1}{\sqrt{3}}}^{\infty} e^{(-i)(8z^3-2z)t}\, e^{(-\sqrt{3z^2-1}+iz)x} \Big(\frac{3z}{\sqrt{3z^2-1}} - i\Big) dz. \tag{55}$$

As in the Appendix, one concludes that

$$I \sim \frac{\sqrt{\pi}\, e^{-i(\frac{\pi}{4} - \frac{x}{\sqrt{3}} + \frac{2t}{3\sqrt{3}})}}{2\sqrt{2t}} \tag{56}$$

as $t \to \infty$. The lemma is thereby established. $\square$

3.4. **The linear BBM equation.** Consider the IBVP

$$\begin{aligned} &u_t + u_x - u_{xxt} = 0 \quad \text{for } x, t \geq 0, \\ &u(x, 0) = 0 \quad \text{and} \quad u(0, t) = g(t) \end{aligned} \tag{57}$$

for the linear BBM equation. The approach to proving that u is eventually periodic is similar to that in the previous subsection. Taking the Laplace transform with respect to t and using the initial and boundary condition, there appears

$$\sigma v + v' - \sigma v'' = 0$$

with

$$v(0, \sigma) = G(\sigma), \qquad v(\infty, \sigma) = 0$$

where $G(\sigma)$ is, as before, the Laplace transform of g in the temporal variable. Solving this ordinary differential equation, we find that

$$v(x,\sigma) = G(\sigma)e^{R(\sigma)\,x},$$

where $R(\sigma)$ solves the equation

$$\sigma\, R^2 - R - \sigma = 0.$$

It is propitious to write v in the form

$$v(x,\sigma) = G(\sigma)\left[e^{R(\sigma)\,x} - e^{-x}\right] + G(\sigma)\, e^{-x}.$$

Applying the inverse Laplace transform to this formula yields

$$u(x,t) = g(t)\, e^{-x} + \int_0^t M(x,s)\, g(t-s) ds$$

where

$$M(x,t) = \frac{1}{2\pi\, i}\int_{-i\infty}^{i\infty} e^{\sigma t}\left[e^{R(\sigma) x} - e^{-x}\right] d\sigma. \tag{58}$$

As for the linear KdV equation, it is the case that for $0 < \alpha < A < \infty$, there are constants C_j, $j = 0, 1, 2, \cdots$, such that

$$\left|\partial_x^j\, M(x,t)\right| \le \frac{C_j}{t^{3/2}} \tag{59}$$

for $t > 0$ and uniformly for $x \in [\alpha, A]$. To this end, substitute $\sigma = iy$ in (58) to obtain

$$\begin{aligned} M(x,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{iyt}\left[e^{r(y)x} - e^{-x}\right] dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyt}\Lambda(y,x) dy \end{aligned}$$

where $r(y) = R(iy)$. Notice that $\Lambda(y,x) \equiv e^{r(y)x} - e^{-x} \sim cx/y^2$ as $y \to \pm\infty$, so this integral exists as a Lebesgue integral or as an improper Riemann integral, uniformly for x on bounded sets. An integration by parts reveals that

$$M(x,t) = \frac{x}{2\pi t}\int_{-\infty}^{\infty} e^{iyt}\, e^{r(y)x} r'(y) dy.$$

We are interested in the behavior as $t \to \infty$ of the integral in the last formula. The standard methods of asymptotic analysis will come to our rescue after some preliminary machinations. Break the integral into three parts, *viz.*

$$\begin{aligned} &I + II + III \\ &= \int_{-\infty}^{-\frac{1}{2}} e^{iyt}\, e^{r(y)x} r'(y) dy + \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{iyt}\, e^{r(y)x} r'(y) dy + \int_{\frac{1}{2}}^{\infty} e^{iyt}\, e^{r(y)x} r'(y) dy. \end{aligned} \tag{60}$$

Here, $r(y)$ is the root of the equation $iyr^2 - r - iy = 0$ with negative or zero real part. More explicitly, it is given by the formulas

$$\begin{aligned}
r(y) &= -\frac{i}{2y} + \frac{\sqrt{4y^2-1}}{2y} \qquad \text{for } -\infty < y \le -\tfrac{1}{2};\\
r(y) &= -\frac{i}{2y} - \frac{\sqrt{4y^2-1}}{2y} \qquad \text{for } \tfrac{1}{2} \le y < \infty;\\
r(y) &= \frac{\sqrt{1-4y^2}-1}{2y} i \qquad \text{for } -\tfrac{1}{2} \le y \le \tfrac{1}{2}.
\end{aligned}$$

