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The Inviscid Limit of the Complex Ginzburg–Landau Equation

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1. INTRODUCTION

The complex Ginzburg–Landau (CGL for short) equation plays an important role in describing spatial pattern formation and the onset of instabilities in fluid dynamical systems [8]. A general form of the CGL equation without driving is

$$\partial_t u = (a + i\nu)\, \Delta u - (b + i\mu)\, |u|^{2\sigma}\, u, \qquad (x, t) \in \mathbb{R}^n \times (0, \infty) \tag{1.1}$$

where u is a complex-valued function of a space variable $x \in \mathbb{R}^n$ and of a time variable $t \in (0, \infty)$, and $\sigma > 0$, $a > 0$, $b > 0$, $\nu > 0$, μ are real parameters.

By taking $a = b = 0$ in (1.1), the CGL equation formally becomes the nonlinear Schrödinger (NLS for short) equation

$$i\, \partial_t v = -\nu\, \Delta v + \mu\, |v|^{2\sigma}\, v. \tag{1.2}$$

Naturally the question of inviscid limit arises. Does the solution u of the CGL equation (1.1) tend to (in an appropriate space norm) the solution v of the NLS equation (1.2) as the parameters a and b tend to 0? What is the convergence rate? The answers are not immediate especially when the initial data for these equations are not smooth.

Because of its importance in both mathematical theory and physical applications, the inviscid limit has been extensively investigated for many partial differential equations such as Burgers' equation [3], the quasi-geostrophic equation [17] and most notably the Navier–Stokes equations. For smooth initial data and in absence of boundary, the inviscid limit of the Navier–Stokes equations is the corresponding Euler equations and the rate is the optimal $O(\nu)$ where ν is the viscosity coefficient ([2, 4, 13]). But the situation changes if the initial data is not that smooth. It's shown in

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[5] that the convergence rate for vortex patch type data is only $O(\sqrt{\nu})$. If the data is even less smooth, the inviscid limit of the Navier–Stokes equations can be modified Euler equations or other equations we currently do not know ([6, 7]). These inviscid limit results for the Navier–Stokes equations turn out to be crucial in proposing corrections to the "K-41" Kolmogorov theory [1].

The CGL equation, derived from the Navier–Stokes equations via multiple scaling methods in convection [15], has been studied only recently in problems related to existence and properties of solutions ([12, 10]). Although there are claims in physics literature, it seems that the general inviscid limit question has not been addressed directly before and we see no mathematical proof existing. Here we should mention the recent work of Cruz-Pacheco, Levermore and Luce who consider the persistence of particular solutions (almost time-periodic, traveling wave and homoclinic solutions) to the 1-D periodic NLS equation under the CGL perturbation. Their technique is the Melnikov type method [9]. In this paper we are mainly concerned with the global (in time) inviscid limit of the CGL equation (1.1) in L^2, $L^{2\sigma+2}$ and H^1 spaces while the initial data u_0 is taken in $L^2(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$. The global existence results of the CGL equation (1.1) with L^2 or H^1 initial data are newly available [12].

The expected inviscid limit, the NLS equation, is known to have finite time blow-up solutions for critical and supercritical exponent σ in the focusing case ($\mu < 0$). Since our main interest is in global inviscid limit results, the exponent σ is assumed to be subcritical or critical with small initial data when $\mu < 0$. The term "critical" (resp. "subcritical", resp. "supercritical") at the level of L^p indicates $n\sigma = p$ (resp. $n\sigma < p$, resp. $n\sigma > p$) and at the level of H^r indicates $\sigma(n-2r) = 2$ (resp. $\sigma(n-2r) < 2$, resp. $\sigma(n-2r) > 2$). In the subcritical case or the critical case for small data, the H^1-norm can be controlled through the conservation of the L^2-norm and of the energy for H^1 solutions and therefore we have global existence. The parameter dependence of estimates is reflected in our results and further remarks on this point can be found in the sequel.

We approach the inviscid limit problem by employing extensive energy estimates to bound the difference between the solutions of the CGL equation and the NLS equation in terms of the initial data and the parameters. The initial data for the NLS equation is taken at least as regular as in $H^1(\mathbb{R}^n)$ so that the energy estimates make sense (see more details in Remark 3.3). Furthermore, some assumptions on the parameters (like (3.1) in Section 3) are necessary in order to obtain a closed equation for the normed difference. Otherwise, only a hierarchy of differential equations, the so called ladder structure, can be developed [10].

The remainder of this paper is organized as follows. In Section 2 we review the existence and regularity of solutions to the CGL equation and the NLS equation. In Section 3 we establish two global L^2 inviscid limit results. The first theorem states that the L^2 difference between solutions of the CGL equation and the NLS equation is of order $O(\sqrt{a}) + O(b^{(2\sigma+1)/(2(2\sigma+2))})$ if the initial data for the NLS equation is taken in $H^1(\mathbb{R}^n)$ (see Theorem 3.1). The second theorem improves the convergence rate to the optimal $O(a)+O(b)$ by taking $v_0 \in H^2$ (see Theorem 3.8 for details). Section 4 treats the $L^{2\sigma+2}$ inviscid limit and the result is given in Theorem 4.1. The main reason for considering and achieving this type of inviscid limit is the special form of the nonlinear term in (1.1). In Section 5 we investigate the H^1 inviscid limit and obtain a convergence rate depending on $a^{-1}\sqrt{b^2+\mu^2}$ (see Theorem 5.1).

2. PRELIMINARIES

In this section we review the existence results and appropriate properties concerning solutions of the CGL equation with initial data u_0 belonging to $L^2(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$ and of the NLS equation with data in $H^1(\mathbb{R}^n)$ or $H^2(\mathbb{R}^n)$.

THEOREM 2.1. *Let $u_0 \in L^2(\mathbb{R}^n)$. Then the CGL equation* 1.1 *with initial data u_0 has a global (in time) solution u satisfying*

$$u \in C([0,\infty); L^2) \cap L^2_{loc}([0,\infty); H^1) \cap L^{2\sigma+2}_{loc}([0,\infty); L^{2\sigma+2}) \qquad (2.1)$$

with $u(0)=u_0$. Furthermore, u satisfies the energy relation

$$\tfrac{1}{2}\|u(t)\|^2_{L^2} + a\int_0^t \|\nabla u(t')\|^2_{L^2}\,dt' + b\int_0^t \|u(t')\|^{2\sigma+2}_{L^{2\sigma+2}}\,dt' = \tfrac{1}{2}\|u_0\|^2_{L^2} \qquad (2.2)$$

for any $t \in [0,\infty)$.

This theorem is stated in [12] (Proposition 2.1, p. 197) and it can be proved by using either Faedo–Galerkin method or smoothing approximations. The solution u is shown to be unique in the class (2.1) under the assumption

$$\left|1 + i\frac{\mu}{b}\right| \leqslant \frac{\sigma+1}{\sigma}. \qquad (2.3)$$

See [12] (Proposition 3.1, p. 201) for a proof. The assumption (2.3) on b and μ turns out to be also important in showing inviscid limits in the subsequent sections.

