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Well-Posedness of a Semilinear Heat Equation with Weak Initial Data

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ABSTRACT. This article mainly consists of two parts. In the first part the initial value problem (IVP) of the semilinear heat equation

$$\begin{aligned} \partial_t u - \Delta u &= |u|^{k-1}u, \quad on \quad \mathbb{R}^n \times (0,\infty), \quad k \ge 2 \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}^n \end{aligned}$$

with initial data in $\dot{L}_{r,p}$ is studied. We prove the well-posedness when

$$1 < p < \infty, \quad \frac{2}{k(k-1)} < \frac{n}{p} \le \frac{2}{k-1}, \quad and \quad r = \frac{n}{p} - \frac{2}{k-1} \quad (\le 0)$$

and construct non-unique solutions for

$$1 < p < \frac{n(k-1)}{2} < k+1, \quad and \quad r < \frac{n}{p} - \frac{2}{k-1}\,.$$

In the second part the well-posedness of the above IVP for $k = 2$ with $u_0 \in H^s(\mathbb{R}^n)$ is proved if

$$-1 < s, \quad for\, n = 1, \qquad \frac{n}{2} - 2 < s, \quad for\, n \ge 2\,.$$

and this result is then extended for more general nonlinear terms and initial data. By taking special values of r, p, s, and u_0, these well-posedness results reduce to some of those previously obtained by other authors [4, 14].

1. Introduction

This article can be divided into two main parts. In the first part we consider the IVP for the semilinear heat equation:

$$\partial_t u - \Delta u = |u|^{k-1}u, \quad \text{on} \quad \mathbb{R}^n \times (0,\infty)\,, \tag{1.1}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n\,, \tag{1.2}$$

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where $k \geq 2$ is a fixed parameter. We are mainly interested in the well-posedness result for initial data u_0 in the homogeneous Lebesgue spaces, $u_0 \in \dot{L}_{r,p}(\mathbb{R}^n)$ (defined below). By well-posedness we mean existence, uniqueness, and persistence (i.e., the solution describes a continuous curve belonging to the same space as does the initial data) and continuous dependence on the data.

Here the homogeneous Lebesgue space $\dot{L}_{s,q}(\mathbb{R}^n)$ consists of all v such that

$$(-\Delta)^{s/2} v \in L^q, \quad s \in \mathbb{R}, \quad 1 \leq q < \infty$$

and the standard norm is given by

$$\|v\|_{s,q} = \left\|(-\Delta)^{s/2} v\right\|_{L^q}$$

These spaces are also called the spaces of Riesz potentials. Kato and Ponce [11] consider the Navier-Stokes equations with initial data in this type of space.

The main results of this part include two theorems. The first theorem shows that the IVP of (1.1) and (1.2) is locally well-posed in $\dot{L}_{r,p}$ if r and p satisfy

$$1 < p < \infty, \quad \frac{2}{k(k-1)} < \frac{n}{p} \leq \frac{2}{k-1}, \quad \text{and} \quad r = \frac{n}{p} - \frac{2}{k-1} \quad (\leq 0)$$

while in the second theorem we construct non-unique solutions to the IVP (1.1) and (1.2) with initial data zero for

$$1 < p < \frac{n(k-1)}{2} < k+1, \quad \text{and} \quad r < \frac{n}{p} - \frac{2}{k-1}.$$

The non-uniqueness indicates in some sense the sharpness of our well-posedness results. The precise statements of the main results are given in Theorem 1 of Section 2 and Theorem 2 of Section 3.

Dimensional analysis might be employed to explain "why" the index $r = \frac{n}{p} - \frac{2}{k-1}$. We only need to notice that if $u(x,t)$ is a solution, then $u_\lambda = \lambda^{\frac{2}{k-1}} u(\lambda x, \lambda^2 t)$ (for any $\lambda > 0$) and

$$\|u_\lambda(\cdot,t)\|_{\dot{L}_{r,p}} = \lambda^{r-\left(\frac{n}{p}-\frac{2}{k-1}\right)} \left\|u(\cdot, \lambda^2 t)\right\|_{\dot{L}_{r,p}}$$

But, unfortunately, we do not know if dimensional analysis always works and how to give a generally applicable criterion to detect the indices.

The problem of well-posedness for semilinear heat equation has attracted the attention of many mathematicians, but not many results related to very weak initial data have been published. The results concerning the IVP (1.1) and (1.2) in L^p setting obtained by Weissler and others are as follows: for $p \geq \frac{n(k-1)}{2}$, there is well-posedness [14, 15]; for $1 \leq p < \frac{n(k-1)}{2}$, non-unique solutions can be constructed [7]. By letting $r = 0$, i.e., $p = \frac{n(k-1)}{2}$, our results reduce to those in the L^p theory. In [1] Baras and Pierre prove that the IVP (1.1) and (1.2) has a solution if and only if the initial measure is not too much concentrated. Clearly Sobolev spaces of negative indices contain distributions. In [13] Kozono and Yamazaki consider the IVP (1.1) and (1.2) with initial data in real interpolation space $\mathcal{N}^r_{p,q,\infty}$ ($r = \frac{n}{p} - \frac{2}{k-1}$), which is slightly larger than $\dot{L}_{r,p}$ (by noting that $\mathcal{N}^r_{p,p,\infty} = \dot{B}^r_{p,\infty} \supseteq \dot{L}_{r,p}$, where $\dot{B}^r_{p,\infty}$ is a homogeneous Besov space, see, e.g., [2, p. 147]. However, they obtain existence under the assumption that $k \leq q \leq p$ and $n(k-1) < 2p < nk(k-1)$, which is slightly more strict than our assumption. Furthermore, we provide an example of non-uniqueness when $r < \frac{n}{p} - \frac{2}{k-2}$.

