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We study the global regularity of solutions to the 2D Boussinesq equations with fractional dissipation, given by $(-\Delta)^{\alpha/2}u$ in the velocity equation and by $(-\Delta)^{\beta/2}\theta$ in the temperature equation. We establish the global regularity for $\frac{2}{3} < \alpha < 1$, $\alpha+\beta > 1$ and $\alpha > \frac{1}{1+\beta}$. This result is for the subcritical regime $\alpha+\beta > 1$ and the point here is to obtain the global regularity for the largest possible range of α.

1. Introduction

This paper examines the global (in time) well-posedness problem on the 2D Boussinesq equations with fractional dissipation. The Boussinesq equations concerned here model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream (see the books by Gill [1982], Majda [2003], Pedlosky [1979]). In addition, the Boussinesq equations also play an important role in the study of Rayleigh–Benard convection [Constantin and Doering 1999]. The standard 2D Boussinesq equations with Laplacian dissipation can be written

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \nu\,\Delta u + \theta e_2, \\ \theta_t + u\cdot\nabla\theta = \kappa\,\Delta\theta, \\ \nabla\cdot u = 0, \end{cases} \tag{1-1}$$

where u denotes the 2D velocity field, p the pressure, θ the temperature in the context of thermal convection and the density in the modeling of geophysical fluids, ν the viscosity, κ the thermal diffusivity, and $e_2 = (0, 1)$ is the unit vector in the vertical direction.

The 2D Boussinesq equations have recently attracted considerable attention in the community of mathematical fluid mechanics due to their mathematical significance. Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain

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some key features of the 3D Euler and Navier–Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows (away from the symmetry axis) (see, e.g., [Majda and Bertozzi 2001]).

Our attention will be focused on the 2D Boussinesq equations with fractional dissipation

$$\begin{cases} u_t + u\cdot\nabla u + \Lambda^\alpha u + \nabla p = \theta e_2, \\ \theta_t + u\cdot\nabla\theta + \Lambda^\beta\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1-2}$$

where $\Lambda = (-\Delta)^{1/2}$ and the general fractional Laplacian operator Λ^α can be defined via the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

This generalization allows us to study a family of equations simultaneously and may be physically relevant. In fact, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian may arise. Flows in the middle atmosphere traveling upward undergo changes due to the changes of atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian (see [Gill 1982; Caputo 1967]).

One of the fundamental problems concerning the Boussinesq system is whether or not its solutions remain smooth for all time or they blow up in a finite time. This problem could be extremely difficult. A standard approach to the global regularity problem is to first obtain the local existence and regularity and then extend the local solution to a global one by establishing global a priori bounds for the solution. Due to the divergence-free condition $\nabla\cdot u = 0$, any solution (u,θ) with sufficiently smooth data admits a global L^2-bound for u and a global L^q-bound for θ ($q \in [1,\infty]$). However, when the dissipation or the thermal diffusion is not sufficient, it can be extremely difficult to obtain global bounds for suitable derivatives of u or θ. When the Boussinesq equations are inviscid (no velocity dissipation or thermal diffusion), the equations of $\omega = \nabla\times u$ and $\nabla^\perp\theta$,

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \partial_{x_1}\theta, \\ \partial_t\nabla^\perp\theta + (u\cdot\nabla)\nabla^\perp\theta = (\nabla^\perp\theta\cdot\nabla)u, \end{cases}$$

resemble the 3D Euler vorticity equation

$$\partial_t\omega^E + (u^E\cdot\nabla)\omega^E = (\omega^E\cdot\nabla)u^E,$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, and u^E and ω^E denote the 3D Euler velocity and the corresponding vorticity, respectively.

When Δu and $\Delta\theta$ are present, the global regularity can then be established following a similar proof as that for the 2D Navier–Stokes equations. The issue that arises naturally is how much dissipation is really needed for the global regularity. This problem has attracted considerable interest recently and important progress has been made (see, e.g., [Adhikari et al. 2010; 2011; 2014; Cao and Wu 2013; Constantin and Vicol 2012; Danchin and Paicu 2011; Hmidi et al. 2010; 2011; Hou and Li 2005; KC et al. 2014; Lai et al. 2011; Larios et al. 2013; Li et al. 2016; Li and Titi 2016; Miao and Xue 2011; Ohkitani 2001; Stefanov and Wu 2018; Wu and Xu 2014; Wu et al. 2016; 2015; Yang et al. 2014; Ye 2017; Ye and Xu 2016; Zhao 2010; Zhou 2018; Zhou and Li 2017]). Various approaches and techniques have been developed to obtain the global regularity for (1-2) with smaller and smaller $\alpha \in (0, 2)$ and $\beta \in (0, 2)$.

As pointed out in [Jiu et al. 2014], it is useful to classify α and β into three categories: the subcritical case when $\alpha+\beta > 1$, the critical case when $\alpha+\beta = 1$ and the supercritical case when $\alpha+\beta < 1$. This classification gives us a sense of the level of difficulty for different parameter ranges. The global regularity problem for the supercritical regime $\alpha+\beta < 1$ appears to be out of reach at this moment. Current results for this regime address the eventual regularity of weak solutions [Yang et al. 2014; Wu et al. 2016]. There are exciting developments for the critical regime. Two special critical cases, $\alpha = 1,\ \beta = 0$ and $\beta = 1, \alpha = 0$, were studied and resolved in [Hmidi et al. 2010; 2011]. More general critical cases with $\alpha+\beta = 1$ and $\alpha \in (0, 1)$ were dealt with by Jiu, Miao, Wu and Zhang [Jiu et al. 2014], who established the global regularity for (1-2) with $\alpha+\beta = 1$ and $1 > \alpha > \alpha_0 \equiv \frac{23-\sqrt{145}}{12} \approx 0.9132$. Stefanov and Wu improved the result of Jiu, Miao, Wu and Zhang by further enlarging the range of α with $\alpha+\beta = 1$ and $1 > \alpha > \frac{\sqrt{1777}-23}{24} \approx 0.7981$ [2018]. A very recent work of Wu, Xu, Xue and Ye assesses the global regularity for $\alpha+\beta = 1$ and $\alpha \in (0.7692, 1)$ [Wu et al. 2016].

This paper focuses on the subcritical regime $\alpha+\beta > 1$. The global regularity problem, even in this regime, can be difficult, and there are ranges of subcritical regime for which the global regularity of (1-2) remains unknown. To give an accurate account of current results, we further divide the subcritical regime into two cases: $\alpha \geq \beta$ and $\alpha < \beta$. We refer to the first case as velocity dissipation dominated and the second case as thermal diffusion dominated. For the velocity dominated case, Miao and Xue [2011] was able to establish the global regularity of (1-2) with

$$\alpha \in \Big(\frac{6-\sqrt{6}}{4},\ 1\Big), \quad \beta \in \Big(1-\alpha,\ \min\Big\{\frac{(7+2\sqrt{6})\alpha}{5}-2,\ \frac{\alpha(1-\alpha)}{\sqrt{6}-2\alpha},\ 2-2\alpha\Big\}\Big).$$

Note that $\frac{6-\sqrt{6}}{4} \approx 0.8876$. Ye [2017] was able to enlarge the range to

$$0.7351 < \alpha < 1, \quad \beta \in \Big(1-\alpha,\ \min\Big\{3-3\alpha,\ \frac{\alpha}{2},\ \frac{3\alpha^2+4\alpha-4}{8(1-\alpha)}\Big\}\Big).$$

For the thermal diffusion dominated case, Constantin and Vicol obtained as a consequence of their nonlinear maximum principle for fractional Laplacian operators the global regularity of (1-2) with $\beta > \frac{2}{2+\alpha}$. In addition, Yang, Jiu and Wu [Yang et al. 2014] obtained the global regularity for a larger range of β, and Ye and Xu [2016] made further improvements on the range of β.