The existence result of H^1-solutions of the CGL equation is obtained only very recently in [12] (Proposition 5.1, p. 215) and it states, in particular, that

THEOREM 2.2. *Assume that either $\mu \geqslant 0$ or $(n-2)\sigma < 2$ and if $n\sigma > 2$*

$$\left|1 + i\frac{v}{a}\right| \leqslant \frac{n\sigma}{n\sigma - 2}.$$

Let $u_0 \in H^1(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n)$. Then the CGL equation 1.1 *has a unique solution u satisfying*

$$u \in C([0, \infty); H^1 \cap L^{2\sigma+2}) \cap L^2_{loc}([0, \infty); H^2) \cap L^{4\sigma+2}_{loc}([0, \infty); L^{4\sigma+2})$$

with $u(0) = u_0$, and u satisfies (2.2).

The existence of solutions to the NLS equation with L^2, H^1 or H^2 data is now well-documented in monographs and survey papers (see e.g. [14, 11]). We shall only need the results concerning H^1 and H^2 solutions.

THEOREM 2.3. *Let $(n-2)\sigma \leqslant 2$ for $n \geqslant 3$. Then for every $v_0 \in H^1(\mathbb{R}^n)$, there is a $T^* = T^*(\|v_0\|_{H^1}) > 0$ and a solution v to the NLS equation* 1.2 *on $[0, T^*)$ such that*

$$v \in C([0, T^*); H^1) \cap C^1([0, T^*); H^{-1}) \cap L^{2\sigma+2}_{loc}([0, T^*); L^{2\sigma+2})$$

with $v(0) = v_0$. Furthermore, for any $t < T^*$,

$$E(v(t)) = E(v_0), \qquad \|v(t)\|_{L^2} = \|v_0\|_{L^2} \tag{2.4}$$

where $E(v)$ is the Hamiltonian

$$E(v) = \frac{v}{2} \|\nabla v\|^2_{L^2} + \frac{\mu}{2\sigma + 2} \|v\|^{2\sigma+2}_{L^{2\sigma+2}}. \tag{2.5}$$

In addition, in the defocusing case ($\mu \geqslant 0$) or in the L^2-subcritical focusing case ($\mu < 0$ and $\sigma < 2/n$) or in the L^2-critical focusing case ($\mu < 0$ and $\sigma = 2/n$) with small data $\|v_0\|_{L^2}$, the local solution is actually global (i.e., $T^* = \infty$).

THEOREM 2.4. *Assume $(n-4)\sigma \leqslant 2$ for $n \geqslant 5$. Then for any $v_0 \in H^2(\mathbb{R}^n)$, there is a $T^* = T^*(\|v_0\|_{H_2}) > 0$ and a unique solution $u \in C([0, T^*); H^2)$ to the NLS equation* 1.2 *with $v(0) = v_0$. Furthermore, for any $T < T^*$,*

$$\|v(t)\|_{H^2} \leqslant K \|v_0\|_{H^2}, \qquad t \leqslant T \tag{2.6}$$

where K is a constant depending on T and $\sup\{\|u(t)\|_{H^1}, t \leqslant T\}$.

In addition, in the defocusing case ($\mu \geqslant 0$) *or in the L^2-subcritical focusing case* ($\mu < 0$ *and* $\sigma < 2/n$) *or in the L^2-critical focusing case* ($\mu < 0$ *and* $\sigma = 2/n$) *with small data* $\|v_0\|_{L^2}$, *the local solution is actually global* (*i.e.*, $T^* = \infty$).

We notice that in Theorem 2.3 (resp. Theorem 2.4) the time T^* of local solutions can be estimated in terms of the H^1 (resp. H^2) norm of v_0 alone. This implies that extending the local (in time) solutions to global ones relies on apriori estimates of $\|v\|_{H^1}$ (resp. $\|v\|_{H^2}$). The assumptions associated with the global results in these theorems are sufficient in controlling the H^1 norm through the conservation laws. The H^2 solution is global under the same assumptions imposed for global H^1 solution because $\|v\|_{H^2}$ is totally controlled by $\|v\|_{H^1}$ in view of (2.6). As we shall see, these conditions are also imposed in the inviscid limit results so that they are all global (in time).

In order to obtain inviscid limit results presented in the next few sections, we use extensively the Gagliardo–Nirenberg inequalities [16] (p. 125) which allow the control of L^p norm through L^q norm of higher derivatives. To illustrate, we state a particular important special case as a lemma and then apply it to bound $\int |v|^{2\sigma+2}$.

LEMMA 2.5. *If*

$$\frac{1}{p} \geqslant \frac{1}{2} - \frac{1}{n}$$
$$\|v\|_{L^p} \leqslant C(p)\, \|\nabla v\|_{L^2}^{\theta}\, \|v\|_{L^2}^{1-\theta}, \qquad \theta = n\left(\frac{1}{2} - \frac{1}{p}\right) \tag{2.7}$$

for some constant $C(p)$.

For $(n-2)\,\sigma \leqslant 2$, $p = 2\sigma + 2$ satisfies (2.7). By Lemma 2.5,

$$\|v\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C(\sigma)\, \|\nabla v\|_{L^2}^{n\sigma}\, \|v\|_{L^2}^{2\sigma+2-n\sigma}$$

which also justifies the definition of $E(v)$ in (2.4) for $v \in H^1$.

3. L^2 INVISCID LIMIT

In this section we consider the inviscid limit of the CGL equation 1.1 with initial data $u_0 \in L^2$ and we obtain two L^2 inviscid limit results. First the initial data v_0 for the NLS equation is taken in $H^1(\mathbb{R}^n)$, which ensure that energy estimates make sense. We then obtain the optimal convergence rate by taking $v_0 \in H^2$.

We start by stating the first inviscid limit result.

THEOREM 3.1. *Assume that $\sigma \leqslant 2/(n-2)$ (for $n \geqslant 3$) if $\mu \geqslant 0$ and $\sigma \leqslant 2/n$ if $\mu < 0$, and b, μ satisfy*

$$\left|1 + i\frac{\mu}{b}\right| \leqslant \frac{\sigma+1}{\sigma}. \tag{3.1}$$

Let $u_0 \in L^2(\mathbb{R}^n)$ and $v_0 \in H^1(\mathbb{R}^n)$ ($\|v_0\|_{L^2}$ should also be small if $\mu < 0$ and $\sigma = 2/n$). Consider the difference

$$w(x, t) = u(x, t) - v(x, t)$$

between a solution u of the CGL equation 1.1 *with $u(x, 0) = u_0(x)$ and a solution v of the NLS equation* 1.2 *with $v(x, 0) = v_0$. Then w obeys the estimate for any $T < \infty$ and $t \leqslant T$*

$$\|u(t) - v(t)\|_{L^2}^2 \leqslant \|u_0 - v_0\|_{L^2}^2 + a\mathscr{F}(v_0)t + C_1(\sigma)\, b^{(2\sigma+1)/(2\sigma+2)}\, (1+b)\, \mathscr{G}(v_0)t + C_2(\sigma)\, b^{(2\sigma+1)/(2\sigma+2)}\, \|u_0\|_{L^2}^2 \tag{3.2}$$

where C_1, C_2 are constants depending on σ only, $\mathscr{F}(v_0)$ and $\mathscr{G}(v_0)$ (independent of a and b) are bounds for $\|\nabla v\|_{L^2}^2$ and $\|v\|_{L^{2\sigma+2}}^{2\sigma+2}$ in terms of v_0 (given explicitly below).