We prove the well-posedness result by contraction mapping arguments while non-unique solutions are constructed by seeking solutions of self-similar form and using a result in L^p setting obtained by Haraux and Weissler [7].

In the second part we seek solutions to the IVP (1.1) and (1.2) for $k = 2$, i.e.,

$$\partial_t u - \Delta u = u^2, \quad \text{on} \quad \mathbb{R}^n \times (0,\infty) \tag{1.3}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{1.4}$$

with initial data in H^s. Here H^s is the standard inhomogeneous Sobolev space consisting of all v such that

$$\|v\|_{H^s} \equiv \left\| \left(1+|\xi|^2\right)^{s/2} \hat{v}(\xi) \right\|_{L^2} < \infty .$$

We find that a family of specially weighted Banach spaces $BC_s((0,T],H^r)$ are quite appropriate for our situation. Introduced by Dix [6] in solving the IVP for the nonlinear Burgers' equations, $BC_s((0,T],H^r)(s \le r, s, r \in \mathbb{R}, T > 0)$ denotes the class of all functions $u \in C([0,T],H^s) \cap C((0,T],H^r)$ that also satisfy the condition

$$\|u\|_{BC_s((0,T],H^r)} \equiv \sup_{t\in[0,T]} \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \left(1+|\xi|^2 t\right)^{\frac{r-s}{2}} \hat{u}(t)(\xi) \right\|_{L^2} < \infty .$$

Here $\hat{f}$ denotes the Fourier transform of f.

We prove that if $u_0 \in H^s$ with s satisfying

$$-1 < s, \quad \text{for } n = 1, \qquad \frac{n}{2} - 2 < s, \quad \text{for } n \ge 2 , \tag{1.5}$$

then the IVP (1.3) and (1.4) is locally well posed and for some $T > 0$ the solution $u \in BC_s((0,T],H^r)$ for any $r \ge s$. See Theorem 3 of Section 4 for a precise statement. Actually the arguments employed in proving this result can be extended to establish well-posedness of the IVP (1.1) and (1.2) with nonlinear term u^k in homogeneous Lebesgue space $L_{s,p}$. This more general result states that if $k \ge 2$ is an integer and $u_0 \in L_{s,p}$ with

$$p = \frac{k}{k-1}, \qquad s > -\frac{2}{k}, \qquad s > \frac{n}{p} - \frac{2}{k-1}$$

Then we have local well-posedness. Further details are given in Theorem 4.

As we explained before, the index $s = \frac{n}{2} - 2$ for $n \ge 2$ is exactly the number from dimensional analysis. But for $n = 1$, our method only allows us to prove well-posedness for $s > -1$ and fails to extend to $s > -3/2$. It would be desirable to show that $s = -1$ is actually sharp by providing a counter example.

By taking special values of s and u_0, the well-posedness in this part reduces to some of those previously obtained by other authors [4, 14]. Letting $s = 0$, our result (for $n \le 4$) reduces to the L^p theory of Weissler and others [14, 15]. In [4] Brezis and Friedman prove for $u_0 = \delta(x)$ that the solution exists for $0 < k < 1 + \frac{2}{n}$ and does not exist for $k \ge 1 + \frac{2}{n}$. For $n = 1$, $\delta(x) \in H^s(\mathbb{R})$ for any $s < -\frac{1}{2}$, our result reduces, by taking $u_0 = \delta$, to Brezis and Friedman's result in 1-D. When $n \ge 2$, $2 \ge 1 + \frac{2}{n}$, the IVP (1.3) and (1.4) with $u_0 = \delta(x)$ has no solution as indicated by Brezis and Friedman, but our results implies that the IVP is well posed for $u_0 \in H^s$ with $s > \frac{n}{2} - 2$, which is slightly more regular than $\delta(x)$ since $\delta(x) \notin H^s(\mathbb{R}^n)$ for $s > \frac{n}{2} - 2$. The fact that for $n = 2$, $\delta(x) \notin H^{-1}(\mathbb{R}^2)$, but $\delta(x) \in H^{-1-\epsilon}(\mathbb{R}^2)$ for any $\epsilon > 0$, combined with the non-existence result implies that our well-posedness result in 2-D is actually sharp.

The well-posedness result in this part is again proved by the contraction mapping arguments and we only deal with the IVP (1.3) and (1.4) for $s \le 0$. The proof for $s > 0$ can be given in a similar (and actually easier) way.

2. Well-Posedness in $\dot{L}_{r,p}$

First we define the spaces of weighted continuous functions in time, which have been introduced by Kato, Ponce and others in solving the Navier-Stokes equations [9, 11, 12].

Definition 1.

Suppose $T > 0$ and $\alpha \geq 0$ are real numbers. The spaces $C_{\alpha,s,q}$ and $\dot{C}_{\alpha,s,q}$ are defined as

$$C_{\alpha,s,q} \equiv \left\{ f \in C((0,T), \dot{L}_{s,q}), \quad \|f\|_{\alpha,s,q} < \infty \right\}$$

where the norm is given by

$$\|f\|_{\alpha,s,q} = \sup\left\{ t^{\alpha}\|f\|_{s,q}, \quad t \in (0,T) \right\}$$

$\dot{C}_{\alpha,s,q}$ is a subspace of $C_{\alpha,s,q}$:

$$\dot{C}_{\alpha,s,q} \equiv \left\{ f \in C_{\alpha,s,q}, \quad \lim_{t\to 0} t^{\alpha}\|f\|_{s,q} = 0 \right\}$$

When $\alpha = 0$, $\bar{C}_{s,q}$ are used for $BC([0,T), \dot{L}_{s,q})$. □

These spaces are important in uniqueness and local existence problems [9, 11, 12]. $f \in C_{\alpha,s,q}$ (resp. $f \in \dot{C}_{\alpha,s,q}$) implies that $\|f(t)\|_{s,q} = O(t^{-\alpha})$ (resp. $o(t^{-\alpha})$).