Thus, the three integrals in (60) can be written in the more detailed form

$$\begin{aligned}
I &= \int_{-\infty}^{-\frac{1}{2}} e^{iyt} \exp\left(\left[-\frac{i}{2y} + \frac{\sqrt{4y^2-1}}{2y}\right] x\right) \left[\frac{2}{\sqrt{4y^2-1}} + \frac{i - \sqrt{4y^2-1}}{2y^2}\right] dy,\\
II &= i \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{iyt} \exp\left(\left[\frac{\sqrt{1-4y^2}-1}{2y}\right] ix\right) \left[\frac{-2}{\sqrt{1-4y^2}} + \frac{1-\sqrt{1-4y^2}}{2y^2}\right] dy,\\
III &= \int_{\frac{1}{2}}^{\infty} e^{iyt} \exp\left(\left[-\frac{i}{2y} - \frac{\sqrt{4y^2-1}}{2y}\right] x\right) \left[\frac{-2}{\sqrt{4y^2-1}} + \frac{i+\sqrt{4y^2-1}}{2y^2}\right] dy.
\end{aligned}$$

To deal with III, make the change of variables

$$4y^2 - 1 = z^2 \qquad \text{for} \quad y \ge \tfrac{1}{2}$$

to obtain

$$\begin{aligned}
III &= \int_0^\infty e^{i\frac{1}{2}\sqrt{1+z^2}t} e^{-\frac{z+i}{\sqrt{1+z^2}}x} \left[\frac{2(z+i)}{1+z^2} - \frac{2}{z}\right] \frac{z}{2\sqrt{z^2+1}} dz\\
&= \int_0^\infty e^{i\frac{1}{2}\sqrt{1+z^2}t} e^{-\frac{z+i}{\sqrt{1+z^2}}x} \left[\frac{z(z+i)}{(1+z^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{1+z^2}}\right] dz\\
&= \int_0^\infty e^{i\frac{1}{2}\sqrt{1+z^2}t} e^{-\frac{z+i}{\sqrt{1+z^2}}x} \frac{-1+iz}{(1+z^2)^{\frac{3}{2}}} dz = III(t,x).
\end{aligned}$$

The integral over $(-\infty, -\frac{1}{2})$ can be transformed in a similar manner to reach the expression

$$I = -\int_0^\infty e^{-i\frac{1}{2}\sqrt{1+z^2}t} \, e^{-\frac{z+i}{\sqrt{1+z^2}}x} \left[\frac{-1+iz}{(1+z^2)^{\frac{3}{2}}}\right] dz.$$

A further, straightforward integration by parts does not appear to be in the cards, so the integral is analyzed directly. Write it as

$$III(t,x) = \int_0^\infty e^{ip(y)t} q(y) dy$$

where

$$p(y) = \frac{1}{2}\sqrt{1+y^2} \quad \text{and} \quad q(y) = e^{-\frac{y+i}{\sqrt{1+y^2}}x} \left[\frac{-1+iy}{(1+y^2)^{\frac{3}{2}}}\right].$$

A straightforward application of the method of the stationary phase (see Appendix) allows the inference

$$III(t,x) \sim -e^{i\frac{\pi}{4}} \frac{e^{-ix}}{2} \Gamma\left(\frac{1}{2}\right) \frac{e^{it/2}}{(t/4)^{\frac{1}{2}}} = \frac{-\sqrt{\pi}\, e^{i(\frac{\pi}{4} - x + \frac{t}{2})}}{\sqrt{t}} \tag{61}$$

as $t \to \infty$. Similar considerations apply to the integral I.