In particular, if $\|u_0 - v_0\|_{L^2}^2$ is of order $O(a) + O(b^{(2\sigma+1)/(2\sigma+2)})$, then

$$\|u - v\|_{L^2} = O(\sqrt{a}) + O(b^{(2\sigma+1)/(2(2\sigma+2))}) \tag{3.3}$$

for small a and b.

We make several remarks about this theorem.

Remark 3.2. The assumptions on σ in this theorem is inherited from Theorem 2.3 to guarantee the global existence of solutions to the NLS equation. The above theorem holds as long as the solution to the NLS equation exists, therefore this inviscid limit result is global (in time).

Remark 3.3. In this theorem the initial data $v_0 \in H^1$. It seems that this is the minimal regularity assumption on the initial data v_0 such that the energy estimates make sense. In fact, for $v_0 \in H^1$, the solution of the NLS equation $v \in C([0, T]; H^1)$ and $\partial_t v$ and the NLS equation itself holds in $C([0, T]; H^{-1})$. The scalar product in L^2 of the NLS equation with v is well defined. If $v_0 \in L^2$ only, the solution v is only a mild solution and the regularity available may not be enough even if we go through a mollifying and limiting process.

Remark 3.4. In general the assumption of the type (3.1) on b, μ is necessary in order to obtain a "closed" equation for $\int |w|^2$. Otherwise, only the "ladder" structure can be developed (see e.g. [10]).

Remark 3.5. The L^2 convergence rate given in (3.3) is of order $\sqrt{a}$ (resp. $b^{(2\sigma+1)/(2(2\sigma+2))}$) in a (resp. b). This rate can be improved to its optimal $O(a)+O(b)$ if we assume more regularity for the initial data of the NLS equation, say, $v_0 \in H^2(\mathbb{R}^n)$. The optimal rate result will be given in Theorem 3.8 below.

In order to prove Theorem 3.1, we need to estimate $\|\nabla v\|_{L^2}$ and $\|v\|_{L^{2\sigma+2}}$. We have the following proposition

PROPOSITION 3.6. *Assume $\sigma > 0$, $\sigma \leqslant 2/(n-2)$ for $n \geqslant 3$ if $\mu \geqslant 0$ and $\sigma \leqslant 2/n$ if $\mu < 0$. Let $v_0 \in H^1(\mathbb{R}^n)$ ($\|v_0\|_{L^2}$ should also be small if $\mu < 0$ and $\sigma = 2/n$). Then the solution of the NLS equation with the initial data v_0 satisfies*

$$\int |\nabla v|^2 \leqslant \mathcal{F}(v_0), \tag{3.4}$$

$$\int |v|^{2\sigma+2} \leqslant \mathcal{G}(v_0) \tag{3.5}$$

where $\mathcal{F}(v_0)$ and $\mathcal{G}(v_0)$ are determined by the initial v_0 but independent of a and b (see their explicit expressions in the proof below).

Proof of Proposition 3.6. The idea of showing (3.4) and (3.5) is to use both the conservation of the L^2-norm and of the energy (see (2.4)). But we need to distinguish between the defocusing ($\mu \geqslant 0$) and the focusing ($\mu < 0$) case.

For $\mu \geqslant 0$, both the H^1-norm and the $L^{2\sigma+2}$-norm are easily controlled by using the conservation laws. In fact, using Lemma 2.5,

$$\|\nabla v\|_{L^2}^2 \leqslant \frac{2}{\nu} E(v_0)$$

$$\int |v|^{2\sigma+2} \leqslant C(\sigma)\, \|\nabla v\|_{L^2}^{n\sigma}\, \|v\|_{L^2}^{2\sigma+2-n\sigma}$$

$$\leqslant C(\sigma) \left(\frac{E(v_0)}{\nu}\right)^{n\sigma/2} \|v_0\|_{L^2}^{2\sigma+2-n\sigma}$$

where $C(\sigma)$ is a constant depending on σ.

For $\mu<0$, the bounds can still be obtained in the L^2-subcritical case ($\sigma<2/n$) and in the L^2-critical case ($\sigma=2/n$) provided $\|v_0\|_{L^2}$ is small enough. Indeed, using Lemma 2.5,

$$E(v_0)=\frac{\nu}{2}\int|\nabla v|^2+\frac{\mu}{2\sigma+2}\int|v|^{2\sigma+2}$$
$$\geqslant\frac{\nu}{2}\int|\nabla v|^2+C(\sigma)\,\mu\left(\int|\nabla v|^2\right)^{n\sigma/2}\|v\|_{L^2}^{2\sigma+2-n\sigma}. \tag{3.6}$$

If $n\sigma=2$ and $\|v_0\|_{L^2}$ is small enough, say,

$$\frac{\nu}{2}+C(\sigma)\,\mu\,\|v_0\|_{L^2}^{2\sigma+2-n\sigma}>0.$$

Then it follows from (3.6) that

$$\int|\nabla v|^2\leqslant\left(\frac{\nu}{2}+C(\sigma)\,\mu\,\|v_0\|_{L^2}^{2\sigma+2-n\sigma}\right)^{-1}E(v_0).$$

If $n\sigma<2$, we use the following simple lemma

LEMMA 3.7. *Let P, Q and $\beta<2$ are all positive numbers. If $y\geqslant 0$ satisfies*

$$y^2-Py^{\beta}\leqslant Q.$$

Then y is bounded by

$$y\leqslant\max\{(2P)^{1/(2-\beta)},\sqrt{2Q}\}.$$

Applying Lemma 3.7 to (3.6)

$$\int|\nabla v|^2\leqslant\max\{(C(\sigma)\,\nu^{-1}(-\mu)\,\|v_0\|_{L^2}^{2\sigma+2-n\sigma})^{2/(2-n\sigma)},4\nu^{-1}E(v_0)\}.$$

The proof of this proposition is concluded if we denote by $\mathscr{F}(v_0)$ and $\mathscr{G}(v_0)$ the bounds for $\int|\nabla v|^2$ and $\int|v|^{2\sigma+2}$ in either the defocusing case or the focusing case. ▮

Proof of Lemma 3.7. The proof is easy. Suppose not. Then

$$y>(2P)^{1/(2-\beta)},\qquad y>\sqrt{2Q}.$$

But $y > (2P)^{1/(2-\beta)}$ implies $Py^{\beta-2} < 1/2$ and thus

$$y^2 \leqslant Py^\beta + Q \leqslant Py^{\beta-2}y^2 + Q < \tfrac{1}{2}y^2 + Q$$

which contradicts $y > \sqrt{2Q}$. ∎

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let u satisfy the CGL equation 1.1 and v satisfy the NLS equation 1.2. Then the difference $w = u - v$ satisfies

$$\partial_t w = (a + iv)\, \Delta w + a\, \Delta v - (b + i\mu)(f(u) - f(v)) - bf(v) \tag{3.7}$$

where $f(u) = |u|^{2\sigma}\, u$.

