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The effect of dissipation on solutions of the complex KdV equation

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Abstract

It is known that some periodic solutions of the complex KdV equation with smooth initial data blow up in finite time. In this paper, we investigate the effect of dissipation on the regularity of solutions of the complex KdV equation. It is shown here that if the initial datum is comparable to the dissipation coefficient in the L^2-norm, then the corresponding solution does not develop any finite-time singularity. The solution actually decays exponentially in time and becomes real analytic as time elapses. Numerical simulations are also performed to provide detailed information on the behavior of solutions in different parameter ranges.

Keywords: Complex KdV–Burgers equation; Dissipation; Global well-posedness

1. Introduction

Consider the complex KdV equation with a periodic boundary condition

$$u_t + 2uu_x + \delta u_{xxx} = 0, \qquad u(x,t) = u(x+1,t), \tag{1.1}$$

where $\delta > 0$ is a parameter and $u = u(x,t)$ is complex-valued. It is known that there exists a smooth initial datum u_0 such that the solution of the initial-value problem (IVP) of (1.1) with

$$u(x,0) = u_0(x) \tag{1.2}$$

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blows up in a finite time. In fact, a class of elliptic solutions represented by the Weierstrass p-function develop finite-time singularities [1,2]. Numerical computations of [6] also indicate finite-time singularities for a special series-type solution.

It is worth pointing out that the complex KdV equation is much more sophisticated than its real counterpart. For the real KdV equation, the hierarchy of infinite conservation laws provides global in time bounds for its solutions in any Sobolev space H^k with $k \geq 0$ [5]. Thus, no finite-time singularity is possible and the real KdV equation is well-posed. In contrast, the complex KdV equation is equivalent to a system of two nonlinearly coupled equations and the conservation laws no longer allow the deduction of global bounds. In fact, we do not know if a solution of the IVP (1.1) and (1.2) has a finite L^2-norm for all time even if it is initially in L^2. This is precisely where the problem arises.

In this paper, we conduct a theoretical and numerical study to investigate the effects of dissipation on the regularity of solutions of the complex KdV equation. More precisely, we examine solutions of the dissipative complex KdV equation of the form

$$u_t + 2uu_x + \delta u_{xxx} + \nu(-\Delta)^\alpha u = 0, \qquad u(x,t) = u(x+1,t), \tag{1.3}$$

where $\nu > 0$ and $(-\Delta)^\alpha$ is a Fourier multiplier operator, namely for each $l \in \mathbb{Z}$

$$\widehat{(-\Delta)^\alpha u}(l) = |2\pi l|^{2\alpha}\hat{u}(l),$$

where the Fourier transform $\hat{u}$ is defined by

$$\hat{u}(l,t) = \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi x l}\, u(x,t)\,\mathrm{d}x,$$

and u can be recovered from $\hat{u}$ through the inverse transform, namely

$$u(x,t) = \sum_{l=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi l x}\hat{u}(l,t).$$

When $\alpha = 1$, the dissipation becomes the Burgers term $-\nu u_{xx}$.

It was shown in a previous work [6] that the L^2-norm of any solution of the complex KdV–Burgers equation dominates its norms in H^k for any $k \geq 1$. Therefore, any solution u of the complex KdV–Burgers equation with $u_0 \in H^k$ remains in H^k as long as its L^2-norm remains bounded. This result suggests that attention be focused on the L^2-norm. In this paper, we prove rigorously that if ν and the initial datum u_0 satisfy for some constant C

$$\nu \geq C\|u_0\|_{L^2}, \tag{1.4}$$

then the L^2-norm of the solution of (1.3) with $u(x,0) = u_0(x)$ is bounded uniformly for all time. When (1.4) holds, the solution decays exponentially in time and becomes real analytic for large time.

Systematic numerical computations are also performed on the complex KdV–Burgers equation to provide detailed information on the behavior of its solutions in different parameter ranges. The numerical results are consistent with our theory.

