THE QUASI-GEOSTROPHIC EQUATION AND ITS TWO REGULARIZATIONS

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ABSTRACT

We consider the quasi-geostrophic model and its two different regularizations. Global regularity results are established for the regularized models with critical or sub-critical indices. The proof[4,13] of Onsager's conjecture[19] concerning weak solutions of the 3D Euler equations and the notion of dissipative solutions of Duchon and Robert[12] are extended to weak solutions of the quasi-geostrophic equation.

Key Words: Dissipative quasi-geostrophic equation; Onsager's conjecture; Regularized quasi-geostrophic equation; Weak solutions; Global regularity

AMS (MOS) Numbers: 86A05; 35K55; 35Q35; 76U05

1 INTRODUCTION

Consider the two dimensional (2D) quasi-geostrophic (QG) equation

$$\theta_t + u \cdot \nabla\theta = 0 \tag{1.1}$$

and its two different regularizations

$$\theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha\theta = 0 \tag{1.2}$$

and

$$\theta_t + u \cdot \nabla\theta + \mu(-\Delta)^\alpha\theta_t = 0, \tag{1.3}$$

where $\theta(x, t)$ is a real-valued function of x and t, $0 \le \alpha \le 1$, $\kappa > 0$ and $\mu > 0$ are real numbers. The advective velocity u in these equations is determined from θ by a stream function ψ via the auxiliary relations

$$u = (u_1, u_2) = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right) \quad \text{and} \quad (-\Delta)^{1/2}\psi = -\theta. \tag{1.4}$$

Interest will mainly focus on the behavior of solutions of the initial value problems (IVP) for these equations wherein

$$\theta(x, 0) = \theta_0(x), \quad \text{is specified.} \tag{1.5}$$

To avoid questions regarding boundaries, we will assume periodic boundary conditions with period box $\Omega = [0, 2\pi]^2$. Without loss of generality we will restrict the discussion to initial data θ_0 with zero mean, namely $(2\pi)^{-2}\int_\Omega \theta_0(x)dx = 0$.

Equations (1.1) and (1.2) are special cases of the general quasi-geostrophic approximations[20] for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. The variable θ represents potential temperature, u is the fluid velocity and ψ can be identified with the pressure. Equation (1.1) is an important example of a 2D active scalar with a specific structure most closely related to the 3D Euler equations. The equation in (1.2) with $\alpha = 1/2$ is derived from the 3D Navier-Stokes equations under the special circumstance of constant potential vorticity and constant buoyancy frequency and constitutes a dimensionally correct analogue of the 3D Navier-Stokes equations. These equations have recently been intensively investigated because of both their mathematical importance and their potential for applications in meteorology and oceanography.[5–7,10,15,18,20,21]

In modeling long waves in nonlinear dispersive media, Benjamin, Bona and Mahony[1] introduced the BBM equation

$$u_t + u_x + uu_x - u_{xxt} = 0$$

as an alternative to the KdV equation

$$u_t + u_x + u u_x + u_{xxx} = 0.$$

Equation (1.3) to (1.2) is like the BBM to the KdV equation and our motivation for proposing such a model for study comes from the effects of regularizations on the global regularity of weak solutions and the potential applications of this new model in geophysics.

These QG models appear to be simpler than the 3D hydrodynamics equations, but contain many of their difficult features. For instance, solutions of Eqs. (1.1) and (1.2) exhibit strong nonlinear behavior, strikingly analogous to that of the potentially singular solutions of the 3D hydrodynamics equations.[5] Although progress has been made in the past several years,[5–7,10,18,21] the theory remains fundamentally incomplete. In particular, it is not known whether or not weak solutions of Eq. (1.2) are regular for all time when α is equal to the critical index $1/2$. The critical case regularity issue turns out to be extremely difficult and was labeled by Klainerman[16] as one of the most challenging PDE problems of the 21st Century. In Section 2 we explore how far one can go toward a regularity proof in the critical case and what are the weakest assumptions needed to fill the gap.

In Section 3 we solve the global regularity problem for Eq. (1.3) with $\alpha \geq 1/2$. We first construct a local solution θ in H^s with $s > 1$ and then derive explicit bounds on the norms of all derivatives of the solution. From this we infer that for $\alpha > 1/2$ the local solution θ remains bounded in H^s for all time and thus no finite-time singularity can occur in this case. For the critical index $\alpha = 1/2$, global smoothness results are established under assumptions that are much weaker than those needed to guarantee regularity for Eq. (1.2). This leads us to conclude that solutions of Eq. (1.3) are better behaved and thus Eq. (1.3) constitutes a reasonable alternative to Eq. (1.2).

