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Dissipative quasi-geostrophic equations with L^p data *

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Abstract

We seek solutions of the initial value problem for the 2D dissipative quasi-geostrophic (QG) equation with L^p initial data. The 2D dissipative QG equation is a two dimensional model of the 3D incompressible Navier-Stokes equations. We prove global existence and uniqueness of regular solutions for the dissipative QG equation with sub-critical powers. For the QG equation with critical or super-critical powers, we establish explicit global L^p bounds for its solutions and conclude that any possible finite time singularity must occur in the first order derivative.

1 Introduction

We study in this paper the 2D dissipative quasi-geostrophic (QG) equation

$$\partial_t\theta + u\cdot\nabla\theta + \kappa(-\Delta)^{\alpha}\theta = f, \quad x\in\mathbb{R}^2, \quad t>0, \tag{1.1}$$

where $\kappa>0$ is the diffusivity coefficient, $\alpha\in[0,1]$ is a fractional power, and $u=(u_1,u_2)$ is the velocity field determined from θ by a stream function ψ via the auxiliary relations

$$u=(u_1,u_2)=\left(-\frac{\partial\psi}{\partial x_2},\frac{\partial\psi}{\partial x_1}\right) \quad \text{and} \quad (-\Delta)^{1/2}\psi=-\theta. \tag{1.2}$$

A fractional power of the Laplacian $(-\Delta)^{\beta}$ is defined by

$$\widehat{(-\Delta)^{\beta}}f(\xi)=(2\pi\,|\xi|)^{2\beta}\widehat{f}(\xi),$$

where $\widehat{f}$ denotes the Fourier transform of f. One may consult the book of Stein [6, p.117] for more details. For notational convenience, we will denote $(-\Delta)^{1/2}$ by Λ. The relation in (1.2) can then be identified with

$$u=\left(\partial_{x_2}\Lambda^{-1}\theta,\,-\partial_{x_1}\Lambda^{-1}\theta\right)=(-\mathcal{R}_2\theta,\,\mathcal{R}_1\theta),$$

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where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the Riesz transforms [6, p.57].

Equation (1.1) is the dissipative version of the inviscid QG equation derived by reducing the general QG models describing atmospheric and oceanic fluid flow under special circumstances of physical interest ([4],[1]). Physically, the scalar θ represents the potential temperature, u is the fluid velocity and ψ can be identified with the pressure. Mathematically, the 2D QG equation serves as a lower dimensional model of the 3D Navier-Stokes equations because of the striking similarity between the behavior of its solution and that of the potentially singular solutions of the 3D hydrodynamic equations.

Our aim of this paper is to establish global existence and uniqueness results for the initial-value problem (IVP) for the QG equation (1.1) with the initial condition

$$\theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^2. \tag{1.3}$$

We seek solutions of the IVP (1.1) and (1.3) in $L^q([0,T];L^p)$ for initial data $\theta_0 \in L^r(\mathbb{R}^2)$. The notation L^r is standard while $L^q([0,T];L^p)$ stands for the space of functions f of x and t satisfying

$$\|f\|_{L^q([0,T];L^p)} = \Big(\int_0^T \big(\int_{\mathbb{R}^2} |f(x,t)|^p dx\big)^{q/p} dt\Big)^{1/q} < \infty.$$

We distinguish between two cases: $\alpha > 1/2$ (the "sub-critical" case) and $\alpha \le 1/2$ (the "critical" or " super-critical" case). In the $\alpha > 1/2$ case, we establish that the IVP (1.1) and (1.3) has a unique global (in time) and regular solution in $L^q([0,T];L^p)$. Precise statements are presented in Section 3. It is not clear in the $\alpha \le \frac{1}{2}$ case whether regular solutions develop finite time singularities. But We show in Section 4 that any singularity must occur in the first derivative if there is a singularity. This is achieved by obtaining explicit L^p bounds for all high order derivatives of any function solving the IVP (1.1) and (1.3).

In preparation, we provide in Section 2 properties of the solution operator for the linear QG equation and show its boundedness when acting on L^p spaces.

2 The solution operator for the linear equation

Consider the solution operator for the linear QG equation

$$\partial_t \theta + \kappa\, \Lambda^{2\alpha} \theta = 0, \quad x \in \mathbb{R}^2, \quad t > 0,$$

where $\kappa > 0$, Λ denotes $(-\Delta)^{1/2}$ and $\alpha \in [0,1]$. For a given initial data θ_0, the solution of this equation is given by

$$\theta = G_\alpha(t)\, \theta_0 = e^{-\kappa\, \Lambda^{2\alpha} t}\, \theta_0,$$

where $G_\alpha(t) \equiv e^{-\kappa \Lambda^{2\alpha} t}$ is a convolution operator with its kernel g_α being defined through the Fourier transform

$$\widehat{g_\alpha}(\xi, t) = e^{-\kappa |\xi|^{2\alpha} t}.$$

The kernel g_α possesses similar properties as the heat kernel does. For example, for $\alpha \in [0,1]$ and $t>0$, $g_\alpha(x,t)$ is a nonnegative and non-increasing radial function, and satisfies the dilation relation

$$g_\alpha(x,t) = t^{-1/\alpha} g_\alpha(xt^{-1/(2\alpha)}, 1). \tag{2.1}$$

Furthermore, the operators G_α and ∇G_α are bounded on L^p. To prove this fact, we need the following lemma.