The second term

$$II = i\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{iyt} e^{-i\frac{1-\sqrt{1-4y^2}}{2y}x} \frac{\sqrt{1-4y^2}-1}{2y^2\sqrt{1-4y^2}}dy$$

is a little more interesting. Notice there is a square root singularity at $\pm\frac{1}{2}$. Break the integral up by integrating over $[-\frac{1}{2},0]$ and $[0,\frac{1}{2}]$ separately. By making the change of variables

$$z^2 = \frac{1}{4} - y^2, \quad 0 \le y \le \frac{1}{2},$$

the integral over $[0,\frac{1}{2}]$ becomes

$$II_2(t,x) = \int_0^{\frac{1}{2}} e^{it\sqrt{\frac{1}{4}-z^2}}\, e^{\frac{2z-1}{2\sqrt{\frac{1}{4}-z^2}}ix} \frac{1}{(1+2z)\sqrt{\frac{1}{4}-z^2}}dz.$$

The integral over $\left[-\frac{1}{2},0\right]$ can be similarly transformed. Both integrals are of the form

$$\int_0^{\frac{1}{2}} e^{ip(z)t}\, q(z)dz.$$

Elementary considerations (see the Appendix) then demonstrate that

$$II(t,x) \sim \frac{e^{-\frac{\pi}{4}i}e^{-ix}\Gamma\left(\frac{1}{2}\right)e^{it/2}}{t^{1/2}} = \frac{\sqrt{\pi}e^{-i(\frac{\pi}{4}+x-t/2)}}{\sqrt{t}} \tag{62}$$

as $t \to \infty$.

With the inequality for the kernel M (59) at our disposal, the following theorem concerning solutions of the BBM equation emerges via arguments similar to those in the previous subsection.

Theorem 3.6. *Suppose $g \in C^1(\mathbb{R}^+)$ to be periodic of period $T > 0$. Then any solution u of the linearized BBM equation*

$$u_t + u_x - u_{xxt} = 0, \quad \textit{for } x,t \ge 0$$

with $u(x,0) = 0$ and $u(0,t) = g(t)$ converges as $t \to \infty$ to a function $u_\infty(x,t)$ which is a strictly periodic function of t of period T, and which takes on the boundary value g at $x = 0$.

In particular, u is eventually periodic in the sense that

$$u(\cdot,t+T) - u(\cdot,t) \to 0$$

as $t \to \infty$, uniformly on compact subsets of $(0,\infty)$.

4. **Conclusion.** To recapitulate, analysis of the initial- and boundary-value problems

$$u_t + u_x + u^p u_x - \nu u_{xx} - u_{xxt} = 0 \quad \text{for } x,t \ge 0,$$
$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0 \quad \text{for } x,t \ge 0$$

has been offered that bears upon a number of interesting questions. These problems arise in wave propagation in nonlinear, dispersive media when long-crested (one-dimensional) waves enter from the left a semi-infinite stretch of the medium. The waves might be incoming waves from other parts of the system or waves generated by a wavemaker at the left-hand end of the physical domain of propagation.

In Section 2, the issue of the growth of energy in the wavefield is considered. If we agree to initiate the system with a disturbance of finite energy, as measured by the $H^1(\mathbb{R}^+)$-norm, and to make suitably bounded disturbances at the left, then a natural question arises. How, if at all, does the total energy in the channel grow as a function of time?

We are able to provide temporal growth rates for a very restricted range of p and with $\nu = 0$ or $\nu > 0$ in both the KdV and the BBM cases. These are obtained via energy estimates. The energy growths are all algebraic and in the linear case ($p = 0$), the theory generally gives growth rates of order $t^{\frac{1}{2}}$ as $t \to \infty$, whether or not dissipation is present. These are probably sharp and are in contrast to the uniform boundedness one obtains for the damping μu featured in equations (4) and (5). However, when the nonlinearity is included, say at the quadratic level, the simple energy methods give larger growth rates, and, what is surprising, the growth rates obtained with $\nu > 0$ are larger than those established for $\nu = 0$. This seems counterintuitive and is perhaps a consequence of the methods. An obvious avenue for further investigation is to understand what are the sharp growth rates as p varies, and to extend the range of p for which one has theory.

Section 3 is concerned with the issue of what has been called eventual periodicity. This is very clearly observed in water wave experiments in a laboratory setting (cf. [9]) and begs for theoretical confirmation. It has been established in the context of the equations (4) and (5) when $\mu > 0$. Our results supplement those derived in [11] by including the case $\mu = 0$. With the damping $-\nu u_{xx}$, which is much weaker for long waves than is μu, we are at present only able to handle the linear equations (37) and (47) with $\nu \geq 0$. Another very interesting issue worth further consideration is to establish eventual periodicity when nonlinearity is included as in (1) and (2). This is likely to be a little harder in this case than it was for (4) and (5) because very long waves are not much damped with the Burgers'-type dissipation, so resulting in less useful *a priori* bounds of various sorts.