This paper focuses on the velocity dissipation dominated case, $\alpha \geq \beta$. Our primary goal has been to obtain the global regularity for the smallest possible $\alpha \in (0, 1)$ with $\alpha + \beta > 1$ and $\alpha > \beta > 0$. Our main result is stated in Theorem 1.2. A slightly weaker result with a smaller range of α is stated in Theorem 1.1. The main reason for keeping Theorem 1.1 is that Theorem 1.2 is built upon Theorem 1.1 and its proof.

Theorem 1.1. *Let $s > 2$. Assume that $u_0 \in H^s(\mathbb{R}^2)$ and $\nabla \cdot u_0 = 0$, and $\theta_0 \in H^s(\mathbb{R}^2)$. Consider the fractional Boussinesq equations (1-2) with α and β satisfying*

$$0 < \alpha, \beta < 1, \qquad \alpha > \frac{2}{\beta+2}, \tag{1-3}$$

then (1-2) has a unique global (in time) solution (u, θ) satisfying

$$(u, \theta) \in C([0, T]; H^s(\mathbb{R}^2)).$$

Theorem 1.2. *Let $s > 2$. Assume that $u_0 \in H^s(\mathbb{R}^2)$ and $\nabla \cdot u_0 = 0$, and $\theta_0 \in H^s(\mathbb{R}^2)$. Consider the fractional Boussinesq equations (1-2) with α and β satisfying*

$$\frac{2}{3} < \alpha < 1, \quad 0 < \beta < 1, \quad \alpha > \frac{1}{\beta+1}, \tag{1-4}$$

then (1-2) has a unique global (in time) solution (u, θ) satisfying

$$(u, \theta) \in C([0, T]; H^s(\mathbb{R}^2)).$$

The proof of Theorem 1.1 relies on the equation for a combined quantity and the nonlinear maximum principle for fractional Laplacian operators developed by Córdoba and Córdoba [2004] and by Constantin and Vicol [2012]. Due to the presence of the "vortex stretching" term $\partial_{x_1}\theta$, energy estimates on the vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\omega + \Lambda^\alpha \omega = \partial_{x_1}\theta$$

with $\alpha \in (0, 1)$ would not yield any global bound on ω. A well-known practice is to eliminate $\partial_{x_1}\theta$ by considering the combined quantity

$$G = \omega - \mathcal{R}_\alpha \theta \quad \text{with} \quad \mathcal{R}_\alpha = \partial_1 \Lambda^{-\alpha},$$

which satisfies

$$G_t + u \cdot \nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u \cdot \nabla]\theta + \Lambda^{\beta-\alpha}\partial_1\theta,$$

where $[\mathcal{R}_\alpha, u \cdot \nabla]\theta$ denotes the standard commutator. Combining this equation with that of $\nabla\theta$, applying the nonlinear maximum principle for fractional Laplacian operators and invoking commutator estimates, one derives differential inequalities for $\|G(t)\|_{L^\infty}$ and $\|\nabla\theta(t)\|_{L^\infty}$, which yields Theorem 1.1. Theorem 1.2 involves improved arguments. Its proof makes use of the global L^2 bound for G whenever $\alpha > \frac{2}{3}$ and $\alpha+\beta > 1$, and the pointwise lower bound

$$f(x)\cdot\Lambda^\alpha f(x) \ge \frac{1}{2}\Lambda^\alpha|f(x)|^2 + \frac{|f(x)|^{2+p\alpha/d}}{c\|f\|_{L^p}^{p\alpha/d}}.$$

This lower bound is in terms of the L^p-norms of the functions instead of the L^p-norm of the antiderivative of f, and thus has a higher power than the corresponding lower bound in terms of the L^p-norm of the antiderivative.

The rest of this paper is divided into two sections. Section 2 proves Theorem 1.1 while Section 3 proves Theorem 1.2. Two appendices are also attached. The first one provides the frequency localization operators and Besov spaces, and related facts. Appendix B supplies the proofs for some of the facts used in Sections 2 and 3.

2. Proof of Theorem 1.1

This section proves Theorem 1.1. To do so, we make several preparations. The first is a pointwise inequality for fractional Laplacian operators in [Constantin and Vicol 2012; Córdoba and Córdoba 2004].

Lemma 2.1. *Let $\alpha \in (0,2)$ and $q \in [1,\infty]$. There exists $C = C(d,\alpha,q)$ such that, for any function $f = f(x)$ with $x \in \mathbb{R}^d$ that is sufficiently smooth and decays at infinity,*

$$\nabla f(x)\cdot\Lambda^\alpha\nabla f(x) \ge \frac{1}{2}\Lambda^\alpha|\nabla f(x)|^2 + \frac{|\nabla f(x)|^{2+q\alpha/(d+q)}}{C\,\|f\|_{L^q}^{q\alpha/d+q}}, \quad x \in \mathbb{R}^d.$$

The next lemma states an interpolation inequality between Besov spaces (see, e.g., [Bahouri et al. 2011; Miao et al. 2012; Hajaiej et al. 2011]). The definition of Besov spaces is provided in Appendix A.

Lemma 2.2. *Let $s_1 < s_2$ be real numbers and let $\gamma \in (0,1)$. Let $p \in [1,\infty]$. Then, there exists a constant $C = C(s_1, s_2, \gamma)$ such that*

$$\|f\|_{\dot B^{\gamma s_1+(1-\gamma)s_2}_{p,1}} \le C\,\|f\|^{\gamma}_{\dot B^{s_1}_{p,\infty}}\,\|f\|^{1-\gamma}_{\dot B^{s_2}_{p,\infty}}.$$

In particular, for any $\sigma \in (0,1)$ and $p \in [1,\infty]$,

$$\|\Lambda^\sigma f\|_{L^p} \le \|f\|_{\dot B^{\sigma}_{p,1}} \le C\,\|f\|^{1-\sigma}_{B^0_{p,\infty}}\,\|f\|^{\sigma}_{B^1_{p,\infty}} \le C\,\|f\|^{1-\sigma}_{L^p}\,\|\nabla f\|^{\sigma}_{L^p}.$$

We will also need the commutator estimates stated in the following lemma. This lemma is taken from [Li et al. 2016, Lemma 2.2].

