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A DUAL-PETROV-GALERKIN METHOD FOR TWO INTEGRABLE FIFTH-ORDER KDV TYPE EQUATIONS

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Abstract. This paper extends the dual-Petrov-Galerkin method proposed by Shen [21], further developed by Yuan, Shen and Wu [27] to general fifth-order KdV type equations with various nonlinear terms. These fifth-order equations arise in modeling different wave phenomena. The method is implemented to compute the multi-soliton solutions of two representative fifth-order KdV equations: the Kaup-Kupershmidt equation and the Caudry-Dodd-Gibbon equation. The numerical results imply that this scheme is capable of capturing, with very high accuracy, the details of these solutions such as the nonlinear interactions of multi-solitons.

1. **Introduction.** Numerical simulation is an indispensable tool in the study of many nonlinear dispersive partial differential equations. In [21] Shen proposed the dual-Petrov-Galerkin method for the third and higher odd-order equations such as the KdV and higher-order KdV type equations. This is a spectral Galerkin method with innovative choices of test and trial function spaces. Numerical tests performed on soliton solutions of the KdV equation indicate that the dual-Petrov-Galerkin method is very efficient and accurate [21]. In a recent work of Yuan, Shen and Wu [27] a numerical scheme based on the dual-Petrov-Galerkin method was implemented for the Kawahara and modified Kawahara equations, two fifth-order KdV type equations. Numerical experiments involving some computationally challenging solitary and oscillatory solitary waves demonstrate that this method can compute the solutions of these equations accurately and efficiently.

This paper intends to further develop the dual-Petrov-Galerkin method to cover more general fifth-order KdV type equations of the form

$$u_t - u_{xxxxx} = F(x, t, u, u_x, u_{xx}, u_{xxx}).$$

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Fifth-order KdV type equations arise naturally in modeling many different wave phenomena such as gravity-capillary waves, the propagation of shallow water waves over a flat surface and magneto-sound propagation in plasmas (see e.g. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 23, 24]). The numerical scheme is implemented on two representative fifth-order KdV equations: the Kaup-Kupershmidt (KK) equation

$$u_t + u_{xxxxx} + 10u\, u_{xxx} + 25u_x\, u_{xx} + 20\, u^2\, u_x = 0 \tag{1}$$

and the Caudrey-Dodd-Gibbon (CDG) equation

$$u_t + u_{xxxxx} + 30u\, u_{xxx} + 30u_x\, u_{xx} + 180u^2\, u_x = 0. \tag{2}$$

There is a large literature on these equations ([3, 4, 12, 14, 15, 16, 18, 19, 20, 23]). We remark that although these two equations have close resemblance, they are actually quite different and can not be converted into each other through scaling.

Numerical tests are focused on the multi-soliton solutions of these equations. Various methods have been employed to find the analytic formulas for multi-soliton solutions of these equations ([7, 8, 9, 10, 12, 18, 19, 20, 23, 25]). However, for $N \geq 3$, the analytic computations of N-soliton solutions are complicated and extremely lengthy. The dual Petrov-Galerkin scheme presented in this paper provides a very efficient and accurate numerical method for simulating these multi-soliton solutions. We are not aware of any other effective numerical methods for these fifth-order equations.

To simulate the multi-soliton solutions of (1) and of (2), we approximate the initial-value problem by an initial- and boundary-value problem (IBVP) for $x \in [-L, L]$ as long as the solitons does not reach the boundaries. By setting $(\tilde{x}, \tilde{t}) = (-L^{-1}x, L^{-1}t)$, it suffices to consider the IBVP for the equation

$$u_t - \frac{1}{L^4}\, u_{xxxxx} - \frac{\beta}{L^2}\, u\, u_{xxx} - \frac{\gamma}{L^2}\, u_x\, u_{xx} - \nu\, u^2\, u_x = 0 \tag{3}$$

in the space-time domain $[-1, 1] \times [0, T]$ with the initial and boundary data of the form

$$\begin{aligned} &u(-1,t) = g(t),\ u_x(-1,t) = h(t),\ u(1,t) = u_x(1,t) = u_{xx}(1,t) = 0,\ t \in [0,T],\\ &u(x,0) = u_0(x), \quad x \in [-1,1]. \end{aligned} \tag{4}$$