2. Global existence result

The goal of this section is to prove a global existence and uniqueness result for the IVP of the complex KdV–Burgers type equation

$$u_t + 2uu_x + \delta u_{xxx} + \nu(-\Delta)^\alpha u = 0, \qquad u(x,t) = u(x+1,t), \qquad u(x,0) = u_0(x). \tag{2.1}$$

The initial datum u_0 is periodic, square integrable and has mean zero, namely

$$\int_0^1 u_0(x)\,\mathrm{d}x = 0.$$

Theorem 2.1. *Let $\delta \geq 0$, $\nu > 0$ and $\alpha \geq 1$. Assume that u_0 is in H^s with $s \geq 1$, periodic and has mean zero. Assume that ν and u_0 further satisfy*

$$\nu > C_\alpha \|u_0\|_{L^2}, \tag{2.2}$$

where $C_\alpha = 2^{1-\alpha}/\pi^{\alpha-1/2}$. Then the IVP (2.1) has a unique global solution u satisfying

$$u \in L^\infty([0,\infty); H^s) \cap L^2([0,\infty); H^{\alpha+s}).$$

Remark. Note that the condition (2.2) does not depend on δ. Thus, this theorem also holds for the case $\delta = 0$, namely the complex Burgers equation.

Proof. The tool is the Galerkin approximation [4]. Let $N \geq 1$ and $|l| \leq N$. Consider the sequence $\{\hat{u}_N(l,t)\}$ solving the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_N(l,t) = -\nu|2\pi l|^{2\alpha}\hat{u}_N(l,t) + \mathrm{i}\delta(2\pi l)^3\hat{u}_N(l,t) - 4\pi\mathrm{i}\sum_{l_1+l_2=l}\hat{u}_N(l_1,t)l_2\hat{u}_N(l_2,t) \tag{2.3}$$

with the initial condition

$$\hat{u}_N(l,0) = \hat{u}_{0N}(l), \quad |l| \leq N,$$

where the sum in (2.3) is restricted to $|l_1| \leq N$ and $|l_2| \leq N$. This is a finite system of ordinary differential equations (ODE) for $\hat{u}_N(l,t)$. Since the right-hand side of (2.3) is a locally Lipschitz function of $\hat{u}_N(l,t)$, the basic ODE theory asserts that there exists a $T > 0$ such that there is a unique continuous function $\hat{u}_N(l,t)$ on $[0,T]$ solving (2.3).

To establish global existence for $\hat{u}_N(l,t)$, we now show that $\hat{u}_N(l,t)$ is bounded uniformly. Dotting (2.3) with $\hat{u}_N(l,t)^*$, summing over l and adding the complex conjugate, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_l |\hat{u}_N(l,t)|^2 + 2\nu\sum_l |2\pi l|^{2\alpha}|\hat{u}_N(l,t)|^2 = K$$

with

$$K = -8\pi\mathrm{i}\mathcal{R}\sum_l \hat{u}_N(l,t)^* \sum_{l_1+l_2=l}\hat{u}_N(l_1,t)l_2\hat{u}_N(l_2,t),$$

where $\mathcal{R}$ denotes the real part. Now define the periodic, mean zero function $u^N(x,t)$ by its Fourier transform $\hat{u}_N(l,t)$,

$$u_N(x,t) = \sum_{|l|\le N} \mathrm{e}^{\mathrm{i}2\pi l x}\, \hat{u}_N(l,t). \tag{2.4}$$

Then K can be written as

$$K = -4\mathcal{R}\int_0^1 u^N(x,t)^*(u^N(x,t)u_x^N(x,t))\,\mathrm{d}x.$$

Integrating by parts yields

$$K = -4\int_0^1 |u^N(x,t)|^2\mathcal{R}u_x^N(x,t)\,\mathrm{d}x = -4\int_0^1((\mathcal{R}u^N)^2+(\mathcal{I}u^N)^2)(\mathcal{R}u^N)_x\,\mathrm{d}x$$
$$= -4\int(\mathcal{I}u^N)^2(\mathcal{R}u^N)_x\,\mathrm{d}x.$$

