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# Bounds and New Approaches for the 3D MHD Equations

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**Summary.** In this paper we establish several properties concerning solutions of the 3D magnetohydrodynamic (MHD) equations including global regularity conditions, a priori bounds, and real analyticity. We also explore two new approaches to the viscous and resistive MHD equations.

**Key words.** MHD equations, global regularity, a priori bounds, real analyticity, Eulerian-Lagrangian approach

**MSC numbers.** 76W05, 35Q35, 76DXX

## 1. Introduction

Magnetohydrodynamics (MHD), the science of the motion of an electrically conducting fluid in the presence of a magnetic field, consists essentially of the interaction between the fluid velocity and the magnetic field. Electric currents induced in the fluid as a result of its motion modify the field; at the same time their flow in the magnetic field leads to mechanical forces which modify the motion. The equations of MHD are the usual hydrodynamic and electromagnetic equations, modified to take this interaction into account. If $u$ is the fluid velocity, $b$ the magnetic field, and $j$ the current density, then

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu_1 \Delta u + j \times b + f, \tag{1.1}$$

$$\partial_t b = \nabla \times (u \times b) + \nu_2 \Delta b + g, \tag{1.2}$$

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where $p$ is the pressure, $\nu_1$ the kinematic viscosity, $\nu_2$ the magnetic diffusivity, $f$ represents volume force applied to the fluid, and $g$ is usually zero when Maxwell's displacement currents are ignored. In the equation of motion (1.1) $j$ and $b$ are related through Ampére's law $\nabla \times b = \mu j$, where the permeability $\mu$ is always a constant and will be set equal to 1 for the sake of clarity. Eliminating $j$ from (1.1) gives

$$\partial_t u + u \cdot \nabla u = -\nabla P + \nu_1 \Delta u + b \cdot \nabla b + f, \tag{1.3}$$

with $P = p + b^2/2$. For an incompressible fluid,

$$\nabla \cdot u = 0, \tag{1.4}$$

and we will also assume that $b$ is divergence-free,

$$\nabla \cdot b = 0, \tag{1.5}$$

which is almost a consequence of Faraday's law [14, p. 11]. The equations (1.2), (1.3), (1.4), and (1.5) constitute a complete set of incompressible MHD equations.

One of the fundamental issues arising naturally in the study of the MHD equations is how this fully nonlinear dynamic system evolves given a particular initial state. In contrast to linear equations, this nonlinear system obeys much more complicated laws that remain to be unearthed. In this paper we focus on two aspects of nonlinear magnetohydrodynamics. First, we consider long time evolution of an initially smooth profile of the velocity $u$ and magnetic field $b$. Mathematically, this is an issue of global (in time) existence for smooth solutions. Second, we construct two new formulations that are actually equivalent to the MHD equations (1.2) and (1.3). These formulations describe the nonlinear MHD from different perspectives and have potential applications in the study of magnetic reconnection [10]. It is likely that the investigation of these two aspects of magnetohydrodynamics will increase our understanding of the nonlinear interaction between the fluid velocity and the magnetic field.

We now outline some of our major results that will be presented in this paper along with their impact and relevant background information. We start with the long time evolution of a given smooth initial state under the nonlinear MHD. In the two-dimensional case, any smooth solution of the MHD equations with $\nu_1 > 0$ and $\nu_2 > 0$ is global in time ([8],[11]). But no such result for three-dimensional (3D) MHD equations is available. In fact, whether smooth solutions of the 3D MHD equations break down in a finite time remains an outstanding open issue. Let us briefly mention some of recent efforts devoted to this open problem. In [3] Caflisch, Klapper, and Steele extended the well-known result of Beale, Kato, and Majda [2] for fluids to MHD to conclude that the maximum norms of the vorticity $\omega$ and the current density $j$ control the breakdown of smooth solutions of the 3D MHD. In [16] we related any possible singularity of $u$ and $b$ to the geometric directions of $\omega$ and $j$ and derived a condition with the geometric interpretation that no finite-time singularity is possible without first developing very small scales. In the two sections that follow, we focus on two very special functionals of $u$ and $b$,

$$\int_0^T (\|\nabla u(\cdot, t)\|_{L^2}^{p_1} + \|\nabla b(\cdot, t)\|_{L^2}^{p_1})\, dt \tag{1.6}$$

and

$$\int_0^T (\|u(\cdot,t)\|_{L^\infty}^{p_2} + \|b(\cdot,t)\|_{L^\infty}^{p_2})\,dt, \tag{1.7}$$

where $T > 0$, $p_1$ and $p_2$ are real indices. We show in Section 2 that if (1.6) with $p_1 = 4$ or (1.7) with $p_2 = 2$ is finite, i.e., if

$$\int_0^T (\|\nabla u(\cdot,t)\|_{L^2}^4 + \|\nabla b(\cdot,t)\|_{L^2}^4)\,dt < \infty \tag{1.8}$$

or

$$\int_0^T (\|u(\cdot,t)\|_{L^\infty}^2 + \|b(\cdot,t)\|_{L^\infty}^2)\,dt < \infty, \tag{1.9}$$

then $u$ and $b$ remain smooth over $[0, T]$ and no breakdown can happen in this time interval. Intuitively this result states that derivatives of $u$ and $b$ at any order can be majorized by either (1.6) with $p_1 = 4$ or (1.7) with $p_2 = 2$. The condition in (1.9) falls into the category of Serrin type for fluids ([6],[12],[13]); it is significant because it does not impose any restriction on derivatives of $u$ and $b$.

It is currently not known if (1.8) or (1.9) holds for all $T > 0$. Efforts devoted to verifying (1.8) or (1.9) by direct manipulation of the MHD equations have so far been futile; the difficulty comes from the nonlinear coupling between the equation of motion (1.3) and that of electromagnetics (1.2). However, (1.6) with $p_1 = 2$ and (1.7) with $p_2 = 1$ are indeed bounded a priori, namely, for any $T > 0$,

$$\int_0^T (\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2)\,dt < \infty$$

and

$$\int_0^T (\|u(\cdot,t)\|_{L^\infty} + \|b(\cdot,t)\|_{L^\infty})\,dt < \infty.$$

Details concerning these a priori bounds will be given in Section 3. Now the important issue of optimal indices naturally arises: What are the largest indices $p_1$ and $p_2$ such that (1.6) and (1.7) are bounded a priori for any $T > 0$? If the maximal index for (1.6) to hold was 4 or for (1.7) to hold was 2, then we could deduce that smooth solutions of the 3D MHD equations were global in time.

Smooth solutions of the MHD equations with $\nu_1 > 0$ and $\nu_2 > 0$ are actually real analytic, as we shall conclude in Section 4. This mathematically rigorous result reflects the physical fact that the kinematic viscosity and the magnetic diffusivity smooth out spatial variations of velocity $u$ and magnetic field $b$. In the case of periodic boundary conditions, one consequence of the analyticity is that the Fourier modes $\hat{u}(k,t)$ and $\hat{b}(k,t)$ are shown to decay exponentially as $k$ increases. The work here was partially inspired by the result of Foias and Temam on the Navier-Stokes equations [9].