Lemma 2.1 *For $t>0$, $\|g_\alpha(\cdot,t)\|_{L^1} = 1$ and for $1 \le p < \infty$*

$$|g_\alpha(\cdot,t) * f|^p \le g_\alpha(\cdot,t) * |f|^p.$$

Proof. For any $t>0$, $\|g_\alpha(\cdot,t)\|_{L^1} = \widehat{g_\alpha}(0,t) = 1$. By Hölder's inequality,

$$\begin{aligned} |g_\alpha(\cdot,t) * f|^p &= \left| \int_{\mathbb{R}^2} g_\alpha^{1/q}(x-y,t) \cdot g_\alpha^{1/p}(x-y,t) f(y) dy \right|^p \\ &\le \|g_\alpha(\cdot,t)\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^2} g_\alpha(x-y,t)|f(y)|^p dy = g_\alpha(\cdot,t) * |f|^p, \end{aligned}$$

where $(1/q)+(1/p)=1$.

Proposition 2.2 *Let $1 \le p \le q \le \infty$. For any $t>0$, the operators $G_\alpha(t)$ and $\nabla G_\alpha(t)$ are bounded operators from L^p to L^q. Furthermore, we have for any $f \in L^p$,*

$$\|G_\alpha(t) f\|_{L^q} \le C t^{-\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p} \tag{2.2}$$

and

$$\|\nabla G_\alpha(t) f\|_{L^q} \le C t^{-\left(\frac{1}{2\alpha}+\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \|f\|_{L^p} \tag{2.3}$$

where C is a constant depending on α, p and q only.

Proof. We first prove (2.2). For $p=q=\infty$, we have

$$\|G_\alpha(t) f\|_{L^\infty} \le \|g_\alpha(\cdot,t)\|_{L^1} \|f\|_{L^\infty} = \|f\|_{L^\infty}.$$

For $p=q<\infty$, we combine Lemma 2.1 and Young's inequality to obtain

$$\begin{aligned} \|G_\alpha(t) f\|_{L^p}^p &= \|g_\alpha(\cdot,t) * f\|_{L^p}^p \le \int_{\mathbb{R}^2} g_\alpha(\cdot,t) * |f|^p dx \\ &\le \|g_\alpha(\cdot,t)\|_{L^1} \|f\|_{L^p}^p = \|f\|_{L^p}^p \end{aligned}$$

To prove the general case, we first estimate $\|G_\alpha(t) f\|_{L^\infty}$. Without loss of generality, we consider $G_\alpha(t) f$ at $x=0$.

$$\begin{aligned} |(G_\alpha(t) f)(0)|^p &\le \int_{\mathbb{R}^2} g_\alpha(|x|,t)|f(x)|^p dx = \int_0^\infty g_\alpha(\rho,t) dr(\rho) \\ &\le \int_0^\infty |g_\alpha'(\rho,t)| r(\rho) d\rho \le \|f\|_{L^p}^p \cdot \int_0^\infty |g_\kappa'(\rho,t)| d\rho \end{aligned} \tag{2.4}$$

where $r(\rho) = \int_{|y|\le\rho} |f(y)|^p dy$ and $g'_\alpha = \frac{\partial g_\alpha}{\partial\rho}$. Using (2.1), one easily sees that for some constant C

$$\int_0^\infty |g'_\kappa(\rho,t)|d\rho = C\,t^{-1/\alpha}$$

and therefore (2.4) becomes (since $x=0$ is not special !)

$$\|G_\alpha(t)f\|_{L^\infty} \le C\,t^{-\frac{1}{p\alpha}}\|f\|_{L^p}.$$

We now estimate $\|G_\alpha(t)f\|_{L^q}$ in terms of $\|f\|_{L^p}$ for $1\le p\le q<\infty$.

$$\|G_\alpha(t)f\|^q_{L^q} \le C\|G_\alpha(t)f\|^{q-p}_{L^\infty}\|G_\alpha(t)f\|^p_{L^p} \le Ct^{-\frac{1}{p\alpha}(q-p)}\|f\|^{q-p}_{L^p}\cdot\|f\|^p_{L^p}.$$

That is, $\|G_\alpha(t)f\|_{L^q} \le Ct^{-\frac{1}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p}$.