The main result of this section is the well-posedness theorem which states the following.

Theorem 1.

Assume $u_0 \in \dot{L}_{r,p}$ with p and r satisfying

$$1 < p < \infty, \quad \frac{2}{k(k-1)} < \frac{n}{p} \leq \frac{2}{k-1} \tag{2.1}$$

$$r = \frac{n}{p} - \frac{2}{k-1} \quad (\leq 0) \tag{2.2}$$

Then for some $T = T(u_0) > 0$, there exists a unique solution $u = \mathcal{U}(u_0)$ to the IVP (1.1) and (1.2) such that

$$u \in Y_T \equiv \left(\cap_{p\leq q<\infty} \bar{C}_{\frac{n}{q}-\frac{2}{k-1},q} \right) \cap \left(\cap_{p\leq q<\infty} \cap_{s>\frac{n}{q}-\frac{2}{k-1}} \dot{C}_{\left(s-\frac{n}{q}+\frac{2}{k-1}\right)/2,s,q} \right) \tag{2.3}$$

In particular, (2.3) implies that

$$u \in BC\left([0,T), \dot{L}_{r,p}\right) \cap \left(\cap_{s>r} C((0,T), \dot{L}_{s,p}) \right)$$

Furthermore, the mapping

$$\mathcal{U} : \Lambda \longmapsto Y_T : \quad u_0 \longmapsto u$$

is Lipschitz in some neighborhood Λ of u_0.

We make several remarks concerning this theorem.

Remark 1.

It is easy to see from our proof of this theorem that if $\|u_0\|_{\dot{L}_{r,p}}$ is sufficiently small, we may take $T = \infty$. □

Remark 2.

The homogeneous spaces $\dot{L}_{s,q}$ can be replaced by inhomogeneous spaces $L_{s,q}$ (i.e., spaces of Bessel potentials):

$$L_{s,q} \equiv \left\{ v : \|v\|_{s,q} \equiv \left\| (1-\Delta)^{s/2} u \right\|_{L^q} < \infty \right\}$$

to obtain quite similar well-posedness results. □

We prove Theorem 1 by the method of integral equation and contraction-mapping arguments. This method has been extensively used by Kato, Ponce and others to prove the well-posedness of the

Navier-Stokes equations in various type of functional spaces [8, 9, 10, 11, 12]. First we write (1.1) in the integral form

$$u = Uu_0(t) + G\left(|u|^{k-1}u\right)(t) \equiv e^{-\Delta t}u_0 + \int_0^t e^{-\Delta(t-\tau)}\left(|u|^{k-1}u\right)(\tau)d\tau \, .$$

Then we estimate the operators U and G separately. The main estimates are established in the propositions that follow.

Proposition 1.

(1) *If $s \in \mathbb{R}$ and $q \in [1, \infty)$, then*

$$Uu_0(t) \to u_0, \quad \text{in } \dot{L}_{s,q} \quad \text{as } t \to 0 \, .$$

(2) *If $s_1 \le s_2$, $q_1 \le q_2$, and*

$$\alpha_2 = \left(s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right)/2 \, ,$$

then U maps continuously from $\dot{L}_{s_1,q_1}$ into $\dot{C}_{\alpha_2,s_2,q_2}$ (When $\alpha_2 = 0$, $\dot{C}_{\alpha_2,s_2,q_2}$ should be replaced by $\bar{C}_{s_2,q_2}$).

Proof. The proof of (1) involves the definitions of the norms and the dominated convergence theorem. See [10] and [11] for the proof of (2). □

Now we give the estimates for the operator G:

$$Gg(t) = \int_0^t e^{-\Delta(t-\tau)}g(\tau)d\tau$$

Proposition 2.

If $q_1, q_2, \alpha_1, \alpha_2, s_1$, and s_2 satisfy

$$\begin{aligned} q_1 &\le q_2 \\ \alpha_1 < 1, \quad \alpha_2 &= \alpha_1 - 1 + \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right] \\ 0 &\le s_2 - s_1 < 2 - n\left(\frac{1}{q_1} - \frac{1}{q_2}\right) , \end{aligned}$$

then G maps continuously from $\dot{C}_{\alpha_1,s_1,q_1}$ to $\dot{C}_{\alpha_2,s_2,q_2}$.

Proof. The proof of this proposition is quite similar to that of Lemma 2.3 in [11]. We just want to point out that the restrictions

$$\alpha_1 < 1, \quad s_2 - s_1 < 2 - n\left(\frac{1}{q_1} - \frac{1}{q_2}\right)$$

are imposed to guarantee the finiteness of a Beta function involved in the estimates of G. □

Now we turn to the proof of Theorem 1.