We take a nonnegative, smooth cutoff function ϕ, identically equal to 1 for $|x| \leqslant 1$ and to 0 for $|x| \geqslant 2$. We multiply the equation (3.7) by $2\bar{w}\phi_R^2$ where $\phi_R(x) = \phi(x/R)$ and $R > 0$. Integrating in space we obtain:

$$\begin{aligned}\partial_t \int \phi_R^2\, |w|^2 &= 2Re \int \phi_R^2\, \partial_t w\bar{w} \\ &= 2Re((a+iv)(\phi_R^2\, \Delta w,\, w)) + 2aRe(\phi_R^2\, \Delta v,\, w) \\ &\quad - 2Re((b+i\mu)(\phi_R^2(f(u) - f(v)),\, w)) - 2bRe(\phi_R^2\, f(v),\, w)\end{aligned} \tag{3.8}$$

where $(F, G) = \int_{\mathbb{R}^n} F\bar{G}$, $\bar{G}$ is the complex conjugate of G and Re denotes the real part.

For simplicity of notation, we denote by I, II, III, IV the four terms on the RHS of (3.8) and now estimate them separately.

$$\mathrm{I} = -4Re((a+iv)(\phi_R\, \nabla w,\, w\, \nabla\phi_R) - 2Re((a+iv)(\phi_R^2\, \nabla w,\, \nabla w))$$

and therefore

$$|\mathrm{I}| \leqslant a\, \|\phi_R \nabla w\|_{L^2}^2 + 4\, \frac{a^2+v^2}{a}\, \|w\, \nabla\phi_R\|_{L^2}^2 - 2a\, \|\phi_R\, \nabla w\|_{L^2}^2.$$

The second term can be estimated similarly

$$|\mathrm{II}| \leqslant (a+\varepsilon)\, \|\phi_R \nabla v\|_{L^2}^2 + 4a^2\varepsilon^{-1}\, \|w\, \nabla\phi_R\|_{L^2}^2 + a\, \|\phi_R\, \nabla w\|_{L^2}^2$$

where $\varepsilon > 0$ is small. Adding the estimates for I and II and using $\|\nabla\phi_R\|_{L^\infty} \leqslant R^{-1}\, \|\nabla\phi\|_{L^\infty}$,

$$\begin{aligned}|\mathrm{I}| + |\mathrm{II}| &\leqslant (a+\varepsilon)\, \|\phi_R\, \nabla v\|_{L^2}^2 \\ &\quad + \left(4\, \frac{a^2+v^2}{a} + 4a^2\varepsilon^{-1}\right) R^{-2}\, \|\nabla\phi\|_{L^\infty}^2\, \|w\|_{L^2}^2.\end{aligned} \tag{3.9}$$

Under the assumption

$$\left|1+i\frac{\mu}{b}\right| \leqslant \frac{\sigma+1}{\sigma}$$

we can show

$$\mathrm{III} = -2Re((b+i\mu)(\phi_R^2(f(u)-f(v)), w)) \leqslant 0. \tag{3.10}$$

In fact, noticing $f(u)=|u|^{2\sigma}u$ and using

$$f(u)-f(v) = \int_0^1 [(\sigma+1)(u-v)\,|Z|^2\sigma + \sigma(\bar{u}-\bar{v})\,Z^2\,|Z|^{2\sigma-2}]\,d\lambda$$

where $Z=\lambda u+(1-\lambda)v$, we rewrite III as

$$\begin{aligned}\mathrm{III} &= -2Re\left((b+i\mu)\int_0^1 d\lambda \int \phi_R^2[(\sigma+1)\,|w|^2\,|Z|^{2\sigma} + \sigma\bar{w}^2 Z^2\,|Z|^{2\sigma-2}]\,dx\right)\\ &\leqslant 2\sigma b \max\left\{0, \left|1+i\frac{\mu}{b}\right| - \frac{\sigma+1}{\sigma}\right\}\int_0^1\int \phi_R^2\,|w|^2\,|Z|^2 = 0.\end{aligned}$$

For the term IV, we use the Young inequality $AB \leqslant (A^p/p)+(B^\varrho/\varrho)$ for $A, B \geqslant 0$ and $(1/p)+(1/\varrho)=1$ to obtain (noticing that $f(v)=|v|^{2\sigma}\,v$)

$$\begin{aligned}|\mathrm{IV}| &= 2b\,|(\phi_R^2 f(v), w)|\\ &\leqslant \frac{2\sigma+1}{\sigma+1}\,b^{(2\sigma+1)/(2\sigma+2)}\int (\phi_R^{2/(2\sigma+1)}\,|v|)^{2\sigma+2}\\ &\quad + \frac{1}{\sigma+1}\,b^{(2\sigma+1)/(2\sigma+2)+1}\int |w|^{2\sigma+2}.\end{aligned} \tag{3.11}$$

Collecting the above estimates (3.9), (3.10), (3.11) and letting $R\to\infty$ and $\varepsilon\to 0$, we obtain

$$\begin{aligned}\partial_t \int |w|^2 &\leqslant a\,\|\nabla v\|_{L^2}^2 + \frac{2\sigma+1}{\sigma+1}\,b^{(2\sigma+1)/(2\sigma+2)}\int |v|^{2\sigma+2}\\ &\quad + \frac{1}{\sigma+1}\,b^{(2\sigma+1)/(2\sigma+2)+1}\int |w|^{2\sigma+2}.\end{aligned} \tag{3.12}$$

Using the estimates in Proposition 3.6 and integrating in t

$$\|w\|_{L^2}^2 \leqslant \|u_0 - v_0\|_{L^2}^2 + a\mathscr{F}(v_0)\, t + C_1(\sigma)\, b^{(2\sigma+1)/(2\sigma+2)}(1+b)\, \mathscr{G}(v_0) t$$
$$+ C_2(\sigma)\, b^{(2\sigma+1)/(2\sigma+2)} \left(b \int_0^t \|u(t')\|_{L^{2\sigma+2}}^{2\sigma+2}\, dt' \right) \tag{3.13}$$

where C_1, C_2 are constants depending on σ only. The expected estimate (3.2) is obtained after applying (2.2) to the last term of (3.13). This concludes the proof of Theorem 3.1. ▮

The inviscid limit results for the Navier–Stokes equations ([2, 5, 7]) and for the quasi-geostrophic equation [17] suggest that the convergence become faster if the initial data become smoother. The L^2 inviscid limit result of Theorem 3.1 can indeed be improved to the optimal rate $O(a) + O(b)$ if the initial data for the NLS equation $v_0 \in H^2(\mathbb{R}^n)$. Here we need to caution the reader that the "linear" appearance of a small parameter in a general non-linear (even linear) equation does not necessarily imply the difference between solutions is also "linear". This is clear if we notice that the L^2 convergence rate of the inviscid limit for the linear heat equation with non-smooth data can be sublinear [5]. In the following we obtain the optimal rate by modifying the proof of Theorem 3.1.