Estimate (2.3) can be proved similarly by using the identity

$$\partial_x g_\alpha(x,t) = t^{-1/(2\alpha)}\tilde{g}_\alpha(x,t)$$

where $\tilde{g}_\alpha$ is another radial function enjoying the same properties as g_α does.

The following lemma provides point-wise bounds for ∇g_α.

Lemma 2.3 *Let $\alpha\in(0,1]$. Then for any $x\in\mathbb{R}^2\setminus\{0\}$, $t>0$, $j=1$ or 2,*

$$|\partial_{x_j} g_\alpha(x,t)| \le \begin{cases} \frac{C}{|x|\,t^{\frac{1}{\alpha}}}, \\ \frac{C}{|x|^2\,t^{\frac{1}{2\alpha}}}, \\ \frac{C}{|x|^3\,t}, \end{cases} \tag{2.5}$$

where C is an explicit constant depending on α only.

Proof. Consider the Fourier transform of $F(x,t) = x_i\partial_{x_j} g_\alpha(x,t)$:

$$\begin{aligned} \widehat{F}(\xi,t) &= i\frac{\partial}{\partial\xi_i}\left(i\xi_j\widehat{g_\alpha}(\xi,t)\right) = (-1)\frac{\partial}{\partial\xi_i}\left(\xi_j\,e^{-\kappa|\xi|^{2\alpha}t}\right) \\ &= \left(-\delta_{ij} + 2\kappa\alpha\,t\,\xi_i\,\xi_j|\xi|^{2\alpha-2}\right)\,e^{-\kappa|\xi|^{2\alpha}t}, \end{aligned}$$

where δ_{ij} is the Kronecker delta. Therefore, for $x\in\mathbb{R}^2$ and $t>0$,

$$\begin{aligned} |x_i\partial_{x_j} g_\alpha(x,t)| &= |F(x,t)| \le \|\widehat{F}(\cdot,t)\|_{L^1} \le \int_{\mathbb{R}^2}(1+2\kappa\alpha|\xi|^{2\alpha}t)e^{-\kappa|\xi|^{2\alpha}t}d\xi \\ &= 2\pi\int_0^\infty(1+2\kappa\alpha\rho^{2\alpha}t)e^{-\kappa\rho^{2\alpha}t}\rho\,d\rho = C\,t^{-1/\alpha}. \end{aligned}$$

where $C = \frac{\pi}{\alpha}\int_0^\infty(1+2\kappa\alpha\,r)\,r^{\frac{1}{\alpha}-1}\,e^{-\kappa r}dr$. This proves the first inequality in (2.5). The next two inequalities can be established in a similar fashion by considering $F(x,t) = x_i\,x_k\,\partial_{x_j}\,g_\alpha(x,t)$ and $F(x,t) = x_l\,x_i\,x_k\,\partial_{x_j}\,g_\alpha(x,t)$, respectively, where the indices $i,j,k,l = 1$ or 2.

We will need the Hardy-Littlewood-Sobolev inequality, which we now recall. It states that the fractional integral

$$Tf(x) = \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{n-\gamma}} dy, \quad 0 < \gamma < n$$

is a bounded operator from L^p to L^q if p and q satisfies

$$1 \le p < q < \infty, \quad \frac{1}{q} + \frac{\gamma}{n} = \frac{1}{p}.$$

One can find the Hardy-Littlewood-Sobolev inequality in [6, p.119].

3 Global existence and uniqueness in the $\alpha > 1/2$ case

In this section we consider the IVP for the dissipative QG equation

$$\begin{gathered} \theta_t + u \cdot \nabla\theta + \kappa\Lambda^{2\alpha}\theta = f, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty), \\ u = (u_1, u_2) = (-\mathcal{R}_2\theta,\, \mathcal{R}_1\theta), \quad (x,t) \in \mathbb{R}^2 \times [0,\infty), \\ \theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^2, \end{gathered} \tag{3.1}$$

where $\kappa > 0$ and $\alpha \in [0,1]$. Our major result is that the IVP (3.1) with $\alpha > 1/2$, $\theta_0 \in L^r$ and $f \in L^{q'}([0,T]; L^{r_1})$ has a unique global (in time) solution in $L^q([0,T]; L^p)$ for proper p, q, q', r and r_1. Furthermore, the solution is shown to be smooth if θ_0 and f are sufficiently smooth. Precise statements will be presented in Theorem 3.4 and Theorem 3.5.