There are two distinct kinds of specifications for the flow field: the Eulerian specification and the Lagrangian specification [1]. The Eulerian specification provides a picture of the spatial distribution of fluid velocity and of other fluid quantities at each instant during the motion. In (1.2) and (1.3) $u$ and $b$ are described in the Eulerian specification. The Lagrangian specification makes use of the fact that physical quantities refer not only to a certain position in space but also to identifiable pieces of matter. The flow quantities in the Lagrangian specification are defined as functions of time and the choice of a material element of fluid, and describe the dynamical history of the selected fluid element. The Lagrangian specification is useful in certain special contexts, and the ideal fluid equations in the Lagrangian specification have been studied by several authors ([7],[13]). Recently Constantin invented the Eulerian-Lagrangian approach to reformulate the incompressible fluid equations ([4],[5],[6]). This approach, in Constantin's own words, "phrases the fluid equations in unbiased Eulerian coordinates, yet describes objects that have Lagrangian significance." One goal here is to extend Constantin's approach to the 3D MHD equations. We remark that such an extension is not straightforward. We will make use of Elsasser's symmetric variables and introduce a new pair of active scales to overcome some of the obstacles. The Eulerian-Lagrangian description of the MHD equations plays a key role in constructing representations for vortex and magnetic field lines and has potential applications in the study of magnetic reconnection [10].

We will also generalize the formulation of Zakharov and Kuznetsov for the ideal MHD equations to the MHD equations with viscosity and resistance. This is accomplished in Section 6. In [17] Zakharov and Kuznetsov reexpressed the ideal MHD equations in terms of a pair of canonical variables. The advantage of this representation of the ideal MHD equations is that it contains all vector Lagrangian invariants, which cannot be expressed in terms of the velocity and magnetic field in the Eulerian specification. In order to generalize the result of Zakharov and Kuznetsov to the MHD equations with viscosity and resistance, we define two new vector fields with physical significance and then derive a system of equations equivalent to the MHD equations.

## 2. Global Regularity Conditions

Although finite time singularities for smooth solutions of the 3D MHD equations have not been ruled out, several physically interesting quantities controlling possible singularity development have been obtained ([3],[15],[16]). In this section we present two new global regularity conditions.

**Theorem 2.1.** *Assume that the initial velocity $u_0$ and magnetic field $b_0$ are both in $H^s$ with $s \geq 3$, and $f$ and $g$ are in $L^2([0,T];L^2)$. If*

$$\int_0^T (\|\nabla u(\cdot,t)\|_{L^2}^4 + \|\nabla b(\cdot,t)\|_{L^2}^4)\, dt < \infty, \tag{2.1}$$

*then all spatial derivatives of the solution $u$ and $b$ to the MHD equations (1.3) and (1.2) are square integrable for any $t \in [0,T]$.*

*Proof.* Our first step is to show that the first-order derivatives of $u$ and $b$ are bounded as long as (2.1) holds. In fact, we establish that

$$\begin{aligned}
&\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2 \\
&\quad + \int_0^t e^{G(t-\tau)}(\nu_1\|\Delta u(\cdot,\tau)\|_{L^2}^2 + \nu_2\|\Delta b(\cdot,\tau)\|_{L^2}^2)\,d\tau \\
&\le (\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2)\,e^{G(t)} \\
&\quad + C\int_0^t e^{G(t-\tau)}\left(\frac{1}{\nu_1}\|f(\cdot,\tau)\|_{L^2}^2 + \frac{1}{\nu_2}\|g(\cdot,\tau)\|_{L^2}^2\right)d\tau, \qquad (2.2)
\end{aligned}$$

where $C$ is a pure constant and

$$G(t) = C\max\{\nu_1^{-3}, \nu_2^{-3}\}\int_0^t (\|\nabla u(\cdot,\tau)\|_{L^2}^4 + \|\nabla b(\cdot,\tau)\|_{L^2}^4)\,d\tau.$$

To this end, we consider the evolution equations for $\|\nabla u\|_{L^2}^2$ and $\|\nabla b\|_{L^2}^2$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu_1\|\Delta u\|_{L^2}^2 &= \int \nabla u\cdot\nabla f\,dx - \int \partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\,dx \\
&\quad + \int \partial_i u_j\cdot\partial_i b_k\cdot\partial_k b_j\,dx \\
&\quad + \int b_k\cdot\partial_k\partial_i b_j\cdot\partial_i u_j\,dx \qquad (2.3)\\
\frac{1}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2 + \nu_2\|\Delta b\|_{L^2}^2 &= \int \nabla b\cdot\nabla g\,dx - \int \partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j\,dx \\
&\quad + \int \partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j\,dx \\
&\quad + \int b_k\cdot\partial_k\partial_i u_j\cdot\partial_i b_j\,dx, \qquad (2.4)
\end{aligned}$$

where the repeated indices are summed. Adding (2.3) and (2.4) and using the fact that the last term of (2.3) and that of (2.4) add up to zero, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \nu_1\|\Delta u\|_{L^2}^2 + \nu_2\|\Delta b\|_{L^2}^2 \\
&\quad = \int \nabla u\cdot\nabla f\,dx + \int \nabla b\cdot\nabla g\,dx - \int \partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\,dx \\
&\qquad - \int \partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j\,dx + \int \partial_i u_j\cdot\partial_i b_k\cdot\partial_k b_j\,dx + \int \partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j\,dx.
\end{aligned}$$

The first two terms on the right can be bounded as follows:

$$\begin{aligned}
\int \nabla u\cdot\nabla f\,dx &= -\int f\cdot\Delta u\,dx \le \frac{\nu_1}{10}\|\Delta u\|_{L^2}^2 + \frac{C}{\nu_1}\|f\|_{L^2}^2, \\
\int \nabla b\cdot\nabla g\,dx &= -\int g\cdot\Delta b\,dx \le \frac{\nu_2}{8}\|\Delta b\|_{L^2}^2 + \frac{C}{\nu_2}\|g\|_{L^2}^2,
\end{aligned}$$

where $C$'s are pure constants. Using Hölder's inequality, the Gagliardo-Nirenberg inequality ($d$ being the spatial dimension)

$$\|F\|_{L^3} \le C\,\|F\|_{L^2}^{1-\frac{d}{6}}\,\|\nabla F\|_{L^2}^{\frac{d}{6}}$$

and the following generalized version of Young's inequality

$$\prod_{k=1}^{m} a_k \le \sum_{k=1}^{m} \frac{a_k^{p_k}}{p_k} \quad \text{for} \quad m \ge 2 \quad \text{and} \quad \sum_{k=1}^{m} \frac{1}{p_k} = 1,$$

we have

$$\begin{aligned}
-\int \partial_i u_k \cdot \partial_k u_j \cdot \partial_i u_j \, dx \;&\le\; \|\nabla u\|_{L^3}^3 \le C\|\nabla u\|_{L^2}^{\frac{3}{2}}\,\|\Delta u\|_{L^2}^{\frac{3}{2}} \\
&\le\; \frac{\nu_1}{10}\|\Delta u\|_{L^2}^2 + C(\nu_1)\|\nabla u\|_{L^2}^6, \\
-\int \partial_i u_k \cdot \partial_k b_j \cdot \partial_i b_j \, dx \;&\le\; C\|\nabla u\|_{L^3}\|\nabla b\|_{L^3}^2 \\
&\le\; C\|\nabla u\|_{L^2}^{\frac{1}{2}}\,\|\Delta u\|_{L^2}^{\frac{1}{2}}\,\|\nabla b\|_{L^2}\,\|\Delta b\|_{L^2} \\
&\le\; \frac{\nu_1}{10}\|\Delta u\|_{L^2}^2 + \frac{\nu_2}{8}\|\Delta b\|_{L^2}^2 + C(\nu_1)\|\nabla u\|_{L^2}^6 \\
&\quad\; + C(\nu_2)\|\nabla b\|_{L^2}^6, \\
\int \partial_i u_j \cdot \partial_i b_k \cdot \partial_k b_j \, dx \;&\le\; \|\nabla u\|_{L^3}\,\|\nabla b\|_{L^3}^2 \\
&\le\; \frac{\nu_1}{10}\|\Delta u\|_{L^2}^2 + \frac{\nu_2}{8}\|\Delta b\|_{L^2}^2 + C(\nu_1)\|\nabla u\|_{L^2}^6 \\
&\quad\; + C(\nu_2)\|\nabla b\|_{L^2}^6,
\end{aligned}$$

