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Small global solutions to the damped two-dimensional Boussinesq equations

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Abstract

The two-dimensional (2D) incompressible Euler equations have been thoroughly investigated and the resolution of the global (in time) existence and uniqueness issue is currently in a satisfactory status. In contrast, the global regularity problem concerning the 2D inviscid Boussinesq equations remains widely open. In an attempt to understand this problem, we examine the damped 2D Boussinesq equations and study how damping affects the regularity of solutions. Since the damping effect is insufficient in overcoming the difficulty due to the "vortex stretching", we seek unique global small solutions and the efforts have been mainly devoted to minimizing the smallness assumption. By positioning the solutions in a suitable functional setting (more precisely, the homogeneous Besov space $\mathring{B}^1_{\infty,1}$), we are able to obtain a unique global solution under a minimal smallness assumption.

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1. Introduction

This paper examines the global (in time) existence and uniqueness problem on the incompressible 2D Boussinesq equations with damping

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nu u = -\nabla p + \theta \mathbf{e}_2, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t\theta + (u\cdot\nabla)\theta + \lambda\theta = 0, & x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot u = 0, & x\in\mathbb{R}^2,\ t>0,\\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), & x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where u represents the fluid velocity, p the pressure, $\mathbf{e}_2$ the unit vector in the vertical direction, θ the temperature in thermal convection or the density in geophysical flows, and $\nu>0$ and $\lambda>0$ are real parameters. When νu is replaced by $-\nu\Delta u$ and $\lambda\theta$ by $-\lambda\Delta\theta$, (1.1) becomes the standard viscous Boussinesq equations. (1.1) with $\nu=0$ and $\lambda=0$ reduces to the inviscid 2D Boussinesq equations. If θ is identically zero, (1.1) degenerates to the 2D incompressible Euler equations.

The Boussinesq equations model many geophysical flows such as atmospheric fronts and ocean circulations (see, e.g., [10,16,25,31]). Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier–Stokes equations such as the vortex stretching mechanism. The vortex stretching term is the greatest obstacle in dealing with the global regularity issue concerning the Boussinesq equations. When suitable partial dissipation or fractional Laplacian dissipation with sufficiently large index is added, the vortex stretching can be controlled and the global regularity can be established (see, e.g., [1,2,5,6,8,9,11,14,17–24,27, 29,34,35,37]). In contrast, the global regularity problem on the inviscid Boussinesq equations appears to be out of reach in spite of the progress on the local well-posedness and regularity criteria (see, e.g., [7,12,13,15,26,29,30,36]). This work is partially aimed at understanding this difficult problem by examining how damping affects the regularity of the solutions to the Boussinesq equations.

As we know, the issue of global existence and uniqueness relies crucially on whether or not one can obtain global bounds on the solutions. Thanks to the divergence-free condition $\nabla\cdot u=0$, global *a priori* bounds for θ in any Lebesgue space L^q and u in L^2 follow directly from simple energy estimates,

$$\|\theta(t)\|_{L^q} \leqslant \|\theta_0\|_{L^q}, \qquad \|u(t)\|_{L^2} \leqslant \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}$$

for $1\leqslant q\leqslant\infty$. However, global bounds for (u,θ) in any Sobolev space, say H^1, cannot be easily achieved and the difficulty comes from the vortex stretching term. More precisely, if we resort to the equations of the vorticity ω and $\nabla^\perp\theta$

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega + \nu\omega = \partial_{x_1}\theta,\\ \partial_t(\nabla^\perp\theta) + (u\cdot\nabla)(\nabla^\perp\theta) + \lambda\nabla^\perp\theta = (\nabla^\perp\theta\cdot\nabla)u, \end{cases} \tag{1.2}$$

we unavoidably have to deal with the "vortex stretching term" $(\nabla^\perp\theta\cdot\nabla)u$, which appears to elude any suitable bound. Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. The damping terms are not sufficient to overcome this difficulty. Therefore damping does not appear to make a big difference in dealing with solutions emanating from a general data.

The aim here is to study the global existence and uniqueness of small solutions. We remark that, if we consider solutions of (1.1) with initial data in the classical setting, say $(u_0, \theta_0) \in H^s$ with $s > 2$, then it is not difficult to prove the global existence of solutions when we impose a very strong smallness condition such as

$$1 + \|u_0\|_{H^s} + \|\theta_0\|_{H^s} \leqslant C \min\{\nu, \lambda\}, \tag{1.3}$$

where C is a suitable constant independent of ν and λ. In fact, the global regularity follows easily from the local well-posedness in H^s and the global energy inequality for $Y(t) \equiv \|u(t)\|_{H^s} + \|\theta(t)\|_{H^s}$,

$$\frac{d}{dt} Y(t) + \min\{\nu, \lambda\} Y(t) \leqslant C\big(1 + \|u\|_{H^s} + \|\theta\|_{H^s}\big) Y(t). \tag{1.4}$$

However, the smallness assumption (1.3) appears to be too restrict. In particular, it forces ν and λ to be of order 1. It appears that (1.3) cannot be easily weakened if we seek solutions in the classical functional setting. This is due to the presence of the forcing term $\theta \mathbf{e}_2$ in the velocity equation and the growth of $\|u(t)\|_{H^s}$ in time. In fact, as shown by Brandolese and Schonbek for the 3D viscous Boussinesq equations, the L^2-norm of u may grow in time when the spatial integral of θ_0 is not zero and when θ does not decay sufficiently fast in time [5]. Our efforts have been devoted to seeking a suitable functional setting so that (1.3) can be relaxed. The right functional space is the homogeneous Besov space $\mathring{B}^1_{\infty,1}$ and our main result can be stated as follows.

Theorem 1.1. *Consider (1.1) with $\nu > 0$ and $\lambda > 0$. Assume that $(u_0, \theta_0) \in L^2$ obeys the smallness conditions*

$$\|\nabla u_0\|_{\mathring{B}^0_{\infty,1}} < A_0 \equiv \min\left\{\frac{\nu}{2C_0}, \frac{\lambda}{C_0}\right\} \quad \textit{and} \quad \|\nabla \theta_0\|_{\mathring{B}^0_{\infty,1}} < B_0 \equiv \frac{\nu}{2C_0} \|\nabla u_0\|_{\mathring{B}^0_{\infty,1}} \tag{1.5}$$

for a suitable constant C_0 independent of ν and λ. Then (1.1) has a unique global solution (u, θ) satisfying

$$(u, \theta) \in L^\infty\big([0, \infty); L^2\big), \quad \nabla u, \nabla \theta \in L^\infty\big([0, \infty); \mathring{B}^0_{\infty,1}\big). \tag{1.6}$$

In addition,

$$\sup_{t \geqslant 0} \big\|\nabla u(t)\big\|_{\mathring{B}^0_{\infty,1}} < A_0 \quad \textit{and} \quad \sup_{t \geqslant 0} \big\|\nabla \theta(t)\big\|_{\mathring{B}^0_{\infty,1}} < B_0.$$

More details on the homogeneous Besov space can be found in Appendix A. We remark that the smallness condition (1.5) is weaker than (1.3) in two senses: first, the norm in $\mathring{B}^1_{\infty,1}$ (the smallness for $(\nabla u, \nabla \theta)$ in $\mathring{B}^0_{\infty,1}$ is equivalent to the smallness of (u, θ) in $\mathring{B}^1_{\infty,1}$) is weaker than the norm in H^s due to the embedding $H^s(\mathbb{R}^2) \hookrightarrow \mathring{B}^1_{\infty,1}(\mathbb{R}^2)$ for $s > 2$; and second, (1.5) does not include the factor 1, as in (1.3). $\mathring{B}^1_{\infty,1}$ appears to be a very natural setting if one wants to ensure the uniqueness of the solutions. It may be difficult to further weaken the functional setting.