The theorems of this section are proved by the method of integral equations and the contraction mapping argument. To proceed, we write the QG equation into the integral form

$$\theta(t) = G_\alpha(t)\,\theta_0 + \int_0^t G_\alpha(t-\tau)\,(f - u\cdot\nabla\theta)(\tau)d\tau, \tag{3.2}$$

We observe that $u \cdot \nabla\theta = \nabla \cdot (u\theta)$ because $\nabla \cdot u = 0$. The nonlinear term can then be alternatively written as

$$B(u,\theta)(t) \equiv \int_0^t \nabla G_\alpha(t-\tau)(u\theta)(\tau)d\tau.$$

We will solve (3.2) in $L^p([0,T]; L^q)$ and the following estimates for the operator B acting on this type of spaces will be used.

Proposition 3.1 *Let $\alpha > 1/2$ and $T > 0$. Assume that u and θ are in $L^q([0,T]; L^p)$ with p and q satisfying*

$$p > \frac{2}{2\alpha - 1}, \quad \frac{1}{p} + \frac{\alpha}{q} = \alpha - \frac{1}{2}.$$

Then the operator B is bounded in $L^q([0,T];L^p)$ with

$$\|B(u,\theta)\|_{L^q([0,T];L^p)} \le C\|u\|_{L^q([0,T];L^p(\mathbb{R}^2))}\cdot\|\theta\|_{L^q([0,T];L^p(\mathbb{R}^2))}.$$

where C is a constant depending on α, p and q only.

Proof. For $p > \frac{2}{2\alpha-1} \ge 2$, we obtain after applying (2.3) of Proposition 2.2

$$\begin{aligned}
\|B(u,\theta)\|_{L^p} &\le \int_0^t \|\nabla G_\alpha(t-\tau)(u\,\theta)(\tau)\|_{L^p}d\tau \\
&\le C\int_0^t \frac{1}{|t-\tau|^{\frac{1}{2\alpha}+\frac{1}{\alpha}\left(\frac{2}{p}-\frac{1}{p}\right)}}\,\|u\,\theta(\cdot,\tau)\|_{L^{p/2}}d\tau \qquad (3.3)\\
&\le C\int_0^t \frac{1}{|t-\tau|^{\frac{1}{2\alpha}+\frac{1}{p\,\alpha}}}\|u(\cdot,\tau)\|_{L^p}\,\|\theta(\cdot,\tau)\|_{L^p}d\tau
\end{aligned}$$

for some constant C depending on α and p only. For $\alpha > 1/2$ and $p > \frac{2}{2\alpha-1}$, we have

$$0 < \frac{1}{2\alpha} + \frac{1}{p\,\alpha} < 1.$$

Applying the Hardy-Littlewood-Sobolev inequality to (3.3) with

$$\frac{1}{q} + \frac{1-\frac{1}{2\alpha}-\frac{1}{p\,\alpha}}{1} = \frac{2}{q}, \quad i.e., \quad \frac{1}{p}+\frac{\alpha}{q} = \alpha - \frac{1}{2},$$

we obtain

$$\begin{aligned}
\|B(u,\theta)\|_{L^q([0,T];L^p)} &\le C\|\left(\|u(\cdot,t)\|_{L^p}\|\theta(\cdot,t)\|_{L^p}\right)\|_{L^{q/2}([0,T])}\\
&\le C\|u\|_{L^q([0,T];L^p(\mathbb{R}^2))}\cdot\|\theta\|_{L^q([0,T];L^p(\mathbb{R}^2))}.
\end{aligned}$$

The next two lemmas detail how G_α behaves when acting on θ_0 and f.

Lemma 3.2 *Let $1/2 < \alpha \le 1$, $T > 0$, and p and q satisfy*

$$p > \frac{2}{2\alpha-1}, \quad \frac{1}{p}+\frac{\alpha}{q} = \alpha-\frac{1}{2}.$$

Assume that $\theta_0 \in L^r(\mathbb{R}^2)$ with $\frac{2}{2\alpha-1} < r \le p$. Then we have

$$\|G_\alpha(t)\theta_0\|_{L^q([0,T];L^p)} \le C\,T^{1-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r}\right)}\,\|\theta_0\|_{L^r},$$

where C is a constant depending on α, p, q and r only.

Proof. By (2.2),

$$\begin{aligned}\|G_\alpha(t)\theta_0\|_{L^q([0,T];L^p)} &= \Big[\int_0^T \|G_\alpha(t)\theta_0\|^q_{L^p} dt\Big]^{1/q}\\ &\le \Big[\int_0^T t^{-\frac{1}{\alpha}(\frac{1}{r}-\frac{1}{p})\cdot q}\|\theta_0\|^q_{L^r}\, dt\Big]^{1/q}\\ &= CT^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{r})}\,\|\theta_0\|_{L^r}.\end{aligned}$$

Lemma 3.3 *Let $1/2<\alpha\le 1$, $T>0$, and p and q satisfy*

$$p>\frac{2}{2\alpha-1},\quad \frac{1}{p}+\frac{\alpha}{q}=\alpha-\frac{1}{2}.$$

Assume $f\in L^{q'}([0,T];L^{r_1})$ with q' being the conjugate of q (i.e., 1/q'+1/q=1) and r_1 satisfying $\frac{2}{2\alpha-1}<r_1\le p$. Then

$$\Big\|\int_0^t G_\alpha(t-\tau)f(\tau)d\tau\Big\|_{L^q([0,T];L^p)} \le CT^{1+\frac{1}{q}-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r_1}\right)}\,\|f\|_{L^{q'}([0,T];L^{r_1})},$$

where C is a constant depending on α, p, q and r_1 only.