Lemma 2.3. *Let $j \ge 0$ be an integer. Let $\alpha \in (0,2)$. Assume $q \in [2,\infty]$ and $q_1, q_2 \in [2,\infty]$ satisfy $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Assume $\nabla \cdot u = 0$. Then*

$$\begin{aligned} \|\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q} \le{}& C\, 2^{(1-\alpha)j}\, \|\nabla u\|_{L^{q_1}}\, \|\Delta_j\theta\|_{L^{q_2}} \\ &+ C\,\|\nabla u\|_{L^{q_1}} \sum_{k\le j-1} 2^{k-j} 2^{(1-\alpha)k}\, \|\Delta_k\theta\|_{L^{q_2}} \\ &+ C\,\|\nabla u\|_{L^{q_1}} \sum_{k\ge j-1} 2^{(2-\alpha)(j-k)} 2^{(1-\alpha)k} \|\Delta_k\theta\|_{L^{q_2}} \\ &+ C\,\|\nabla u\|_{L^{q_1}} \sum_{k\ge j-1} 2^{j-k}\, 2^{(1-\alpha)k} \|\Delta_k\theta\|_{L^{q_2}}, \end{aligned} \tag{2-1}$$

where C's are constants. In addition, (2-1) still holds if $\mathcal{R}_\alpha$ is replaced by $\Lambda^{1-\alpha}$. A special consequence of (2-1) is the bound

$$\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q} \le C\,\|\nabla u\|_{L^{q_1}}\, \|\theta\|_{B^{1-\alpha}_{q_2,1}}. \tag{2-2}$$

Similarly,

$$\|[\Lambda^{1-\alpha}, u\cdot\nabla]\theta\|_{L^q} \le C\,\|\nabla u\|_{L^{q_1}}\, \|\theta\|_{B^{1-\alpha}_{q_2,1}}.$$

Alternatively, the commutator can also be bounded as follows. A proof is provided in Appendix B.

Lemma 2.4. *Let $\alpha \in (0,1)$. Then,*

$$\|[\partial_1\Lambda^{-\alpha}, u\cdot\nabla]\theta\|_{B^0_{\infty,1}} \le C(\|\omega\|_2 + \|\omega\|_\infty)\|\theta\|_{B^{1-\alpha+\epsilon}_{\infty,1}} + C\|u\|_2\|\theta\|_2.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. . It suffices to establish a global a priori bound on $\|(u,\theta)\|_{H^s}$. As we know, if one of the global bounds, for any $t > 0$,

$$\int_0^t \|\nabla\omega(\tau)\|_{L^\infty}\,d\tau < \infty \quad \text{or} \quad \int_0^t \|\nabla\theta(\tau)\|_{L^\infty}\,d\tau < \infty \tag{2-3}$$

holds, then $\|(u,\theta)(t)\|_{H^s}$ is globally bounded. The rest of the proof verifies the bounds in (2-3).

The following global bounds follow easily from (1-2):

$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q} \quad \text{for any } q \in [1,\infty],$$

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}} u(\tau)\|_{L^2}^2\,d\tau \le (\|u_0\|_{L^2} + t\,\|\theta_0\|_{L^2})^2.$$

However, direct energy estimates on (1-2) or on the equation of the vorticity $\omega = \nabla\times u$,

$$\begin{cases} \omega_t + u\cdot\nabla\omega + \Lambda^\alpha\omega = \partial_1\theta, \\ \theta_t + u\cdot\nabla\theta + \Lambda^\beta\theta = 0, \end{cases}$$

would not yield the desired global bound in (2-3), due to the vortex stretching term $\partial_1\theta$. As in [Hmidi et al. 2010; 2011; Miao and Xue 2011; Jiu et al. 2014], the idea is to eliminate $\partial_1\theta$ and work with the combined quantity

$$G = \omega - \mathcal{R}_\alpha\theta \quad \text{with} \quad \mathcal{R}_\alpha = \partial_1\Lambda^{-\alpha}, \tag{2-4}$$

which satisfies

$$G_t + u\cdot\nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u\cdot\nabla]\theta + \Lambda^{\beta-\alpha}\partial_1\theta, \tag{2-5}$$

where we have used the standard commutator notation

$$[\mathcal{R}_\alpha, u\cdot\nabla]\theta = \mathcal{R}_\alpha(u\cdot\nabla\theta) - u\cdot\nabla\mathcal{R}_\alpha\theta.$$

Following the idea of [Constantin and Vicol 2012], we obtain the differential inequality for $\|G(t)\|_{L^\infty}$,

$$\frac{d}{dt}\|G\|_{L^\infty} + C\frac{\|G\|_{L^\infty}^{1+\alpha/2}}{(\|u\|_{L^2}+\|\Lambda^{-\alpha}\theta\|_{L^2})^{\alpha/2}} \le \|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^\infty} + \|\Lambda^{\beta-\alpha}\partial_1\theta\|_{L^\infty} \tag{2-6}$$

and for $\|\nabla\theta\|_{L^\infty}$,

$$\frac{d}{dt}\|\nabla\theta\|_{L^\infty} + C\,\frac{\|\nabla\theta\|_{L^\infty}^{1+\beta}}{\|\theta\|_{L^\infty}^{\beta}} \le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty}. \tag{2-7}$$

We briefly explain the derivation of (2-6). Without loss of generality, we assume G is smooth and decays to zero at infinity. Multiplying (2-5) by G and applying Lemma 2.1 with $q = 2$, we have

$$\begin{aligned}\partial_t|G|^2 + u\cdot\nabla|G|^2 + \Lambda^\alpha|G|^2 + C\frac{|G|^{1+\alpha/2}}{(\|u\|_{L^2} + \|\Lambda^{-\alpha}\theta\|_{L^2})^{\alpha/2}}&\\ \le 2(\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^\infty} + \|\Lambda^{\beta-\alpha}\partial_1\theta\|_{L^\infty})|G|.&\end{aligned} \tag{2-8}$$

For each $t > 0$, there exists $\bar{x} = \bar{x}(t) \in \mathbb{R}^2$ such that

$$G(\bar{x}(t), t) = \|G(t)\|_{L^\infty} = \max_{x\in\mathbb{R}^2}|G(x, t)|.$$

As explained in [Córdoba and Córdoba 2004] and [Constantin et al. 2015, Appendix B],

$$(\partial_t|G|)(\bar{x}(t), t) = \frac{d}{dt}G(\bar{x}(t), t) = \frac{d}{dt}\|G(t)\|_{L^\infty}.$$

In addition, we recall the facts that $(u\cdot\nabla)|G|(\bar{x}(t), t) = 0$ and $(\Lambda^\alpha|G|^2)(\bar{x}(t), t) \ge 0$. Therefore, setting $x = \bar{x}(t)$ in (2-8) and invoking the aforementioned facts yields (2-6). The inequality (2-7) is obtained in a similar fashion.

The terms in (2-6) can be further bounded as follows:

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + t\,\|\theta_0\|_{L^2}, \qquad \|\Lambda^{-\alpha}\theta\|_{L^2} \le C\,\|\theta\|_{L^{2/(1+\alpha)}} \le C\,\|\theta_0\|_{L^{2/(1+\alpha)}}.$$

By (2-2) of Lemma 2.3 and Lemma 2.2,

$$\begin{aligned}\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^\infty} &\le C\,\|\nabla u\|_{L^\infty}\,\|\theta\|_{B^{1-\alpha}_{\infty,1}}\\ &\le C\,\|\nabla u\|_{L^\infty}\,\|\theta\|^{\alpha}_{B^0_{\infty,\infty}}\,\|\theta\|^{1-\alpha}_{B^1_{\infty,\infty}}\\ &\le C\,\|\nabla u\|_{L^\infty}\,\|\theta\|^{\alpha}_{L^\infty}\,\|\nabla\theta\|^{1-\alpha}_{L^\infty}\end{aligned}$$

and

$$\|\Lambda^{\beta-\alpha}\partial_1\theta\|_{L^\infty} \le \|\Lambda^{\beta-\alpha}\partial_1\theta\|_{\dot B^0_{\infty,1}} \le C\,\|\theta\|^{\alpha-\beta}_{L^\infty}\|\nabla\theta\|^{1+\beta-\alpha}_{L^\infty}.$$