and

$$\begin{aligned}
\int \partial_i b_k \cdot \partial_k u_j \cdot \partial_i b_j \, dx \;&\le\; \frac{\nu_1}{10}\|\Delta u\|_{L^2}^2 + \frac{\nu_2}{8}\|\Delta b\|_{L^2}^2 + C(\nu_1)\|\nabla u\|_{L^2}^6 \\
&\quad\; + C(\nu_2)\|\nabla b\|_{L^2}^6,
\end{aligned}$$

where $C(\nu_1) = \frac{C}{\nu_1^3}$ and $C(\nu_2) = \frac{C}{\nu_2^3}$ for some pure constant $C$.

Consequently,

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \nu_1\|\Delta u\|_{L^2}^2 + \nu_2\|\Delta b\|_{L^2}^2 \\
&\qquad \le \frac{C}{\nu_1}\|f\|_{L^2}^2 + \frac{C}{\nu_2}\|g\|_{L^2}^2 + C\left(\frac{1}{\nu_1^3}\|\nabla u\|_{L^2}^6 + \frac{1}{\nu_2^3}\|\nabla b\|_{L^2}^6\right),
\end{aligned} \tag{2.5}$$

where $C$'s are pure constants. The use of Gronwall's inequality finishes the proof of (2.2).

Higher order derivatives of $u$ and $b$ can be handled in a similar fashion, but their bounds may now involve lower order derivatives. Thus, one has to go through an inductive procedure. The details are routine and thus omitted. This completes the proof of Theorem 2.1. □

**Theorem 2.2.** *Assume that the initial velocity $u_0$ and magnetic field $b_0$ are both in $H^s$ with $s \geq 3$, and $f, g \in L^2([0,T]; L^2)$. If*

$$\int_0^T (\|u(\cdot,t)\|_{L^\infty}^2 + \|b(\cdot,t)\|_{L^\infty}^2)\, dt < \infty, \tag{2.6}$$

*then any spatial derivative of the solution $u$ and $b$ to the MHD equations (1.3) and (1.2) is square integrable for any $t \in [0,T]$.*

*Proof.* It suffices to verify that $\nabla u(x,t)$ and $\nabla b(x,t)$ are in $L^2$ for any $t \leq T$. Using (1.3), one easily checks that $\|\nabla u\|_2^2$ satisfies the evolution equation

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu_1\|\Delta u\|_{L^2}^2 = \int u\cdot\nabla u\cdot\Delta u - \int b\cdot\nabla b\cdot\Delta u - \int \Delta u\cdot f.$$

By Hölder's inequality,

$$\begin{aligned}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu_1\|\Delta u\|_{L^2}^2 \ \leq\ & \frac{1}{\nu_1}\|u(\cdot,t)\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \frac{1}{\nu_1}\|f\|_{L^2}^2 \\ & + \frac{2}{\nu_1}\|b(\cdot,t)\|_{L^\infty}^2\|\nabla b\|_{L^2}^2 + \frac{\nu_1}{8}\|\Delta u\|_{L^2}^2.\end{aligned} \tag{2.7}$$

Similarly, $\nabla b$ obeys

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla b\|_{L^2}^2 + \frac{5}{8}\nu_2\|\Delta b\|_{L^2}^2 \ \leq\ & \frac{2}{\nu_2}\|g\|_{L^2}^2 + \frac{2}{\nu_2}\|u(\cdot,t)\|_{L^\infty}^2\|\nabla b\|_{L^2}^2 \\ & + \frac{2}{\nu_2}\|b(\cdot,t)\|_{L^\infty}^2\|\nabla u\|_{L^2}^2.\end{aligned} \tag{2.8}$$

Now let

$$Y(t) = \|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2.$$

Combining (2.7) and (2.8), we obtain

$$\begin{aligned}&\frac{d}{dt}Y(t) + \nu_1\|\Delta u\|_{L^2}^2 + \nu_2\|\Delta b\|_{L^2}^2 \\ &\qquad \leq\ C(\nu_1,\nu_2)\,(\|u(\cdot,t))\|_{L^\infty}^2 + \|b(\cdot,t)\|_{L^\infty}^2)\,Y(t) + \frac{1}{\nu_1}\|f\|_{L^2}^2 + \frac{2}{\nu_2}\|g\|_{L^2}^2,\end{aligned}$$

where $C(\nu_1,\nu_2) = \max\{\nu_1^{-1}, 2\nu_2^{-2}\}$. It then follows from applying Gronwall's inequality that (2.6) implies for any $t \in [0,T]$

$$Y(t) < \infty, \quad \text{i.e.,} \quad \|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2 < \infty.$$ □

## 3. A Priori Bounds

The goal of this section is to prove that the quantity

$$\int_0^T (\|u(\cdot,t)\|_{L^\infty} + \|b(\cdot,t)\|_{L^\infty})\,dt$$

is bounded a priori for any $T > 0$. The simple bound

$$\int_0^T (\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2)\,dt < \infty$$

will be provided first.

**Theorem 3.1.** *Consider the MHD equations (1.3) and (1.2) either with periodic boundary conditions or defined in the whole space with sufficient decay at infinity. If the initial velocity $u_0$ and magnetic field $b_0$ are in $L^2$, and the forces $f$ and $g$ are in $L^2([0,T];L^2) \cap L^2([0,T];H^{-1})$, then any solution $(u,b)$ satisfies*

$$\|u(\cdot,t)\|_{L^2}^2 + \|b(\cdot,t)\|_{L^2}^2 + \int_0^t (\nu_1\,\|\nabla u(\cdot,\tau)\|_{L^2}^2 + \nu_2\,\|\nabla b(\cdot,\tau)\|_{L^2}^2)\,d\tau \le E, \tag{3.1}$$

*where $E = \min\{E_1, E_2\}$ with*

$$E_1 = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + \frac{1}{\nu_1}\int_0^t \|\Delta^{-\frac{1}{2}} f(\cdot,\tau)\|_{L^2}^2\,d\tau + \frac{1}{\nu_2}\int_0^t \|\Delta^{-\frac{1}{2}} g(\cdot,\tau)\|_{L^2}^2\,d\tau$$

*and*

$$E_2 = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + 4t\int_0^t \|f(\cdot,\tau)\|_{L^2}^2\,d\tau + 4t\int_0^t \|g(\cdot,\tau)\|_{L^2}^2\,d\tau.$$

*Proof.* We will make use of the evolution equation for $\|u\|_{L^2}^2$ and $\|b\|_{L^2}^2$,

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \nu_1\|\nabla u\|_{L^2}^2 + \nu_2\|\nabla b\|_{L^2}^2 = \int u\cdot f\,dx + \int b\cdot g\,dx.$$