May it be possible to sharpen the result of Theorem 1.1 by removing one of the damping terms νu or $\lambda\theta$? This problem appears to be extremely challenging. For the 2D Boussinesq equations, it appears that any functional setting guaranteeing the uniqueness of solutions necessarily involves the derivatives of the functions. If $\lambda\theta$ is not present, the norm of θ in such a functional setting may grow exponentially (in time) at the rate of $\|\nabla u\|_{L^\infty}$. Consequently the norm of u may grow even when the velocity equation has the damping term νu. It is also clear that, if νu is missing, then the norm of u is expected to grow. Therefore, when any one of the damping terms is removed, the small data well-posedness problem becomes as difficult as the well-posedness problem for a general initial data.

The rest of this paper consists of a section that proves Theorem 1.1 and an appendix that provides the definitions of Besov spaces and related facts.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The proof is lengthy and consists of five major steps. The first step constructs a sequence of approximate smooth solutions while the second step shows that these approximate solutions obey global (in time) bounds in the functional setting of the initial data. One key component leading to the global bounds is a global differential inequality, which we establish as a proposition. The third step is to show that the sequence of approximate solutions consists of a strongly convergent subsequence. The fourth step shows that the limit of the convergence actually solves the Boussinesq equations in a suitable functional setting. This step involves extensive applications of the Besov space techniques. The last step asserts the uniqueness of the solutions.

As a preparation for the proof of Theorem 1.1, we first state and prove a global differential inequality.

Proposition 2.1. *Consider (1.1) with $\nu > 0$ and $\lambda > 0$. Assume $(u_0, \theta_0) \in L^2$ and $(\nabla u_0, \nabla\theta_0) \in \mathring{B}^0_{\infty,1}$. Assume that (u, θ) is the corresponding solution. Then (u, θ) satisfies the differential inequalities, for any $q \in [2, \infty]$,*

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u\|_{\mathring{B}^0_{q,1}} + \nu\|\nabla u\|_{\mathring{B}^0_{q,1}} \leqslant C_0\|\nabla u\|_{\mathring{B}^0_{\infty,1}}\|\nabla u\|_{\mathring{B}^0_{q,1}} + C_0\|\nabla\theta\|_{\mathring{B}^0_{q,1}}, \\
&\frac{d}{dt}\|\nabla\theta\|_{\mathring{B}^0_{q,1}} + \lambda\|\nabla\theta\|_{\mathring{B}^0_{q,1}} \leqslant C_0\|\nabla u\|_{\mathring{B}^0_{\infty,1}}\|\nabla\theta\|_{\mathring{B}^0_{q,1}},
\end{aligned} \tag{2.1}$$

where $C_0 > 0$ is a constant independent of q, ν and λ.

Proof. Let $2 \leqslant q \leqslant \infty$ be arbitrarily fixed. We now derive the differential inequalities for $\|\nabla\theta\|_{\mathring{B}^0_{q,1}}$ and $\|\nabla u\|_{\mathring{B}^0_{q,1}}$. Let j be an integer. Applying Δ_j to the equation of $\nabla\theta$ yields

$$\partial_t\Delta_j\nabla\theta + \Delta_j\nabla\big((u\cdot\nabla)\theta\big) + \lambda\Delta_j\nabla\theta = 0.$$

Multiplying by $\Delta_j\nabla\theta|\Delta_j\nabla\theta|^{q-2}$ and integrating in space, we have

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\nabla\theta\|^q_{L^q} + \lambda\|\Delta_j\nabla\theta\|^q_{L^q} = K_1 + K_2, \tag{2.2}$$

where

$$K_1 = -\int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot \Delta_j(\nabla u \cdot \nabla\theta),$$
$$K_2 = -\int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot \Delta_j(u \cdot \nabla\nabla\theta).$$

To bound K_1, we use the notion of paraproducts to write K_1 into three terms,

$$K_1 = K_{11} + K_{12} + K_{13},$$

where

$$K_{11} = -\sum_{|k-j|\leqslant 2} \int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot \Delta_j(S_{k-1}\nabla u \cdot \Delta_k \nabla\theta),$$
$$K_{12} = -\sum_{|k-j|\leqslant 2} \int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot \Delta_j(\Delta_k \nabla u \cdot S_{k-1}\nabla\theta),$$
$$K_{13} = -\sum_{k\geqslant j-1} \int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot \Delta_j(\Delta_k \nabla u \cdot \widetilde{\Delta}_k \nabla\theta)$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. By Hölder's inequality,

$$|K_{11}| \leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \|S_{j-1}\nabla u\|_{L^\infty} \|\Delta_j \nabla\theta\|_{L^q} \leqslant C\|\nabla u\|_{L^\infty} \|\Delta_j \nabla\theta\|_{L^q}^q,$$

where C is a constant independent of q. Here we have applied the simple fact that, for fixed j, the summation in K_{11} is for a finite number of k's satisfying $|k-j| \leqslant 2$ and the estimate for the term with the index k is only a constant multiple of the bound for the term with the index j. By Hölder's inequality,

$$|K_{12}| \leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \|\Delta_j \nabla u\|_{L^\infty} \sum_{m\leqslant j-1} \|\Delta_m \nabla\theta\|_{L^q},$$

where again C is independent of q. Thanks to $\nabla \cdot u = 0$ and by Hölder's inequality and Bernstein's inequality,

$$|K_{13}| \leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \sum_{k\geqslant j-1} 2^j \|\Delta_k \nabla u\|_{L^\infty} \|\Delta_k \theta\|_{L^q}$$
$$\leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \sum_{k\geqslant j-1} 2^{j-k} \|\Delta_k \nabla u\|_{L^\infty} \|\Delta_k \nabla\theta\|_{L^q}.$$