Inserting the bounds above in (2-6) yields

$$\begin{aligned}\frac{d}{dt}\|G\|_{L^\infty} + C_1(t)\,\|G\|^{1+\alpha/2}_{L^\infty} &\le C\,\|\nabla u\|_{L^\infty}\,\|\nabla\theta\|^{1-\alpha}_{L^\infty} + C\,\|\nabla\theta\|^{1+\beta-\alpha}_{L^\infty},\\ \frac{d}{dt}\|\nabla\theta\|_{L^\infty} + C_2\,\|\nabla\theta\|^{1+\beta}_{L^\infty} &\le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty},\end{aligned}$$

where

$$C_1(t) = \frac{1}{(\|u_0\|_{L^2} + t\,\|\theta_0\|_{L^2} + \|\theta_0\|_{L^{2/(1+\alpha)}})^{\alpha/2}}.$$

Furthermore, according to Constantin and Vicol [2012],

$$\begin{aligned}\|\nabla u(t)\|_{L^\infty} \le\ & C\,(1+\|\omega(t)\|_{L^\infty}) + C\,\|\omega(t)\|_{L^\infty})\\ & \log\Bigl(1+\int_0^t (1+\|u(\tau)\|_{L^2}+\|\omega(\tau)\|_{L^\infty}+\|\nabla\theta(\tau)\|_{L^2})^{\gamma(\alpha,\beta)}\,d\tau\Bigr),\end{aligned}\tag{2-9}$$

where $\gamma(\alpha,\beta) > 0$ is a constant depending on α and β. Due to

$$\|\omega\|_{L^\infty} \le \|G\|_{L^\infty} + \|\mathcal{R}_\alpha\theta\|_{L^\infty} \le \|G\|_{L^\infty} + C\,\|\theta\|^{\alpha}_{L^\infty}\,\|\nabla\theta\|^{1-\alpha}_{L^\infty},\tag{2-10}$$

we obtain

$$\begin{aligned}\frac{d}{dt}\|G\|_{L^\infty} + C_1\|G\|^{1+\alpha/2}_{L^\infty} \le\ & C_2\|G\|_{L^\infty}\,\|\nabla\theta\|^{1-\alpha}_{L^\infty}\,L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty})\\ & + C_3\,\|\nabla\theta\|^{2-2\alpha}_{L^\infty}\,L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) + C_4\|\nabla\theta\|^{1+\beta-\alpha}_{L^\infty},\end{aligned}\tag{2-11}$$

$$\begin{aligned}\frac{d}{dt}\|\nabla\theta\|_{L^\infty} + C_5\,\|\nabla\theta\|^{1+\beta}_{L^\infty} \le\ & C_6\,\|G\|_{L^\infty}\,\|\nabla\theta\|_{L^\infty}\,L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty})\\ & + C_7\|\nabla\theta\|^{2-\alpha}_{L^\infty}\,L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}),\end{aligned}\tag{2-12}$$

where, for notational convenience, we have written

$$L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) = 1+\log\Bigl(1+\int_0^t (1+\|G\|_{L^\infty}+\|\nabla\theta\|_{L^\infty})^{\gamma(\alpha,\beta)}ds\Bigr).\tag{2-13}$$

We combine (2-11) and (2-12) to prove the global bound (2-3). The argument is as follows. For each $t \ge 0$, we consider two cases:

$$\tfrac12 C_5\|\nabla\theta\|^{\beta}_{L^\infty} > C_6\,\|G\|_{L^\infty}\ L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty})\tag{2-14}$$

and

$$\tfrac{1}{2}C_5\|\nabla\theta\|_{L^\infty}^{\beta} \le C_6\,\|G\|_{L^\infty}\; L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}). \tag{2-15}$$

We start with the first case when (2-14) holds. We split this case into two cases, either

$$C_7\,\|\nabla\theta\|_{L^\infty}^{2-\alpha}\, L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) \le \tfrac{1}{2}C_5\,\|\nabla\theta\|_{L^\infty}^{1+\beta} \tag{2-16}$$

or

$$C_7\,\|\nabla\theta\|_{L^\infty}^{2-\alpha}\, L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) > \tfrac{1}{2}C_5\,\|\nabla\theta\|_{L^\infty}^{1+\beta}. \tag{2-17}$$

When (2-16) is valid, then (2-12) becomes

$$\frac{d}{dt}\|\nabla\theta\|_{L^\infty} + \big(\tfrac{1}{2}C_5\,\|\nabla\theta\|_{L^\infty}^{1+\beta} - C_6\,\|G\|_{L^\infty}\,\|\nabla\theta\|_{L^\infty}\, L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty})\big) < 0,$$

which, due to (2-14), implies that $\|\nabla\theta\|_{L^\infty} < \infty$. Then (2-11) implies $\|G\|_{L^\infty} < \infty$. When (2-17) is valid,

$$C_7\, L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) > \tfrac{1}{2}C_5\,\|\nabla\theta\|_{L^\infty}^{1+\beta-(2-\alpha)}.$$

Since $1+\beta-(2-\alpha) = \alpha+\beta-1 > 0$, we have

$$\|\nabla\theta\|_{L^\infty}^{1+\beta-(2-\alpha)} \le 2C_5^{-1}\, L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}). \tag{2-18}$$

Due to (2-14), L only grows logarithmically in $\|\nabla\theta\|_{L^\infty}$ and thus (2-18) implies that $\|\nabla\theta\|_{L^\infty} < \infty$. Then (2-11) implies $\|G\|_{L^\infty} < \infty$. We now turn to the second case when (2-15) holds. We also split this case into two cases: either

$$L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) \le C\,\|G\|_{L^\infty}^{\epsilon} \tag{2-19}$$

or

$$L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) > C\,\|G\|_{L^\infty}^{\epsilon}, \tag{2-20}$$

where $\epsilon > 0$ is small such that (2-22) below holds. When (2-19) holds, (2-11) becomes

$$\begin{aligned} \frac{d}{dt}\|G\|_{L^\infty} + C_1\|G\|_{L^\infty}^{1+\alpha/2} \\ \le \widetilde{C}_2\,\|G\|_{L^\infty}^{1+(1-\alpha/\beta)+\tilde{\epsilon}} + \widetilde{C}_3\,\|G\|_{L^\infty}^{(2-2\alpha/\beta)+\tilde{\epsilon}} + \widetilde{C}_4\|G\|_{L^\infty}^{(1+\beta-\alpha/\beta)+\tilde{\epsilon}}, \end{aligned} \tag{2-21}$$

where $\widetilde{C}_2$, $\widetilde{C}_3$ and $\widetilde{C}_4$ are constants, and

$$\tilde{\epsilon} = \epsilon\ \max\Big\{1+\frac{1-\alpha}{\beta}, 1+\frac{2-2\alpha}{\beta}, \frac{1+\beta-\alpha}{\beta}\Big\}.$$

Due to (1-3) or $\alpha > \frac{2}{2+\beta}$, we can choose $\epsilon > 0$ small such that

$$1+\frac{\alpha}{2} > \max\Big\{1+\frac{1-\alpha}{\beta}, \frac{2-2\alpha}{\beta}, \frac{1+\beta-\alpha}{\beta}\Big\} + \tilde{\epsilon}. \tag{2-22}$$

Then (2-21) implies $\|G\|_{L^\infty} < \infty$ and (2-19) implies $\|\nabla\theta\|_{L^\infty} < \infty$. When (2-20) holds, (2-15) and the logarithmic growth of L in $\|G\|_{L^\infty}$ implies $\|G\|_{L^\infty} < \infty$. Therefore, for each case, the global bounds in (2-3) hold. This argument here can also be understood as a continuation argument. One starts with initial data that falls into one of the cases. Obviously, the corresponding solution can be continued as long as the solution remains in the same case. If, at a certain time, the solution evolves into the opposite case, the solution can also be continued. That is, the solution can be continued forever. The proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We first prove a proposition stating the global L^2-bound for G. This result was obtained by Ye [2017], but we provide a slightly simpler and more transparent proof.