Proof of Theorem 1. We consider two cases: $r < 0$ and $r = 0$. For $r < 0$, we define

$$X = \bar{C}_{r,p} \cap \dot{C}_{-\frac{r}{2},0,p} \, ,$$

with norm for $u \in X$ given by

$$\|u\|_X = \|u - Uu_0\|_{0,r,p} + \|u\|_{-\frac{r}{2},0,p}$$

and the complete metric space X_R to be the closed ball in X of radius R. Consider the operator $\mathcal{A}(u, u_0) : X_R \times \Lambda \longmapsto X$

$$\mathcal{A}(u, u_0)(t) = Uu_0(t) + G\left(|u|^{k-1}u\right)(t), \quad 0 \le t < T$$

where Λ is some neighborhood of u_0 in $\dot{L}_{r,p}$. □

Using Proposition 1 with

$$s_1 = r, \quad s_2 = 0, \quad q_1 = q_2 = p, \quad \alpha_2 = -\frac{r}{2},$$

we find that $Uu_0 \in X_R$ if $T > 0$ is small enough and Λ chosen properly. To estimate G, we use Proposition 2 with

$$q_1 = \frac{p}{k}, \quad q_2 = p, \quad \alpha_1 = -\frac{kr}{2}, \quad \alpha_2 = \frac{l}{k}, \quad s_1 = 0, \quad s_2 = \frac{2l}{k} + r,$$

we obtain

$$\left\|G(|u|^{k-1}u)\right\|_{\frac{l}{k}, \frac{2l}{k}+r, p} \le c\left\||u|^{k-1}u\right\|_{-\frac{kr}{2}, 0, \frac{p}{k}} \le \|u\|^k_{-\frac{r}{2}, 0, p} \le cR^k$$

for all $l \in [0, -\frac{k^2}{2}r)$. Here it is important to notice that the restrictions on p and r (2.1) and (2.2) are necessary in order to apply Proposition 2.

Furthermore,

$$\begin{aligned} \|\mathcal{A}(u, u_0) - \mathcal{A}(\tilde{u}, u_0)\|_X &\le \left\|G\left(|u|^{k-1}u\right) - G\left(|\tilde{u}|^{k-1}\tilde{u}\right)\right\|_X \\ &\le \left\|G\left(|u-\tilde{u}|^{k-1}u\right)\right\|_X + \left\|G\left(|u-\tilde{u}||\tilde{u}|^{k-1}\right)\right\|_X . \end{aligned}$$

Using Proposition 2 again

$$\begin{aligned} \|\mathcal{A}(u, u_0) - \mathcal{A}(\tilde{u}, u_0)\|_X &\le 2\left\||u-\tilde{u}|^{k-1}u\right\|_{-\frac{kr}{2}, 0, \frac{p}{k}} + 2\left\||u-\tilde{u}||\tilde{u}|^{k-1}\right\|_{-\frac{kr}{2}, 0, \frac{p}{k}} \\ &\le c\|u\|_X\|u-\tilde{u}\|_X^{k-1} + c\|u-\tilde{u}\|_X\|\tilde{u}\|_X^{k-1} . \end{aligned}$$

So, if we choose T to be small and R properly, then $\mathcal{A}(u, u_0)$ maps X_R into itself and is a contraction map when $k \ge 2$. Consequently there exists a unique fixed point $u \in X_R$: $u = \mathcal{U}(u_0)$ satisfying $u = \mathcal{A}(u, u_0)$. It is easy to see from the above estimates that the uniqueness can be extended to $X_{R'}$ for all R' by reducing the time interval and thus to the whole X.

To show that u is in the class of Y_T (defined in Theorem 1), we notice that

$$u(t) = \mathcal{A}(u, u_0)(t) \equiv Uu_0(t) + G\left(|u|^{k-1}u\right)(t), \quad t \in [0, T) .$$

We apply Proposition 1 twice to Uu_0 to show that

$$Uu_0 \in \bar{C}_{\frac{n}{q}-\frac{2}{k-1}, q}, \qquad Uu_0 \in \dot{C}_{\left(s-\frac{n}{q}+\frac{2}{k-1}\right)/2, s, q}$$

for any $p \le q < \infty$ and $s > \frac{n}{q} - \frac{2}{k-1}$. To show the second part

$$G\left(|u|^{k-1}u\right) \in \bar{C}_{\frac{n}{q}-\frac{2}{k-1}, q}, \quad p \le q < \infty$$

we use Proposition 2 with

$$q_1 = \frac{p}{k}, \quad q_2 = q, \quad \alpha_1 = -\frac{kr}{2}, \quad \alpha_2 = 0, \quad s_1 = 0, \quad s_2 = \frac{n}{q} - \frac{2}{k-1}$$

and obtain
$$\left\|G\left(|u|^{k-1}u\right)\right\|_{0,\frac{n}{q}-\frac{2}{k-1},q} \le c\left\||u|^{k-1}u\right\|_{-\frac{kr}{2},0,\frac{p}{k}} \le c\|u\|^k_{-\frac{r}{2},0,p} .$$

We apply Proposition 2 once again with
$$\begin{aligned} q_1 &= \frac{p}{k}, \quad q_2 = q, \quad s_1 = 0, \quad s_2 = s, \\ \alpha_1 &= -\frac{kr}{2}, \quad \alpha_2 = \left(s - \frac{n}{q} + \frac{2}{k-1}\right)/2 \end{aligned}$$
to show that
$$G\left(|u|^{k-1}u\right) \in \dot{C}_{\left(s-\frac{n}{q}+\frac{2}{k-2}\right)/2,s,q}, \quad s > \frac{n}{q} - \frac{2}{k-1} \tag{2.4}$$
but s should also satisfy
$$s < 2 - \left(\frac{kn}{p} - \frac{n}{q}\right)$$
which is required by Proposition 2. For large s, (2.4) can be shown by an induction process (see an analogous argument in [9, p. 60]).