If $v_0 \in H^2$, then the solution v of the NLS equation with data v_0 is in $C([0, T); H^2)$ and satisfies the estimate (2.6) according to Theorem 2.4. This fact allows us to estimate II and IV differently.

$$\begin{aligned} |\mathrm{II}| &= 2a\, |Re(\phi_R^2 \Delta v, w)| \leqslant \int \phi_R^2\, |w|^2 + a^2 \int \phi_R^2\, |\Delta v|^2 \\ |\mathrm{IV}| &= 2b \int \phi_R^2\, |f(v)|\, |w| \leqslant \int \phi_R^2\, |w|^2 + b^2 \int |v|^{4\sigma+2}. \end{aligned} \tag{3.14}$$

Applying Gagliardo–Nirenberg's inequality to $\int |v|^{4\sigma+2}$

$$\|v\|_{L^{4\sigma+2}} \leqslant C(\sigma)\, \|\Delta v\|_{L^2}^{\theta}\, \|v\|_{L^2}^{1-\theta}$$

where $C(\sigma)$ is a constant depends on σ, $0 \leqslant \theta$ is given by

$$\theta = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{4\sigma+2}\right)$$

and $\theta \leqslant 1$ if we assume $\sigma \leqslant 2/(n-4)$. The estimates for I, III remain unchanged. Collecting the estimates and letting $R \to \infty$, we obtain

$$\partial_t \int |w|^2 \leqslant 2 \int |w|^2 + a^2 \int |\Delta v|^2 + b^2 C(\sigma)\, \|\Delta v\|_{L^2}^{n\sigma}\, \|v_0\|_{L^2}^{4\sigma+2-n\sigma}. \tag{3.15}$$

To bound $\|\Delta v\|_{L^2}$, we use (2.6). For $t \leqslant T$,

$$\|\Delta v(t)\|_{L^2}^2 \leqslant K^2 \|v_0\|_{H^2}^2 \equiv \mathscr{A}(v_0) \tag{3.16}$$

where K is a bound in terms of T and $\mathscr{F}(v_0)$. Letting

$$\mathscr{D}(v_0) \equiv C(\sigma)\, \mathscr{A}(v_0)^{n\sigma/2} \|v_0\|_{L^2}^{4\sigma+2-n\sigma} \tag{3.17}$$

and integrating (3.15) in t, we obtain

THEOREM 3.8. *Assume $\sigma > 0$, $\sigma \leqslant 2/(n-4)$ for $n \geqslant 5$ if $\mu \geqslant 0$ and $\sigma \leqslant 2/n$ if $\mu < 0$. Assume b, μ satisfy the condition* (3.1). *Let $u_0 \in L^2(\mathbb{R}^n)$ and $v_0 \in H^2(\mathbb{R}^n)$ ($\|v_0\|_{L^2}$ should also be small if $\mu < 0$ and $\sigma = 2/n$). Consider the difference*

$$w(x, t) = u(x, t) - v(x, t)$$

between a solution u of the CGL equation 1.1 *with $u(x, 0) = u_0(x)$ and a solution v of the NLS equation* 1.2 *with $v(x, 0) = v_0$. Then w obey the estimate*

$$\|u(t) - v(t)\|_{L^2}^2 \leqslant \|u_0 - v_0\|_{L^2}^2 e^{2t} + \tfrac{1}{2} a^2 \mathscr{A}(v_0)(e^{2t} - 1) + \tfrac{1}{2} b^2 \mathscr{D}(v_0)(e^{2t} - 1)$$

for $t < T$ with any $T < \infty$, where $\mathscr{A}(v_0)$ is the bound for $\|v\|_{C([0, T]; H^2)}$ in terms of the initial data v_0 (see (3.16)*) and $\mathscr{D}(v_0)$ is given in* (3.17).

In particular, for small a and b and $\|u_0 - v_0\|_{L^2} = O(a) + O(b)$,

$$\|u - v\|_{L^2} = O(a) + O(b).$$

4. $L^{2\sigma+2}$ INVISCID LIMIT

As stated in Theorem 2.1 and Theorem 2.3, the solution u of the CGL equation with $u_0 \in L^2(\mathbb{R}^n)$ and v of the NLS equation with $v_0 \in H^1(\mathbb{R}^n)$ are both in $L^{2\sigma+2}_{loc}([0, \infty); L^{2\sigma+2})$. It seems reasonable to consider the difference $w = u - v$ in $L^{2\sigma+2}(\mathbb{R}^n)$ and we do have the following inviscid limit result.

THEOREM 4.1. *Assume that $\sigma \leqslant 2/(n-4)$ (for $n \geqslant 5$) if $\mu \geqslant 0$ and $\sigma \leqslant 2/n$ if $\mu < 0$, and*

$$(2\sigma+1)(2\sigma+2) \leqslant \frac{2n}{n-4}, \qquad \text{for} \quad n \geqslant 5. \tag{4.1}$$

Assume in addition that a, ν *and* b, μ *satisfy*

$$|1+i\nu/a| < \frac{\sigma+1-\delta}{\sigma}, \qquad |1+i\mu/b| \leqslant \frac{\sigma+1}{\sigma} \tag{4.2}$$

where $\delta \in (0,1)$ *is an arbitrary parameter. Let* $u_0 \in L^2(\mathbb{R}^n) \cap L^{2\sigma+2}(\mathbb{R}^n)$ *and* $v_0 \in H^2(\mathbb{R}^n)$ ($\|v_0\|_{L^2}$ *is sufficiently small if* $\mu < 0$ *and* $n\sigma = 2$). *Then the difference*

$$w = u - v$$

between a solution u *to the CGL equation* 1.1 *with* $u(0) = u_0$ *and* v *to the NLS equation* 1.2 *with* $v(0) = v_0$ *obeys the* $L^{2\sigma+2}$ *estimate*

$$\begin{aligned} \|w(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant \|u_0 - v_0\|_{L^{2\sigma+2}}^{2\sigma+2} e^{c(\sigma)t} + c_1(\sigma)\, b^{2\sigma+2} \mathscr{H}(v_0)(e^{c(\sigma)t} - 1) \\ + c_2(\sigma)(a\delta^{-1})^{\sigma+1}\, \mathscr{K}(v_0)(e^{c(\sigma)t} - 1) \end{aligned} \tag{4.3}$$

for any $T < \infty$ *and* $t \leqslant T$, *where* c, c_1, c_2 *are constants depending only on* σ, $\mathscr{H}$ *and* $\mathscr{K}$ *are bounds in terms of the initial data* v_0 *but independent of* a *and* b (*see* (4.10) *and* (4.11) *for their definitions*).

In particular, (4.3) *indicates that the* $L^{2\sigma+2}$ *converge rate is of order* $O(\sqrt{a}) + O(b)$ *if* δ *is near* 1.

Remark 4.2. Part of the assumptions imposed on σ is for the global existence of solutions v to the NLS equation and the control of $\|\nabla v\|_{L^2}$. The condition (4.1) arises from applying the Gagliardo–Nirenberg inequality and can be eliminated by more regularity assumption on v_0. The assumption (4.2) is necessary in obtaining a "closed" equation for $\|w\|_{L^{2\sigma+2}}$. Otherwise, only the "ladder" structure can be developed.