By the Schwarz inequality and the estimate that $\|f\|_{L^\infty}^2 \le 2\|f\|_{L^2}\,\|f_x\|_{L^2}$ for any f with mean zero,

$$|K| \le 4\|\mathcal{I}u^N\|_{L^\infty}\,\|\mathcal{I}u^N\|_{L^2}\|(\mathcal{R}u^N)_x\|_{L^2} \le 4\sqrt{2}\|\mathcal{I}u^N\|_{L^2}^{3/2}\,\|(\mathcal{I}u^N)_x\|_{L^2}^{1/2}\|(\mathcal{R}u^N)_x\|_{L^2}.$$

Because $\hat{u}_N(0,t)=0$, the Poincaré inequality implies that

$$\|\mathcal{I}u^N\|_{L^2} \le \frac{1}{2\pi}\|(\mathcal{I}u^N)_x\|_{L^2}.$$

Therefore,

$$|K| \le \frac{4}{\sqrt{\pi}}\,\|\mathcal{I}u^N\|_{L^2}\,\|(\mathcal{I}u^N)_x\|_{L^2}\,\|(\mathcal{R}u^N)_x\|_{L^2} \le \frac{2}{\sqrt{\pi}}\,\|\mathcal{I}u^N\|_{L^2}\,\|u_x^N\|_{L^2}^2.$$

Identifying

$$\|u^N(\cdot,t)\|_{L^2}^2 = \sum_l|\hat{u}_N(l,t)|^2 \quad\text{and}\quad \|(-\Delta)^{\alpha/2}u^N\|_{L^2}^2 = \sum_l|2\pi l|^{2\alpha}|\hat{u}_N(l,t)|^2,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u^N(\cdot,t)\|_{L^2}^2 + 2\nu\,\|(-\Delta)^{\alpha/2}u^N\|_{L^2}^2 \le \frac{2}{\sqrt{\pi}}\,\|\mathcal{I}u^N\|_{L^2}\,\|u_x^N\|_{L^2}^2. \tag{2.5}$$

Applying Poincaré's inequality

$$\|u_x^N\|_{L^2}^2 \le \frac{1}{(2\pi)^{\alpha-1}}\,\|(-\Delta)^{\alpha/2}u^N\|_{L^2}^2, \tag{2.6}$$

(2.5) can then be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^N(\cdot,t)\|_{L^2}^2 \le -2(\nu - C_\alpha\|\mathcal{I}u\|_{L^2})\,\|(-\Delta)^{\alpha/2}u^N\|_{L^2}^2, \tag{2.7}$$

where $C_\alpha = 2^{1-\alpha}/\pi^{\alpha-1/2}$. When u_0 satisfies (2.2), then

$$\nu > C_\alpha\,\|\mathcal{I}u_0\|_{L^2},$$

and (2.7) implies that $\|u^N(\cdot,t)\|_{L^2}$ decreases as t grows. Thus,

$$\|u^N(\cdot,t)\|_{L^2} \le \|u_0^N\|_{L^2} \le \|u_0\|_{L^2}.$$

That is, $\|u^N\|_{L^2}$ is bounded uniformly in t and in the order of the approximation N.

In addition, the following inequality holds

$$\int_0^t \|(-\Delta)^{\alpha/2}u^N(\cdot,\tau)\|_{L^2}^2\,\mathrm{d}\tau \le C_0, \tag{2.8}$$

where C_0 is a constant depending on α and ν only. Thus, for any $t > 0$, there is a subsequence u^{N_j} and a L^2 function $u(x,t)$ such that u^{N_j} converges weakly to $u(x,t)$. It is easily verified that $u(x,t)$ is a weak solution of (2.1), namely $u(x,t)$ satisfies (2.1) in the distributional sense.