In addition, if u solves (3) and (4), then

$$w(x,t) = u(x,t) - v(x,t)$$

with

$$v(x,t) = \frac{(1-x)^3}{8}\left[\left(h(t) + \frac{3}{2}g(t)\right)(x+1) + g(t)\right]$$

solves an IBVP with homogeneous boundary conditions. Therefore, it is enough to consider the following IBVP with homogeneous boundary condition

$$\begin{aligned} &\alpha\, u_t - \frac{1}{L^4}\, u_{xxxxx} - \frac{\beta}{L^2}\, u\, u_{xxx} - \frac{\gamma}{L^2}\, u_x\, u_{xx} - \nu\, u^2\, u_x = 0, \quad x \in I,\ t \in (0,T],\\ &u(\pm 1,t) = u_x(\pm 1,t) = u_{xx}(1,t) = 0, \quad t \in [0,T],\\ &u(x,0) = u_0(x), \quad x \in I, \end{aligned} \tag{5}$$

where $I = [-1, 1]$. The dual Petrov-Galerkin method is implemented on the IBVP (5). A crucial component of this method is the choice of the test and trial function spaces and their basis functions. The trial functions satisfy the boundary conditions of the differential equations while the test functions satisfy the “dual” boundary conditions so that no additional boundary terms are generated when integrating by parts. By choosing "compact" combinations of Legendre polynomials, the resulting linear problem is compactly sparse and well conditioned. The time is discretized by the Crank-Nicholson-leap-frog scheme.

The method is applied to compute 1-soliton, 2-soliton and 3-soliton solutions of (1) and 1-soliton and 2-soliton solutions of (2). Besides plotting the graphs of the numerical simulation of these solutions, we also track the L^2-errors between the analytic solutions and their corresponding numerical solutions for a range of time steps. The tables constructed for these errors clearly demonstrate that the scheme is second-order in time. In the case of multi-soliton solutions, special attention is paid to the phase shifts of the solitons after the their interactions. As a measure of the accuracy of the method, we compare the theoretical phase shifts against the corresponding numerical phase shifts and find that they are very close. This is an indication that this method can capture the nonlinear behavior of these multi-solitons to extreme accuracy.

The rest of this paper is divided into two sections. Section 2 describes the dual-Petrov-Galerkin method while Section 3 presents the numerical results.

2. **Numerical methods.**

We start with some notation. Let $I = (-1, 1)$ and let $\omega = \omega(x)$ be a positive function in I. For $m = 0, \pm 1, \cdots$, we use $H^m_\omega(I)$ to denote the weighted Sobolev space of order m with its norm given by $\|\cdot\|_{m,\omega}$. For example,

$$H^0_\omega(I) = L^2_\omega(I) \equiv \{u : (u, u)_\omega := \int_I u^2(x)\omega(x)dx < +\infty\}$$

with $\|\cdot\|_\omega = (u, u)^{\frac{1}{2}}_\omega$. For any constants α and β, let $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ be the Jacobi weight function with index (α, β). We define a set of non-uniformly weighted Sobolev spaces as follows:

$$H^m_{\omega^{\alpha,\beta}}(I) = \{u \in L^2_{\omega^{\alpha,\beta}}(I) : \partial^l_x u \in L^2_{\omega^{l-\alpha,l-\beta}}(I),\ 1 \le l \le m\}. \tag{6}$$

Let P_N denote the space of polynomials of degree $\le N$ and set

$$\begin{aligned} W_N &= \{u \in P_N :\ u(\pm 1) = u_x(\pm 1) = u_{xx}(1) = 0\}, \\ W^*_N &= \{u \in P_N :\ u(\pm 1) = u_x(\pm 1) = u_{xx}(-1) = 0\}. \end{aligned} \tag{7}$$

Let Π_N be the orthogonal projection from $L^2_{\omega^{-3,-2}}$ onto W_N defined by

$$(u - \Pi_N u, v_N)_{\omega^{-3,-2}} = 0 \quad \text{for any } v_N \in W_N.$$

With these notations at our disposal, we are ready to provide the numerical scheme for the IBVP (5). The numerical scheme consists of a dual-Petrov-Galerkin method in space and the second-order Crank-Nicholson-leap-frog discretization in time. The dual-Petrov-Galerkin method generates a sequence of approximate solutions that satisfy a weak form of the original differential equations as tested against polynomials in a dual space. Assume (5) admits a unique solution u satisfying

$$u \in C^3([0, T]; L^2_{\omega^{2,2}}(I)) \cap C^1([0, T]; H^m_{\omega^{-3,-2}}(I)) \quad \text{with } m \ge 3.$$