Integrating the above over $[0,t]$ and estimating the terms on the right-hand side in the following two different ways:

$$\int u\cdot f\,dx \;\le\; \frac{\nu_1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2\nu_1}\|\Delta^{-\frac{1}{2}} f\|_{L^2}^2,$$

$$\int b\cdot g\,dx \;\le\; \frac{\nu_2}{2}\|\nabla b\|_{L^2}^2 + \frac{1}{2\nu_2}\|\Delta^{-\frac{1}{2}} g\|_{L^2}^2,$$

and

$$\int_0^t\int u\cdot f\,dx\,d\tau \;\le\; \sup_{\tau\in[0,t]}\|u(\cdot,\tau)\|_{L^2}\cdot\left(t\int_0^t\int f^2\,dx\,d\tau\right),$$

$$\int_0^t\int b\cdot g\,dx\,d\tau \;\le\; \sup_{\tau\in[0,t]}\|b(\cdot,\tau)\|_{L^2}\cdot\left(t\int_0^t\int g^2\,dx\,d\tau\right),$$

we then prove (3.1). □

**Theorem 3.2.** *If the initial velocity $u_0$ and magnetic field $b_0$ are in $L^2$, and the forces $f$ and $g$ are in $L^2([0,T];L^2)$, then any solution $(u,b)$ of (1.3) and (1.2) satisfies for any $T>0$*

$$\int_0^T (\|u(\cdot,t)\|_{L^\infty} + \|b(\cdot,t)\|_{L^\infty})\,dt < \infty.$$

*Proof.* We start with the inequality (2.5) obtained in Section 2. Letting

$$K(t) = 1 + \|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2,$$

and dividing both sides of (2.5) by $K^2$ and integrating over $[0,t]$, we obtain

$$\begin{aligned}
&\frac{1}{1+\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2} - \frac{1}{1+\|\nabla u(\cdot,t)\|_{L^2}^2 + \|\nabla b(\cdot,t)\|_{L^2}^2}\\
&\qquad + \int_0^t (\nu_1\|\Delta u(\cdot,\tau)\|_{L^2}^2 + \nu_2\|\Delta b(\cdot,\tau)\|_{L^2}^2)\cdot K^{-2}(\tau)\,d\tau\\
&\quad \le \frac{C}{\nu_1}\int_0^t \|f(\cdot,\tau)\|_{L^2}^2\,d\tau + \frac{C}{\nu_2}\int_0^t \|g(\cdot,\tau)\|_{L^2}^2\,d\tau\\
&\qquad + C\int_0^t \left(\frac{1}{\nu_1^3}\|\nabla u(\cdot,\tau)\|_{L^2}^2 + \frac{1}{\nu_2^3}\|\nabla b(\cdot,\tau)\|_{L^2}^2\right)d\tau,
\end{aligned}$$

where $C$'s are pure constants. Thus,

$$\begin{aligned}
&\int_0^t \left(\nu_1\|\Delta u(\cdot,\tau)\|_{L^2}^2 + \nu_2\|\Delta b(\cdot,\tau)\|_{L^2}^2\right)\cdot K^{-2}(\tau)\,d\tau\\
&\qquad \le 1 + \frac{E}{\nu_1^4} + \frac{E}{\nu_2^4} + \frac{C}{\nu_1}\int_0^t \|f(\cdot,\tau)\|_{L^2}^2\,d\tau + \frac{C}{\nu_2}\int_0^t \|g(\cdot,\tau)\|_{L^2}^2\,d\tau.
\end{aligned}$$

By the Gagliardo-Nirenberg inequality and then the Hölder inequality, we obtain

$$\begin{aligned}
\int_0^t \|u(\cdot,\tau)\|_{L^\infty}\,d\tau &\le C\int_0^t \|\nabla u(\cdot,\tau)\|_{L^2}^{\frac12}\,\|\Delta u(\cdot,\tau)\|_{L^2}^{\frac12}\,d\tau\\
&\le C\left(\int_0^t \|\nabla u(\cdot,\tau)\|_{L^2}^2\,d\tau\right)^{\frac14}\left(\int_0^t \|\Delta u(\cdot,\tau)\|_{L^2}^{\frac23}\,d\tau\right)^{\frac34}\\
&\le C\left(\int_0^t \|\nabla u(\cdot,\tau)\|_{L^2}^2\,d\tau\right)^{\frac14}\\
&\qquad \times\left(\int_0^t \|\Delta u(\cdot,\tau)\|_{L^2}^2\cdot K^{-2}(\tau)\,d\tau\right)^{\frac14}\left(\int_0^t K\,ds\right)^{\frac12},
\end{aligned}$$

where $C$ is a pure constant. Combining this estimate with a similar one for $\int_0^t \|b(\cdot,\tau)\|_{L^\infty}\,d\tau$, we have

$$\int_0^t (\|u(\cdot,\tau)\|_{L^\infty} + \|b(\cdot,\tau)\|_{L^\infty})\,d\tau \le C(\nu_1^{-\frac12} + \nu_2^{-\frac12})\,K_1^{\frac14}\,K_2^{\frac14}\left(\int_0^t K\,ds\right)^{\frac12},$$

where $C(\nu_1, \nu_2)$ is a constant depending on $\nu_1$ and $\nu_2$ only,

$$K_1 = \int_0^t (\nu_1 \|\nabla u(\cdot, \tau)\|_{L^2}^2 + \nu_2 \|\nabla b(\cdot, \tau)\|_{L^2}^2)\, d\tau$$

and

$$K_2 = \int_0^t (\nu_1 \|\Delta u(\cdot, \tau)\|_{L^2}^2 + \nu_2 \|\Delta b(\cdot, \tau)\|_{L^2}^2) \cdot K^{-2}(\tau)\, d\tau.$$

Using (3.1), we conclude that

$$\int_0^t (\|u(\cdot, \tau)\|_{L^\infty} + \|b(\cdot, \tau)\|_{L^\infty})\, d\tau \le G(\nu_1, \nu_2, E, f, g, t),$$

where the bound $G$ is explicit,

$$G = C E^{\frac{1}{4}} (\nu_1^{-\frac{1}{2}} + \nu_2^{-\frac{1}{2}}) \left( t + \frac{E}{\nu_1} + \frac{E}{\nu_2} \right)^{\frac{1}{2}} \times \left( 1 + \frac{E}{\nu_1^4} + \frac{E}{\nu_2^4} + \int_0^t \left[ \frac{1}{\nu_1} \|f(\cdot, \tau)\|_{L^2}^2 + \frac{1}{\nu_2} \|g(\cdot, \tau)\|_{L^2}^2 \right] d\tau \right)^{\frac{1}{4}}$$

for some pure constant $C$. □

## 4. Real Analyticity

In this section we restrict ourselves to periodic boundary conditions and show that solutions of the viscous MHD equations exhibit real analyticity. To simplify the presentation, we will assume that $f$ and $g$ are both zero.