Finally, the same range of questions naturally present themselves for more general dispersion relations than exhibited by the KdV or BBM equations (see, e.g. [1] and the references contained therein).

Appendix

A slightly more expanded commentary on the asymptotic analysis of the various oscillatory integrals arising in subsections 3.3 and 3.4 is now offered. The analysis relies upon standard results in the theory of stationary phase, as expounded in Theorem 13.1 in F. Olver's book [16], for example. For the readers' convenience, this theory is summarized below.

Suppose that in the integral

$$I(t) = \int_a^b e^{itp(y)}\, q(y)\, dy$$

the limits a and b are independent of t, a being finite and $b\,(> a)$ finite or infinite. The functions $p(y)$ and $q(y)$ are independent of t, $p(y)$ being real and $q(y)$ either real or complex. Assume that the only point at which $p'(y)$ vanishes is a. Without loss of generality, both t and $p'(y)$ are taken to be positive; cases in which one of them is negative can be handled by changing the sign of i throughout. We require the following.

(i) In (a, b), the functions $p'(y)$ and $q(y)$ are continuous, $p'(y) > 0$, and $p''(y)$ and $q'(y)$ have at most a finite number of discontinuities and infinities.

(ii) As $y \to a+$,

$$p(y) - p(a) \sim P(y-a)^{\mu}, \qquad q(y) \sim Q(y-a)^{\lambda-1}, \tag{A.1}$$

the first of these relations being differentiable. Here P, μ and λ are positive constants, and Q is a real or complex constant.

(iii) For each $\epsilon \in (0, b-a)$,

$$\mathcal{V}_{a+\epsilon,b}\Big\{\frac{q(y)}{p'(y)}\Big\} \equiv \int_{a+\epsilon}^{b} \left|\Big(\frac{q(y)}{p'(y)}\Big)'\right| dy < \infty.$$

(iv) As $t \to b-$, the limit of $q(y)/p'(y)$ is finite, and this limit is zero if $p(b) = \infty$.

With these conditions, the nature of the asymptotic approximation to $I(t)$ for large t depends on the sign of $\lambda - \mu$. In the case $\lambda < \mu$, the following result obtains.

Theorem A.1. *In addition to the above conditions, assume that $\lambda < \mu$, the first of (A.1) is twice differentiable and the second of (A.1) is differentiable. Then, the asymptotic condition*

$$I(t) \sim e^{\lambda\pi i/(2\mu)} \frac{Q}{\mu}\Gamma\Big(\frac{\lambda}{\mu}\Big) \frac{e^{itp(a)}}{(Pt)^{\lambda/\mu}}$$

holds as $t \to \infty$.

With this result in hand, the details leading to (54) can be made explicit. Apply Theorem A.1 to the integral

$$II_1 = \int_{-\frac{1}{\sqrt{3}}}^{0} e^{(-i)p(z)t}\, q(z)\, dz$$

with

$$p(z) = -z^3 + z \quad \text{and} \qquad q(z) = e^{izx}.$$

The reason that we write $-i$ instead of i in this integral is to insure $p'(z) > 0$ for $z \in (-\frac{1}{\sqrt{3}}, 0]$. Thus,

$$p'(z) = -3z^2 + 1, \qquad p''(z) = -6z, \quad q'(z) = ixe^{izx}.$$

In addition, as $z \to -\frac{1}{\sqrt{3}}+$,

$$p(z) - p\Big(-\frac{1}{\sqrt{3}}\Big) \sim \frac{1}{\sqrt{3}}\Big(z + \frac{1}{\sqrt{3}}\Big)^2, \qquad q(z) \to e^{-i\frac{1}{\sqrt{3}}x}.$$

Now, notice that

$$\frac{q}{p'} = \frac{e^{izx}}{-3z^2+1} \to 1 \quad \text{as } z \to 0$$

and that $\mathcal{V}_{-\frac{1}{\sqrt{3}}+\epsilon,0}$ is finite for any fixed $\epsilon > 0$. In fact,

$$\frac{p'q' - qp''}{(p')^2} = e^{ixz}\frac{ix(-3z^2+1) + 6z}{(3z^2-1)^2}$$

and thus

$$\mathcal{V}_{-\frac{1}{\sqrt{3}}+\epsilon,0} = \int_{-\frac{1}{\sqrt{3}}+\epsilon}^{0} \left|\frac{p'q' - qp''}{(p')^2}\right| dz < \infty.$$