Proof of Theorem 4.1. The difference $w = u - v$ satisfies the equation

$$\partial_t w = (a + i\nu)\,\Delta w + a\,\Delta v - (b + i\mu)(f(u) - f(v)) - bf(v)$$

where $f(u) = |u|^{2\sigma} u$. Let $\phi_R(x)$ be the cutoff function as before and we find

$$\begin{aligned} \frac{1}{2(\sigma+1)}\,\partial_t \int \phi_R^2\, |w|^{2\sigma+2} = Re(a+i\nu) \int \phi_R^2\, |w|^{2\sigma}\, \Delta w \bar{w} + aRe \int \phi_R^2\, |w|^{2\sigma}\, (\Delta v)\, \bar{w} \\ - Re(b+i\mu) \int \phi_R^2\, |w|^{2\sigma}\, (f(u) - f(v))\, \bar{w} \\ - bRe \int \phi_R^2\, |w|^{2\sigma} f(v)\, \bar{w}. \end{aligned}$$

For simplicity of notation, the four terms on the RHS are written as I, II, III and IV. Integrating by parts,

$$\mathrm{I} = -Re(a+iv)\int \phi_R^2\,[(\sigma+1)\,|w|^{2\sigma}\,|\nabla w|^2 + \sigma\,|w|^{2\sigma-2}\,(\bar{w}\,\nabla w)^2]$$
$$-Re(a+iv)\int \phi_R\,|w|^{2\sigma}\,\bar{w}\,\nabla w\,\nabla\phi_R.$$

Choose $\varepsilon > 0$ such that

$$|1+iv/a| \leqslant \frac{\sigma+1-\varepsilon-\delta}{\sigma}$$

and divide I into I_1 and I_2 with

$$\mathrm{I}_1 = -Re(a+iv)\left(\int \phi_R^2[(\sigma+1-\varepsilon-\delta)\,|w|^{2\sigma}\,|\nabla w|^2 + \sigma\,|w|^{2\sigma-2}\,(\bar{w}\,\nabla w)^2]\right).$$

It is easy to check that

$$|\mathrm{I}_1| \leqslant \sigma a \max\left\{0,\,|a+iv| - \frac{\sigma+1-\delta-\varepsilon}{\sigma}\right\}\int \phi_R^2\,|w|^{2\sigma}\,|\nabla w|^2 = 0$$

$$\mathrm{I}_2 = -a(\delta+\varepsilon)\int \phi_R^2\,|w|^{2\sigma}\,|\nabla w|^2 - Re(a+iv)\int \phi_R\,|w|^{2\sigma}\,\bar{w}\,\nabla w\,\nabla\phi_R.$$

Applying Young's inequality to the second term in I_2,

$$|\mathrm{I}_2| \leqslant -a\delta\int \phi_R^2\,|w|^{2\sigma}\,|\nabla w|^2 + \frac{|a+iv|^2}{4a\varepsilon}\int |w|^{2\sigma+2}\,|\nabla\phi_R|^2.$$

Integration by parts in II gives

$$\mathrm{II} = -aRe\int \phi_R^2[(\sigma+1)\,|w|^{2\sigma}\,\nabla v\,\nabla\bar{w} + \sigma\,|w|^{2\sigma-2}\,\bar{w}^2\,\nabla v\,\nabla w]$$

$$-2aRe\int \phi_R\,|w|^{2\sigma}\,\nabla\phi_R\,\nabla v\bar{w}.$$

Applying Young's inequality,

$$|\mathrm{II}| \leqslant C_1(\sigma)\int \phi_R^2\,|w|^{2\sigma+2} + C_2(\sigma)(a\delta^{-1})^{\sigma+1}\int |\nabla v|^{2\sigma+2} + a\delta\int \phi_R^2\,|w|^{2\sigma}\,|\nabla w|^2$$

$$+a\int |\nabla\phi_R|\,|w|^{2\sigma+2} + a\int |\nabla\phi_R|\,|\nabla v|^{2\sigma+2}.$$

Noting $\|\nabla\phi_R\|_{L^\infty} \leqslant R^{-1}\,\|\nabla\phi\|_{L^\infty}$

$$|\mathrm{I}+\mathrm{II}| \leqslant C_1(\sigma)\int \phi_R^2\,|w|^{2\sigma+2} + (C_2(\sigma)(a\delta^{-1})^{\sigma+1} + aR^{-1}\,\|\nabla\phi\|_{L^\infty})\int |\nabla v|^{2\sigma+2}$$

$$+\left(\frac{|a+iv|^2}{4a\varepsilon}\,R^{-2}\,\|\nabla\phi\|^2_{L^\infty} + aR^{-1}\,\|\nabla\phi\|_{L^\infty}\right)\int |w|^{2\sigma+2}. \tag{4.4}$$

The term III can be dealt with similarly as in the proof of Theorem 3.1 and the conclusion is

$$\mathrm{III} \leqslant 0, \qquad \text{if} \quad |1+i\mu/b| \leqslant \frac{\sigma+1}{\sigma} \tag{4.5}$$

Now we turn to term IV.

$$\mathrm{IV} = -bRe\left(\int \phi_R^2\,|w|^{2\sigma}\,|v|^{2\sigma}\,v\bar{w}\right)$$

Using the Young inequality,

$$|\mathrm{IV}| \leqslant \frac{2\sigma+1}{2\sigma+2}\int |w|^{2\sigma+2} + \frac{1}{2\sigma+2}\,b^{2\sigma+2}\int |v|^{(2\sigma+1)(2\sigma+2)} \tag{4.6}$$

Applying the Gagliardo–Nirenberg inequality

$$\int |v|^{(2\sigma+1)(2\sigma+2)} \leqslant C_3(\sigma)(\|\varDelta v\|^{\theta}_{L^2}\,\|v\|^{1-\theta}_{L^2})^{(2\sigma+1)(2\sigma+2)} \tag{4.7}$$

$$\int |\nabla v|^{2\sigma+2} \leqslant C_4(\sigma)(\|\varDelta v\|^{\theta_1}_{L^2}\,\|v\|^{1-\theta_1}_{L^2})^{(2\sigma+2)} \tag{4.8}$$

where C_3, C_4 are constants depending only on σ and $\theta, \theta_1 \geqslant 0$ are given by

$$\theta = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{(2\sigma+1)(2\sigma+2)}\right), \qquad \theta_1 = \frac{n}{2}\left(\frac{1}{2} + \frac{1}{n} - \frac{1}{2\sigma+2}\right)$$

By the assumptions on σ, $\theta, \theta_1 \leqslant 1$. Using (3.16) of Section 3,

$$\int |v|^{(2\sigma+1)(2\sigma+2)} \leqslant \mathscr{H}(v_0), \qquad \int |\nabla v|^{2\sigma+2} \leqslant \mathscr{K}(v_0) \tag{4.9}$$

with $\mathscr{H}(v_0)$, $\mathscr{K}(v_0)$ given by

$$\mathscr{H}(v_0) \equiv C_3(\sigma)\, \mathscr{A}(v_0)^{\theta(2\sigma+1)(\sigma+1)}\, \|v_0\|_{L^2}^{(1-\theta)(2\sigma+1)(2\sigma+2)} \tag{4.10}$$

$$\mathscr{K}(v_0) \equiv C_4(\sigma)\, \mathscr{A}(v_0)^{\theta_1(2\sigma+2)}\, \|v_0\|_{L^2}^{(1-\theta_1)(2\sigma+2)}. \tag{4.11}$$

Collecting the estimates (4.4)–(4.11), integrating in t, and letting $R \to \infty$ and then $\varepsilon \to 0$, we obtain

$$\int |w|^{2\sigma+2} \leqslant e^{C(\sigma)t} \int |u_0 - v_0|^{2\sigma+2} + C_5(\sigma)\, b^{2\sigma+2} \mathscr{H}(v_0)(e^{C(\sigma)t} - 1)$$
$$+ C_6(\sigma)(a\delta^{-1})^{\sigma+1}\, \mathscr{K}(v_0)(e^{C(\sigma)t} - 1)$$

where C, C_5, C_6 are constants depending only on σ. This concludes the proof of Theorem 4.1. ∎

5. H^1 INVISCID LIMIT

Motivated by the inviscid limit results concerning the derivatives of solutions to the Navier–Stokes equations [6], we consider in this section the H^1 inviscid limit of the CGL equation.