Proof. The result is a consequence of direct computation. By (2.2) and then Hölder's inequality,

$$\begin{aligned}&\|\int_0^t G_\alpha(t-\tau)f(\tau)d\tau\|_{L^q([0,T];L^p)}\\ &\le \Big[\int_0^T \Big(\int_0^t (t-\tau)^{-\frac{1}{\alpha}\left(\frac{1}{r_1}-\frac{1}{p}\right)}\|f(\cdot,\tau)\|_{L^{r_1}}d\tau\Big)^q dt\Big]^{1/q}\\ &\le \Big[\int_0^T \int_0^t (t-\tau)^{-\frac{1}{\alpha}\left(\frac{1}{r_1}-\frac{1}{p}\right)\cdot q}\,d\tau\cdot\Big(\int_0^t \|f(\cdot,\tau)\|^{q'}_{L^{r_1}}d\tau\Big)^{q/q'}\, dt\Big]^{1/q}\\ &\le CT^{1+\frac{1}{q}-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r_1}\right)}\|f\|_{L^{q'}([0,T];L^{r_1})}.\end{aligned}$$

Now we state and prove the main theorem.

Theorem 3.4 *Let $1/2<\alpha\le 1$, $T>0$, and p and q satisfy*

$$p>\frac{2}{2\alpha-1},\quad \frac{1}{p}+\frac{\alpha}{q}=\alpha-\frac{1}{2}.$$

Assume that $\theta_0\in L^r(\mathbb{R}^2)$ with $\frac{2}{2\alpha-1}<r\le p$ and $f\in L^{q'}([0,T];L^{r_1})$ with $\frac{2}{2\alpha-1}<r_1\le p$, where q' denotes the conjugate of q (i.e., 1/q'+1/q=1). Then there exists a constant C such that for any θ_0 and f satisfying

$$T^{1-\frac{1}{\alpha}(\frac{1}{2}+\frac{1}{r})}\,\|\theta_0\|_{L^r}+T^{1+\frac{1}{q}-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r_1}\right)}\,\|f\|_{L^{q'}([0,T];L^{r_1})}\le C,$$

there exists a unique strong solution $\theta\in L^q([0,T];L^p)$ for the IVP (3.1) in the sense of (3.2).

Proof. We write the integral equation (3.2) symbolically as $\theta = A\theta$. The operator A is seen as a mapping of the space $E \equiv L^q([0,T];L^p)$ into itself. Let

$$b = T^{1-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r}\right)}\|\theta_0\|_{L^r} + T^{1+\frac{1}{q}-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r_1}\right)}\|f\|_{L^{q'}([0,T];L^{r_1})}$$

and set $R = 2b$. Define B_R to be the closed ball with radius R centered at the origin in E. We now show that if b is bounded by an appropriate constant, then A is a contraction map on B_R. Let θ and $\bar\theta$ be any two elements of B_R. Then we have

$$\|A\,\theta - A\,\bar\theta\|_E = \Big\| \int_0^t G_\alpha(t-\tau)(u\cdot\nabla\theta)d\tau - \int_0^t G_\alpha(t-\tau)(\bar u\cdot\nabla\bar\theta)d\tau \Big\|_E,$$

where u and $\bar u$ are determined by θ and $\bar\theta$, respectively, through the second relation in (3.1). Recalling the notation B, we have

$$\begin{aligned}\|A\,\theta - A\,\bar\theta\|_E &= \big\|B(u-\bar u,\theta) + B(\bar u,\theta-\bar\theta)\big\|_E \\ &\le \|B(u-\bar u,\theta)\|_E + \|B(\bar u,\theta-\bar\theta)\|_E.\end{aligned}$$

It then follows from applying Proposition 3.1 that

$$\|A\,\theta - A\,\bar\theta\|_E \le C\|u-\bar u\|_E\|\theta\|_E + C\|\bar u\|_E\|\theta-\bar\theta\|_E,$$

where C is a constant depending on α, p and q only. Since u and $\bar u$ are Riesz transforms of θ and $\bar\theta$, respectively, the classical Calderon-Zygmund singular integral estimates imply that

$$\|u\|_E \le C\|\theta\|_E, \quad \|\bar u\|_E \le C\|\bar\theta\|_E.$$

One can consult the book of Stein [6] for more details on Riesz transforms. Therefore,

$$\|A\,\theta - A\,\bar\theta\|_E \le C(\|\theta\|_E + \|\bar\theta\|_E)\,\|\theta-\bar\theta\|_E \le C\,R\,\|\theta-\bar\theta\|_E.$$