Theorem A.1 then implies that as $t \to \infty$,

$$II_1 = \int_{-\frac{1}{\sqrt{3}}}^{0} e^{(-i)p(z)t}\, q(z)\, dz \ \sim \ e^{-i\pi/4}\, \frac{e^{-i\frac{1}{\sqrt{3}}x}}{2} \Gamma\Big(\frac{1}{2}\Big) \frac{e^{-it(-\frac{2}{3\sqrt{3}})}}{(t/\sqrt{3})^{1/2}},$$

which can be simplified to yield (54).

Attention is now turned to the asymptotic approximation (56). It suffices to check the conditions of Theorem A.1 for the integral defined in (55). For this integral, the auxiliary parameters and functions have the form

$$a = \frac{1}{\sqrt{3}}, \quad b = \infty, \quad p(z) = 8z^3 - 2z, \quad q(z) = e^{(-\sqrt{3z^2-1}+iz)x} \Big(\frac{3z}{\sqrt{3z^2-1}} - i\Big);$$

consequently,

$$p'(z) = 24z^2 - 2, \quad p''(z) = 48z,$$

$$q'(z) = e^{(-\sqrt{3z^2-1}+iz)x} \left(-x\Big(\frac{3z}{\sqrt{3z^2-1}} - i\Big)^2 - \frac{3}{\sqrt{(3z^2-1)^3}} \right).$$

As $z \to \frac{1}{\sqrt{3}}+$,

$$p(z) - p\Big(\frac{1}{\sqrt{3}}\Big) \ \sim \ 6\Big(z - \frac{1}{\sqrt{3}}\Big), \qquad q(z) \to \frac{\sqrt{3}}{2}\, e^{i\frac{1}{\sqrt{3}}x} \Big(z - \frac{1}{\sqrt{3}}\Big)^{-1/2}.$$

Furthermore, as $z \to \infty$, $p(z) \to \infty$ and $q/p' \to 0$. For any $\epsilon > 0$, it is easily checked that

$$\mathcal{V}_{\frac{1}{\sqrt{3}}+\epsilon,\infty}\left\{\frac{q}{p'}\right\} = \int_{\frac{1}{\sqrt{3}}+\epsilon}^{\infty} \left| \frac{p'q' - qp''}{(p')^2} \right| dz < \infty.$$

The asymptotic relation (56) is then a consequence of Theorem A.1.

Here are the details leading to the conclusion expressed in (61). As before, the relevant integral is of the form $I(t)$ where

$$p'(y) = \frac{1}{2}\frac{y}{\sqrt{1+y^2}}, \quad p''(y) = \frac{1}{2}\frac{1}{(1+y^2)^{\frac{3}{2}}},$$

$$q'(y) = e^{-\frac{y+i}{\sqrt{1+y^2}}x} \left\{ x\left[\frac{-1+iy}{(1+y^2)^{\frac{3}{2}}}\right]^2 + \frac{-2iy^2+3y+i}{(1+y^2)^{\frac{3}{2}}} \right\}.$$

As $y \to 0$,

$$p(y) - p(0) = \frac{1}{2}(\sqrt{1+y^2} - 1) \sim \frac{1}{4}y^2, \qquad q(t) \sim -e^{-ix}.$$

As $y \to \infty$,

$$p(y) \to \infty, \quad p'(y) \to \frac{1}{2}, \quad q(y) \to 0 \quad \text{and} \quad \frac{q(y)}{p'(y)} \to 0.$$

It then suffices to verify that for any $\epsilon > 0$,

$$\mathcal{V}_{\epsilon,\infty}\left(\frac{q}{p'}\right) < \infty.$$

to this end, compute

$$p'q' - qp'' = \frac{1}{2}\frac{y}{\sqrt{1+y^2}}\, xe^{-\frac{y+i}{\sqrt{1+y^2}}x} \left\{ \left[\frac{-1+iy}{(1+y^2)^{\frac{3}{2}}}\right]^2 + \frac{-2iy^2+3y+i}{(1+y^2)^{\frac{3}{2}}} \right\}$$

$$-e^{-\frac{y+i}{\sqrt{1+y^2}}x}\left[\frac{-1+iy}{(1+y^2)^{\frac{3}{2}}}\right]\frac{1}{2}\frac{1}{(1+y^2)^{\frac{3}{2}}}$$

and

$$(p')^2=\frac{1}{4}\frac{y^2}{1+y^2}.$$

It is easily checked that $|(p'q'-qp'')/(p')^2|$ behaves like $\frac{1}{y^2}$ at both $y=0$ and $y=\infty$. Therefore, $\mathcal{V}_{\epsilon,\infty}\left(\frac{q}{p'}\right)<\infty$.