Proposition 3.1. *Consider the equation of G in (2-5). Assume that α and β satisfy*

$$\tfrac{2}{3} < \alpha < 1, \quad 0 < \beta < 1, \quad \alpha + \beta > 1.$$

Then we have the following global bounds, for any $t > 0$,

$$\|G(t)\|_{L^2} < \infty, \qquad \int_0^t \|\Lambda^{\alpha/2} G(\tau)\|_{L^2}^2 \, d\tau < \infty,$$

$$\sup_{j \ge -1} \int_0^t 2^{2\beta j} \|\Delta_j \theta(\tau)\|^2 \, d\tau < \infty, \quad \textit{especially}, \quad \int_0^t \|\Lambda^\sigma \theta(\tau)\|_{L^2}^2 \, d\tau < \infty,$$

where $0 < \sigma < \beta$.

In order to prove Proposition 3.1, we state a lemma and its corollary first.

Lemma 3.2. *Assume $\beta > 0$. Assume θ solves*

$$\theta_t + u \cdot \nabla\theta + \Lambda^\beta \theta = 0, \quad \theta(x, 0) = \theta_0(x).$$

Then,

$$\sup_{j \ge -1} \int_0^t 2^{2\beta j} \|\Delta_j \theta(\tau)\|^2 \, d\tau \le C \, \|\theta_0\|_{B^{\beta/2}_{2,\infty}}^2 + \widetilde{C} \int_0^t \|\omega(\tau)\|_{L^2}^2 \, d\tau, \tag{3-1}$$

where ω denotes the vorticity, and C, $\widetilde{C}$ are constants depending on the initial data.

A special consequence of Lemma 3.2 is the following corollary.

Corollary 3.3. *Assume that α and β satisfy*

$$0 < \alpha, \beta < 1, \quad \alpha + \beta > 1.$$

Then

$$\sup_{j \ge -1} \int_0^t 2^{2\beta j} \|\Delta_j \theta(\tau)\|^2 \, d\tau \le C(t, \|(u_0, \theta_0)\|_{H^1}) + C \int_0^t \|G(\tau)\|_{L^2}^2 \, d\tau. \tag{3-2}$$

In particular, *for any* $0 < \sigma < \beta$,

$$\int_0^t \|\Lambda^\sigma \theta(\tau)\|_{L^2}^2 \, d\tau \le C(t, \|(u_0, \theta_0)\|_{H^1}) + C \int_0^t \|G(\tau)\|_{L^2}^2 \, d\tau. \tag{3-3}$$

We provide the proof of Lemma 3.2 and Corollary 3.3.

Proof of Lemma 3.2 and Corollary 3.3. Applying the Fourier localization operator Δ_j with $j \in \mathbb{Z}$ and $j \ge -1$ to the equation of θ and then dotting the resulting equation with $\Delta_j \theta$ yields

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta\|_{L^2}^2 + 2^{2\beta j}\|\Delta_j\theta\|_{L^2}^2 = -\int \Delta_j\theta\,[\Delta_j, u\cdot\nabla\theta]\,dx \le \|\Delta_j\theta\|_{L^2}\,\|[\Delta_j, u\cdot\nabla\theta]\|_{L^2}.$$

Applying a standard commutator estimate (see, e.g, [Hmidi et al. 2011, p. 443])

$$\|[\Delta_j, u\cdot\nabla]\theta\|_{L^2} \le C\,\|\theta\|_{B^0_{\infty,\infty}}\,\|\nabla u\|_{L^2},$$

we obtain

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^2} + 2^{\beta j}\,\|\Delta_j\theta\|_{L^2} \le C\,\|\theta_0\|_{L^\infty}\,\|\omega\|_{L^2}.$$

Integrating in time yields

$$\|\Delta_j\theta(t)\|_{L^2} \le C\,e^{-2^{\beta j}t}\,\|\Delta_j\theta_0\|_{L^2} + C\int_0^t e^{-2^{\beta j}(t-\tau)}\|\omega(\tau)\|_{L^2}\,d\tau.$$

Taking the L^2-norm in time and applying Young's inequality for convolution, we have

$$\left[\int_0^t \|\Delta_j\theta(\tau)\|_{L^2}^2\,d\tau\right]^{\frac{1}{2}} \le C\,2^{-\frac{1}{2}\beta j}\,\|\Delta_j\theta_0\|_{L^2} + C\,2^{-\beta j}\left[\int_0^t \|\omega(\tau)\|_{L^2}^2\,d\tau\right]^{\frac{1}{2}}.$$

Multiplying each side by $2^{\beta j}$ and then squaring each side yields

$$\int_0^t 2^{2\beta j}\|\Delta_j\theta(\tau)\|_{L^2}^2\,d\tau \le C\,2^{\beta j}\,\|\Delta_j\theta_0\|_{L^2}^2 + C_0\int_0^t \|\omega(\tau)\|_{L^2}^2\,d\tau.$$

Taking the supremum with respect to j yields (3-1). To show (3-2), we note that

$$\|\omega\|_{L^2} \le \|G\|_{L^2} + \|\Lambda^{1-\alpha}\theta\|_{L^2}. \tag{3-4}$$

For any $\sigma < \beta$, we choose a large integer j_0 such that

$$\sum_{j \ge j_0+1} 2^{2(\sigma-\beta)j} < \frac{1}{4\widetilde{C}}.$$

Then

$$
\begin{aligned}
\|\Lambda^\sigma\theta\|_{L^2}^2 &= \sum_{j\le j_0} 2^{2\sigma j}\|\Delta_j\theta\|_{L^2}^2 + \sum_{j\ge j_0+1} 2^{2(\sigma-\beta)j}\,2^{2\beta j}\|\Delta_j\theta\|_{L^2}^2 \\
&\le C(j_0, \|\theta_0\|_{L^2}) + \frac{1}{4\widetilde{C}}\sup_{j\ge -1} 2^{2\beta j}\|\Delta_j\theta\|_{L^2}^2.
\end{aligned} \tag{3-5}
$$

Inserting (3-4) and (3-5) in (3-1) yields (3-2), and (3-3) follows from (3-5). This completes the proof of Lemma 3.2 and Corollary 3.3. □

We also need the following lemma (see [Stefanov and Wu 2018]).