We now turn to K_2. We decompose it into five terms via the notion of paraproducts,

$$K_2 = K_{21} + K_{22} + K_{23} + K_{24} + K_{25},$$

where

$$K_{21} = -\sum_{|k-j|\leqslant 2} \int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot [\Delta_j, S_{k-1}u \cdot \nabla]\Delta_k \nabla\theta,$$

$$K_{22} = -\sum_{|k-j|\leqslant 2} \int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot (S_{k-1}u - S_j u) \cdot \nabla\Delta_j \Delta_k \nabla\theta,$$

$$K_{23} = -\int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot S_j u \cdot \nabla\Delta_j \nabla\theta,$$

$$K_{24} = -\sum_{|k-j|\leqslant 2} \int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \nabla\theta),$$

$$K_{25} = -\sum_{k\geqslant j-1} \int \Delta_j \nabla\theta |\Delta_j \nabla\theta|^{q-2} \cdot \Delta_j (\Delta_k u \cdot \nabla\widetilde{\Delta}_k \nabla\theta).$$

Thanks to the divergence-free condition, $\nabla \cdot u = 0$, we have $K_{23} = 0$. By Hölder's inequality and a standard commutator estimate (see, e.g., [3, p. 110]),

$$|K_{21}| \leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \|\nabla S_{j-1}u\|_{L^\infty} \|\Delta_j \nabla\theta\|_{L^q} \leqslant C\|\nabla u\|_{L^\infty} \|\Delta_j \nabla\theta\|_{L^q}^q,$$

where C is a constant independent of q. It is easy to see from Hölder's inequality and Bernstein's inequality that

$$|K_{22}| \leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \|\Delta_j \nabla u\|_{L^\infty} \|\Delta_j \nabla\theta\|_{L^q}.$$

Again, by Hölder's inequality and Bernstein's inequality,

$$|K_{24}| \leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \|\Delta_j \nabla u\|_{L^\infty} \sum_{m\leqslant j-1} \|\Delta_m \nabla\theta\|_{L^q}.$$

Due to the divergence-free condition,

$$|K_{25}| \leqslant C\|\Delta_j \nabla\theta\|_{L^q}^{q-1} \sum_{k\geqslant j-1} 2^{j-k} \|\nabla\Delta_k u\|_{L^\infty} \|\nabla\Delta_k \theta\|_{L^q}.$$

Inserting the estimates for K_1 and K_2 above in (2.2), we obtain

$$\frac{d}{dt}\|\Delta_j \nabla\theta\|_{L^q} + \lambda\|\Delta_j \nabla\theta\|_{L^q} \leqslant C\|\nabla u\|_{L^\infty} \|\Delta_j \nabla\theta\|_{L^q} + C\|\Delta_j \nabla u\|_{L^\infty} \sum_{m\leqslant j-1} \|\Delta_m \nabla\theta\|_{L^q} + C\sum_{k\geqslant j-1} 2^{j-k} \|\nabla\Delta_k u\|_{L^\infty} \|\nabla\Delta_k \theta\|_{L^q}.$$

Summing over all integer j and applying Young's inequality for series convolution, we obtain

$$\frac{d}{dt}\|\nabla\theta\|_{\mathring{B}^0_{q,1}}+\lambda\|\nabla\theta\|_{\mathring{B}^0_{q,1}}\leqslant C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{\mathring{B}^0_{q,1}}+C\|\nabla u\|_{\mathring{B}^0_{\infty,1}}\|\nabla\theta\|_{\mathring{B}^0_{q,1}}.$$

Invoking the simple fact

$$\|\nabla u\|_{L^\infty}\leqslant\|\nabla u\|_{\mathring{B}^0_{\infty,1}},$$

we obtain

$$\frac{d}{dt}\|\nabla\theta\|_{\mathring{B}^0_{q,1}}+\lambda\|\nabla\theta\|_{\mathring{B}^0_{q,1}}\leqslant C\|\nabla u\|_{\mathring{B}^0_{\infty,1}}\|\nabla\theta\|_{\mathring{B}^0_{q,1}} \tag{2.3}$$

for a pure constant C independent of q.

We now derive a differential inequality for $\|\nabla u\|_{\mathring{B}^0_{q,1}}$. The process is similar to that for $\|\nabla\theta\|_{\mathring{B}^0_{q,1}}$, but we need to deal with the pressure term. Applying Δ_j to the equation of ∇u yields

$$\partial_t\Delta_j\nabla u+\Delta_j\nabla\big((u\cdot\nabla)u\big)+\nu\Delta_j\nabla u=-\Delta_j\nabla\nabla p+\Delta_j\nabla(\theta\mathbf{e}_2).$$

Multiplying by $\Delta_j\nabla u|\Delta_j\nabla u|^{q-2}$ and integrating in space, we have

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\nabla u\|^q_{L^q}+\nu\|\Delta_j\nabla u\|^q_{L^q}=L_1+L_2+L_3+L_4, \tag{2.4}$$

where

$$\begin{aligned}
L_1&=-\int\Delta_j\nabla u|\Delta_j\nabla u|^{q-2}\cdot\Delta_j(\nabla u\cdot\nabla u),\\
L_2&=-\int\Delta_j\nabla u|\Delta_j\nabla u|^{q-2}\cdot\Delta_j(u\cdot\nabla\nabla u),\\
L_3&=-\int\Delta_j\nabla u|\Delta_j\nabla u|^{q-2}\cdot\Delta_j\nabla\nabla p,\\
L_4&=-\int\Delta_j\nabla u|\Delta_j\nabla u|^{q-2}\cdot\Delta_j\nabla(\theta\mathbf{e}_2).
\end{aligned}$$

L_1 and L_2 can be estimated in a similar fashion as K_1 and K_2, respectively. They obey the following bounds,

$$\begin{aligned}
|L_1|,|L_2|\leqslant{}& C\|\nabla u\|_{L^\infty}\|\Delta_j\nabla u\|^q_{L^q}+C\|\Delta_j\nabla u\|^{q-1}_{L^q}\|\Delta_j\nabla u\|_{L^\infty}\|\nabla u\|_{\mathring{B}^0_{q,1}}\\
&+C\|\Delta_j\nabla u\|^{q-1}_{L^q}\sum_{k\geqslant j-1}2^{j-k}\|\Delta_k\nabla u\|_{L^\infty}\|\Delta_k\nabla u\|_{L^q}.
\end{aligned}$$

To bound L_3, we first apply the divergence-free condition to obtain

$$-\Delta p=\partial_l u_m\partial_m u_l-\partial_2\theta,$$

where the Einstein summation convention is invoked. Therefore,

$$\Delta_j \nabla\nabla p = \nabla\nabla(-\Delta)^{-1}\Delta_j(\partial_l u_m \partial_m u_l) - \nabla\nabla(-\Delta)^{-1}\Delta_j \partial_2\theta.$$