We first state the main result.

THEOREM 5.1. *Let $n\sigma \leqslant 2$. Assume $u_0 \in H^1(\mathbb{R}^n)$ ($\|u_0\|_{L^2}$ is small if $n\sigma = 2$) and $v_0 \in H^2(\mathbb{R}^n)$ ($\|v_0\|_{L^2}$ is small if $\mu < 0$ and $n\sigma = 2$). Consider the difference*

$$w(x, t) = u(x, t) - v(x, t)$$

between a solution u of the CGL equation 1.1 *with $u(0) = u_0$ and v of the NLS equation* 1.2 *with $v(0) = v_0$. Then w obeys for any $T < \infty$ and $t < T$*

$$\|\nabla w(t)\|_{L^2}^2 \leqslant \|\nabla u_0 - \nabla v_0\|_{L^2}^2 + 2a\mathscr{A}(v_0)\, t + 4a^{-1}(b^2 + \mu^2)\, \mathscr{D}(v_0)\, t$$
$$+ 2a^{-1}(b^2 + \mu^2)\, \mathscr{L}(v_0, a, b, t) \tag{5.1}$$

where $\mathscr{A}(v_0)$, $\mathscr{D}(v_0)$ given previously in (3.16), (3.17) *are bounds in terms of v_0 but independent of a, b, and $\mathscr{L}(v_0, a, b, t)$ depends on v_0, a, b, t (defined in* (5.10), (5.11) *below) and is of order $a^{(-n\sigma)/2}$ if $b^2 + \mu^2$ is of order a^2 or higher for small a.*

In particular, (5.1) *implies that if $u_0 = v_0$ and $b^2 + \mu^2 = O(a^2)$ or higher order, then we have the inviscid limit result*

$$\|\nabla(u - v)(t)\|_{L^2}^2 = O(a) + O((b^2 + \mu^2)\, a^{-1-(n\sigma/2)}).$$

Before we prove the theorem, we obtain an estimate for $\int_0^t \|\Delta u(t')\|_{L^2}^2 \, dt'$, which will be used in the proof of the theorem. The point here is that the bound for $\int_0^t \|\Delta u(t')\|_{L^2}^2 \, dt'$ depends explicitly on a and b.

PROPOSITION 5.2. *Assume $\sigma \leqslant 2/(n-4)$ for $n \geqslant 5$. Let u be a solution of the CGL equation with $u_0 \in H^1(\mathbb{R}^n)$. Then u obey the estimate*

$$\|\nabla u(t)\|_{L^2}^2 + a \int_0^t \|\Delta u(t')\|_{L^2}^2 \, dt' - \left(C(\sigma) \frac{b^2+\mu^2}{a} \|u_0\|_{L^2}^{4\sigma+2-n\sigma} \right) \int_0^t \|\Delta u(t')\|_{L^2}^{n\sigma} \, dt' \leqslant \|\nabla u_0\|_{L^2}^2 \quad (5.2)$$

where $C(\sigma)$ is a constant. Furthermore, if $n\sigma = 2$ and $\|u_0\|_{L^2}$ is small enough, say, $\|u_0\|_{L^2}^{4\sigma+2-n\sigma} < C^{-1}(\sigma)\, a^2 (b^2+\mu^2)^{-1}$ then

$$\int_0^t \|\Delta u(t')\|_{L^2}^2 \, dt' \leqslant (a - C(\sigma)\, a^{-1}(b^2+\mu^2)\, \|u_0\|_{L^2}^{4\sigma+2-n\sigma})^{-1} \, \|\nabla u_0\|_{L^2}^2 \quad (5.3)$$

and if $n\sigma < 2$, then

$$\int_0^t \|\Delta u(t')\|_{L^2}^2 \, dt' \leqslant \max\{2a^{-1} \|\nabla u_0\|_{L^2}^2, [C(\sigma)\, a^{-2}(b^2+\mu^2)\, \|u_0\|_{L^2}^{2-(n-4)\sigma}]^{2/(2-n\sigma)} \, t\}. \quad (5.4)$$

Proof of Proposition 5.1. Let $\phi_R^2(x) (R > 0)$ be the cutoff function as defined in the proof of Theorem 3.1. It is easy to see that

$$\partial_t \int \phi_R^2 \, |\nabla u|^2 = 2Re \int \phi_R^2 (\partial_t \nabla u) \, \nabla \bar{u} = \mathrm{I} + \mathrm{II}$$

where I and II are given by

$$\mathrm{I} = 2Re(a+iv) \int \phi_R^2 \, \Delta(\nabla u) \, \nabla \bar{u}$$

$$\mathrm{II} = -2Re(b+i\mu) \int \phi_R^2 \, \nabla(|u|^{2\sigma} u) \, \nabla \bar{u}.$$

Integrating by parts,

$$\mathrm{I} = -2a \int \phi_R^2\, |\Delta u|^2 - 4Re(a+iv) \int (\phi_R\, \Delta u)(\nabla \phi_R\, \nabla \bar{u})$$

$$\mathrm{II} = 2Re(b+i\mu) \int \phi_R^2\, |u|^{2\sigma}\, u\, \Delta \bar{u} + 4Re(b+i\mu) \int \phi_R\, |u|^{2\sigma}\, u\, \nabla \phi_R\, \nabla \bar{u}.$$

Using Young's inequality to split the terms in I and II as in the proof of Theorem 3.1, adding them and letting $R \to \infty$,

$$\partial_t \int |\nabla u|^2 \leqslant -a \int |\Delta u|^2 + \frac{8(b^2+\mu^2)}{a} \int |u|^{4\sigma+2}.$$

Applying Gagliardo–Nirenberg's inequality to $\int |u|^{4\sigma+2}$

$$\|u\|_{L^{4\sigma+2}} \leqslant C(\sigma)\, \|\Delta u\|_{L^2}^{\theta}\, \|u\|_{L^2}^{1-\theta}$$

where $C(\sigma)$ is a constant depends on σ and $0 \leqslant \theta$ is given by

$$\theta = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{4\sigma+2}\right)$$

and $\theta \leqslant 1$ because of the assumption $\sigma \leqslant 2/(n-4)$. Therefore,

$$\partial_t \int |\nabla u|^2 + a\, \|\Delta u\|_{L^2}^2 - \frac{C(\sigma)(b^2+\mu^2)}{a}\, \|\Delta u\|_{L^2}^{n\sigma}\, \|u_0\|_{L^2}^{4\sigma+2-n\sigma} \leqslant 0.$$

We obtain (5.2) after integrating in t. (5.3) follows easily from (5.2). (5.4) is obtained from (5.2) and using Lemma 3.7. ∎

We now prove Theorem 5.