This regularity assumption is backed by the rigorous theory of Goubet and Shen [6] on the KdV equation and of Khanal, Wu and Yuan on fifth-order KdV type equations [14]. For a given Δt, we set $t_k = k\Delta t$ and let $u^0_N = \Pi_N u_0$ and u^1_N be a suitable approximation of $u(\cdot, t_1)$. Then, the second-order Crank-Nicolson-leap-frog scheme in time with a dual-Petrov-Galerkin approximation in space reads: for $k = 1, 2, \cdots, [T/\Delta t] - 1$, find $u_N^{k+1} \in W_N$ such that

$$\begin{aligned} &\frac{\alpha}{2\Delta t}(u_N^{k+1} - u_N^{k-1}, \eta_N) - \frac{1}{2}\left(\partial_x^2(u_N^{k+1} + u_N^{k-1}), \partial_x^3 \eta_N\right) \\ &\quad = \frac{\nu}{3}\left((u_N^k)^3, \partial_x \eta_N\right) + \beta\left(u_N^k \partial_x^2 u_N^k, \partial_x \eta_N\right) + \frac{\gamma - \beta}{2}((\partial_x u_N^k)^2, \partial_x \eta_N) \end{aligned} \tag{8}$$

for any $\eta_N \in W_N^*$. Notice that for any $v_N \in W_N$, we have $\omega^{-1,1} v_N \in W_N^*$. Thus, the dual-Petro-Galerkin formulation in (8) is equivalent to the following weighted spectral-Galerkin approximation: Find $u_N \in W_N$ such that

$$\begin{aligned} &\frac{\alpha}{2\Delta t}(u_N^{k+1} - u_N^{k-1}, v_N)_{\omega^{-1,1}} - \frac{1}{2}\left(\partial_x^2(u_N^{k+1} + u_N^{k-1}), \partial_x^3(v_N \omega^{-1,1})\right) \\ &\quad = \frac{\nu}{3}\left((u_N^k)^3, \partial_x(v_N\, \omega^{-1,1})\right) + \beta\left(u_N^k\, \partial_x^2\, u_N^k, \partial_x(v_N \omega^{-1,1})\right) \\ &\qquad + \frac{\gamma - \beta}{2}((\partial_x u_N^k)^2, \partial_x(v_N \omega^{-1,1})) \end{aligned} \tag{9}$$

for any $v_N \in W_N$. The dual-Petrov-Galerkin formulation (8) is most suitable for implementation while the weighted Galerkin formulation (9) is more convenient for error analysis.

3. **Numerical results.**

This section presents the numerical results on N-soliton solutions of the IBVPs for (1) and (2). Computed solutions are plotted and the L^2-errors between the numerical 1-soliton solutions and their analytic counterparts at various times and corresponding to different time steps are recorded to test the accuracy and the convergence rate of the scheme. In the case of multi-soliton solutions, theoretical phase shifts of the solitons after their interaction are compared against the numerical ones.

3.1. N-soliton solutions of the KK equation.

We compute the 1-soliton, 2-soliton and 3-soliton solutions of the KK equation

$$u_t + u_{xxxxx} + 10u\, u_{xxx} + 25 u_{xx}\, u_x + 20\, u^2\, u_x = 0 \tag{10}$$

and compare them with the corresponding analytic formulas.

3.1.1. 1-*soliton solution.* The analytic formula for 1-soliton solution has previously been obtained in many papers (see e.g. [7, 12, 16, 18, 19]). Following Hereman and Nuseir [7], the 1-soliton solution is given by

$$u = \frac{3}{2}\frac{\partial^2}{\partial x^2} \ln f, \qquad f = 1 + \exp(\theta) + \frac{1}{16}\exp(2\theta), \tag{11}$$

where $\theta = k\,x - k^5\,t + \delta$ with k and δ being constants. More explicitly, setting $\delta = 0$, the 1-soliton solution u can be written as

$$u_{\mathrm{KK1}}(x, t, k) = \frac{24k^2 \exp(k\,x - k^5\,t)\,[4\exp(k\,x - k^5\,t) + \exp(2(k\,x - k^5\,t)) + 16]}{[16\exp(k\,x - k^5\,t) + \exp(2(k\,x - k^5\,t)) + 16]^2}. \tag{12}$$

Our attention will be focused on the special case when $k = 1$,

$$u_{\mathrm{KK1S}}(x, t) = \frac{24\exp(x - t)\,[4\exp(x - t) + \exp(2(x - t)) + 16]}{[16\exp(x - t) + \exp(2(x - t)) + 16]^2}. \tag{13}$$

As mentioned in the introduction, in order to apply the dual-Petrov-Galerkin scheme, we rescale (10) with $(\tilde{x}, \tilde{t}) = (-L^{-1}x, L^{-1}t)$ and still use (x, t) to denote $(\tilde{x}, \tilde{t})$. Then we are led to consider the following IBVP:

$$\begin{aligned}
&u_t - \frac{1}{L^4}\, u_{xxxxx} - \frac{10}{L^2}\, u u_{xxx} - \frac{25}{L^2}\, u_x\, u_{xx} - 20\, u^2 u_x = 0, \quad x \in (-1, 1),\\
&u(\pm 1, t) = u_x(\pm 1, t) = u_{xx}(1, t) = 0,\\
&u(x, 0) = u_{\mathrm{KK1S}}(-L\, x, 0) = \frac{24 \exp(-Lx)\, [4 \exp(-Lx) + \exp(-2Lx) + 16]}{[16 \exp(-Lx) + \exp(-2Lx) + 16]^2}.
\end{aligned} \tag{14}$$