We now show that $u(x,t)$ actually satisfies (2.1) in the classical sense. The first step is to bound u_x. Recall that u^N defined in (2.4) satisfies

$$\partial_t u^N + 2P^N(u^N\cdot u_x^N) + \delta u_{xxx}^N + \nu(-\Delta)^\alpha u_{xx}^N = 0, \qquad u^N(x,0) = u_0^N(x), \tag{2.9}$$

where P_N denotes the projection onto the space spanned by the modes $\{\mathrm{e}^{\mathrm{i}2\pi lx}\}_{|l|\le N}$. Now, differentiate (2.9) and take the L^2-norm,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x^N\|_{L^2}^2 + 2\nu\|(-\Delta)^{\alpha/2}u_x^N\|_{L^2}^2 = 2\int \overline{u}_x^N\,|u_x^N|^2\,\mathrm{d}x - 4\int u_x^N\overline{u}_{xx}^N\mathcal{I}u_x^N\,\mathrm{d}x. \tag{2.10}$$

For notational convenience, we label the terms on the right-hand side as I and II. Applying the Poincaré inequality

$$\|u_{xx}^N\|_{L^2} \le \frac{1}{(2\pi)^{\alpha-1}}\,\|(-\Delta)^{\alpha/2}u_x^N\|_{L^2},$$

we obtain

$$|\mathrm{I}| \le 2\|u_x^N\|_{L^3}^3 \le 2\sqrt{2}\|u_x^N\|_{L^2}^{5/2}\,\|u_{xx}^N\|_{L^2}^{1/2} \le C_1\|u_x^N\|_{L^2}^{5/2}\,\|(-\Delta)^{\alpha/2}u_x^N\|_{L^2}^{1/2},$$

where $C_1 = 2\sqrt{2/(2\pi)^{\alpha-1}}$. By Young's inequality,

$$|\mathrm{I}| \le \frac{\nu}{2}\|(-\Delta)^{\alpha/2}u_x^N\|_{L^2}^2 + C_2\|u_x^N\|_{L^2}^{10/3}, \tag{2.11}$$

where C_2 is a constant depending on α and ν only.

$$|\mathrm{II}| \le 4\sqrt{2}\|u^N\|_{L^2}^{1/2}\,\|u_x^N\|_{L^2}^{3/2}\,\|u_{xx}^N\|_{L^2} \le \frac{\nu}{2}\|(-\Delta)^{\alpha/2}u_x^N\|_{L^2}^2 + C_3\|u^N\|_{L^2}\,\|u_x^N\|_{L^2}^3, \tag{2.12}$$

Inserting the estimates (2.11) and (2.12) in (2.10), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x^N\|_{L^2}^2 + \nu\|(-\Delta)^{\alpha/2}u_x^N\|_{L^2}^2 \le C_2\,\|u_x^N\|_{L^2}^{10/3} + C_3\|u^N\|_{L^2}\|u_x^N\|_{L^2}^3,$$

which, in particular, implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x^N\|_{L^2} \le C_4\|u_x^N\|_{L^2}^{7/3} + C_5\|u^N\|_{L^2}\,\|u_x^N\|_{L^2}^2.$$

Integrating with respect to t and noticing (2.6) and (2.8), we obtain for any $t > 0$

$$\sup_{\tau\le t}\|u_x^N(\cdot,\tau)\|_{L^2} \le \|u_{0x}^N\|_{L^2} + C_6\sup_{\tau\le t}\|u_x^N(\cdot,\tau)\|_{L^2}^{1/3} + C_7.$$

This inequality allows us to conclude that

$$\|u_x^N(\cdot,t)\|_{L^2} \le C_8,$$

where C_8 is a constant depending on α, ν and $\|u_{0x}\|_{L^2}$ only. Thus, we conclude that u, the limit of u^N, must also has square integrable derivatives at any time t.