We now estimate $\|A\theta\|_E$. By Lemma 3.2 and Lemma 3.3, the norm of

$$A\,0 = G_\alpha(t)\theta_0 + \int_0^t G_\alpha(t-\tau)f(\tau)d\tau$$

in E can be bounded by

$$\|A\,0\|_E \le C\,T^{1-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r}\right)}\|\theta_0\|_{L^r} + CT^{1+\frac{1}{q}-\frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{r_1}\right)}\|f\|_{L^{q'}([0,T];L^{r_1})} = b$$

Therefore,

$$\|A\,\theta\|_E = \|A\,\theta - A\,0 + A\,0\|_E \le \|A\,\theta - A\,0\|_E + \|A\,0\|_E \le C\,R\,\|\theta\|_E + b.$$

If $2C\,b \le \frac{1}{2}$, then $C\,R = 2C\,b \le \frac{1}{2}$ and we have

$$\|A\,\theta - A\,\bar\theta\|_E \le \frac{1}{2}\|\theta-\bar\theta\|_E, \quad \text{and} \quad \|A\,\theta\|_E \le R.$$

It follows from the contraction mapping principle that there exists a unique $\theta \in E = L^q([0,T]; L^p)$ solving (3.2). This finishes the proof of Theorem 3.4.

We now show that the solution obtained in the previous theorem is actually smooth. We introduce a notation. For a non-negative multi-index $k = (k_1, k_2)$, we define

$$D^k = \left(\frac{\partial}{\partial_{x_1}}\right)^{k_1} \left(\frac{\partial}{\partial_{x_2}}\right)^{k_2}.$$

and $|k| = k_1 + k_2$.

Theorem 3.5 *Let $1/2 < \alpha \leq 1$, $T > 0$, and p and q satisfy*

$$p > \frac{2}{2\alpha - 1}, \quad \frac{1}{p} + \frac{\alpha}{q} = \alpha - \frac{1}{2}.$$

Assume that for a non-negative multi-index k

$$D^k\,\theta_0 \in L^r(\mathbb{R}^2) \quad \text{and} \quad D^k\,f \in L^{q'}([0,T]; L^{r_1}(\mathbb{R}^2)), \tag{3.4}$$

where $\frac{2}{2\alpha-1} < r \leq p$, $\frac{2}{2\alpha-1} < r_1 \leq p$ and q' denotes the conjugate of q. Then for any non-negative multi-index j with $|j| \leq |k|$

$$D^j\,\theta \in L^q([0,T]; L^p). \tag{3.5}$$

Furthermore, for each j with $0 \leq |j| \leq |k| - 2$ and almost every $t \in [0,T]$

$$\partial_t D^j\,\theta \in L^p(\mathbb{R}^2). \tag{3.6}$$

Proof. The basic tool of establishing (3.5) is still the contraction mapping argument. We first consider the case $|j| = 1$. Taking D of (3.2), we obtain

$$D\,\theta(t) = G_\alpha(t)(D\theta_0) + \int_0^t G_\alpha(t-\tau)(Df(\tau))d\tau + B(D\,u, \theta) + B(u, D\,\theta). \tag{3.7}$$

This integral equation can then be viewed as $(D\theta) = \tilde{A}(D\theta)$ and $\tilde{A}$ is seen as a mapping of the space E consisting of functions θ such that

$$\theta \in L^q([0,T]; L^p) \quad \text{and} \quad D\theta \in L^q([0,T]; L^p).$$

The norm in E is given by

$$\|\theta\|_E = \|\theta\|_{L^q([0,T];L^p)} + \|D\theta\|_{L^q([0,T];L^p)}.$$

For θ_0 and f satisfying (3.4), the first two terms are bounded in E. The two nonlinear terms acting on E have similar bounds as stated in Proposition 3.1. As in the proof of Theorem 3.4, we can then show that $\tilde{A}$ is a contraction mapping of E into itself. Therefore $\tilde{A}$ has a fixed point in E. The uniqueness result of Theorem 3.4 indicates that this θ is just the original θ. Thus we have shown that $D\theta \in L^q([0,T]; L^p)$. The proof of (3.5) for $|j| = 2, 3, \cdots, |k|$ is similar and we thus omit details.