The exact solution of (14) is given by

$$u(x, t) = u_{\mathrm{KK1S}}(-L\, x, L\, t).$$

We present the numerical results for $L = 40$. By taking $N = 1000$ in the dual-Petrov-Galerkin scheme, the spatial error is negligible and the error is dominated by the time discretization error. In Table 1, we list the L^2-errors at different times with two different time steps. The rate "4" in each row of the table is the approximate ratio of the two L^2-errors in that row. As the table shows, the L^2-errors quadruple when the time steps double. Therefore, Table 1 indicates that the Crank-Nicholson-leap-frog scheme is of second-order in time. In Figure 1, we

Time	L^2-error(Δt=1.0E-4)	L^2-error(Δt=2.0E-4)	rate
0.125	2.641E-6	1.057 E-5	4.0
0.25	2.881E-6	1.152E-5	4.0
0.375	3.007E-6	1.203E-5	4.0

TABLE 1. Errors for the 1-soliton solution of KK equation

plot the computed and exact solutions for the IBVP (14). The computed solutions and the exact solutions are virtually indistinguishable. We remark that the actual physical time corresponding to the plot marked "$t = 0.375$" is $L\, t = 40 * 0.375 = 15$.

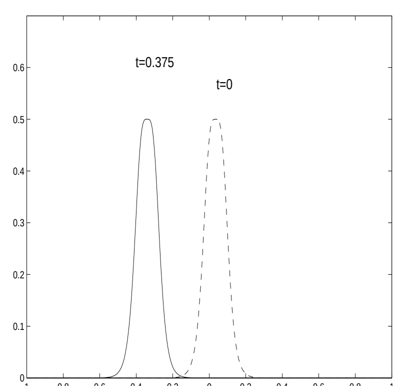

FIGURE 1. 1-soliton solution of (14) with Δt=1.0E-4

3.1.2. 2-*soliton solution.* The analytic formula for the 2-soliton solution is much more complex than the 1-soliton solution and is given by

$$u_{\mathrm{KK2}}(x,t,k_1,k_2) = \frac{3}{2}\frac{\partial^2}{\partial x^2}\ln f \tag{15}$$

with

$$\begin{aligned} f = 1 + \exp(\theta_1) + \exp(\theta_2) + \frac{1}{16}\,(\exp(2\theta_1) + \exp(2\theta_2)) + a_{12}\exp(\theta_1+\theta_2) \\ + b_{12}\,(\exp(2\theta_1+\theta_2) + \exp(\theta_1+2\theta_2)) + b_{12}^2\exp(2\theta_1+2\theta_2), \end{aligned} \tag{16}$$

where $\theta_i = (k_i x - k_i^5 t) + \delta_i$ for i=1, 2 and $k_2 > k_1 > 0$, and

$$a_{12} = \frac{2k_1^4 - k_1^2k_2^2 + 2k_2^4}{2(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)}, \tag{17}$$

$$b_{12} = \frac{(k_1-k_2)^2(k_1^2-k_1k_2+k_2^2)}{16(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)}. \tag{18}$$

It is easy to show that, for $k_2 > k_1 > 0$,

$$u_{\mathrm{KK2}}(x,t,k_1,k_2) \approx u_{\mathrm{KK1}}(x,t,k_1) + u_{\mathrm{KK1}}(x-\eta_2,t,k_2) \quad \text{as } t\to\infty, \tag{19}$$

$$u_{\mathrm{KK2}}(x,t,k_1,k_2) \approx u_{\mathrm{KK1}}(x-\eta_1,t,k_1) + u_{\mathrm{KK1}}(x,t,k_2) \quad \text{as } t\to-\infty, \tag{20}$$

where η_1 and η_2 are the phase shift constants,

$$\eta_i = -\frac{\ln(16\,b_{12})}{k_i}, \quad i=1,2. \tag{21}$$

That is, the collision between two solitons cause the large soliton to shift forward by η_2 along the x-direction and the small soliton to shift backward by η_1. This type of interaction is referred as the nonlinear interaction.