The boundedness for the general norm $\|u\|_{H^s}$ can be proved by iteration. We omit the details. □

Global solutions in Theorem 2.1 actually decay exponentially in time.

Theorem 2.2. *Let $\delta \ge 0$, $\nu > 0$ and $\alpha \ge 1$. Assume that $u_0 \in H^s$ $(s \ge 1)$ is periodic and has mean zero. If u_0 further satisfies*

$$\nu > C_\alpha\,\|u_0\|_{L^2},$$

then the corresponding solution u of (2.1) decay exponentially in L^2

$$\|u(\cdot,t)\|_{L^2} \le \mathrm{e}^{-\tilde{C}t}\,\|u_0\|_{L^2} \tag{2.13}$$

for all $t \ge 0$, where $\tilde{C} = (2\pi)^\alpha(\nu - C_\alpha\|\mathcal{I}u_0\|_{L^2})$. In addition, there exist a time t_0 and a constant $\bar{C}$ such that for any $t \ge t_0$,

$$\|u_x(\cdot,t)\|_{L^2} \le \mathrm{e}^{-\bar{C}t}\,\|u_{0x}\|_{L^2}. \tag{2.14}$$

Proof. Combining (2.5) with (2.6) and letting $N \to \infty$, we find that u satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|_{L^2}^2 + 2\nu\|(-\Delta)^{\alpha/2}u\|_{L^2}^2 \le 2C_\alpha\|\mathcal{I}u\|_{L^2}\,\|(-\Delta)^{\alpha/2}u\|_{L^2}^2.$$

Applying Poincaré's inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(\cdot,t)\|_{L^2}^2 + 2\tilde{C}\,\|u(\cdot,t)\|_{L^2}^2 \le \frac{\mathrm{d}}{\mathrm{d}t}\,\|u(\cdot,t)\|_{L^2}^2 + 2(\nu - C_\alpha\,\|\mathcal{I}u_0\|_{L^2})\,\|(-\Delta)^{\alpha/2}u\|_{L^2}^2 \le 0,$$

which immediately implies the inequality in (2.13).

To show (2.14), we recall that there exists a constant C_9 depending on α, ν and $\|u_{0x}\|_{L^2}$ only such that

$$\|u_x(\cdot,t)\|_{L^2} \le C_9 \quad \text{and} \quad \int_0^t \|u_{xx}(\cdot,\tau)\|_{L^2}^2\,\mathrm{d}\tau \le C_9.$$

Since u satisfies (2.1), u_x obeys

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u_x(\cdot,t)\|_{L^2}^2 + 2\nu\|(-\Delta)^{\alpha/2}u_x\|_{L^2}^2 = 2\int \bar{u}_x|u_x|^2\,\mathrm{d}x - 4\int u_x\bar{u}_{xx}\mathcal{I}u_x\,\mathrm{d}x.$$

To bound the terms J_1 and J_2 on the right-hand side, we use the Gagliardo–Nirenberg type inequalities followed by Poincaré's inequality

$$J_1 \le 2\|u_x\|_{L^3}^3 \le C_{10}\|u\|_{L^2}^{5/4}\,\|u_{xx}\|_{L^2}^{7/4} \le C_{11}\,\|u\|_{L^2}\,\|u_{xx}\|_{L^2}^2,$$

$$J_2 \le 4\|u_{xx}\|_{L^2}\|u_x\|_{L^4}^2 \le C_{12}\|u_{xx}\|_{L^2}\,\|u\|_{L^2}^{1/2}\,\|u_x\|_{L^2}^{1/2}\,\|u_{xx}\|_{L^2} = C_{12}\,\|u\|_{L^2}^{1/2}\,\|u_x\|_{L^2}^{1/2}\,\|u_{xx}\|_{L^2}^2 \le C_{13}\,\|u\|_{L^2}^{1/2}\,\|u_{xx}\|_{L^2}^2.$$