Since $\nabla\nabla(-\Delta)^{-1}$ are two Riesz transforms, $\nabla\nabla(-\Delta)^{-1}\Delta_j(\partial_l u_m \partial_m u_l)$ admits the same bound as $\Delta_j(\partial_l u_m \partial_m u_l)$ in any L^q with $q \in [1,\infty]$. The reason for the boundedness in the case of $p = 1$ or $p = \infty$ is that Δ_j is a homogeneous localization operator. Therefore, L_3 can be handled similarly as L_1 and

$$\begin{aligned}|L_3| \leqslant{}& C\|\nabla u\|_{L^\infty}\|\Delta_j \nabla u\|_{L^q}^q + C\|\Delta_j \nabla u\|_{L^q}^{q-1}\|\Delta_j \nabla u\|_{L^\infty}\|\nabla u\|_{\mathring{B}^0_{q,1}} \\ &+ C\|\Delta_j \nabla u\|_{L^q}^{q-1}\sum_{k\geqslant j-1} 2^{j-k}\|\Delta_k \nabla u\|_{L^\infty}\|\Delta_k \nabla u\|_{L^q} + \|\Delta_j \nabla u\|_{L^q}^{q-1}\|\Delta_j \nabla\theta\|_{L^q}.\end{aligned}$$

Applying Hölder's inequality to L_4 yields

$$|L_4| \leqslant C\|\Delta_j \nabla u\|_{L^q}^{q-1}\|\Delta_j \nabla\theta\|_{L^q}.$$

Inserting the estimates for $|L_1|$, $|L_2|$, $|L_3|$ and $|L_4|$ in (2.4), we find

$$\begin{aligned}\frac{d}{dt}\|\Delta_j \nabla u\|_{L^q} + \nu\|\Delta_j \nabla u\|_{L^q} \leqslant{}& C\|\nabla u\|_{L^\infty}\|\Delta_j \nabla u\|_{L^q} + C\|\Delta_j \nabla u\|_{L^\infty}\|\nabla u\|_{\mathring{B}^0_{q,1}} \\ &+ C\sum_{k\geqslant j-1} 2^{j-k}\|\Delta_k \nabla u\|_{L^\infty}\|\Delta_k \nabla u\|_{L^q} + C\|\Delta_j \nabla\theta\|_{L^q}.\end{aligned}$$

Summing over all integer j and by Young's inequality for series convolution, we have

$$\frac{d}{dt}\|\nabla u\|_{\mathring{B}^0_{q,1}} + \nu\|\nabla u\|_{\mathring{B}^0_{q,1}} \leqslant C_0\|\nabla u\|_{\mathring{B}^0_{\infty,1}}\|\nabla u\|_{\mathring{B}^0_{q,1}} + C_0\|\nabla\theta\|_{\mathring{B}^0_{q,1}} \tag{2.5}$$

for a constant C_0 independent of q, ν and λ. (2.3) and (2.5) yield (2.1). This completes the proof of Proposition 2.1. □

To prove Theorem 1.1, we need a few notations and list several facts. For $N > 0$, we denote by J_N the Fourier multiplier operator defined by

$$\widehat{J_N f}(\xi) = \chi_{B(0,N)}(\xi)\widehat{f}(\xi),$$

where $B(0,N)$ denotes the closed ball centered at the origin with radius N and $\chi_{B(0,N)}$ the characteristic function on $B(0,N)$. Let $\mathbb{P}$ denote the Leray projection onto divergence-free vector fields. More precise definition of $\mathbb{P}$ can be found in the book of Majda and Bertozzi [26, p. 35, p. 99]. The following simple properties of J_N and $\mathbb{P}$ will be used.

Lemma 2.2. *Let J_N with $N > 0$ and $\mathbb{P}$ denote the aforementioned operators. Then the following properties hold:*

(1) *For any $s \geqslant 0$,*

$$\|J_N f\|_{H^s} \leqslant \|f\|_{H^s}, \qquad \|\mathbb{P} f\|_{H^s} \leqslant \|f\|_{H^s};$$

(2) *Assume that* $f \in L^2$ *and* $s \geqslant 0$, *then* $J_N f \in H^s$ *and*

$$\|J_N f\|_{H^s} \leqslant C N^s \|f\|_{L^2};$$

(3) *Assume that* $f \in H^s$, *then*

$$\|J_N f - f\|_{H^{s-1}} \leqslant \frac{C}{N}\|f\|_{H^s}, \qquad \|J_N f - f\|_{H^s} \to 0 \quad \text{as } N \to \infty.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof of this theorem is long. For the sake of clarity, we divided it into five major steps.

Step 1 *(Construction of approximate solutions).* Let $N > 0$ be an integer. In this step, we construct a smooth global solution (u^N, θ^N) satisfying

$$\begin{cases} \partial_t u^N + \mathbb{P}J_N\big(\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N u^N\big) + \nu \mathbb{P}J_N u^N = \mathbb{P}J_N\big(\theta^N \mathbf{e}_2\big), \\ \partial_t \theta^N + J_N\big(\mathbb{P}J_N u^N \cdot \nabla J_N \theta^N\big) + \lambda J_N \theta^N = 0, \\ u^N(x,0) = J_N u_0(x), \qquad \theta^N(x,0) = J_N \theta_0(x). \end{cases} \tag{2.6}$$

It follows from Picard's theorem [26, p. 100] that, for fixed $N > 0$, (2.6) has a unique global smooth solution (u^N, θ^N) satisfying, for any $T > 0$,

$$\big(u^N, \theta^N\big) \in C\big([0,T]; H^s\big(\mathbb{R}^2\big)\big)$$

for any $s > 0$. In particular,

$$\nabla u^N, \nabla \theta^N \in C\big([0,T]; \mathring{B}^0_{\infty,1}\big(\mathbb{R}^2\big)\big).$$

It is easily checked that, if (u^N, θ^N) solves (2.6), then $(J_N u^N, J_N \theta^N)$ and $(\mathbb{P}u^N, \theta^N)$ also solve (2.6). By the uniqueness,

$$J_N u^N = \mathbb{P}u^N = u^N, \qquad J_N \theta^N = \theta^N.$$

$\mathbb{P}u^N = u^N$ implies $\nabla \cdot u^N = 0$. Consequently (2.6) is reduced to

$$\begin{cases} \partial_t u^N + \mathbb{P}J_N\big(u^N \cdot \nabla u^N\big) + \nu u^N = \mathbb{P}J_N\big(\theta^N \mathbf{e}_2\big), \\ \partial_t \theta^N + J_N\big(u^N \cdot \nabla \theta^N\big) + \lambda \theta^N = 0, \\ \nabla \cdot u^N = 0. \end{cases} \tag{2.7}$$