Here are the details verifying the conditions in Theorem A.1 in pursuit of establishing that (62) is correct. The integral under consideration is

$$\int_0^{\frac{1}{2}} e^{(-i)p(z)t}\,q(z)dz$$

with

$$p(z)=-\sqrt{\frac{1}{4}-z^2},\qquad q(z)=e^{\frac{2z-1}{2\sqrt{\frac{1}{4}-z^2}}ix}\frac{1}{(1+2z)\sqrt{\frac{1}{4}-z^2}}.$$

Therefore,

$$p'(z)=\frac{z}{\sqrt{\frac{1}{4}-z^2}},\qquad p''(z)=\frac{1}{4\left(\frac{1}{4}-z^2\right)^{3/2}},$$

$$q'(z)=e^{\frac{2z-1}{2\sqrt{\frac{1}{4}-z^2}}ix}\left\{ix\left[\frac{1}{(1+2z)\sqrt{\frac{1}{4}-z^2}}\right]^2+\frac{4z^2+z+\frac{1}{2}}{(1+2z)^2\left(\frac{1}{4}-z^2\right)^{3/2}}\right\}.$$

As $z\to 0+$,

$$p(z)-p(0)=-\sqrt{\frac{1}{4}-z^2}+\frac{1}{2}\;\sim\;z^2,\qquad q(z)\;\sim\;2e^{-ix}.$$

As $z\to\frac{1}{2}-$,

$$\frac{q(x)}{p'(x)}=\left(e^{\frac{2z-1}{2\sqrt{\frac{1}{4}-z^2}}ix}\frac{1}{(1+2z)\sqrt{\frac{1}{4}-z^2}}\right)\Big/\left(\frac{z}{\sqrt{\frac{1}{4}-z^2}}\right)\;\to\;1.$$

Finally, consider the integral

$$\mathcal{V}_{\epsilon,\frac{1}{2}}\left(\frac{q}{p'}\right)=\int_\epsilon^{\frac{1}{2}}\left|\frac{p'q'-qp''}{(p')^2}\right|dz$$

for fixed $\epsilon>0$. Elementary calculations show that

$$
\begin{aligned}
&p'q' - qp'' \\
=&\frac{-z}{\sqrt{\frac{1}{4}-z^2}}\, e^{\frac{2z-1}{2\sqrt{\frac{1}{4}-z^2}}ix} \left\{ ix \left[\frac{1}{(1+2z)\sqrt{\frac{1}{4}-z^2}} \right]^2 + \frac{4z^2+z-\frac{1}{2}}{(1+2z)^2\left(\frac{1}{4}-z^2\right)^{3/2}} \right\} \\
&- e^{\frac{2z-1}{2\sqrt{\frac{1}{4}-z^2}}ix}\, \frac{1}{(1+2z)\sqrt{\frac{1}{4}-z^2}}\, \frac{-1}{4\left(\frac{1}{4}-z^2\right)^{3/2}} \\
=&e^{\frac{2z-1}{2\sqrt{\frac{1}{4}-z^2}}ix} \left\{ \frac{-ixz}{(1+2z)^2\left(\frac{1}{4}-z^2\right)^{3/2}} + \frac{2(8z^2+6z+1)}{(1+2z)^4\,\left(\frac{1}{2}-z\right)} \right\}
\end{aligned}
$$

and that

$$
(p')^2 = \left(\frac{-z}{\sqrt{\frac{1}{4}-z^2}} \right)^2 .
$$

In summary, $(p'q'-qp'')/(p')^2$ behaves like $1/z^2$ at $z=0$ and behaves like $1/\sqrt{\frac{1}{4}-z^2}$ near $z=\frac{1}{2}$. Thus, the integral $\mathcal{V}_{\epsilon,\frac{1}{2}}$ is bounded.

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