To prove the Lipschitz continuity of $\mathcal{U}$, let $u = \mathcal{U}(u_0)$ and $v = \mathcal{U}(v_0)$ for $u_0, v_0 \in \Lambda$. Then
$$\begin{aligned} \|u - v\|_X &= \|\mathcal{A}(u, u_0) - \mathcal{A}(v, v_0)\|_X \\ &\le \|\mathcal{A}(u, u_0) - \mathcal{A}(v, u_0)\|_X + \|\mathcal{A}(v, u_0) - \mathcal{A}(v, v_0)\|_X \\ &\le \gamma\|u - v\|_X + \|U(u_0 - v_0)\|_X \end{aligned}$$

For small T and properly chosen Λ, $\gamma < 1$ since the mapping is a contraction and we obtain asserted result by using Proposition 1 to the second term.

In the case $r = 0$, we define
$$X = \bar{C}_{0,p} \cap \dot{C}_{\frac{1}{4(k-1)},0,\frac{4p}{3}}$$
with the norm
$$\|u\|_X = \|u - Uu_0\|_{0,0,p} + \|u\|_{\frac{1}{4(k-1)},0,\frac{4p}{3}}$$
and X_R is again the closed ball in X of radius R. By Proposition 2,
$$\begin{aligned} \left\|G\left(|u|^{k-1}u\right)\right\|_X &= \left\|G\left(|u|^{k-1}u\right)\right\|_{0,0,p} + \left\|G\left(|u|^{k-1}u\right)\right\|_{\frac{1}{4(k-1)},0,\frac{4p}{3}} \\ &\le c\left\|u^k\right\|_{\frac{k}{4(k-1)},0,\frac{4p}{3k}} \le c\|u\|^k_{\frac{1}{4(k-1)},0,\frac{4p}{3}} \le cR^k \end{aligned}$$
and the rest of the proof reduces to the previous case. This completes the proof of Theorem 1.

3. Non-Uniqueness for $r < \frac{n}{p} - \frac{2}{k-1}$

In this section we consider the situation when
$$1 < p < \frac{n(k-1)}{2} < k+1$$
and prove that if
$$r < \frac{n}{p} - \frac{2}{k-1} ,$$

then the solution of the IVP:

$$\partial_t u - \Delta u = |u|^{k-1}u, \qquad x \in \mathbb{R}^n, \quad t \ge 0 \tag{3.1}$$

$$u \to 0, \qquad \text{in} \quad \dot{L}_{r,p} \quad \text{as } t \to 0 \tag{3.2}$$

is not unique. This is accomplished by constructing a non-trivial solution of the above IVP. The precise statement of our result is the following.

Theorem 2.

Assume that

$$1 < p < \frac{n(k-1)}{2} < k+1 \quad \textit{and} \quad r < \frac{n}{p} - \frac{2}{k-1}. \tag{3.3}$$

Then for some $T > 0$, there exists at least one non-trivial solution Φ to the IVP (3.1) and (3.2) such that

$$\Phi \in C\left([0,T), \dot{L}_{r,p}\right) \cap C\left((0,T), \dot{C}_{-r/2,0,p}\right)$$

Thus we get at least three different solutions Φ, $-\Phi$ and 0, corresponding to the same initial data 0.

We seek solutions to (3.1) of the self-similar form

$$\Phi(x,t) = t^{-\frac{1}{k-1}}\omega\left(\frac{x}{\sqrt{t}}\right)$$

Then (3.1), which Φ should satisfy, reduces to an O.D.E. of ω,

$$\Delta\omega(x) + \frac{x}{2}\cdot\nabla\omega(x) + \frac{\omega(x)}{k-1} + |\omega(x)|^{k-1}\omega(x) = 0, \quad x \in \mathbb{R}^n$$

By assuming ω is radial, i.e., $\omega(x) = v(|x|)$ with $v : [0,\infty) \longmapsto \mathbb{R}$, the equation is further reduced to

$$v''(x) + \left(\frac{n-1}{x} + \frac{x}{2}\right)v'(x) + \frac{v(x)}{k-1} + |v(x)|^{k-1}v(x) = 0, \quad x > 0 \tag{3.4}$$

Haraux and Weissler [7] consider the solutions of (3.4) and we need to use the following result of theirs.

Proposition 3.

Let $k > 1$ and $n \ge 1$. If

$$1 < \frac{n(k-1)}{2} < k+1$$

then for some v_0, there is a unique solution $v \in C^2([0,\infty))$ to (3.4) with $v(0) = v_0$ and $v'(0) = 0$ such that

$$\lim_{x\to\infty} x^m v(x) = 0, \quad \textit{for all } m > 0.$$

To prove the theorem, we only need to prove the following assertions about the solution $\Phi = t^{-\frac{1}{k-1}}v(\frac{|x|}{\sqrt{t}})$ constructed above.

Proposition 4.

Assume that the indices k, n, p, and r satisfy (3.3). Then

(1)

$$\Phi(t) \to 0, \qquad \textit{in} \quad S'\left(\mathbb{R}^n\right) \quad \textit{as} \quad t \to 0$$

where $S'(\mathbb{R}^n)$ is the space of tempered distributions.