We focus on the special 2-soliton $u_{\mathrm{KK2S}}(x,t)$ given by (15), (16), (17) and (18) with $k_1 = 1$, $k_2 = 1.5$, $\delta_1 = 0$ and $\delta_2 = -100$. To compute this 2-soliton solution, we apply the dual Petrov-Galerkin method to the IBVP:

$$\begin{aligned} & u_t - \frac{1}{L^4}\,u_{xxxxx} - \frac{10}{L^2}\,uu_{xxx} - \frac{25}{L^2}\,u_x\,u_{xx} - 20\,u^2u_x = 0, \quad x\in(-1,1), \\ & u(\pm1,t) = u_x(\pm1,t) = u_{xx}(1,t) = 0, \\ & u(x,0) = u_{\mathrm{KK2S}}(-L\,x,0). \end{aligned} \tag{22}$$

The exact solution of this IBVP is given by

$$u(x,t) = u_{\mathrm{KK2S}}(-L\,x,\,L\,t).$$

We set $L = 100$ and choose $N = 2500$ in the dual-Petrov-Galerkin method. In Figure 2, we plot the graph at $t = 0$ and that of the numerical solution of (22) at $t = 0.3$ (solid lines). We also plot the two solitons without phase shift (dashed lines). These plots clearly show that the colliding waves undergo phase shifts.

According to (21), the phase shift of the large soliton (shift in the x-direction) is given by

$$\eta_2 = \frac{1}{Lk_2}\,\ln(16\,b_{12}) = \frac{1}{150}\,\ln(0.014737) = -0.028116$$

and that of the small soliton (shift in the x-direction) is

$$\eta_1 = -\frac{1}{Lk_1}\,\ln(16\,b_{12}) = -\frac{1}{100}\,\ln(0.014737) = 0.042174.$$

As a comparison, we measure the corresponding numerical phase shifts to be 0.028098 and 0.042117, and the errors are 1.8×10^{-5} and 5.7×10^{-5}, respectively.

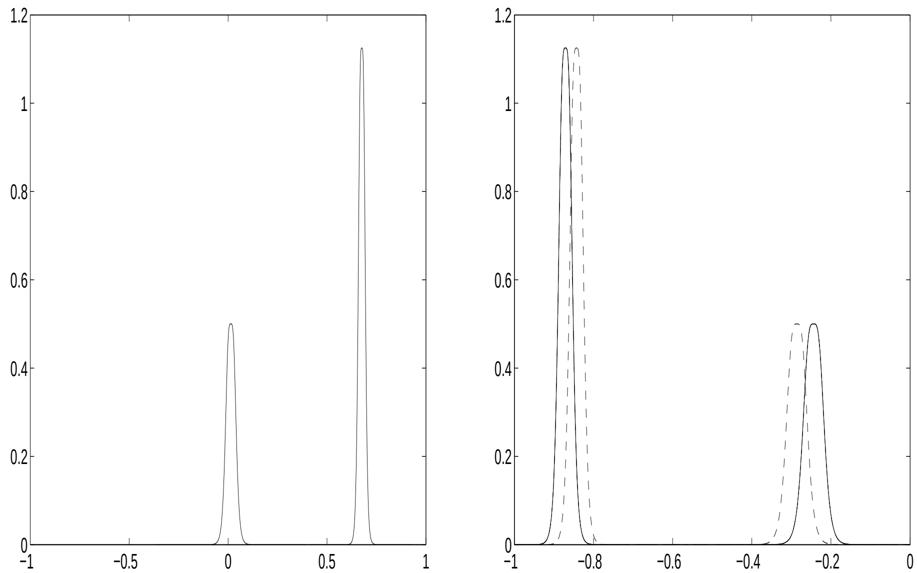

FIGURE 2. Two-soliton solutions of KK at $t = 0$ (left), with linear interaction (- -) and with nonlinear interaction (-) at $t = 0.3$ with Δt=1.0E-6 (right).

3.1.3. *3-soliton solution.* The formula for 3-soliton solution can be found in ([7, 18, 19]). Following [7], the 3-soliton solution can be written as

$$u_{\mathrm{KK3}} = \frac{3}{2}\frac{\partial^2}{\partial x^2} \ln f, \qquad f = 1 + f^{(1)} + f^{(2)} + f^{(3)} + f^{(4)} + f^{(5)} + f^{(6)} \tag{23}$$

with

$$\begin{aligned}
f^{(1)} &= \sum_{i=1}^{3} \exp(\theta_i),\\
f^{(2)} &= \frac{1}{16}\sum_{i=1}^{3} \exp(2\theta_i) + \sum_{1\le i<j\le 3} a_{ij}\exp(\theta_i+\theta_j),\\
f^{(3)} &= \sum_{1\le i<j\le 3} b_{ij}[\exp(2\theta_i+\theta_j) + \exp(\theta_i+2\theta_j)] + c_{123}\exp(\theta_1+\theta_2+\theta_3),\\
f^{(4)} &= \sum_{1\le i<j\le 3} b_{ij}^2 \exp(2\theta_i+2\theta_j) + 16[a_{23}b_{12}b_{13}\exp(2\theta_1+\theta_2+\theta_3)\\
&\quad + a_{13}b_{12}b_{23}\exp(\theta_1+2\theta_2+\theta_3) + a_{12}b_{13}b_{23}\exp(\theta_1+\theta_2+3\theta_3)],\\
f^{(5)} &= 16^2 b_{12}b_{13}b_{23}[b_{12}\exp(2\theta_1+2\theta_2+\theta_3) + b_{13}\exp(2\theta_1+\theta_2+2\theta_3)\\
&\quad + b_{23}\exp(\theta_1+2\theta_2+2\theta_3)],\\
f^{(6)} &= 16(16b_{12}b_{13}b_{23})^2\exp(2\theta_1+2\theta_2+2\theta_3).
\end{aligned}$$