**Theorem 4.1.** *Assume that $u_0$ and $b_0$ are in $H^1$ and divergence free. If a solution $(u, b)$ of the MHD equations (1.3) and (1.2) satisfies either*

$$M_1 \equiv \int_0^T (\|u(\cdot, \tau)\|_{L^\infty}^2 + \|b(\cdot, \tau)\|_{L^\infty}^2)\, d\tau < \infty \tag{4.1}$$

*or*

$$M_2 \equiv \int_0^T (\|\nabla u(\cdot, \tau)\|_{L^2}^4 + \|\nabla b(\cdot, \tau)\|_{L^2}^4)\, d\tau < \infty, \tag{4.2}$$

*then $(u, b)$ is analytic on [0,T]. In fact, for any $t \in [0, T]$,*

$$\sum_k e^{2|k|t} |k|^2 \left( |\hat{u}(k, t)|^2 + |\hat{b}(k, t)|^2 \right) < C \tag{4.3}$$

*for some constant C depending only on $u_0$, $b_0$, and $M_1$ or $M_2$.*

*Proof.* To eliminate $P$ from (1.3) and (1.2), we project (1.3) and (1.2) onto the space of divergence-free vector fields. In terms of Fourier modes, the projected equations can then be rewritten as

$$\begin{aligned}\frac{d}{dt}\hat{u}(k,t)+\nu_1|k|^2\hat{u}(k,t) \;=\; & i\,\mathbb{P}(k)\sum_{k'+k''=k}\hat{u}(k',t)\cdot k''\hat{u}(k'',t)\\ & - i\,\mathbb{P}(k)\sum_{k'+k''=k}\hat{b}(k',t)\cdot k''\hat{b}(k'',t)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\hat{b}(k,t)+\nu_2|k|^2\hat{b}(k,t) \;=\; & i\,\mathbb{P}(k)\sum_{k'+k''=k}\hat{u}(k',t)\cdot k''\hat{b}(k'',t)\\ & - i\,\mathbb{P}(k)\sum_{k'+k''=k}\hat{b}(k',t)\cdot k''\hat{u}(k'',t),\end{aligned}$$

where $\mathbb{P}(k)=I-\frac{k\otimes k}{|k|^2}$ and $k\otimes k$ is the dyadic product. It then follows that

$$\begin{aligned}&\frac{d}{dt}\sum_k e^{2|k|t}|k|^2|\hat{u}(k,t)|^2+\nu_1\sum_k e^{2|k|t}|k|^4|\hat{u}(k,t)|^2\\ &\quad=\sum_k e^{2|k|t}|k|^3|\hat{u}(k,t)|^2\\ &\qquad+Re\left\{i\,\mathbb{P}(k)\sum_k e^{t|k|}|k|^2\hat{u}^*(k,t)\cdot\sum_{k'+k''=k}e^{t|k|}\hat{u}(k',t)\cdot k''\hat{u}(k'',t)\right\}\\ &\qquad-Re\left\{i\,\mathbb{P}(k)\sum_k e^{t|k|}|k|^2\hat{u}^*(k,t)\cdot\sum_{k'+k''=k}e^{t|k|}\hat{b}(k',t)\cdot k''\hat{b}(k'',t)\right\}\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dt}\sum_k e^{2|k|t}|k|^2|\hat{b}(k,t)|^2+\nu_2\sum_k e^{2|k|t}|k|^4|\hat{b}(k,t)|^2\\ &\quad=\sum_k e^{2|k|t}|k|^3|\hat{b}(k,t)|^2\\ &\qquad+Re\left\{i\,\mathbb{P}(k)\sum_k e^{t|k|}|k|^2\hat{b}^*(k,t)\cdot\sum_{k'+k''=k}e^{t|k|}\hat{u}(k',t)\cdot k''\hat{b}(k'',t)\right\}\\ &\qquad-Re\left\{i\,\mathbb{P}(k)\sum_k e^{t|k|}|k|^2\hat{b}^*(k,t)\cdot\sum_{k'+k''=k}e^{t|k|}\hat{b}(k',t)\cdot k''\hat{u}(k'',t)\right\}.\end{aligned}$$

To simplify the notation, we define two functions via their Fourier transforms:

$$U(x,t)=\sum_k e^{t|k|}\hat{u}(k,t)e^{ikx},\qquad B(x,t)=\sum_k e^{t|k|}\hat{b}(k,t)e^{ikx}.$$

Using the fact that $e^{t|k|} \le e^{t|k'|}e^{t|k''|}$ for $k' + k'' = k$, we then deduce from the equations above that

$$\frac{d}{dt}\|\Delta^{\frac{1}{2}}U\|_{L^2}^2 + \nu_1\|\Delta U\|_{L^2}^2 \le \|\Delta^{\frac{3}{4}}U\|_{L^2}^2 + \left|\int (\Delta U)\cdot U\cdot(\Delta^{\frac{1}{2}}U)\right| + \left|\int (\Delta U)\cdot B\cdot(\Delta^{\frac{1}{2}}B)\right| \tag{4.4}$$

and

$$\frac{d}{dt}\|\Delta^{\frac{1}{2}}B\|_{L^2}^2 + \nu_2\|\Delta B\|_{L^2}^2 \le \|\Delta^{\frac{3}{4}}B\|_{L^2}^2 + \left|\int (\Delta B)\cdot U\cdot(\Delta^{\frac{1}{2}}B)\right| + \left|\int (\Delta B)\cdot B\cdot(\Delta^{\frac{1}{2}}U)\right|. \tag{4.5}$$

We now estimate the terms on the right of (4.4):

$$\|\Delta^{\frac{3}{4}}U\|_{L^2}^2 \le \frac{\nu_1}{2}\|\Delta U\|_{L^2}^2 + \frac{1}{2\nu_1}\|\Delta^{\frac{1}{2}}U\|_{L^2}^2,$$

$$\left|\int (\Delta U)\cdot U\cdot(\Delta^{\frac{1}{2}}U)\right| \le C\|U\|_{L^\infty}\|\Delta U\|_{L^2}\|\Delta^{\frac{1}{2}}U\|_{L^2} \le C\|\Delta U\|_{L^2}^{\frac{3}{2}}\|\Delta^{\frac{1}{2}}U\|_{L^2}^{\frac{3}{2}},$$

$$\left|\int (\Delta U)\cdot B\cdot(\Delta^{\frac{1}{2}}B)\right| \le C\|\Delta U\|_{L^2}\|\Delta^{\frac{1}{2}}B\|_{L^2}^{\frac{3}{2}}\|\Delta B\|_{L^2}^{\frac{1}{2}}.$$

Inserting these estimates in (4.4), we then have

$$\frac{d}{dt}\|\Delta^{\frac{1}{2}}U\|_{L^2}^2 + \frac{\nu_1}{2}\|\Delta U\|_{L^2}^2 \le \frac{1}{2\nu_1}\|\Delta^{\frac{1}{2}}U\|_{L^2}^2 + C\|\Delta U\|_{L^2}^{\frac{3}{2}}\|\Delta^{\frac{1}{2}}U\|_{L^2}^{\frac{3}{2}} + C\|\Delta U\|_{L^2}\|\Delta^{\frac{1}{2}}B\|_{L^2}^{\frac{3}{2}}\|\Delta B\|_{L^2}^{\frac{1}{2}}. \tag{4.6}$$

In a similar fashion, we have from (4.5)

$$\frac{d}{dtt}\|\Delta^{\frac{1}{2}}B\|_{L^2}^2 + \frac{\nu_2}{2}\|\Delta B\|_{L^2}^2 \le \frac{1}{2\nu_2}\|\Delta^{\frac{1}{2}}B\|_{L^2}^2 + C\|\Delta^{\frac{1}{2}}U\|_{L^2}^{\frac{1}{2}}\|\Delta U\|_{L^2}^{\frac{1}{2}}\|\Delta^{\frac{1}{2}}B\|_{L^2}\|\Delta B\|_{L^2} + C\|\Delta^{\frac{1}{2}}U\|_{L^2}\|\Delta^{\frac{1}{2}}B\|_{L^2}^{\frac{1}{2}}\|\Delta B\|_{L^2}^{\frac{3}{2}}. \tag{4.7}$$