Step 2 *(Uniform global bounds).* In this step, we establish uniform global bounds for (u^N, θ^N). Simple energy estimates combined with $\nabla \cdot u^N = 0$ yield

$$\begin{aligned} \big\|\theta^N(t)\big\|_{L^2} &= \|J_N \theta_0\|_{L^2} e^{-\lambda t} \leqslant \|\theta_0\|_{L^2} e^{-\lambda t}, \\ \big\|u^N(t)\big\|_{L^2} &\leqslant \|u_0\|_{L^2} e^{-\nu t} + \frac{1}{\nu}\|\theta_0\|_{L^2}. \end{aligned} \tag{2.8}$$

Furthermore, a similar procedure as in the proof of Proposition 2.1 implies

$$\frac{d}{dt}\left\|\nabla u^N\right\|_{\mathring{B}^0_{\infty,1}} + \nu\left\|\nabla u^N\right\|_{\mathring{B}^0_{\infty,1}} \leqslant C_0\left\|\nabla u^N\right\|^2_{\mathring{B}^0_{\infty,1}} + C_0\left\|\nabla\theta^N\right\|_{\mathring{B}^0_{\infty,1}}, \tag{2.9}$$

$$\frac{d}{dt}\left\|\nabla \theta^N\right\|_{\mathring{B}^0_{\infty,1}} + \lambda\left\|\nabla \theta^N\right\|_{\mathring{B}^0_{\infty,1}} \leqslant C_0\left\|\nabla u^N\right\|_{\mathring{B}^0_{\infty,1}}\left\|\nabla\theta^N\right\|_{\mathring{B}^0_{\infty,1}}, \tag{2.10}$$

where C_0 is a constant independent of N, ν and λ. We claim that these differential inequalities yield the global bounds, for large N and any $t > 0$,

$$\left\|\nabla u^N(t)\right\|_{\mathring{B}^0_{\infty,1}} < A_0, \qquad \left\|\nabla\theta^N\right\|_{\mathring{B}^0_{\infty,1}} < B_0. \tag{2.11}$$

To see this, we first choose a large N such that

$$\|\nabla J_N u_0\|_{\mathring{B}^0_{\infty,1}} < A_0, \qquad \|\nabla J_N \theta_0\|_{\mathring{B}^0_{\infty,1}} < B_0. \tag{2.12}$$

(2.12) is realized as follows. Since $u_0 \in \mathring{B}^0_{\infty,1}$ satisfies (1.5), we can choose N sufficiently large such that

$$\|\nabla u_0\|_{\mathring{B}^0_{\infty,1}} + \|\Delta_{j_0+1} J_N \nabla u_0\|_{L^\infty} + \|\Delta_{j_0+2} J_N \nabla u_0\|_{L^\infty} < A_0,$$

where j_0 is an integer such that $2^{j_0+1} \leqslant N < 2^{j_0+2}$. By the definition of $\mathring{B}^0_{\infty,1}$ and J_N,

$$\begin{aligned}
\|\nabla J_N u_0\|_{\mathring{B}^0_{\infty,1}} &\leqslant \sum_{j=-\infty}^{j_0} \|\Delta_j \nabla J_N u_0\|_{L^\infty} + \|\Delta_{j_0+1}\nabla J_N u_0\|_{L^\infty} + \|\Delta_{j_0+2}\nabla J_N u_0\|_{L^\infty} \\
&= \sum_{j=-\infty}^{j_0} \|\Delta_j \nabla u_0\|_{L^\infty} + \|\Delta_{j_0+1}\nabla J_N u_0\|_{L^\infty} + \|\Delta_{j_0+2}\nabla J_N u_0\|_{L^\infty} \\
&< \|\nabla u_0\|_{\mathring{B}^0_{\infty,1}} + \|\Delta_{j_0+1}\nabla J_N u_0\|_{L^\infty} + \|\Delta_{j_0+2}\nabla J_N u_0\|_{L^\infty} \\
&< A_0.
\end{aligned}$$

Here we have used the facts that $\Delta_j J_N = \Delta_j$ for $j \leqslant j_0$ and $\Delta_{j_0+3} J_N u_0 = 0$ due to

$$\widehat{\Delta_j J_N u_0} = \widehat{\Phi}_j(\xi)\chi_{B(0,N)}(\xi)\widehat{u}_0(\xi) = \widehat{\Phi}_j(\xi)\widehat{u}_0(\xi) = \widehat{\Delta_j u_0},$$

$$\Delta_{j_0+3}\widehat{J_N} u_0(\xi) = \widehat{\Phi}_{j_0+3}(\xi)\chi_{B(0,N)}(\xi)\widehat{u}_0(\xi) \equiv 0.$$

More details Δ_j and $\widehat{\Phi}_j$ can be found in Appendix A. Similarly, for sufficiently large N,

$$\|\nabla J_N \theta_0\|_{\mathring{B}^0_{\infty,1}} < B_0.$$

Now suppose (2.11) is not true and $T^* > 0$ is the first time such that at least one of the inequalities in (2.11) is violated. That is,

$$\left\|\nabla u^N\left(T^*\right)\right\|_{\mathring{B}^0_{\infty,1}} = A_0 \quad \text{or} \quad \left\|\nabla \theta^N\left(T^*\right)\right\|_{\mathring{B}^0_{\infty,1}} = B_0 \tag{2.13}$$

and, for $t \in (0, T^*)$,

$$\left\|\nabla u^N(t)\right\|_{\mathring{B}^0_{\infty,1}} < A_0 \quad \text{and} \quad \left\|\nabla \theta^N(t)\right\|_{\mathring{B}^0_{\infty,1}} < B_0. \tag{2.14}$$

A contradiction then easily follows from (2.9) and (2.10). In fact, (2.10), (2.14) and the definition of A_0 in (1.5) imply

$$\begin{aligned}\left\|\nabla \theta^N\left(T^*\right)\right\|_{\mathring{B}^0_{\infty,1}} &\leqslant \|\nabla J_N\theta_0\|_{\mathring{B}^0_{\infty,1}} \exp\left(-\int_0^{T^*}\left(\lambda - C_0\left\|\nabla u^N(t)\right\|_{\mathring{B}^0_{\infty,1}}\right)dt\right) \\ &\leqslant \|\nabla J_N\theta_0\|_{\mathring{B}^0_{\infty,1}} < B_0.\end{aligned} \tag{2.15}$$

By (2.9), (2.14) and the definitions of A_0 and B_0 in (1.5),

$$\begin{aligned}\left\|\nabla u^N\left(T^*\right)\right\|_{\mathring{B}^0_{\infty,1}} &\leqslant \|\nabla J_N u_0\|_{\mathring{B}^0_{\infty,1}} \exp\left(-\int_0^{T^*}\left(\nu - C_0\left\|\nabla u^N(t)\right\|_{\mathring{B}^0_{\infty,1}}\right)dt\right) \\ &\quad + \int_0^{T^*} \exp\left(-\int_t^{T^*}\left(\nu - C_0\left\|\nabla u^N(s)\right\|_{\mathring{B}^0_{\infty,1}}\right)ds\right)\left\|\nabla \theta^N(t)\right\|_{\mathring{B}^0_{\infty,1}} dt \\ &\leqslant \|\nabla J_N u_0\|_{\mathring{B}^0_{\infty,1}} \exp\left(-\nu T^*/2\right) + \frac{2}{\nu}\left(1 - \exp\left(-\nu T^*/2\right)\right)\|\nabla J_N\theta_0\|_{\mathring{B}^0_{\infty,1}} \\ &< A_0 \exp\left(-\nu T^*/2\right) + A_0\left(1 - \exp\left(-\nu T^*/2\right)\right) = A_0.\end{aligned} \tag{2.16}$$

Clearly (2.15) and (2.16) contradict (2.13).