Proof of Theorem 5.1. The difference $w = u - v$ satisfies the equation

$$\partial_t w = (a+iv)\, \Delta w + a\, \Delta v - (b+i\mu)(f(u) - f(v)) - bf(v)$$

where $f(u) = |u|^{2\sigma}\, u$. Let $\phi_R(x)$ be the cutoff function as before and we find

$$\partial_t \int \phi_R^2\, |\nabla w|^2 = 2Re(a+iv) \int \phi_R^2 \Delta(\nabla w)\, \nabla \bar{w} + 2aRe \int \phi_R^2 \Delta(\nabla v)\, \nabla \bar{w}$$

$$-2Re(b+i\mu) \int \phi_R^2\, \nabla(f(u) - f(v))\, \nabla \bar{w} - 2bRe \int \phi_R^2\, \nabla f(v)\, \nabla \bar{w}.$$

We mark the four terms on the RHS as I, II, III, IV and estimate them separately.

$$\mathrm{I} = -2a \int \phi_R^2\, |\Delta w|^2 - 4Re(a+iv) \int \phi_R \Delta w (\nabla \phi_R\, \nabla \bar{w})$$

$$\mathrm{II} = -2aRe \int \phi_R^2\, \Delta v\, \Delta \bar{w} - 4aRe \int \phi_R\, \Delta v (\nabla \phi_R\, \nabla \bar{w})$$

Using Young's inequality to split the terms in I and II in the same way as in the proof of Theorem 3.1, we obtain

$$|\mathrm{I}| + |\mathrm{II}| \leqslant -a \int \phi_R^2\, |\Delta w|^2 + (2a+\varepsilon)\, \|\phi_R\, \Delta v\|_{L^2}^2$$
$$+ 8 \left(\frac{a^2+v^2}{a} + a^2 \varepsilon^{-1} \right) R^{-2}\, \|\nabla \phi\|_{L^\infty}^2\, \|\nabla w\|_{L^2}^2 \tag{5.5}$$

where $\varepsilon > 0$ is small. Integrating by parts,

$$\mathrm{III} = 2Re(b+i\mu) \int \phi_R^2 (f(u) - f(v))\, \Delta \bar{w}$$
$$+ 4Re(b+i\mu) \int \phi_R (f(u) - f(v))\, \nabla \phi_R\, \nabla \bar{w}$$

$$\mathrm{IV} = 2bRe \int \phi_R^2\, f(v)\, \Delta \bar{w} + 4bRe \int \phi_R\, f(v)\, \nabla \phi_R\, \nabla \bar{w}.$$

Breaking the terms in III and IV we find that

$$|\mathrm{III} + \mathrm{IV}| \leqslant a \int \phi_R^2\, |\Delta w|^2 + 2a^{-1}(b^2+\mu^2) \int \phi_R^2\, |f(u)|^2$$
$$+ 2a^{-1}(2b^2+\mu^2) \int \phi_R^2 |f(v)|^2$$
$$+ 8\, |b+i\mu|\, R^{-1}\, \|\nabla \phi\|_{L^\infty} \left(\int (|f(u)|^2 + |f(v)|^2) \right)^{1/2} \left(\int |\nabla w|^2 \right)^{1/2}. \tag{5.6}$$

Collecting the estimates (5.5), (5.6), integrating in t and letting $R \to \infty$ and $\varepsilon \to 0$,

$$\int |\nabla(u(t)-v(t))|^2 \leqslant \int |\nabla(u_0-v_0)|^2 + 2a\int_0^t \|\Delta v(t')\|_{L^2}^2\, dt'$$
$$+4a^{-1}(b^2+\mu^2)\int_0^t \|v(t')\|_{L^{4\sigma+2}}^{4\sigma+2}\, dt'$$
$$+2a^{-1}(b^2+\mu^2)\int_0^t \|u(t')\|_{L^{4\sigma+2}}^{4\sigma+2}\, dt'. \tag{5.7}$$

As shown in the proof of Theorem 3.8,

$$\|\Delta v\|_{L^2}^2 \leqslant \mathscr{A}(v_0), \qquad \|v\|_{L^{4\sigma+2}}^{4\sigma+2} \leqslant \mathscr{D}(v_0) \tag{5.8}$$

with $\mathscr{A}(v_0)$ and $\mathscr{D}(v_0)$ given in (3.16), (3.17). The estimate for $\int |u|^{4\sigma+2}$ is already given in the proof of Proposition 5.2:

$$\int |u|^{4\sigma+2} \leqslant C(\sigma)\, \|\Delta u\|_{L^2}^{n\sigma}\, \|u\|_{L^2}^{2-(n-4)\sigma}$$

where C depends on σ only. Therefore,

$$\int_0^t \|u(t')\|_{L^{4\sigma+2}}^{4\sigma+2}\, dt' \leqslant C(\sigma)\, \|u_0\|_{L^2}^{2-(n-4)\sigma} \left(\int_0^t \|\Delta u(t')\|_{L^2}^2\, dt'\right)^{n\sigma/2} t^{1-(n\sigma/2)}. \tag{5.9}$$

It then follows from Proposition 5.2 that if $n\sigma = 2$ and $\|u_0\|_{L^2}$ is small

$$\int_0^t \|u(t')\|_{L^{4\sigma+2}}^{4\sigma+2}\, dt'$$
$$= C(\sigma)\, \|u_0\|_{L^2}^{2-(n-4)\sigma}\, \|\nabla u_0\|_{L^2}^2\, (a - C(\sigma)\, a^{-1}(b^2+\mu^2)\, \|u_0\|_{L^2}^{4\sigma+2-n\sigma})^{-1} \tag{5.10}$$

and if $n\sigma < 2$

$$\int_0^t \|u(t')\|_{L^{4\sigma+2}}^{4\sigma+2}\, dt' \leqslant \max\{C(\sigma)\, a^{-(n\sigma)/2}\, \|\nabla u_0\|_{L^2}^{n\sigma}\, \|u_0\|_{L^2}^{2-(n-4)\sigma}\, t^{1-(n\sigma/2)},$$
$$C(\sigma)(a^{-2}(b^2+\mu^2))^{n\sigma/(2-n\sigma)}\, \|u_0\|_{L^2}^{2+(8\sigma/(2-n\sigma))}\, t\}. \tag{5.11}$$

The bounds in (5.10) and (5.11) will be denoted by the notation $\mathscr{L}(a, b, v_0, t)$. Clearly if $b^2+\mu^2 = O(a^2)$ for small a, $\mathscr{L}(a, b, v_0, t)$ is of order $a^{-n\sigma/2}$.

The proof of this theorem is completed after inserting (5.8), (5.9) with (5.10) and (5.11) into (5.7). ∎

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