We now prove (3.6) and start with the case $|j| = 0$. Because θ satisfies

$$\partial_t \theta = f - u\cdot\nabla\theta - \kappa\Lambda^{2\alpha}\theta,$$

it suffices to show that the terms on the right are in L^p for almost every t. Since for almost every t

$$f\in L^{r_1},\quad Df\in L^{r_1},\quad u\in L^p,\quad Du\in L^p,\quad D\theta\in L^p,\quad D^2\theta\in L^p,$$

we obtain by applying the Gagliardo-Nirenberg inequality

$$\|f(\cdot,t)\|_{L^p}\le C\|f(\cdot,t)\|_{L^{r_1}}^{1-\sigma}\,\|Df(\cdot,t)\|_{L^{r_1}}^{\sigma},\quad \sigma=\frac{2}{r_1}-\frac{2}{p}.$$

$$\begin{aligned}\|(u\cdot\nabla\theta)(\cdot,t)\|_{L^p} &\le C\|u(\cdot,t)\|_{L^{2p}}\,\|D\theta(\cdot,t)\|_{L^{2p}}\\ &\le C\|u(\cdot,t)\|_{L^p}^{1-\epsilon}\,\|Du(\cdot,t)\|_{L^p}^{\epsilon}\,\|D\theta(\cdot,t)\|_{L^p}^{1-\epsilon}\,\|D^2\theta(\cdot,t)\|_{L^p}^{\epsilon},\end{aligned}$$

where $\epsilon = 1/p$. Therefore, for almost every t

$$f(\cdot,t)\in L^p,\quad u\cdot\nabla\in L^p,\quad \text{and}\quad \Lambda^{2\alpha}\theta\in L^p$$

and this in turn implies $\partial_t\theta\in L^p$. The proof for $\partial_t D^j\theta\in L^p$ with $|j|>0$ is similar. This completes the proof of (3.6).

4 L^p bounds in the $\alpha\le 1/2$ case

For $\alpha > 1/2$, the issue of global existence, uniqueness and regularity concerning the IVP (3.1) with L^r initial data is resolved in Section 3. Our major interest of this section is in the $\alpha\le\frac{1}{2}$ case although all theorems to be presented hold for any $\alpha\in[0,1]$. We conclude that any possible finite time singularity must occur in the first derivative. This is achieved by bounding the L^p norms of all high order derivatives of θ by the initial L^p norms and a magic quantity.

Lemma 4.1 *Let $\alpha\in[0,1]$, $p\in(1,\infty)$ and k be a nonnegative multi-index. Then for any sufficiently smooth θ, we have for any $t\ge 0$*

$$\int_{\mathbb{R}^2}|D^k\theta|^{p-2}(x,t)\,\left(D^k\theta(x,t)\right)\,\Lambda^{2\alpha}D^k\theta(x,t)dx\ge 0.$$

Proof. Let $g_\alpha(x,s)$ be the kernel of the solution operator for the linear QG equation, as defined in the previous section. Then $\Theta(x,s)\equiv g_\alpha(\cdot,s)*(D^k\theta)$ satisfies the equation

$$\partial_s\Theta+\kappa\Lambda^{2\alpha}\,\Theta=0 \tag{4.1}$$

and $\Theta(x,s)\to D^k\theta$ as $s\to 0$. Multiplying both sides of (4.1) by $p|\Theta|^{p-2}\Theta$ and integrate over $\mathbb{R}^2$, we obtain

$$\frac{d}{ds}\int_{\mathbb{R}^2}|\Theta|^p dx+p\,\kappa\int_{\mathbb{R}^2}|\Theta|^{p-2}\,\Theta\Lambda^{2\alpha}\,\Theta dx=0.$$

Integrating the above over $[s_1, s_2]$ with respect to s, we have

$$\int_{\mathbb{R}^2} |\Theta|^p(x,s_2)dx - \int_{\mathbb{R}^2} |\Theta|^p(x,s_1)dx = -p\,\kappa \int_{s_1}^{s_2}\int_{\mathbb{R}^2} |\Theta|^{p-2}\Theta\Lambda^{2\alpha}\,\Theta dxds, \quad (4.2)$$

where s_1 and s_2 are arbitrarily fixed. Applying (2.2) of Proposition 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\Theta|^p(x,s_2)dx &= \|g_\alpha(\cdot,s_2) * (D^k\theta)\|_{L^p}^p \\ &= \|g_\alpha(\cdot,s_2-s_1) * [g_\alpha(\cdot,s_1) * (D^k\theta)]\|_{L^p}^p \\ &\leq \|g_\alpha(\cdot,s_1) * (D^k\theta)\|_{L^p}^p = \int_{\mathbb{R}^2} |\Theta|^p(x,s_1)dx.\end{aligned}$$

That is, the left hand side of (4.2) is not positive. Therefore

$$\int_{s_1}^{s_2}\int_{\mathbb{R}^2} |\Theta|^{p-2}\Theta\Lambda^{2\alpha}\,\Theta dxds \geq 0.$$