Step 3 *(Extraction of a strongly convergent subsequence).* We show here that we can extract a subsequence of (u^N, θ^N), still denoted by (u^N, θ^N), such that

$$\left\|\left(u^N, \theta^N\right) - (u, \theta)\right\|_{L^2} \to 0, \quad \text{as } N \to \infty, \tag{2.17}$$

where $(u, \theta) \in L^2$. This is achieved by showing that (u^N, θ^N) is a Cauchy sequence in L^2, namely

$$\left\|\left(u^N(t), \theta^N(t)\right) - \left(u^{N'}(t), \theta^{N'}(t)\right)\right\|_{L^2} \to 0 \tag{2.18}$$

as N and N' tend to infinity. With the global bounds (2.8) and (2.11) at our disposal, it is not hard to verify (2.18) by performing energy estimates on (2.7). We omit the details.

Step 4 *(Verifying that* (u,θ) *solves (1.1))*. Now we show that (u,θ) solves the 2D Boussinesq equations (1.1) in the sense of $\mathring{H}_*^{-\sigma}$ for any $\sigma\in(0,1)$, where $\mathring{H}_*^{-\sigma}$ denotes a subspace of $\mathring{H}^{-\sigma}$,

$$\mathring{H}_*^{-\sigma}=\left\{f\in\mathring{H}^{-\sigma}\;\middle|\;\|f\|_{\mathring{H}_*^{-\sigma}}<\infty\right\}$$

with

$$\|f\|^2_{\mathring{H}_*^{-\sigma}}=\sum_{j=-\infty}^{0}\|\Delta_j f\|^2_{L^2}+\sum_{j=1}^{\infty}2^{-2\sigma j}\|\Delta_j f\|^2_{L^2}.$$

We take the limit of (2.7) as $N\to\infty$. Trivially

$$\nu u^N\to\nu u,\qquad \mathbb{P}J_N\big(\theta^N\mathbf{e}_2\big)\to\mathbb{P}(\theta\mathbf{e}_2)\quad\text{in }\mathring{H}_*^{-\sigma}. \tag{2.19}$$

Our main effort is devoted to showing the convergence of the nonlinear term. We consider the difference $\|\mathbb{P}J_N(u^N\cdot\nabla u^N)-\mathbb{P}(u\cdot\nabla u)\|_{\mathring{H}_*^{-\sigma}}$. By Lemma 2.2,

$$\begin{aligned}
&\big\|\mathbb{P}J_N\big(u^N\cdot\nabla u^N\big)-\mathbb{P}(u\cdot\nabla u)\big\|_{\mathring{H}_*^{-\sigma}}\\
&\quad\leqslant\big\|\mathbb{P}J_N\big(u^N\cdot\nabla u^N-u\cdot\nabla u\big)\big\|_{\mathring{H}_*^{-\sigma}}+\big\|\mathbb{P}(J_N-1)(u\cdot\nabla u)\big\|_{\mathring{H}_*^{-\sigma}}\\
&\quad\leqslant\big\|u^N\cdot\nabla u^N-u\cdot\nabla u\big\|_{\mathring{H}_*^{-\sigma}}+\big\|\mathbb{P}(J_N-1)(u\cdot\nabla u)\big\|_{L^2}\\
&\quad\leqslant\big\|u^N-u\big\|_{L^2}\big\|\nabla u^N\big\|_{L^\infty}+\big\|u\cdot\nabla\big(u^N-u\big)\big\|_{\mathring{H}_*^{-\sigma}}+\big\|(J_N-1)(u\cdot\nabla u)\big\|_{L^2}.
\end{aligned}\tag{2.20}$$

Due to the embedding $\mathring{B}^0_{\infty,1}\hookrightarrow L^\infty$ and the bound in (2.11),

$$\|u\cdot\nabla u\|_{L^2}\leqslant\|u\|_{L^2}\|\nabla u\|_{L^\infty}\leqslant\|u_0\|_{L^2}A_0.$$

Therefore, (2.11), (2.17) and Lemma 2.2 imply that, as $N\to\infty$,

$$\big\|u^N-u\big\|_{L^2}\big\|\nabla u^N\big\|_{L^\infty}\leqslant A_0\big\|u^N-u\big\|_{L^2}\to 0,\qquad \big\|(J_N-1)(u\cdot\nabla u)\big\|_{L^2}\to 0.$$

To show that $\|u\cdot\nabla(u^N-u)\|_{\mathring{H}_*^{-\sigma}}\to 0$ as $N\to\infty$, we write by the notion of paraproduct, for any integer j,

$$\Delta_j\big(u\cdot\nabla\big(u^N-u\big)\big)=M_1+M_2+M_3,$$

where

$$\begin{aligned}
M_1&=\sum_{|j-k|\leqslant 2}\Delta_j\big(S_{k-1}u\cdot\nabla\Delta_k\big(u^N-u\big)\big),\\
M_2&=\sum_{|j-k|\leqslant 2}\Delta_j\big(\Delta_k u\cdot\nabla S_{k-1}\big(u^N-u\big)\big),\\
M_3&=\sum_{k\geqslant j-1}\Delta_j\big(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\big(u^N-u\big)\big).
\end{aligned}$$

Letting $r = \frac{2}{\sigma}$ and $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and applying Hölder's inequality, we have

$$\|M_1\|_{L^2} \leqslant C\|S_{j-1}u\|_{L^q} \big\|\nabla\Delta_j\big(u^N - u\big)\big\|_{L^r}.$$

By Bernstein's inequality and an interpolation inequality,

$$\begin{aligned}\|M_1\|_{L^2} &\leqslant C2^j\|u\|_{L^q}\big\|\Delta_j\big(u^N - u\big)\big\|_{L^r} \\ &\leqslant C2^j\|u\|_{L^q}\big\|\Delta_j\big(u^N - u\big)\big\|_{L^\infty}^{1-\frac{2}{r}}\big\|\Delta_j\big(u^N - u\big)\big\|_{L^2}^{\frac{2}{r}}.\end{aligned}$$