(2)
$$\Phi(t) \to 0, \qquad in \quad \dot{L}_{r,p} \quad as \quad t \to 0$$

Proof. To prove assertion (1), we calculate for any $\phi \in S$,

$$\begin{aligned}\lim_{t\to 0}\int \Phi(x,t)\phi(x)dx &= \lim_{t\to 0}\left(t^{-\frac{1}{k-1}}\int v\left(\frac{|x|}{\sqrt{t}}\right)\phi(x)dx\right)\\ &\le \lim_{t\to 0}\left(t^{-\frac{1}{k-1}+\frac{n}{2p}}\|v\|_{L^p}\|\phi\|_{L^q}\right), \quad \text{with} \quad \frac{1}{p}+\frac{1}{q}=1\end{aligned}$$

Since $\|v\|_{L^p}$ is finite as implied by Proposition 3, we conclude that the above limit is zero.

To prove (2), we need the following lemma.

Lemma 1.

Let $q \in (1,\infty)$, $0 < T < \infty$, and $v_0 \in S'(\mathbb{R}^n)$. If $v \in C((0,T), L^q)$ solves the IVP of the linear heat equation

$$\begin{gathered}\partial_t v - \Delta v = 0, \quad x \in \mathbb{R}^n, \quad t \ge 0,\\ v(x,t) \to v_0, \quad in\ S'\left(\mathbb{R}^n\right) \quad as\ t \to 0,\end{gathered}$$

then v is uniquely given by $v = Uv_0$. Especially $v_0 = 0$ implies that $v = 0$.

Now we prove assertion (2). It is easy to check that $\Phi - \mathcal{A}(\Phi, 0)$ solves the linear heat equation with initial data zero and hence, by Lemma 1, must be zero. That is

$$\Phi(t) = G\left(|\Phi|^{k-1}\Phi\right)(t)$$

Using Proposition 2,

$$\begin{aligned}\|\Phi(t)\|_{\dot{L}_{r,p}} &= \left\|G\left(|\Phi|^{k-1}\Phi\right)(t)\right\|_{\dot{L}_{r,p}}\\ &\le c\left\||\Phi^{k-1}\Phi\right\|_{-\frac{kr}{2},0,\frac{p}{k}} \le \|\Phi\|^k_{-\frac{r}{2},0,p}\end{aligned}$$

Therefore,

$$\lim_{t\to 0}\left(\|\Phi(t)\|_{\dot{L}_{r,p}}\right) \le \lim_{t\to 0}\left(t^{-\frac{rk}{2}}\|\Phi\|^k_{L^p}\right) = \lim_{t\to 0}\left(t^{\frac{k}{2}\left(\frac{n}{p}-\frac{2}{k-1}-r\right)}\|v\|^k_{L^p}\right)$$

The above limit is zero for $r < \frac{n}{p} - \frac{2}{k-1}$. This finishes the proof of the theorem. □

4. Well-Posedness in H^s

In this section we first state and prove the main result. Then we extend it to the general situation.

Theorem 3.

Assume $u_0 \in H^s$ with s satisfying

$$-1 < s, \quad for\ n = 1, \qquad \frac{n}{2} - 2 < s, \quad for\ n \ge 2$$

Then for some $T = T(u_0) > 0$, there is a unique solution $u(t)$ of the IVP (1.3) and (1.4) on the time interval $[0,T]$ satisfying

$$u \in BC_s\left((0,T], H^r\right), \quad for\ any\ r \ge 0$$

Furthermore, for any $T' \in (0, T)$, there exists a neighborhood V of u_0 in H^s such that the mapping

$$\Phi : V \longmapsto BC_s\left((0, T'], H^r\right) ,$$

is Lipschitz.

Remark 3.

The theorem remains unchanged if the nonlinear term u^2 in (1.3) is replaced by $-u^2$. At this point the nonlinear heat equation is similar to the nonlinear Schrödinger equation for which the focusing and defocusing cases are the same for short time but quite different for long time results (see, e.g., [3]). $\square$

The proof of this theorem is again based on the contraction mapping principle. We write (1.3) in the integral form

$$u = U(u_0)(t) + G\left(u^2\right)(t) \equiv e^{-\Delta t} u_0 + \int_0^t e^{-\Delta(t-\tau)}\left(u^2\right)(\tau) d\tau .$$

Then we estimate the operators U and G on $BC_s((0, T], H^r)$. The main estimates are established in the propositions that follow.

Proposition 5.

Let $0 < T < \infty$, $s \in \mathbb{R}$ and $u_0 \in H^s$. Then for any $r \geq s$, $u = U(u_0) \in BC_s((0, T], H^r)$ and

$$\|u\|_{BC_s((0,T],H^r)} \leq C_\Delta \|u_0\|_{H^s}$$

where $C_\Delta = \|(1 + |\xi|^2)^{\frac{r-s}{2}} \exp(-|\xi|^2)\|_{L^\infty}$ is a constant.

The proof of this proposition involves merely the definition of $\|u\|_{BC_s((0,T],H^r)}$ and can be found in [6].

Proposition 6.

Let $0 < T \leq 1$ and s be a real number satisfying

$$-1 < s, \quad \text{for } n = 1, \qquad \frac{n}{2} - 2 < s, \quad \text{for } n \geq 2 .$$

Assume $r \geq 0$ and $u \in BC_s((0, T], H^r)$. Then for any q :

$$s \leq q < r + 2 - \frac{n}{2}$$

the function $u_G(t) \equiv G(u^2) \in BC_{s,0}((0, T], H^q)$ and

$$\left\| G\left(u^2\right)\right\|_{BC_s((0,T],H^q)} \leq C_G T^{(s+2-\frac{n}{2})/2} \|u\|^2_{BC_s((0,T],H^r)}$$

where C_G is a constant and $BC_{s,0}((0, T], H^q)$ is a subspace consisting of those $u \in BC_s((0, T], H^q)$ such that $u(0) = 0$.