Lemma 3.4. *Let $1 > \alpha > \frac{1}{2}$, $1 < p_2 < \infty$, $1 < p_1 < \infty$, and $1 < p_3 \le \infty$, so that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. For every s_1 $(0 \le s_1 < 1-\alpha)$ and s_2 $(s_2 > 1-\alpha-s_1)$, there exists a $C = C(p_1, p_2, p_3, s_1, s_2)$, such that*

$$
\left|\int_{\mathbb{R}^d} F[\mathcal{R}_\alpha, u_G\cdot\nabla]\theta\,dx\right| \le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|G\|_{L^{p_3}}. \tag{3-6}
$$

Similarly, for every s_1 $(0 \le s_1 < 1-\alpha)$ and s_2 $(s_2 > 2-2\alpha-s_1)$, we have

$$
\left|\int_{\mathbb{R}^d} F[\mathcal{R}_\alpha, u_\theta\cdot\nabla]\psi\,dx\right| \le C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}}\|\psi\|_{L^{p_3}}. \tag{3-7}
$$

Here u_G denotes the velocity associated with G, namely $u_G = \nabla^\perp(-\Delta)^{-1}G$, and $u_\theta = \nabla^\perp(-\Delta)^{-1}\partial_1\Lambda^{-\alpha}\theta$. The definition of G implies that $u = u_G + u_\theta$.

We now prove Proposition 3.1.

Proof of Proposition 3.1. This proof is obtained by modifying that for the global L^2 bound of G in Stefanov and Wu [2018]. Dotting (2-5) with G and integrating by parts yields

$$
\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + \|\Lambda^{\alpha/2}G\|_{L^2}^2 = J_1 + J_2, \tag{3-8}
$$

where

$$
J_1 = \int G\,\Lambda^{\beta-\alpha}\partial_1\theta\,dx, \qquad J_2 = \int G\,[\mathcal{R}_\alpha, u\cdot\nabla]\theta\,dx.
$$

By Hölder's inequality and Corollary 3.3 with $\sigma = \beta + 1 - \frac{3}{2}\alpha$ $\left(\sigma < \beta \text{ since } \alpha > \frac{2}{3}\right)$,

$$
|J_1| \le \|\Lambda^{\alpha/2}G\|_{L^2}\,\|\Lambda^{\beta+1-3\alpha/2}\theta\|_{L^2} \le \tfrac{1}{4}\|\Lambda^{\alpha/2}G\|_{L^2}^2 + \|\Lambda^{\beta+1-3\alpha/2}\theta\|_{L^2}^2.
$$

As in [Jiu et al. 2014] and [Stefanov and Wu 2018], we write

$$
u = \nabla^\perp\Delta^{-1}\omega = \nabla^\perp\Delta^{-1}G + \nabla^\perp\Delta^{-1}\mathcal{R}_\alpha\theta \equiv u_G + u_\theta.
$$

J_2 is then split into two parts accordingly. The term with u_G part is estimated as in [Stefanov and Wu 2018]. For $\alpha > \frac{2}{3}$, we choose $1-\alpha < s < \frac{\alpha}{2}$ and apply Lemma 3.4,

$$\left| \int G\,[\mathcal{R}_\alpha, u_G \cdot \nabla]\theta\,dx \right| \le C\,\|\theta_0\|_{L^\infty}\,\|G\|_{L^2}\,\|G\|_{H^{\alpha/2}} \le \tfrac{1}{4}\|\Lambda^{\alpha/2}G\|_{L^2}^2 + C\,\|G\|_{L^2}^2.$$

To bound the term associated with u_θ, we apply Lemma 3.4 with $s_1 = \beta + 1 - \frac{3}{2}\alpha$ and $s_2 = \frac{1}{2}(1-\beta)$. Since $\alpha > \frac{2}{3}$ and $\alpha + \beta > 1$, we have $s_1 < \beta$ and $s_2 < \frac{\alpha}{2}$, and $s_1 + s_2 > 2 - 2\alpha$. Therefore

$$\begin{aligned}\left| \int G\,[\mathcal{R}_\alpha, u_\theta \cdot \nabla]\theta\,dx \right| &\le C\,\|\Lambda^{s_1}\theta\|_{L^2}\,\|\theta\|_{L^\infty}\,\|G\|_{H^{s_2}}\\ &\le C\,\|\theta_0\|_{L^\infty}\,\|\Lambda^{\beta+1-3\alpha/2}\theta\|_{L^2}\,\|G\|_{H^{\alpha/2}}\\ &\le \tfrac{1}{4}\,\|\Lambda^{\alpha/2}G\|_{L^2}^2 + C\,\|\Lambda^{\beta+1-3\alpha/2}\theta\|_{L^2}^2.\end{aligned}$$

Inserting the bounds above in (3-8), and applying Corollary 3.3 with $\sigma = \beta + 1 - \frac{3}{2}\alpha$ and Gronwall's inequality yields the desired global bound. □

In order to prove Theorem 1.2, we also need the following lower bound for the fractional Laplacian operator. The proof of this lemma follows the lines of Constantin and Vicol [2012] and will be provided in Appendix B.

Lemma 3.5. *Let $\alpha \in (0, 2)$. For any smooth function f that decays sufficiently fast at infinity, suppose that $\bar{x} \in \mathbb{R}^2$ is a point at which $|f(x)|$ attains its maximum. Then,*

$$f\,\Lambda^\alpha f \ge C\,\frac{|f(\bar{x})|^{2+\alpha}}{\|f\|_{L^2}^\alpha}$$

for a constant $C = C(\alpha)$.

Proof of Theorem 1.2. Making use of Lemma 3.5, we obtain, as in the derivation of (2-6),

$$\frac{d}{dt}\|G\|_{L^\infty} + C\frac{\|G\|_{L^\infty}^{1+\alpha}}{\|G\|_{L^2}^\alpha} \le C\,\|\nabla u\|_{L^\infty}\,\|\nabla\theta\|_{L^\infty}^{1-\alpha} + C\,\|\nabla\theta\|_{L^\infty}^{\beta+1-\alpha},$$

$$\frac{d}{dt}\|\nabla\theta\|_{L^\infty} + C\frac{\|\nabla\theta\|_{L^\infty}^{1+\beta}}{\|\theta\|_{L^\infty}^\beta} \le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty}.$$

We further use (2-9) and (2-10) to obtain

$$\begin{aligned}\frac{d}{dt}\|G\|_{L^\infty} + C_1(t)\,\|G\|_{L^\infty}^{1+\alpha} &\le C_2\,\|G\|_{L^\infty}\,\|\nabla\theta\|_{L^\infty}^{1-\alpha}\,L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty})\\ &\quad + C_3\,\|\nabla\theta\|_{L^\infty}^{2(1-\alpha)}\,L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) + C_4\,\|\nabla\theta\|_{L^\infty}^{1+\beta-\alpha},\end{aligned}$$

$$\frac{d}{dt}\|\nabla\theta\|_{L^\infty} + C_5\|\nabla\theta\|_{L^\infty}^{1+\beta} \le C_6\|G\|_{L^\infty}\|\nabla\theta\|_{L^\infty} L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}) \\ + C_7\|\nabla\theta\|_{L^\infty}^{2-\alpha} L(\|G\|_{L^\infty}, \|\nabla\theta\|_{L^\infty}),$$

where $C_1(t) = C\,(\|G(t)\|_{L^2})^{-1}$, $C_5 = C\,(\|\theta\|_{L^\infty})^{-1}$, and L is as defined in (2-13). We can then argue in a similar fashion as in the proof of Theorem 1.1 that the global bounds $\|G\|_{L^\infty} < \infty$ and $\|\nabla\theta\|_{L^\infty} < \infty$ hold if α and β satisfy (1-4). In fact, if $\frac{2}{3} < \alpha < 1$ and $\alpha > \frac{1}{1+\beta}$, then

$$\alpha > \frac{1-\alpha}{\beta}, \qquad 1+\alpha > \frac{2-2\alpha}{\beta}, \qquad 1+\alpha > \frac{1+\beta-\alpha}{\beta}$$

and the argument in the proof of Theorem 1.1 works here. This completes the proof of Theorem 1.2. □

Appendix A. Frequency localization and Besov spaces

This appendix provides the definition of the Littlewood–Paley decomposition and the definition of Besov spaces. Some related facts used in the previous sections are also included. The material presented in this appendix can be found in several books and many papers (see, e.g., [Bahouri et al. 2011; Bergh and Löfström 1976; Miao et al. 2012; Runst and Sickel 1996; Triebel 1992]).