Thanks to $q > 2$, (2.11) and (2.17), we apply Bernstein's inequality to obtain

$$\begin{aligned}\|u\|_{L^q} \leqslant \|u\|_{B^0_{q,2}} &= \|\Delta_{-1}u\|_{L^2} + \Big[\sum_{j\geqslant 0}\|\Delta_j u\|_{L^q}^2\Big]^{\frac{1}{2}} \\ &= \|\Delta_{-1}u\|_{L^2} + \Big[\sum_{j\geqslant 0}\|\Delta_j u\|_{L^\infty}^{2-\frac{4}{q}}\|\Delta_j u\|_{L^2}^{\frac{4}{q}}\Big]^{\frac{1}{2}} \\ &= \|\Delta_{-1}u\|_{L^2} + \Big[\sum_{j\geqslant 0}\|\Delta_j u\|_{L^\infty}^2\Big]^{1-\frac{2}{q}}\Big[\sum_{j\geqslant 0}\|\Delta_j u\|_{L^2}^2\Big]^{\frac{2}{q}} \\ &\leqslant C\|u\|_{L^2}\big(1 + \|u\|_{\mathring{B}^0_{\infty,1}}\big) < \infty.\end{aligned}$$

By $\nabla \cdot u = 0$, Bernstein's inequality and Hölder's inequality,

$$\begin{aligned}\|M_2\|_{L^2} &\leqslant C2^j\|\Delta_j u\|_{L^\infty}\big\|S_{j-1}\big(u^N - u\big)\big\|_{L^2} \\ &\leqslant C2^j\|\Delta_j u\|_{L^\infty}\big\|u^N - u\big\|_{L^2}\end{aligned}$$

and

$$\|M_3\|_{L^2} \leqslant C\sum_{k\geqslant j-1}2^{j-k}\|\nabla\Delta_k u\|_{L^\infty}\big\|\Delta_k\big(u^N - u\big)\big\|_{L^2}.$$

Therefore, by combining the bounds above, we have

$$\begin{aligned}&\big\|u\cdot\nabla\big(u^N - u\big)\big\|_{\mathring{H}_*^{-\sigma}}^2 \\ &\quad= \sum_{j=-\infty}^{0}\big\|\Delta_j\big(u\cdot\nabla\big(u^N - u\big)\big)\big\|_{L^2}^2 + \sum_{j=1}^{\infty}2^{-2\sigma j}\big\|\Delta_j\big(u\cdot\nabla\big(u^N - u\big)\big)\big\|_{L^2}^2 \\ &\quad\leqslant C\big\|u^N - u\big\|_{L^2}^2\sum_{j=-\infty}^{\infty}2^{2j}\|\Delta_j u\|_{L^\infty}^2\end{aligned}$$

$$+ C \sum_{j=-\infty}^{\infty} \left[\sum_{k \geqslant j-1} 2^{j-k} \|\nabla \Delta_k u\|_{L^\infty} \left\| \Delta_k \left(u^N - u\right) \right\|_{L^2} \right]^2$$

$$+ C \|u\|_{L^q}^2 \sum_{j=-\infty}^{\infty} 2^{2(1-\sigma)j} \left\| \Delta_j \left(u^N - u\right) \right\|_{L^\infty}^{2-\frac{4}{r}} \left\| \Delta_j \left(u^N - u\right) \right\|_{L^2}^{\frac{4}{r}}. \tag{2.21}$$

We further estimate the terms on the right and have

$$\sum_{j=-\infty}^{\infty} 2^{2j} \|\Delta_j u\|_{L^\infty}^2 = \|u\|_{\mathring{B}^1_{\infty,2}}^2 \leqslant C \|\nabla u\|_{\mathring{B}^0_{\infty,1}}^2 < \infty, \tag{2.22}$$

$$\sum_{j=-\infty}^{\infty} \left[\sum_{k \geqslant j-1} 2^{j-k} \|\nabla \Delta_k u\|_{L^\infty} \left\| \Delta_k \left(u^N - u\right) \right\|_{L^2} \right]^2 \leqslant \sum_{j=-\infty}^{\infty} \|\nabla \Delta_j u\|_{L^\infty}^2 \left\| \Delta_j \left(u^N - u\right) \right\|_{L^2}^2$$
$$\leqslant C \left\| u^N - u \right\|_{L^2}^2 \|\nabla u\|_{\mathring{B}^0_{\infty,1}}^2, \tag{2.23}$$

where the Besov embedding $\mathring{B}^0_{\infty,1} \hookrightarrow \mathring{B}^0_{\infty,2}$ is used in the first inequality and Young's inequality for series convolutions is used in the second inequality. Noticing that $\frac{2}{r} = \sigma$, we obtain by Hölder's inequality

$$\sum_{j=-\infty}^{\infty} 2^{2(1-\sigma)j} \left\| \Delta_j \left(u^N - u\right) \right\|_{L^\infty}^{2-\frac{4}{r}} \left\| \Delta_j \left(u^N - u\right) \right\|_{L^2}^{\frac{4}{r}}$$
$$\leqslant \left[\sum_{j=-\infty}^{\infty} 2^{2j} \left\| \Delta_j \left(u^N - u\right) \right\|_{L^\infty}^2 \right]^{1-\sigma} \left[\sum_{j=-\infty}^{\infty} \left\| \Delta_j \left(u^N - u\right) \right\|^2 \right]^{\sigma}$$
$$\leqslant \left\| u^N - u \right\|_{\mathring{B}^1_{\infty,2}}^{1-\sigma} \left\| u^N - u \right\|_{L^2}^{\sigma}$$
$$\leqslant C \left(\left\| \nabla u^N \right\|_{\mathring{B}^0_{\infty,1}} + \|\nabla u\|_{\mathring{B}^0_{\infty,1}} \right)^{1-\sigma} \left\| u^N - u \right\|_{L^2}^{\sigma}. \tag{2.24}$$

Inserting (2.22), (2.23) and (2.24) in (2.21), we obtain

$$\left\| u \cdot \nabla \left(u^N - u\right) \right\|_{\mathring{H}_*^{-\sigma}}^2 \leqslant C \left\| u^N - u \right\|_{L^2}^2 \|\nabla u\|_{\mathring{B}^0_{\infty,1}}^2$$
$$+ C \|u\|_{L^q}^2 \left(\left\| \nabla u^N \right\|_{\mathring{B}^0_{\infty,1}} + \|\nabla u\|_{\mathring{B}^0_{\infty,1}} \right)^{1-\sigma} \left\| u^N - u \right\|_{L^2}^{\sigma}.$$

Therefore, (2.17) implies

$$\left\| u \cdot \nabla \left(u^N - u\right) \right\|_{\mathring{H}_*^{-\sigma}}^2 \to 0, \quad \text{as } N \to \infty.$$

It then follows from (2.20) that

$$\left\| \mathbb{P} J_N \left(u^N \cdot \nabla u^N\right) - \mathbb{P}(u \cdot \nabla u) \right\|_{\mathring{H}_*^{-\sigma}} \to 0, \quad \text{as } N \to \infty. \tag{2.25}$$