We thus have established that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u_x(\cdot,t)\|_{L^2}^2 + \frac{2\nu}{(2\pi)^{\alpha/2}}\,\|u_{xx}\|_{L^2}^2 \le C_{14}(\|u\|_{L^2} + \|u\|_{L^2}^{1/2})\|u_{xx}\|_{L^2}^2.$$

Since $\|u(\cdot,t)\|_{L^2}$ decays exponentially in t, we can choose t_0 such that for any $t \ge t_0$,

$$\frac{2\nu}{(2\pi)^{\alpha/2}} - C_{14}(\|u(\cdot,t)\|_{L^2} + \|u(\cdot,t)\|_{L^2}^{1/2}) > 0. \tag{2.15}$$

Therefore, for $t \ge t_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x(\cdot,t)\|_{L^2}^2 + \bar{C}\,\|u_{xx}(\cdot,t)\|_{L^2}^2 \le 0,$$

where $\bar{C}$ denotes the quantity on the left-hand side of (2.15). This inequality then leads to (2.14). □

The following theorem further asserts that the global solutions become analytic as time elapses. For the sake of a concise presentation, we focus on the case $\alpha = 1$, namely the complex KdV–Burgers equation.

Theorem 2.3. *Let $\delta \ge 0$, $\nu > 0$ and $\alpha = 1$. Let $u_0 \in H^s (s \ge 1)$ be periodic of period 1 and have mean zero. Assume that u_0 satisfies*

$$\nu > C_\alpha\,\|u_0\|_{L^2}.$$

Then there exists $t_0 > 0$ *such that the solution u of* (2.1) *is real analytic for* $t \geq t_0$. *More precisely, u can be expanded into the complex plane such that the extension* $u(z, t)$ *is analytic in*

$$S_t \equiv \{z = x + \mathrm{i}y : x \in [0, 1],\ y \in [-\nu(t - t_0), \nu(t - t_0)]\}. \tag{2.16}$$

Proof. According to Theorem 2.2, there exists $t_0 > 0$ such that $\|u_x(\cdot, t)\|_{L^2}$ decays exponentially in t for $t \geq t_0$. Because

$$\sum_{l \neq 0} |\hat{u}(l, t)| \leq \left(\sum_{l \neq 0} |l|^{-2}\right)^{1/2} \left(\sum_{l \neq 0} |l|^2 |\hat{u}(l, t)|^2\right)^{1/2} = C\|u_x(\cdot, t)\|_{L^2}^2,$$

there exists a time (still denoted t_0) such that

$$\sum_{l \neq 0} |\hat{u}(l, t)| \leq \nu. \tag{2.17}$$

Now, consider the functions

$$Y(t) = \sum_{l \neq 0} |\hat{u}(l, t)|\, \mathrm{e}^{\nu\pi|l|(t - t_0)}, \qquad Z(t) = \sum_{l \neq 0} |l|\, |\hat{u}(l, t)|\, \mathrm{e}^{\nu\pi|l|(t - t_0)}.$$

Noticing that $\hat{u}(l, t)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(l, t) = -\nu|2\pi l|\hat{u}(l, t) + \mathrm{i}\delta(2\pi l)^3\hat{u}(l, t) - 4\pi\mathrm{i} \sum_{l_1 + l_2 = l} \hat{u}(l_1, t) l_2 \hat{u}(l_2, t),$$

we obtain after some manipulations that Y and Z satisfies

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} + Z(t)(\nu - Y(t)) \leq 0.$$

Because of (2.17), namely $Y(t_0) \leq \nu$, this equation then implies that $Y(t)$ is a non-increasing function of t for $t \geq t_0$. We thus conclude that for any $t \geq t_0$

$$Y(t) = \sum_{l \neq 0} |\hat{u}(l, t)|\, \mathrm{e}^{\nu\pi|l|(t - t_0)} \leq \nu.$$

That is, $u(z, t)$, the extension of $u(x, t)$ to the complex plane, is analytic in the set S_t defined in (2.16). □