Applying the following generalized version of Young's inequality

$$\prod_{k=1}^m a_k \le \sum_{k=1}^m \frac{a_k^{p_k}}{p_k} \quad \text{for} \quad m \ge 2 \quad \text{and} \quad \sum_{k=1}^m \frac{1}{p_k} = 1$$

to the terms on the right of (4.6), we get

$$\|\Delta U\|_{L^2}^{\frac{3}{2}}\|\Delta^{\frac{1}{2}}U\|_{L^2}^{\frac{3}{2}} \le \frac{\nu_1}{8}\|\Delta U\|_{L^2}^2 + C(\nu_1)\|\Delta^{\frac{1}{2}}U\|_{L^2}^6,$$

$$\|\Delta U\|_{L^2}\|\Delta^{\frac{1}{2}}B\|_{L^2}^{\frac{3}{2}}\|\Delta B\|_{L^2}^{\frac{1}{2}} \le \frac{\nu_1}{8}\|\Delta U\|_{L^2}^2 + \frac{\nu_2}{8}\|\Delta B\|_{L^2}^2 + C(\nu_1,\nu_2)\|\Delta^{\frac{1}{2}}B\|_{L^2}^6.$$

Consequently, (4.6) becomes

$$\frac{d}{dt}\|\Delta^{\frac{1}{2}}U\|_{L^2}^2+\frac{\nu_1}{4}\|\Delta U\|_{L^2}^2 \le \frac{1}{2\nu_1}\|\Delta^{\frac{1}{2}}U\|_{L^2}^2+\frac{\nu_2}{8}\|\Delta B\|_{L^2}^2 + C(\nu_1)\|\Delta^{\frac{1}{2}}U\|_{L^2}^6+C(\nu_1,\nu_2)\|\Delta^{\frac{1}{2}}B\|_{L^2}^6. \quad (4.8)$$

Similarly, from (4.7),

$$\frac{d}{dt}\|\Delta^{\frac{1}{2}}B\|_{L^2}^2+\frac{\nu_2}{4}\|\Delta B\|_{L^2}^2 \le \frac{1}{2\nu_2}\|\Delta^{\frac{1}{2}}B\|_{L^2}^2+\frac{\nu_1}{8}\|\Delta U\|_{L^2}^2 + C(\nu_1)\|\Delta^{\frac{1}{2}}U\|_{L^2}^6+C(\nu_2)\|\Delta^{\frac{1}{2}}B\|_{L^2}^6. \quad (4.9)$$

Adding (4.8) and (4.9) yields the differential inequality

$$\frac{dy}{dt}\le \max\left\{\frac{1}{2\nu_1},\frac{1}{2\nu_2}\right\}y+C(\nu_1,\nu_2)y^3, \qquad y(t)=\|\Delta^{\frac{1}{2}}U\|_{L^2}^2+\|\Delta^{\frac{1}{2}}B\|_{L^2}^2.$$

Such a differential inequality implies that

$$y(t)\le \frac{e^{c_1 t}\, y(0)}{\sqrt{1-c_2c_1^{-1}\, y(0)^2\left(e^{2c_1 t}-1\right)}}$$

for $t\in[0,T^*(y(0)))$, where

$$c_1 = \max\left\{\frac{1}{2\nu_1},\frac{1}{2\nu_2}\right\}, \qquad c_2=C(\nu_1,\nu_2),$$
$$y(0) = \|\nabla u_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2<\infty,$$

and

$$T^*(y(0)) = \frac{1}{2c_1}\log\left(1+\frac{c_1}{c_2 y(0)^2}\right). \quad (4.10)$$

If either (4.1) or (4.2) holds, Theorem 2.2 and Theorem 2.1 indicate that

$$M\equiv \sup_{t\in[0,T]}(\|\nabla u(\cdot,t)\|_{L^2}^2+\|\nabla b(\cdot,t)\|_{L^2}^2)<\infty.$$

This will enable us to show that (4.3) holds for any $t\in[0,T)$. In fact, if $t<T^*(y(0))$, then the mission has already been fulfilled. Otherwise, we can repeat the argument above at $t_0=t-T^*(M)/2$ and conclude that $y(t)=y(t_0+T^*(M)/2)$ is finite. Thus (4.3) is proved for any $t\in[0,T]$. □

## 5. Constantin's Eulerian-Lagrangian Approach

In this section, Constantin's Eulerian-Lagrangian approach to the Navier-Stokes equations is employed to derive an equivalent system of equations to the viscous MHD equations. Using Elsasser's variables $z^{\pm} = u \pm b$, we write the MHD equations in the symmetric from

$$\partial_t z^- + z^+ \cdot \nabla z^- = -\nabla P + \nu^+ \Delta z^- + \nu^- \Delta z^+, \tag{5.1}$$

$$\partial_t z^+ + z^- \cdot \nabla z^+ = -\nabla P + \nu^+ \Delta z^+ + \nu^- \Delta z^-, \tag{5.2}$$

where $\nu^{\pm} = \nu_1 \pm \nu_2$, and $P = p + b^2/2$ is the total pressure. The MHD equations (1.3) and (1.2) have similar mathematical structures and $u$ and $b$ have the same scaling dimension. This fact is made clearer by representing (1.3) and (1.2) in terms of Elsasser's variables.

To simplify our analysis, we will assume that $\nu_1 = \nu_2$, i.e., $\nu^- = 0$. That means the terms $\nu^- \Delta z^+$ and $\nu^- \Delta z^-$ become zero in the equations above. In astrophysical magnetic phenomena, both the Reynolds number and the magnetic Reynolds number are huge and the difference between $\nu_1$ and $\nu_2$ is not critical. In this aspect our simplification is justified.

To proceed, we split $z^-$ and $z^+$ as the following sums:

$$z^- = (\nabla A^-)^* v^- - \nabla n^- + f^-; \qquad z^+ = (\nabla A^+)^* v^+ - \nabla n^+ + f^+. \tag{5.3}$$

We now introduce the new quantities that have appeared in (5.3). Analytically, $A^-$ and $A^+$ are functions of $x$ and $t$ and satisfy the active vector equations

$$(\partial_t + z^+ \cdot \nabla - \nu^+ \Delta) A^- = 0, \tag{5.4}$$

$$(\partial_t + z^- \cdot \nabla - \nu^+ \Delta) A^+ = 0. \tag{5.5}$$

Both $A^-$ and $A^+$ have a Lagrangian interpretation. In the inviscid case, i.e., $\nu_1 = \nu_2 = 0$, $A^-$ and $A^+$ are inverse maps of the particle trajectories $a \to x = X^{\mp}(a, t)$ with

$$\frac{dX^{\mp}}{dt} = z^{\mp}(X^{\mp}, t), \qquad X^{\mp}(a, 0) = a.$$

Since $A^-$ and $A^+$ have the dimension of length, $\nabla A^-$ and $\nabla A^+$ are both nondimensional. When $\nabla A^{\mp}$ are invertible, we use $Q^{\mp}$ to denote their inverses. That is, $(\nabla A^{\mp}) Q^{\mp} = I$ and $(\nabla A^{\mp})^* (Q^{\mp})^* = I$, or

$$Q^{\mp}_{jk} \frac{\partial A^{\mp}_m}{\partial x_j} = \delta_{km}, \qquad Q^{\mp}_{kj} \frac{\partial A^{\mp}_j}{\partial x_m} = \delta_{km},$$

where $*$ refers to the transpose of a matrix, $I$ denotes the unit matrix, and $\delta_{km}$ stands for the Kronecker delta.