The proof of this proposition is lengthy and similar to that of Theorem 3.4 in [6]. For completeness, a sketch of the proof will be given at the end of this section.

We now sketch the proof of Theorem 3.

Proof of Theorem 3. Let $r \geq 0$ be any real number, $X_T = BC_s((0, T], H^r)$ and $X_{T,R}$ be the closed ball centered at zero of radius R, where T, R are yet to be determined. Define the nonlinear map Φ on $X_{T,R}$

$$\Phi(u)(t) = U(u_0)(t) + G\left(u^2\right)(t)$$

Using the results of Proposition 5 and Proposition 6,

$$\|\Phi(u)\|_{X_T} \leq C_\Delta K + C_G T^{\frac{s+2-\frac{n}{2}}{2}} \|u\|_{X_T}^2 ,$$

where $K = \|u_0\|_{H^s}$. For $u, \tilde{u} \in X_{T,R}$,

$$\|\Phi(u) - \Phi(\tilde{u})\|_{X_T} \leq 2C_G T^{\frac{s+2-\frac{n}{2}}{2}} \|u + \tilde{u}\|_{X_T} \|u - \tilde{u}\|_{X_T}$$

It is not hard to check that if

$$4C_\Delta C_G K T^{\frac{s+2-\frac{n}{2}}{2}} < 1 \tag{4.1}$$

then for some $R > 0$, Φ maps $X_{T,R}$ into $X_{T,R}$ and is a contraction. Thus there is a unique fixed point $u = \Phi(u)$ in $X_{T,R}$. It is clear that by reducing the time interval $(0, T)$, we can extend the uniqueness to $X_{T,R'}$ for any R' and thus to the whole class in X_T.

For $T' \in (0, T)$, we can see from (4.1) that K can be replaced by a larger K' such that (4.1) still holds for K' and T'. That is, Φ is still a contraction map for $v_0 \in V$ where V is some neighborhood of u_0. The Lipschitz continuity of Φ is easily obtained by using the fact that Φ is a contraction map on V. This finishes the proof of this theorem. □

Now we state a lemma that will be used in the proof of Proposition 6. In what follows we will denote $(1 + |\xi|^2)^{1/2}$ by $w(\xi)$ where $\xi \in \mathbb{R}^n$.

Lemma 2.

Let $r \geq 0, a$ be real numbers. If $g, h \in H^r$, then

$$\left\| w(\xi a)^r \widehat{gh}(\xi) \right\|_{L^\infty} \leq \left\| w(\xi a)^r \hat{g}(\xi) \right\|_{L^2} \left\| w(\xi a)^r \hat{h}(\xi) \right\|_{L^2}$$

The proof of this lemma is simple and can be found in [5].

Proof of Proposition 6. First we estimate $\|u_G\|_{BC_s((0,T],H^q)}$. We only need to prove for the case $r \leq q \leq r + 2 - \frac{n}{2}$ since the norm is a nondecreasing function of q. It is easy to check that for $0 < T \leq 1, -2 \leq s \leq 0 \leq r$

$$t^{\frac{|s|}{2}} \leq w(\xi)^s w\left(\xi t^{1/2}\right)^{-s} \leq |\xi|^s + t^{\frac{|s|}{2}}$$

and it then follows that $\|u\|_{BC_s((0,T],H^r)}$ is bounded by

$$\sup_{0<t\leq T} t^{\frac{|s|}{2}} \left\| w\left(\xi t^{1/2}\right)^r \hat{u}(t)(\xi) \right\|_{L^2} \leq \|u\|_{BC_s((0,T],H^r)} \tag{4.2}$$

$$\leq \sup_{0<t\leq T} t^{\frac{|s|}{2}} \left\| w\left(\xi t^{1/2}\right)^r \hat{u}(t)(\xi) \right\|_{L^2} + \sup_{0<t\leq T} \left\| |\xi|^s w\left(\xi t^{1/2}\right)^r \hat{u}(t)(\xi) \right\|_{L^2} = I + II$$

So to bound $\|u_G\|_{BC_s((0,T],H^q)}$, we only need to estimate $I(u_G)$ and $II(u_G)$. We first bound $I(u_G)$:

$$I = t^{|s|/2} \left\| \int_0^t w(\xi\sqrt{t})^q e^{-|\xi|^2(t-\tau)} \widehat{u^2}(\tau)(\xi) d\tau \right\|_{L^2}$$

Clearly, for any $\xi \in \mathbb{R}^n$ and $0 \leq \tau \leq t$,

$$w\left(\xi\sqrt{t}\right) \leq w\left(\xi\sqrt{t-\tau}\right) w(\xi\sqrt{\tau})$$

Therefore,

$$\begin{aligned} I \quad \leq \quad & t^{|s|/2} \int_0^t \left\| w\left(\xi\sqrt{t}\right)^{q-r} w\left(\xi\sqrt{(t-\tau)}\right)^r e^{-|\xi|^2(t-\tau)} \right\|_{L^2} \\ & \times \left\| w\left(\xi\sqrt{\tau}\right)^r \widehat{u^2}(\tau)(\xi) \right\|_{L^\infty} d\tau \end{aligned}$$