where

$$\begin{aligned}
\theta_i &= (k_i x - k_i^5 t) + \delta_i \text{ for } i = 1,\ 2 \text{ and } k_3 \geq k_2 \geq k_1 > 0,\\
a_{ij} &= \frac{2k_i^4 - k_i^2 k_j^2 + 2k_j^4}{2(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)}, \qquad 1 \leq i < j \leq 3,\\
b_{ij} &= \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{16(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)}, \qquad 1 \leq i < j \leq 3,\\
c_{123} &= \frac{1}{D}\Big[(2k_1^4 - k_1^2 k_2^2 + 2k_2^4)(k_3^8 + k_1^4 k_2^4)\\
&\quad + (2k_1^4 - k_1^2 k_3^2 + 2k_3^4)(k_2^8 + k_1^4 k_3^4)\\
&\quad + (2k_2^4 - k_2^2 k_3^2 + 2k_3^4)(k_1^8 + k_2^4 k_3^4)\Big]\\
&\quad - \frac{1}{2D}\Big[(k_1^2 + k_1^2)(k_1^4 + k_2^4)(k_3^6 + k_1^2 k_2^2 k_3^2)\\
&\quad + (k_1^2 + k_3^2)(k_1^4 + k_2^4)(k_2^6 + k_1^2 k_2^2 k_3^2)\\
&\quad + (k_2^2 + k_3^2)(k_2^4 + k_3^4)(k_1^6 + k_1^2 k_2^2 k_3^2) + 12 k_1^4 k_2^4 k_3^4\Big]
\end{aligned}$$

with

$$D = 4 \prod_{1 \leq i < j \leq 3} (k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2).$$

Asymptotically u_{KK3} can be approximated by the superposition of three solitons (modulo phase shifts), namely

$$u_{\text{KK3S}} \quad = \quad \sum_{i=1}^{3} \frac{24 k_i^2 \exp(\theta_i)[4\exp(\theta_i) + \exp(2\theta_i) + 16]}{[16\exp(\theta_i) + \exp(2\theta_i) + 16]^2} \tag{24}$$

and the error between u_{KK3} and u_{KK3S} is exponentially small ([5, 26]). After two interactions, the phase shifts of these solitons are

$$\text{the largest soliton :} \quad -\frac{1}{k_3}\left[\ln(16 b_{13}) + \ln(16\, b_{23})\right], \tag{25}$$

$$\text{the middle soliton :} \quad -\frac{1}{k_2}\left[\ln(16 b_{12}) - \ln(16\, b_{23})\right], \tag{26}$$

$$\text{the smallest soliton :} \quad \frac{1}{k_1}\left[\ln(16 b_{12}) + \ln(16\, b_{13})\right]. \tag{27}$$

We consider the special 3-soliton solution with $k_1 = 1$, $k_2 = 1.8$, $k_3 = 2$, $\delta_1 = 12$, $\delta_2 = -65$ and $\delta_3 = -165$. We apply the dual-Petrov-Galerkin method to the scaled problem as we did in the two-soliton case. Numerical results are presented for $L = 100$ and $N = 3000$. The numerical solutions at $t = 0$ and $t = 0.1$ are plotted in Figure 3. As a way to test the accuracy of the numerical method, we also compare the numerical phase shifts with the phase shifts given by (25), (26) and (27). For the scaled problems, the theoretical phase shifts are

$$\frac{1}{k_3\, L}(\ln(16\, b_{13}) + \ln(16\, b_{23})) = -0.05012329855, \tag{28}$$

$$\frac{1}{k_2\, L}(\ln(16\, b_{12}) - \ln(16\, b_{23})) = 0.01982336314, \tag{29}$$

$$-\frac{1}{k_1\, L}(\ln(16\, b_{12}) + \ln(16\, b_{23})) = 0.06511810919 \tag{30}$$

for the largest, middle and smallest solitons, respectively. The corresponding actual phase shifts in our numerical test are -0.049895498, 0.019727626, 0.06456454347 and the errors are 2.28×10^{-4}, 9.57×10^{-5} and 6.62×10^{-4}, respectively.