The second ingredient in (5.3) is made up of the virtual velocities $v^-$ and $v^+$. They are both vector functions of $x$ and $t$ and obey the evolution equations

$$(\partial_t + z^+ \cdot \nabla - \nu^+ \Delta) v^- = 2\nu^+ Q^- \cdot \nabla(\nabla A^-)(\nabla v^-); \tag{5.6}$$

$$(\partial_t + z^- \cdot \nabla - \nu^+ \Delta) v^+ = 2\nu^+ Q^+ \cdot \nabla(\nabla A^+)(\nabla v^+), \tag{5.7}$$

where the two terms on the right can be made precise by inspecting their $m$-th components,

$$(Q^{\mp} \cdot \nabla(\nabla A^{\mp})(\nabla v^{\mp}))_m = Q^{\mp}_{km} \frac{\partial^2 A^{\mp}_l}{\partial x_k \partial x_i} \frac{\partial v^{\mp}_l}{\partial x_i}.$$

The two scalar functions $n^-(x,t)$ and $n^-(x,t)$ in (5.3) are the so-called Eulerian-Lagrangian potentials, which play a role similar to that of the pressure. $n^-$ and $n^+$ are related through the following relation:

$$\partial_t n^- + z^+ \cdot \nabla n^- - \nu^+ \Delta n^- = \partial_t n^+ + z^- \cdot \nabla n^+ - \nu^+ \Delta n^+. \tag{5.8}$$

We now approach the final component of $z^\mp$ given by the expression in (5.3). As functions of $x$ and $t$, $f^-$ and $f^+$ satisfy the active vector equations

$$\partial_t f^- + z^+ \cdot \nabla f^- - \nu^+ \Delta f^- = -(\nabla z^+)^* f^- + (\nabla z^+)^* z^- \tag{5.9}$$

and

$$\partial_t f^+ + z^- \cdot \nabla f^+ - \nu^+ \Delta f^+ = -(\nabla z^-)^* f^+ + (\nabla z^-)^* z^+. \tag{5.10}$$

Having become familiarized with these quantities, we now show that $z^-$ and $z^+$, when combined as in (5.3), solve the MHD equations of the symmetric form (5.1) and (5.2).

**Theorem 5.1.** *Assume that $A^-$, $A^+$, $v^-$, $v^+$, $f^-$, and $f^+$ satisfy (5.4), (5.5), (5.6), (5.7), (5.9), and (5.10), respectively. Then $z^-$ and $z^+$ given by the formula in (5.3) solve (5.1) and (5.2) with $P$ determined by*

$$P = \partial_t n^- + z^+ \cdot \nabla n^- - \nu^+ \Delta n^- \quad \textit{or} \quad P = \partial_t n^+ + z^- \cdot \nabla n^+ - \nu^+ \Delta n^+, \tag{5.11}$$

*because of the relation (5.8).*

*Proof.* We start by examining the $j$-th component of $\partial_t z^- + z^+ \cdot \nabla z^-$.

$$\begin{aligned}
\partial_t z_j^- + z_k^+ \frac{\partial z_j^-}{\partial x_k} &= \partial_t \left( \frac{\partial A_m^-}{\partial x_j} v_m^- - \frac{\partial n^-}{\partial x_j} + f_j^- \right) + z_k^+ \frac{\partial}{\partial x_k} \left( \frac{\partial A_m^-}{\partial x_j} v_m^- - \frac{\partial n^-}{\partial x_j} + f_j^- \right) \\
&= -\frac{\partial}{\partial x_j}(D_t^+ n^-) + \frac{\partial z_k^+}{\partial x_j} \frac{\partial n^-}{\partial x_k} + D_t^+ f_j^- \\
&\quad + \frac{\partial A_m^-}{\partial x_j}(D_t^+ v_m^-) + v_m^- D_t^+ \left( \frac{\partial A_m^-}{\partial x_j} \right),
\end{aligned}$$

where $D_t^+ = \partial_t + z_k^+ \partial_k$ denotes a generalized material derivative. Taking $\frac{\partial}{\partial x_j}$ of (5.4) yields

$$D_t^+ \left( \frac{\partial A_m^-}{\partial x_j} \right) = \nu^+ \Delta \left( \frac{\partial A_m^-}{\partial x_j} \right) - \frac{\partial z_k^+}{\partial x_j} \frac{\partial A_m^-}{\partial x_k}. \tag{5.12}$$

Using (5.12) and (5.6), we obtain

$$\begin{aligned}
\partial_t z_j^- + z_k^+ \frac{\partial z_j^-}{\partial x_k} &= -\frac{\partial}{\partial x_j}(D_t^+ n^-) + \frac{\partial z_k^+}{\partial x_j} \frac{\partial n^-}{\partial x_k} + D_t^+ f_j^- \\
&\quad + v_m^- \left( \nu^+ \Delta \left( \frac{\partial A_m^-}{\partial x_j} \right) - \frac{\partial z_k^+}{\partial x_j} \frac{\partial A_m^-}{\partial x_k} \right) + \nu^+ \frac{\partial A_m^-}{\partial x_j} \Delta v_m^- \\
&\quad + 2\nu^+ \frac{\partial A_m^-}{\partial x_j} \left( Q_{km}^- \frac{\partial^2 A_l^-}{\partial x_k \partial x_i} \frac{\partial v_l^-}{\partial x_i} \right),
\end{aligned}$$

where $Q^-$ is the inverse of $\nabla A$. It then follows from regrouping the terms that

$$\partial_t z_j^- + z_k^+ \frac{\partial z_j^-}{\partial x_k} = -\frac{\partial}{\partial x_j}(D_t^+ n) + \frac{\partial z_k^+}{\partial x_j}\left(\frac{\partial n}{\partial x_k} - \frac{\partial A_m^-}{\partial x_k} v_m^-\right) + D_t^+ f^- + \nu^+ \left(\frac{\partial A_m^-}{\partial x_j}\Delta v_m^- + v_m^- \Delta\left(\frac{\partial A_m^-}{\partial x_j}\right)\right) + 2\nu^+ \delta_{jk} \frac{\partial^2 A_l^-}{\partial x_k \partial x_i}\frac{\partial v_l^-}{\partial x_i}.$$

Combining the terms and using (5.3) for $z^-$, we have

$$\begin{aligned}\partial_t z_j^- + z_k^+ \frac{\partial z_j^-}{\partial x_k} &= -\frac{\partial}{\partial x_j}(D_t^+ n - \nu^+ \Delta n) \\ &\quad + \left(D_t^+ f^- - (z_k^- + f_k^-)\frac{\partial z_k^+}{\partial x_j} - \nu^+ \Delta f^-\right) \\ &\quad + \nu^+ \Delta z^- - 2\nu^+ \frac{\partial^2 A_l^-}{\partial x_k \partial x_i}\frac{\partial v_l^-}{\partial x_i} + 2\nu^+ \frac{\partial^2 A_l^-}{\partial x_k \partial x_i}\frac{\partial v_l^-}{\partial x_i}.\end{aligned}$$

Then by (5.11) and (5.9),

$$\partial_t z^- + z^+ \cdot \nabla z^- = -\nabla P + \nu^+ \Delta z^-.$$

The same line of argument may be applied to establish

$$\partial_t z^+ + z^- \cdot \nabla z^+ = -\nabla P + \nu^+ \Delta z^+. \qquad \square$$

## 6. Conjugate Vector Fields

We return to the original MHD equations (1.3) and (1.2) either supplemented with periodic boundary conditions or defined in the whole space with sufficient decay at infinity. In this section we introduce a new pair of vector fields, $v$ and $A$, and then show that the 3D viscous MHD equations are equivalent to a new system of equations represented in terms of $v$ and $A$.