The arbitrariness of s_1 and s_2 then implies that for any $s > 0$,

$$\int_{\mathbb{R}^2} |\Theta|^{p-2}(x,s)\,\Theta(x,s)\,\Lambda^{2\alpha}\,\Theta(x,s)dxds \geq 0. \quad (4.3)$$

Letting $s \to 0$ and recalling the definition of Θ, we obtain for any $t \geq 0$

$$\int_{\mathbb{R}^2} |D^k\theta|^{p-2}(x,t)\,(D^k\theta(x,t))\,\Lambda^{2\alpha} D^k\theta(x,t)dx \geq 0.$$

One consequence of the previous lemma is that the L^p-norm ($p \in (1,\infty]$) of any solution θ of the IVP (3.1) is uniformly bounded by the L^p norm of the initial data. Thus finite-time singularity is only possible in the derivatives of θ. The following result was shown in [5] and we now briefly describe it.

Theorem 4.2 *Let $\alpha \in [0,1]$ and $p \in (1,\infty]$. Then any solution θ of the IVP (3.1) satisfies for $t \geq 0$*

$$\|\theta(\cdot,t)\|_{L^p(\mathbb{R}^2)} \leq \|\theta\|_{L^p(\mathbb{R}^2)}.$$

A sketch of the proof for this theorem is given in [2].

We now state and prove our main theorem, in which we establish estimates to bound the L^p norms of derivatives of any solution θ of the IVP (3.1) in terms of ∇u (u is related to θ through the second relation in (3.1)). Roughly speaking, this means that no finite time singularity in high-order derivatives is possible if ∇u does not become infinite first. The role of the forcing term f is not crucial, so we set it equal to zero for the sake of clear presentation.

Theorem 4.3 *Let $\alpha \in [0,1]$. Assume that θ is a solution of the IVP (3.1). Then for any $p \in (1,\infty]$ and a multi-index k with $|k| \geq 1$,*

$$\|D^k\theta(\cdot,t)\|_{L^p} \leq \|D^k\theta_0\|_{L^p} \cdot e^{\int_0^t \|\nabla u(\cdot,\tau)\|_{L^\infty} d\tau} \quad (4.4)$$

holds for any $t \geq 0$, where u is determined by θ through the second relation in (3.1).

Proof. We start with the case $|k|=1$. For $p\in(0,\infty)$, we take D of the first equation in (3.1), multiply by $p\,|D\theta|^{p-2}\,D\theta$ and then integrate over $\mathbb{R}^2$ to obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|D\theta|^p dx+\,p\,\kappa\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\cdot\Lambda^{2\alpha}(D\theta)\,dx\\ &\quad=\quad -p\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\,\cdot D(u\cdot\nabla\theta)dx \end{aligned}\tag{4.5}$$

The right hand side actually consists of two terms

$$-p\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\,\cdot Du\cdot\nabla\theta dx\quad\text{and}\quad -\,p\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\,\cdot u\cdot\nabla(D\theta)dx,$$

but one of them is zero

$$\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\,\cdot u\cdot\nabla(D\theta)dx=\int_{\mathbb{R}^2}u\cdot\nabla\left(|D\theta|^p\right)dx=0$$

because $\nabla\cdot u=0$. Therefore, (4.5) becomes

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|D\theta|^p dx+p\,\kappa\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\cdot\Lambda^{2\alpha}(D\theta)\,dx\\ &\quad=\quad -p\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\,\cdot Du\cdot\nabla\theta dx,\end{aligned}$$

which in turn implies that

$$\frac{d}{dt}\int_{\mathbb{R}^2}|D\theta|^p dx+p\,\kappa\int_{\mathbb{R}^2}|D\theta|^{p-2}\,D\theta\cdot\Lambda^{2\alpha}(D\theta)\,dx\le p\|\nabla u(\cdot,t)\|_{L^\infty}\int_{\mathbb{R}^2}|D\theta|^p dx.$$

By Lemma 4.1, the second term on the left hand side is nonnegative. So

$$\frac{d}{dt}\int_{\mathbb{R}^2}|D\theta|^p dx\le p\,\|\nabla u(\cdot,t)\|_{L^\infty}\int_{\mathbb{R}^2}|D\theta|^p dx.$$

Gronwall's inequality then implies (4.4). Once we have the bound (4.4) for any $p<\infty$, we can then take the limit of (4.4) as $p\to\infty$ to establish (4.4) for $p=\infty$.

The inequality (4.4) for general k can be proved by induction. One needs the Calderon-Zygmund inequality for Riesz transforms

$$\|D^j u(\cdot,t)\|_{L^p}\le C\|D^j\theta(\cdot,t)\|_{L^p},\quad p\in(1,\infty),\quad |j|\le|k|.$$

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