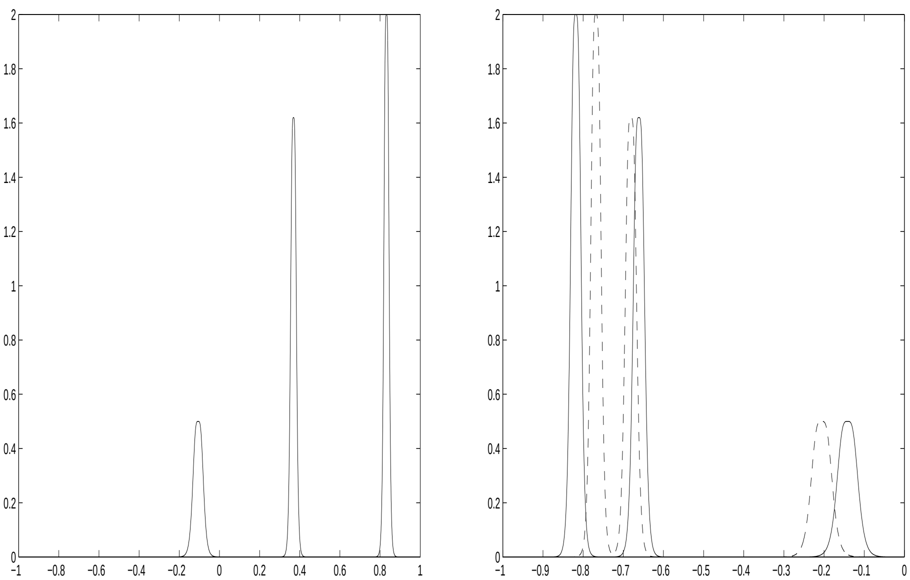

FIGURE 3. Three-soliton solutions of KK at $t = 0$ (left), with linear interaction (- -) and with nonlinear interaction (-) at $t = 0.1$ with $\Delta t = 5.0 \times 10^{-7}$ (right).

3.2. ***N*-soliton solutions of the CDG equation.** This subsection focuses on the 1-soliton and 2-soliton solutions of the CDG equation:

$$u_t + u_{xxxxx} + 30u\, u_{xxx} + 30u_x u_{xx} + 180u^2\, u_x = 0. \tag{31}$$

3.2.1. 1-*soliton.* Following [3, 18, 25], the 1-soliton solution of the CDG equation assumes the form

$$u_{\text{CDG1}}(x,t) = \frac{k_1^2 \exp(k_1(x - k_1^4\, t))}{(1 + \exp(k_1(x - k_1^4\, t)))^2}$$

where k_1 is a constant. We numerically compute the 1-soliton solution by applying the dual-Petrov-Galerkin method to the IBVP for the scaled CDG equation:

$$\begin{aligned} &u_t - 180u^2\, u_x - \frac{30}{L^2} u\, u_{xxx} - \frac{30}{L^2} u_x u_{xx} - \frac{1}{L^4} u_{xxxxx} = 0, \quad x \in (-1,1), \\ &u(\pm 1,t) = u_x(\pm 1,t) = u_{xx}(1,t) = 0, \quad u(x,0) = u_{\text{CDG1}}(-Lx,0). \end{aligned} \tag{32}$$

The exact solution of this problem $u(x,t) = u_{\text{CDG1}}(-Lx, Lt)$. The numerical solution with $L = 100$ for $t = 0$ and $t = 0.04$ are plotted in Figure 4. To check the accuracy of the scheme, we construct a table of L^2-errors at different times for two time steps.

Time	L^2-error(Δt=1.0E-6)	L^2-error(Δt=2.0E-6)	rate
0.01	3.415E-6	1.366E-5	4.0
0.02	4.329E-6	1.733E-5	4.0
0.04	6.323E-6	2.534E-5	4.01

TABLE 2. Errors for the 1-soliton solution of the CDG equation

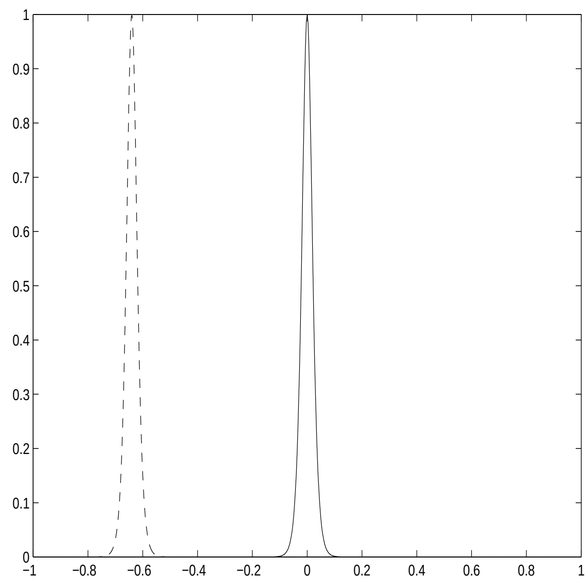

FIGURE 4. The 1-soliton solution of (32) at $t = 0$ (-) and $t = 0.04$ (- -) with Δt=1.0E-6