Using the result of Lemma 2 and (4.2),

$$\begin{aligned} I &\le Ct^{|s|/2}\|u\|^2_{BC_s((0,T],H^r)}\int_0^t \frac{\left\|w\left(\xi\sqrt{t}\right)^{q-r} w\left(\xi\sqrt{\tau}\right)^r e^{-|\xi|^2\tau}\right\|_{L^2}}{(t-\tau)^{|s|}}d\tau \\ &\le Ct^{(s+2-\frac{n}{2})/2}\|u\|^2_{BC_s((0,T],H^r)}\int_0^1 \frac{\left\|w\left(\xi\sigma^{-1/2}\right)^{q-r} w(\xi)^r e^{-|\xi|^2}\right\|_{L^2}}{(1-\sigma)^{|s|}\sigma^{n/4}}d\sigma \end{aligned}$$

The integral in the last inequality is finite if $q-r<2-\frac{n}{2}$ and this follows from the estimate for $0\le\frac{q-r}{2}\le 1$

$$w\left(\xi\sigma^{-1/2}\right)^{q-r} = \left(1+|\xi|^2\sigma^{-1}\right)^{\frac{q-r}{2}} \le 1+|\xi|^{q-r}\sigma^{-\frac{q-r}{2}}$$

and the fact that for $a>0, b>0$ the Beta function

$$B(a,b)=\int_0^1 (1-x)^{a-1}x^{b-1}dx$$

is finite. II can be estimated in a quite similar way and the final result is the same as that of I apart from that the constant C may be different. □

Now we show that $u_G:(0,T]\to H^q$ is continuous. Let $t_1,t_2\in(0,T]$ and we estimate the difference

$$\begin{aligned} \|u_G(t_2)-u_G(t_1)\|_{H^q} &\le \left\|w(\xi)^q\int_{t_1}^{t_2} e^{-|\xi|^2(t_2-\tau)}\widehat{u^2}(\tau)(\xi)d\tau\right\|_{L^2} \\ &\quad + \left\|w(\xi)^q\int_0^{t_1}\left[e^{-|\xi|^2(t_2-\tau)}-e^{-|\xi|^2(t_1-\tau)}\right]\widehat{u^2}(\tau)(\xi)d\tau\right\|_{L^2} \\ &= III+IV \end{aligned}$$

In a similar manner III and IV can be estimated and consequently we can show that for $s\le q\le r+2-\frac{n}{2}$:

$$III,\quad IV\to 0,\quad \text{as}\quad t_2-t_1\to 0$$

Finally, we show that

$$u_G(t)\to 0,\quad \text{in } H^s \text{ as } t\to 0\,.$$

We estimate

$$\|u_G\|_{H^s}=\left\|w(\xi)^s\int_0^t e^{-|\xi|^2(t-\tau)}\widehat{u^2}(\tau)(\xi)d\tau\right\|_{L^2}$$

and this can be done similarly as before. We omit details.

Actually the arguments employed above in proving Theorem 3 can be extended to establish well-posedness of the IVP (1.1) and (1.2) with nonlinear term u^k in weighted Lebesgue space $L_{s,p}$. To state the result, we need to define the general space $BC_s((0,T],L_{s_1,p})$, where $L_{s,p}$ is the weighted Lebesgue space of all v such that $\|w(\xi)^s\hat{v}(\xi)\|_{L^p}<\infty$.

For $T>0, s\le s_1, 1\le p\le\infty$,

$$BC_s\left((0,T],L_{s_1,p}\right) = \left\{u\in C([0,T],L_{s,p})\cap C\left((0,T],L_{s_1,p}\right): \|u\|_{BC_s\left((0,T],L_{s_1,p}\right)}<\infty\right\}$$

where

$$\|u\|_{BC_s\left((0,T],L_{s_1,p}\right)} \equiv sup_{t\in[0,T]} \left\| w(\xi)^s w\left(\xi t^{1/2}\right)^{s_1-s} \hat{u}(\xi) \right\|_{L^p_j}$$

Theorem 4.

Let $k \geq 2$ be an integer and $p = \frac{k}{k-1}$. Assume that $u_0 \in L_{s,p}$ with

$$s > -\frac{2}{k}, \qquad s > \frac{n}{p} - \frac{2}{k-1}$$

Then for some $T = T(u_0) > 0$, there is a unique solution $u(t)$ of the IVP (1.1) and (1.2) on the time interval $[0, T]$ satisfying

$$u \in BC_s\left((0, T], L_{\gamma,p}\right), \quad \text{for any } \gamma \geq 0$$

Furthermore, for any $T' \in (0, T)$, there exists a neighborhood V of u_0 in $L_{s,p}$ such that the mapping

$$\Phi : V \longmapsto BC_s\left((0, T'], L_{\gamma,p}\right) : \quad u_0 \longmapsto u$$

is Lipschitz.

This theorem reduces to Theorem 3 when $k = 2$. We can prove this theorem by generalizing the arguments used in the proof of Theorem 3. For example, Lemma 2 should be extended to the following.

Lemma 3.

Let $\gamma \geq 0$, a be real numbers. If for $1 \leq i \leq k$, $1 < p_i < \infty$,

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = k - 1 ,$$

and $g_i \in L_{\gamma,p_i}$, then

$$\left\| w(\xi a)^\gamma \widehat{g_1 g_2 \cdots g_k}(\xi) \right\|_{L^\infty} \leq C \left\| w(\xi a)^\gamma \widehat{g}_1(\xi) \right\|_{L^{p_1}} \cdots \left\| w(\xi a)^\gamma \widehat{g}_k(\xi) \right\|_{L^{p_k}}$$

where C is a constant.

We omit further details because the modifications are straightforward.

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