We start with several notational conventions. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$,

$$A_j = \{\xi \in \mathbb{R}^d : 2^{j-1} \le |\xi| < 2^{j+1}\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\operatorname{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{A-1}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0 & \text{if } j \le -2, \\ \Psi * f & \text{if } j = -1, \\ \Phi_j * f & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A-2}$$

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k.$$

For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. It is clear from (A-1) that $S_j \to \mathrm{Id}$ as $j \to \infty$ in the distributional sense. In addition, the notation $\widetilde{\Delta}_k$, defined by

$$\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1},$$

is also useful and has been used in the previous sections.

Definition A.1. The inhomogeneous Besov space $B^s_{p,q}$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty,$$

where $\Delta_j f$ is as defined in (A-2).

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.2. *For any $s \in \mathbb{R}$,*

$$H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$B^s_{q,\min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,\max\{q,2\}}.$$

For any noninteger $s > 0$, the Hölder space C^s is equivalent to $B^s_{\infty,\infty}$.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Proposition A.3. *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

(1) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(1/p-1/q)} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(1/p-1/q)} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

Appendix B. Proofs of facts used in the previous sections

This appendix provides the proofs of several facts used in Sections 2 and 3.

We first provide several pointwise inequalities involving fractional Laplacian operators. These lower bounds here are in terms of the L^p-norms of the functions instead of the L^p-norms of the antiderivatives. Therefore, these lower bounds have higher powers than the corresponding lower bounds in terms of the antiderivatives. The proofs of these lower bounds follow the ideas of Constantin and Vicol [2012].

Lemma B.1. *Let $p \in [1, \infty)$. Assume $f \geq 0$, $f \in L^p(\mathbb{R}^d)$ and $f \in C^1(\mathbb{R}^d)$. Suppose that f attains its maximum value at the point $\bar{x}$. Then,*

$$\Lambda^\alpha f(\bar{x}) \geq \frac{f(\bar{x})^{1+\alpha p/d}}{c\|f\|_{L^p}^{\alpha p/d}} \tag{B-1}$$

for some constant $c = c(d, \alpha, p)$.

Proof. Let χ be a radially nondecreasing smooth cut-off function, which vanishes on $|x| \leq 1$ and is identically 1 on $|x| \geq 2$, and $|\nabla \chi| \leq 4$. Let $R > 0$ be a number to be specified later. We estimate

$$\begin{aligned}\Lambda^\alpha f(\bar{x}) &= c_{d,\alpha}\int_{\mathbb{R}^d}\frac{f(\bar{x})-f(\bar{x}-y)}{|y|^{d+\alpha}}\,dy\\ &\geq c_{d,\alpha} f(\bar{x})\int_{\mathbb{R}^d}\frac{\chi(y/R)}{|y|^{d+\alpha}}dy - c_{d,\alpha}\left|\int_{\mathbb{R}^d} f(\bar{x}-y)\frac{\chi(y/R)}{|y|^{d+\alpha}}dy\right|\\ &\geq c_{d,\alpha} f(\bar{x})\int_{|y|\geq 2R}\frac{1}{|y|^{d+\alpha}}dy - c_{d,\alpha}\|f\|_{L^p}\left(\int_{\mathbb{R}^d}\left|\frac{\chi(y/R)}{|y|^{d+\alpha}}\right|^{p'}dy\right)^{1/p'}\\ &\geq c_1\frac{f(\bar{x})}{R^\alpha} - c_2\frac{\|f\|_{L^p}}{R^{\alpha+d/p}},\end{aligned}$$

where $c_1 = c_1(d,\alpha)$, and $c_2 = c_2(d,\alpha,\delta)$ are positive constants, which may be computed explicitly. Letting $R^{d/p} = 2c_2\|f\|_{L^p}/(c_1 f(\bar{x}))$ concludes the proof. $\square$

Lemma B.2. *Let $\alpha\in(0,2)$ and let $p\in[1,\infty)$. Assume $f\in L^p(\mathbb{R}^d)$ and $f\in C^1(\mathbb{R}^d)$. Then we have the pointwise bound*

$$f(x)\cdot\Lambda^\alpha f(x)\geq \tfrac{1}{2}\Lambda^\alpha|f(x)|^2+\frac{|f(x)|^{2+p\alpha/d}}{c\|f\|_{L^p}^{p\alpha/d}} \tag{B-2}$$

for some positive constant $c = c(d,\alpha,p)$.

Proof. Recall the pointwise identity (see [Constantin and Vicol 2012])

$$f(x)\cdot\Lambda^\alpha f(x) = \tfrac{1}{2}\Lambda^\alpha(|f|^2)(x)+\tfrac{1}{2}D, \tag{B-3}$$

where

$$D = c_{d,\alpha}PV\int_{\mathbb{R}^d}\frac{|f(x)-f(x+y)|^2}{|y|^{d+\alpha}}\,dy. \tag{B-4}$$

For χ defined as in the previous proof,

$$\begin{aligned}D &\geq c_{d,\alpha}\int_{\mathbb{R}^d}\frac{|f(x)-f(x+y)|^2}{|y|^{d+\alpha}}\chi(y/R)dy\\ &\geq c_{d,\alpha}|f(x)|^2\int_{\mathbb{R}^d}\frac{\chi(y/R)}{|y|^{d+\alpha}}dy - 2c_{d,\alpha}|f(x)|\left|\int_{\mathbb{R}^d} f(x+y)\frac{\chi(y/R)}{|y|^{d+\alpha}}\,dy\right|\\ &\geq c_{d,\alpha}|f(x)|^2\int_{|y|\geq R}\frac{1}{|y|^{d+\alpha}}dy - 2\,c_{d,\alpha}|f(x)|\,\|f\|_{L^p}\left(\int_{\mathbb{R}^d}\left|\frac{\chi(y/R)}{|y|^{d+\alpha}}\right|^{p'}dy\right)^{1/p'}\\ &\geq c_1\frac{|f(x)|^2}{R^\alpha} - c_2\frac{|f(x)|\,\|f\|_{L^p}}{R^{\alpha+d/p}},\end{aligned}$$

for some positive constants c_1 and c_2 which depend only on d, α, and p. Letting

$$R^{d/p} = \frac{c_2\|f\|_{L^p}}{2c_1|f(x)|}$$

concludes the proof of this lemma. □

A special consequence is the following lower bound.