3.2.2. 2-*soliton.* According to [25], the 2-soliton solution of (31) is given by

$$u_{\text{CDG2}} = \frac{\partial^2}{\partial x^2} \ln f, \tag{33}$$

with

$$f = 1 + \exp(\theta_1) + \exp(\theta_2) + a_{12} \exp(\theta_1 + \theta_2), \tag{34}$$

where

$$a_{12} = \frac{(k_1 - k_2)^2 (k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2 (k_1^2 + k_1 k_2 + k_2^2)}, \quad \theta_i = (k_i x - k_i^5 t) + \delta_i \quad \text{for} \quad i = 1, 2. \tag{35}$$

Our attention is focused on the special 2-soliton solution u_{CDG2S} with $k_1 = 1$ and $k_2 = 2$. The phase shifts after the interaction are given by $-\frac{1}{k_2} \ln(a_{12})$ for the large soliton and $-\frac{1}{k_1} \ln(a_{12})$ for the small soliton.

To numerically compute this 2-soliton solution, we apply the dual-Petrov-Galerkin method to the scaled CDG equation

$$\begin{aligned} &u_t - 180u^2\, u_x - \frac{30}{L^2} u\, u_{xxx} - \frac{30}{L^2} u_x u_{xx} - \frac{1}{L^4} u_{xxxxx} = 0, \quad x \in (-1, 1), \\ &u(\pm 1, t) = u_x(\pm 1, t) = u_{xx}(1, t) = 0, \quad u(x, 0) = u_{\text{CDG2S}}(-Lx, 0). \end{aligned} \tag{36}$$

The exact solution of this IBVP is given by

$$u(x,t) = u_{\text{CDG2S}}(-L\,x,\,L\,t).$$

Setting $L = 100$ and choosing $N = 2000$ in the dual Petrov-Galerkin scheme, we solve (36) and plot the initial data and the solution at $t = 0.08$ (see Figure 5). Special attention is paid to the phase shifts after the interaction of the solitons. The solid lines on the right of Figure 5 represent the numerical solution of (36) and the dashed lines the solitons without phase shifts. The theoretical phase shifts are

$$\frac{1}{k_2\,L}\ln(a_{12}) = 0.01522261219 \quad \text{(forward shift for the large soliton)} \quad (37)$$

$$\frac{1}{k_1\,L}\ln(a_{12}) = 0.03044522438 \quad \text{(backward shift for the small soliton)} (38)$$

and the numerical phase shifts are 0.01526530328 and 0.0304144155, correspondingly. The errors are 4.3×10^{-5} and 3.1×10^{-5}, respectively.

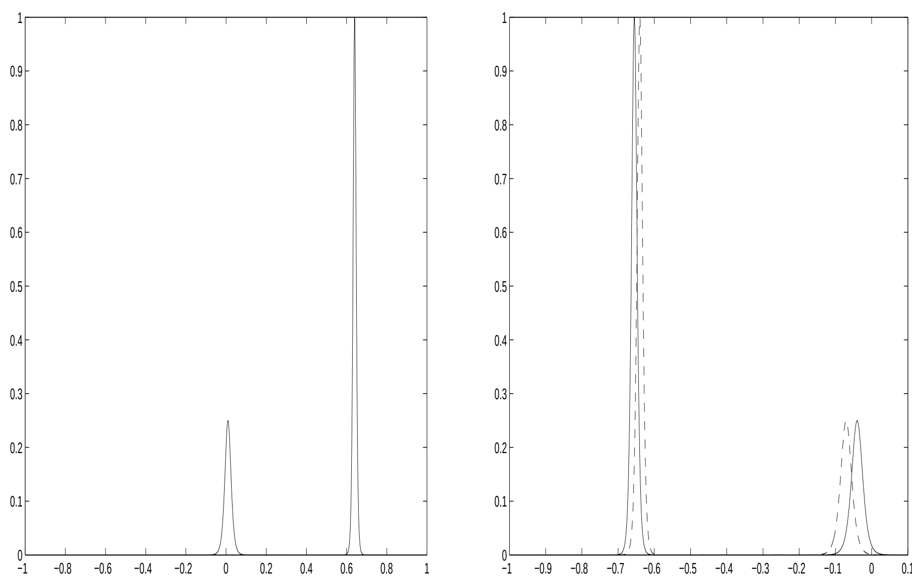

FIGURE 5. Two-soliton solutions of CDG at $t = 0$ (left) and two-soliton solutions of CDG with linear interaction (- -) and with non-linear interaction (-) at $t = 0.08$ (right).

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