For a divergence-free velocity field $u$, we express it as the difference

$$u = v - \nabla\Phi, \tag{6.1}$$

where $\Phi$ is a scalar and $v$ is a vector satisfying

$$\partial_t v + u \cdot \nabla v - \nu_1 \Delta v + (\nabla u)^* v = b \cdot \nabla b. \tag{6.2}$$

Such a decomposition of $u$ is validated by the fact that any vector can be written as a sum of a curl and a gradient. One can see from (6.1) that $u$ and $v$ have the same curl and $u$ can also be recovered from $v$ by projecting (6.1) onto the divergence-free functions.

We now introduce the vector field $A$, which obeys

$$\partial_t A + u \cdot \nabla A - \nu_2 \Delta A + (\nabla u)^* A = 0. \tag{6.3}$$

It will be clear that $\nabla \times A$ recovers the dynamics of the magnetic field $b$.

**Theorem 6.1.** *Assume that u is given by (6.1) and v satisfies (6.2). Then u and* $b = \nabla \times A$ *satisfies the viscous MHD equations*

$$\partial_t u + u \cdot \nabla u - \nu_1 \Delta u \;=\; -\nabla\left(p + \frac{b^2}{2}\right) + b \cdot \nabla b, \tag{6.4}$$

$$\partial_t b + u \cdot \nabla b - \nu_2 \Delta b \;=\; b \cdot \nabla u, \tag{6.5}$$

*where* $p$ *is determined by*

$$p = \partial_t \Phi + u \cdot \nabla \Phi - \nu_1 \Delta \Phi - \frac{b^2 + u^2}{2} + C \tag{6.6}$$

*for a free constant* $C$.

*Proof.* First we verify that $u$ given by the expression in (6.1) satisfies (6.4). In fact,

$$\begin{aligned} &\partial_t u + u \cdot \nabla u - \nu_1 \Delta u + \nabla\left(p + \frac{b^2}{2}\right) - b \cdot \nabla b \\ &\quad= \partial_t v + u \cdot \nabla v - u \cdot \nabla(\nabla \Phi) - \nu_1 \Delta v - b \cdot \nabla b \\ &\qquad + \nabla\left(p + \frac{b^2}{2} - \partial_t \Phi + \nu_1 \Delta \Phi\right). \end{aligned}$$

Since $u \cdot \nabla(\nabla\Phi) = \nabla(u \cdot \nabla\Phi) - (\nabla u)^* \nabla\Phi = \nabla(u \cdot \nabla\Phi) + (\nabla u)^* u - (\nabla u)^* v$,

$$\begin{aligned} &\partial_t u + u \cdot \nabla u - \nu_1 \Delta u + \nabla\left(p + \frac{b^2}{2}\right) - b \cdot \nabla b \\ &\quad= (\partial_t v + u \cdot \nabla v - \nu_1 \Delta v + (\nabla u)^* v - b \cdot \nabla b) \\ &\qquad + \nabla\left(p + \frac{b^2}{2} - \partial_t \Phi - u \cdot \nabla\Phi + \nu_1 \Delta \Phi + \frac{u^2}{2}\right). \end{aligned}$$

Using (6.2) and (6.6), we conclude that the right-hand side is zero.

Now we prove that $b = \nabla \times A$ satisfies (6.5). For this purpose, we take the curl of each term of equation (6.3)

$$\partial_t b + u \cdot \nabla b - \nu_2 \Delta b = -\nabla \times (u \cdot \nabla A) + u \cdot \nabla b - \nabla \times ((\nabla u)^* A).$$

It then suffices to check that the right-hand side of the equation above is the same as that of (6.5), i.e.,

$$-\nabla \times (u \cdot \nabla A) + u \cdot \nabla(\nabla \times A) - \nabla \times ((\nabla u)^* A) = (\nabla \times A) \cdot \nabla u.$$

Using $\nabla \cdot u = 0$ and the basic vector formulae $\nabla \times (\nabla f) = 0$ and $\nabla \times (fG) = f\nabla \times G - G \times \nabla f$ for a scalar $f$ and a vector $G$, we have

$$\begin{aligned} &-\nabla \times (u \cdot \nabla A) + u \cdot \nabla(\nabla \times A) - \nabla \times ((\nabla u)^* A) \\ &\quad= -\partial_j(\nabla \times (u_j A)) + \partial_j(u_j \nabla \times A) - \nabla \times (\nabla u_j A_j) \\ &\quad= \partial_j(A \times \nabla u_j) - (A_j \nabla \times (\nabla u_j)) + \nabla u_j \times \nabla A_j \\ &\quad= -\nabla u_j \times \partial_j A + \nabla u_j \times \nabla A_j = \nabla u_j \times (\nabla A_j - \partial_j A), \end{aligned}$$

where $j$ is a dummy index implying a sum from $j = 1$ to 3. Finally we check that

$$\nabla u_j \times (\nabla A_j - \partial_j A) = (\nabla \times A) \cdot \nabla u, \tag{6.7}$$

and we compare the $l$-th component of both sides. The $l$-th component of the term on the left of (6.7) is given by

$$(\nabla u_j \times (\nabla A_j - \partial_j A))_l = \epsilon_{ikl} \partial_i u_j (\partial_k A_j - \partial_j A_k), \tag{6.8}$$

where $\epsilon_{ikl}$ is the standard permutation symbol. Since $i, k$, and $l$ have to be different for $\epsilon_{ikl}$ not equal to zero, $\epsilon_{ikl}\, \partial_i u_j (\partial_k A_j - \partial_j A_k)$ can be written as a sum of two terms,

$$\epsilon_{ikl} \partial_i u_j (\partial_k A_j - \partial_j A_k) + \epsilon_{ikl} \partial_k u_j (\partial_i A_j - \partial_j A_i),$$

where we sum over $j$, but not over $i$ or $k$. The sum over $j$ can be written more explicitly as

$$\begin{aligned} &\epsilon_{ikl}\{\partial_i u_i (\partial_k A_i - \partial_i A_k) + \partial_i u_k (\partial_k A_k - \partial_k A_k) + \partial_i u_l (\partial_k A_l - \partial_l A_k)\} \\ &\quad + \epsilon_{kil}\{\partial_k u_i (\partial_i A_i - \partial_i A_i) + \partial_k u_k (\partial_i A_k - \partial_k A_i) + \partial_k u_l (\partial_i A_l - \partial_l A_i)\}. \end{aligned}$$

Now regroup the terms and use $\partial_i u_i + \partial_k u_k = -\partial_l u_l$ (i.e., $\nabla \cdot u = 0$) to get

$$\begin{aligned} &\epsilon_{ikl} (\partial_k A_l - \partial_l A_k)\, \partial_i u_l + \epsilon_{kil}\, (\partial_i A_l - \partial_l A_i)\, \partial_k u_l + \epsilon_{ikl}\, (\partial_i A_k - \partial_k A_i) \partial_l u_l \\ &= (\nabla \times A)_i \cdot \partial_i u_l + (\nabla \times A)_k \cdot \partial_k u_l + (\nabla \times A)_l \cdot \partial_l u_l \\ &= (\nabla \times A) \cdot \nabla u_l, \end{aligned}$$

which is exactly the $l$-th component of the term on the right of (6.7). Therefore (6.7) holds and $b = \nabla \times A$ satisfies (6.5). □

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