3. Numerical results

In this section, we present several representative results from our numerical experiments performed on the complex KdV–Burgers equation, namely (2.1) with $\alpha = 1$ and $\delta = 1/4\pi^2$. The initial data are of the

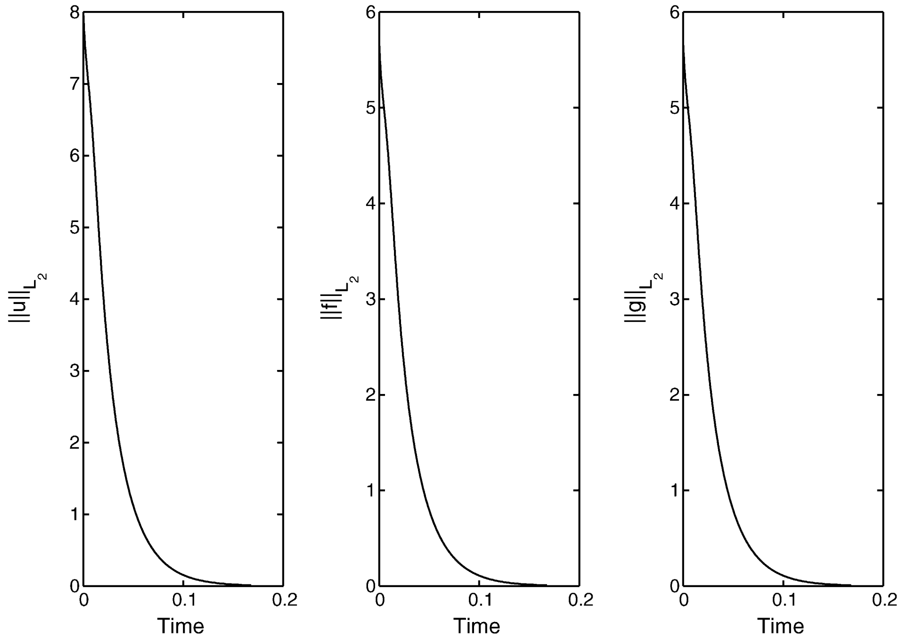

Fig. 1. $a = 8$, $\nu = 1$, $n = 1024$ and $k = 10^{-8}$.

form

$$u_0(x) = a\,\mathrm{e}^{2\pi \mathrm{i} x}.$$

We have used the spectral method with an FFT algorithm designed for a real KdV equation [3]. Appropriate modifications have been made to suit the complex equation. The numerical results are consistent with our theory. We computed the solutions for a range of a's and ν's. When the condition (2.2) is satisfied, the corresponding solutions decay exponentially. However, if ν is sufficiently small, then the solutions appear to blow up.

Table 1
a vs. ν_c ($n = 1024$ and $k = 10^{-8}$)

a	ν_c
6	0.095
8	0.225
10	0.375
12.6	0.55
13.75	0.65
15	0.75
16.25	0.85
17.5	0.925
18.75	1.05
20	1.125