As a consequence of (2.19) and (2.25), as $N \to \infty$,

$$\partial_t u^N = -\mathbb{P} J_N\big(u^N \cdot \nabla u^N\big) - \nu u^N + \mathbb{P} J_N\big(\theta^N \mathbf{e}_2\big)$$

converges strongly in $\mathring{H}_*^{-\sigma}$. On the other hand, since $u^N \to u$ in L^2,

$$\partial_t u^N \to \partial_t u$$

in the distributional sense. Therefore,

$$\big\|\partial_t u^N - \partial_t u\big\|_{\mathring{H}_*^{-\sigma}} \to 0, \quad \text{as } N \to \infty. \tag{2.26}$$

In summary, we have shown that, by letting $N \to \infty$ in (2.7) and invoking the limits in (2.19), (2.25) and (2.26),

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) + \nu u = \mathbb{P}(\theta \mathbf{e}_2) \quad \text{in } \mathring{H}_*^{-\sigma},$$

which can also be written as

$$\partial_t u + u \cdot \nabla u + \nu u = -\nabla p + \theta \mathbf{e}_2, \quad \nabla \cdot u = 0.$$

In a similar manner, we can also show that

$$\partial_t \theta + u \cdot \nabla \theta + \lambda \theta = 0 \quad \text{in } \mathring{H}_*^{-\sigma}.$$

Step 5 *(Uniqueness).* This step is devoted to showing that any two solutions $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ obeying (1.6) must coincide. It is clear that the difference (v, Θ) with

$$v = u^{(1)} - u^{(2)}, \qquad \Theta = \theta^{(1)} - \theta^{(2)}$$

satisfies

$$\begin{cases} \partial_t v + \mathbb{P}\big(u^{(1)} \cdot \nabla v\big) + \mathbb{P}\big(v \cdot \nabla u^{(2)}\big) + \nu v = \mathbb{P}(\Theta \mathbf{e}_2), \\ \partial_t \Theta + u^{(1)} \cdot \nabla \Theta + v \cdot \nabla \theta^{(2)} + \lambda \Theta = 0, \\ v(x,0) = 0, \qquad \Theta(x,0) = 0. \end{cases} \tag{2.27}$$

Taking the inner product with (v, Θ) yields

$$\frac{1}{2}\frac{d}{dt}\big(\|v\|_{L^2}^2 + \|\Theta\|_{L^2}^2\big) + \nu\|v\|_{L^2}^2 + \lambda\|\Theta\|_{L^2}^2 \\ \leqslant \|v\|_{L^2}\|\Theta\|_{L^2} + \left|\int v \cdot \nabla u^{(2)} \cdot v\right| + \left|\int v \cdot \nabla \theta^{(2)} \theta\right|.$$

Bounding the last two terms on the right-hand side by Hölder's inequality and applying the embedding $\mathring{B}^0_{\infty,1} \hookrightarrow L^\infty$ yield

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|_{L^2}^2+\|\Theta\|_{L^2}^2\right)+\nu\|v\|_{L^2}^2+\lambda\|\Theta\|_{L^2}^2$$
$$\leqslant C\left(1+\left\|\left(\nabla u^{(2)},\nabla\theta^{(2)}\right)\right\|_{\mathring{B}^0_{\infty,1}}\right)\left(\|v\|_{L^2}^2+\|\Theta\|_{L^2}^2\right).$$

Gronwall's inequality then implies that $(v,\Theta)\equiv 0$. We have completed all the steps and thus the whole proof of Theorem 1.1. □

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Appendix A. Functional spaces

This appendix provides the definitions of some of the functional spaces and related facts used in the previous sections. Materials presented in this appendix can be found in several books and many papers (see, e.g., [3,4,28,32,33]).

We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0=\left\{\phi\in\mathcal{S}:\ \int_{\mathbb{R}^d}\phi(x)x^\gamma\,dx=0,\ |\gamma|=0,1,2,\dots\right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0'=\mathcal{S}'/\mathcal{S}_0^\perp=\mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials.

To introduce the Littlewood–Paley decomposition, we write for each $j\in\mathbb{Z}$

$$A_j=\left\{\xi\in\mathbb{R}^d:\ 2^{j-1}\leqslant|\xi|<2^{j+1}\right\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}}\in\mathcal{S}$ such that

$$\operatorname{supp}\widehat{\Phi}_j\subset A_j,\qquad \widehat{\Phi}_j(\xi)=\widehat{\Phi}_0\left(2^{-j}\xi\right)\quad\text{or}\quad \Phi_j(x)=2^{jd}\Phi_0\left(2^jx\right),$$

and

$$\sum_{j=-\infty}^{\infty}\widehat{\Phi}_j(\xi)=\begin{cases}1, & \text{if } \xi\in\mathbb{R}^d\setminus\{0\},\\ 0, & \text{if } \xi=0.\end{cases}$$

Therefore, for a general function $\psi\in\mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \quad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \quad j \in \mathbb{Z}. \tag{A.1}$$

Definition A.1. For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \left\| 2^{js} \|\mathring{\Delta}_j f\|_{L^p} \right\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leqslant -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \ldots. \end{cases} \tag{A.2}$$

Definition A.2. The inhomogeneous Besov space $B^s_{p,q}$ with $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \big\|2^{js}\|\Delta_j f\|_{L^p}\big\|_{l^q} < \infty.$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \leqslant p, q \leqslant \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} \, dt \right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} \, dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way.

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.3. *For any $s \in \mathbb{R}$,*

$$\mathring{H}^s \sim \mathring{B}^s_{2,2}, \qquad H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$\mathring{B}^s_{q,\min\{q,2\}} \hookrightarrow \mathring{W}^s_q \hookrightarrow \mathring{B}^s_{q,\max\{q,2\}}.$$

In particular, $\mathring{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \mathring{B}^0_{q,\max\{q,2\}}$.

For notational convenience, we write Δ_j for $\mathring{\Delta}_j$. There will be no confusion if we keep in mind that Δ_j's associated with the homogeneous Besov spaces is defined in (A.1) while those associated with the inhomogeneous Besov spaces are defined in (A.2). Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (A.2). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition A.4. *Let* $\alpha \geqslant 0$. *Let* $1 \leqslant p \leqslant q \leqslant \infty$.

(1) *If* f *satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d\colon\ |\xi| \leqslant K 2^j\},$$

for some integer j *and a constant* $K > 0$, *then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_1 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If* f *satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d\colon\ K_1 2^j \leqslant |\xi| \leqslant K_2 2^j\}$$

for some integer j *and constants* $0 < K_1 \leqslant K_2$, *then*

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leqslant \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_2 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 *and* C_2 *are constants depending on* α, p *and* q *only.*

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