Corollary B.3. *Let $\alpha \in (0, 2)$. Assume f is smooth and decays sufficiently fast at infinity. Assume that $\bar{x} \in \mathbb{R}^d$ is a maximum point at which $|f(x)|$ attains its maximum. Then,*

$$f(\bar{x}) \cdot \Lambda^\alpha f(\bar{x}) \geq \frac{|f(\bar{x})|^{2+p\alpha/d}}{c\|f\|_{L^p}^{p\alpha/d}}$$

where $c = c(d, \alpha, p)$.

Next we provide the proof of Lemma 2.4.

Proof. Write $\mathcal{R}_\alpha = \partial_{x_1}\Lambda^{-\alpha}$, then $\Delta_k\mathcal{R}_\alpha = 2^{(1-\alpha)k}h_k$, where $h_k(x) = 2^{dk}h_0(2^k x)$ and $h_0(x) \in C_0^\infty(\mathbb{R}^d)$. By the notion of paraproducts,

$$\text{(B-5)}\quad \Delta_k[\mathcal{R}_\alpha, u\cdot\nabla]\theta = \sum_{|j-k|\leq 2}\Delta_k[\mathcal{R}_\alpha, S_{j-1}u\cdot\nabla]\Delta_j\theta + \sum_{|j-k|\leq 2}\Delta_k[\mathcal{R}_\alpha, \Delta_j u\cdot\nabla]S_{j-1}\theta \\ + \sum_{j\geq k-4}[\mathcal{R}_\alpha, \Delta_j u\cdot\nabla]\tilde{\Delta}_j\theta := J_1 + J_2 + J_3.$$

We estimate the L^∞-norm of the terms on the right.

$$\begin{aligned}\|J_1\|_{L^\infty} &\leq C2^{(1-\alpha)k}\||x|2^{kd}h_0(2^k x)\|_{L^1}\|\nabla S_{k-1}u\|_{L^\infty}\|\Delta_k\nabla\theta\|_{L^\infty} \\ &\leq C2^{-\alpha k}\|\nabla\Delta_k\theta\|_{L^\infty}\left(\|\nabla\Delta_{-1}\|_{L^\infty} + \sum_{j=0}^{k-2}\|\Delta_j\nabla u\|_{L^\infty}\right) \\ &\leq C2^{-\alpha k}\|\nabla\Delta_k\theta\|_\infty(\|u\|_{L^2} + k\|\omega\|_{L^\infty}).\end{aligned}$$

For J_2 and J_3, we have

$$\begin{aligned}\|J_2\|_{L^\infty} &\leq C\|\Delta_k\mathcal{R}_\alpha(\Delta_k u\cdot\nabla S_{k-1}\theta) - \Delta_k(\Delta_k u\cdot\nabla\mathcal{R}_\alpha(S_{k-1}\theta - \Delta_{-1}))\|_{L^\infty} \\ &\qquad + \|\Delta_k(\Delta_k u\cdot\nabla\mathcal{R}_\alpha\Delta_{-1}\theta)\|_{L^\infty} \\ &\leq C2^{(1-\alpha)k}\||x|2^{dk}h_0(2^k x)\|_{L^1}\|\nabla\Delta_k u\|_{L^\infty}\|\nabla S_{k-1}\theta\|_{L^\infty} \\ &\qquad + C\|\Delta_k u\|_{L^\infty}\|\nabla\mathcal{R}_\alpha\Delta_{-1}\theta\|_{L^\infty} \\ &\leq C2^{-k\alpha}\|\Delta_k\omega\|_{L^\infty}\left(\sum_{j=-1}^{k-2}\|\nabla\Delta_j\theta\|_{L^\infty}\right) + C\|\theta\|_{L^2}\|\Delta_k u\|_{L^\infty}.\end{aligned}$$

$$\begin{aligned}\|J_3\|_{L^\infty} \le & \sum_{j\le 1} \|\nabla\cdot\Delta_k\mathcal{R}_\alpha(\Delta_j u\tilde{\Delta}_j\theta)\|_{L^\infty} + \|\nabla\cdot\Delta_k(\Delta_j u\mathcal{R}_\alpha\tilde{\Delta}_j\theta)\|_{L^\infty} \\ & + \sum_{j\ge\max(2,k-4)} \|\Delta_k\mathcal{R}_\alpha(\Delta_j u\cdot\nabla\tilde{\Delta}_j\theta)\|_{L^\infty} + \|\Delta_k\nabla\cdot(\Delta_j u\cdot\mathcal{R}_\alpha\tilde{\Delta}_j\theta)\|_{L^\infty} \\ \le & C\|u\|_{L^2}\|\theta\|_{L^2} + \sum_{j\ge\max(2,k-1)} 2^{(1-\alpha)k}\|\Delta_j u\|_{L^\infty}\|\nabla\tilde{\Delta}_j\theta\|_{L^\infty} \\ & + 2^k\|\Delta_j u\|_{L^\infty}\|\mathcal{R}_\alpha\tilde{\Delta}_j\theta\|_{L^\infty}.\end{aligned}$$

Therefore,

$$\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{B^0_{\infty,1}} \le \sum_{k\ge -1}\|J_1\|_{L^\infty} + \sum_{k\ge -1}\|J_2\|_{L^\infty} + \sum_{k\ge -1}\|J_3\|_{L^\infty} := I_1 + I_2 + I_3$$

and

$$I_1 \le C(\|\omega\|_{L^2}+\|\omega\|_{L^\infty})\sum_{k\ge -1} 2^{(1-\alpha)+\varepsilon k}\|\Delta_k\theta\|_{L^\infty} \le C(\|\omega\|_{L^2}+\|\omega\|_{L^\infty})\|\theta\|_{B^{1-\alpha+\varepsilon}_{\infty,1}},$$

$$\begin{aligned} I_2 \le & C\sum_{k\ge -1}\|\Delta_k\omega\|_{L^\infty}\sum_{j=-1}^{k-2} 2^{\alpha(j-k)}2^{-\alpha j}\|\nabla\Delta_j\theta\|_{L^\infty} + C\|\theta\|_{L^2}\sum_{k\ge 0}2^{-k}\|\Delta_k\omega\|_{L^\infty} + C \\ \le & C\|\omega\|_{L^\infty}\|\theta\|_{B^{1-\alpha+\epsilon}_{\infty,1}} + C\|\omega\|_{L^\infty}, \end{aligned}$$

$$\begin{aligned} I_3 \le & C\|u||_{L^2}\|\theta\|_{L^2} + C\sum_{k\ge -1}\sum_{j\ge\max(2,k-1)} 2^{(1-\alpha)(k-j)}\|\Delta_j\nabla u\|_{L^\infty}2^{-\alpha j}\|\nabla\tilde{\Delta}_j\theta\|_{L^\infty} \\ & + C\sum_{k\ge -1}\sum_{j\ge\max(2,k-1)} 2^{k-j}\|\nabla\Delta_j u\|_{L^\infty}2^{(1-\alpha)j}\|\tilde{\Delta}_j\theta\|_{L^\infty} \\ \le & C\|u\|_{L^2}\|\theta\|_{L^2} + C\|\omega\|_{L^\infty}\|\theta\|_{B^{1-\alpha}_{\infty,1}}. \end{aligned}$$

Combining these estimates, we have

$$\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{B^0_{\infty,1}} \le C(\|\omega\|_{L^2} + \|\omega\|_{L^\infty})\|\theta\|_{B^{1-\alpha+\varepsilon}_{\infty,1}} + C\|u\|_{L^2}\|\theta\|_{L^2}$$

for any $\varepsilon > 0$. □

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