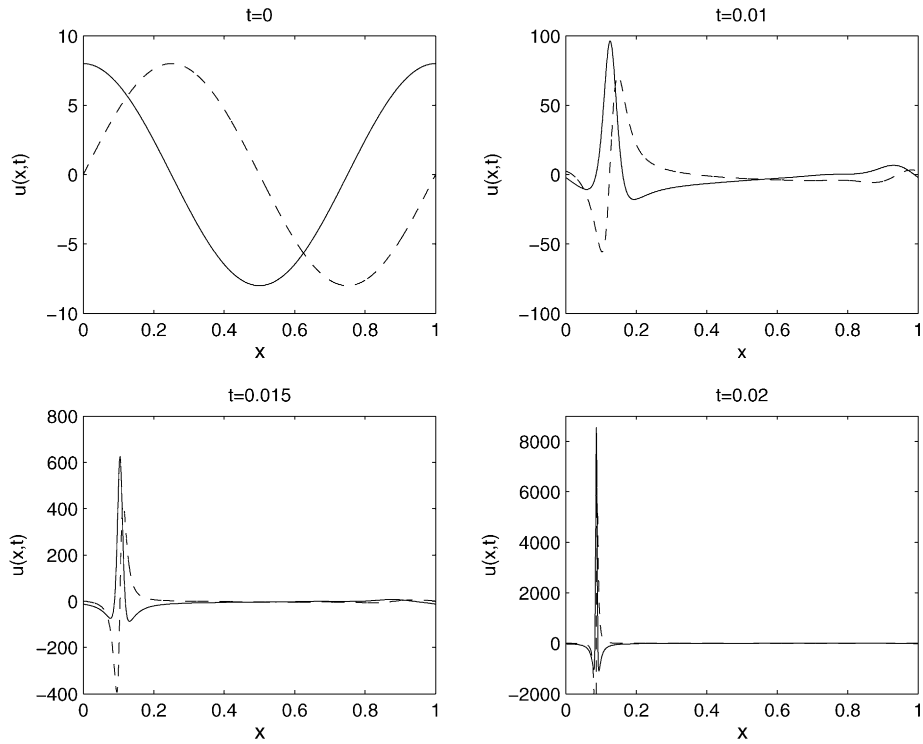

Fig. 2. $a = 8$, $\nu = 0.2$, $n = 1024$ and $k = 10^{-8}$. The solid curves represent the real part f and the dashed curves represent the imaginary part g.

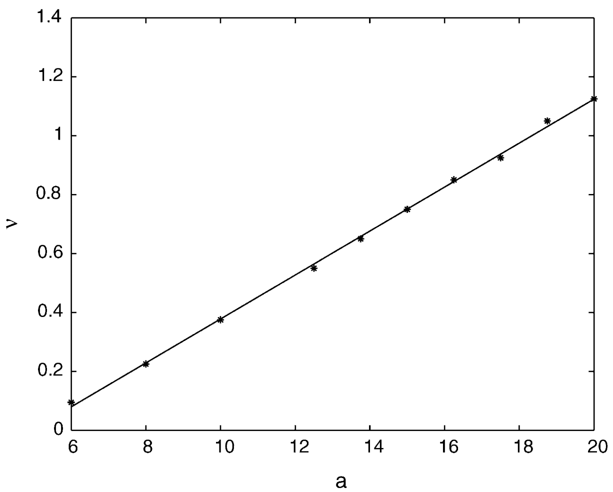

Fig. 3. a vs. ν_c ($n = 1024$ and $k = 10^{-8}$). The line is $\nu_c = -0.368 + 0.0746a$.

In Fig. 1, $a = 8$, $\nu = 1$, $n = 1024$ and $k = 10^{-8}$, where n represents the number of modes and k the time step. The plots clearly indicate that the L^2-norms of u, its real part f and imaginary part g all decay exponentially in time from the beginning. The exponential decay of u is asserted in Theorem 2.2.

In Fig. 2, $a = 8$, $\nu = 0.2$, $n = 1024$ and $k = 10^{-8}$. Obviously, $\nu \ll C_\alpha a$. That is, the condition (2.2) in Theorem 2.1 is violated. Both the real part f and the imaginary part g quickly lose shapes and increase very rapidly. At $t = 0.02$, the numerical instability kicks in and f and g appear to blow up. This computation was then repeated using $n = 2048$ and the same behavior of f and g occurred again.

For a ranging from 6 to 20, we computed the corresponding solutions associated with a range of ν. Our purpose has been to find the critical $\nu_c = \nu_c(a)$ such that the solutions appear to blow up for $\nu < \nu_c$ and the solutions are bounded for $\nu \geq \nu_c$. The results are given in Table 1. To see how ν_c depends on the corresponding a, Table 1 is plotted in Fig. 3. It is clear that ν_c and a, the L^2-norm of the initial data are linearly related.

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