As is well-known, weak solutions of the 3D hydrodynamics equations in general only satisfy an energy inequality rather than equality. But Onsager conjectured in Ref. [19] that weak solutions of the 3D Euler equations in a Hölder space C^γ with exponent $\gamma > 1/3$ should conserve energy. In Ref. [13] Eyink proved energy conservation for weak solutions in a strong form of Hölder space C_*^γ ($\gamma > 1/3$), in which the norm is defined in terms of absolute Fourier coefficients. Constantin et al.[4] provided a proof for the sharp version of Onsager's conjecture. Section 4 is concerned with solutions of the QG Eq. (1.1). First we verify Onsager's conjecture for weak solutions of the QG equation, extending the result of Constantin et al. Then the QG equation is shown to possess the dissipative weak solutions, a notion proposed by Duchon and Robert.[12] Finally the two models Eqs. (1.1) and (1.3)

are proven to be close by considering the limit of Eq. (1.3) as $\mu \to 0$. This provides further evidence for the validity of Eq. (1.3).

We now review the notations used throughout the sequel. The Fourier transform $\widehat{f}$ of a tempered distribution $f(x)$ on Ω is defined as

$$\hat{f}(k) = \frac{1}{(2\pi)^2} \int_\Omega f(x)e^{-ik\cdot x}\, dx.$$

We will denote the square root of the Laplacian $(-\Delta)^{1/2}$ by Λ and obviously

$$\widehat{\Lambda f}(k) = |k|\hat{f}(k).$$

More generally, $\Lambda^\beta f$ for $\beta \in \mathbb{R}$ can be identified with the Fourier series

$$\sum_{k\in\mathbb{Z}^2} |k|^\beta \hat{f}(k)e^{ik\cdot x}.$$

The equality relating u to θ in Eq. (1.4) can be rewritten in terms of periodic Riesz transforms

$$u = \left(\partial_{x_2}\Lambda^{-1}\theta,\ -\partial_{x_1}\Lambda^{-1}\theta\right) = (-\mathcal{R}_2\theta,\ \mathcal{R}_1\theta),$$

where $\mathcal{R}_j, j = 1, 2$ denotes the Riesz transforms defined by

$$\widehat{\mathcal{R}_j f}(k) = -i\frac{k_j}{|k|}\hat{f}(k), \quad k \in Z^2 \setminus \{0\}.$$

$L^p(\Omega)$ denotes the space of the pth-power integrable functions normed by

$$|f|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}.$$

For any tempered distribution f on Ω and $s \in \mathbb{R}$, we define

$$\|f\|_s = |\Lambda^s f|_2 = \left(\sum_{k\in\mathbb{Z}^2} |k|^{2s}|\hat{f}(k)|^2\right)^{1/2}$$

and H^s denotes the Sobolev space of all f for which $\|f\|_s$ is finite. For $1 \le p \le \infty$ and $s \in \mathbb{R}$, the space $\mathcal{L}_s^p(\Omega)$ is a subspace of $L^p(\Omega)$, consisting of all f which can be written in the form $f = \Lambda^{-s}g$, $g \in L^p(\Omega)$ and the $\mathcal{L}_s^p$ norm of f is defined to be the L^p norm of g, i.e.,

$$\|f\|_{p,s} = |g|_p.$$

2 DISSIPATIVE QG EQUATION

In this section we focus on the nonlinear behavior of weak solutions of the initial-value problem (IVP) for the dissipative QG equation

$$
\begin{cases}
\theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = f, & (x,t) \in \Omega \times [0,\infty), \\
u = (u_1, u_2) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), & (x,t) \in \Omega \times [0,\infty), \\
\theta(x,0) = \theta_0(x), & x \in \Omega,
\end{cases}
\tag{2.1}
$$

where $0 \leq \alpha \leq 1$ and $\kappa > 0$ are real numbers. We establish nonlinear estimates which characterize the regularity of weak solutions of the IVP (2.1) with α greater than or equal to the critical index $1/2$.

For $0 \leq \alpha \leq 1$, weak solutions of the IVP (2.1) are known to exist globally in time [21], although the uniqueness of weak solutions is currently open. More precisely, for any $T > 0$, $\theta_0 \in L^2$ and $f \in L^1([0,T]; L^2)$, there exists a weak solution $\theta \in L^\infty([0,T]; L^2) \cap L^2([0,T]; H^\alpha)$ satisfying

$$
|\theta(\cdot,t)|_2^2 + \int_0^t \|\theta(\cdot,\tau)\|_\alpha^2 \, d\tau \leq \left[|\theta_0|_2 + C \int_0^t |f(\cdot,\tau)|_2 \, d\tau \right]^2
$$

Furthermore, if $\theta_0 \in L^q$ and $f \in L^1([0,T]; L^q)$ for $1 < q \leq \infty$, then any weak solution θ satisfies the maximum principle

$$
|\theta(\cdot,t)|_q \leq |\theta_0|_q + \int_0^t |f(\cdot,\tau)|_q \, d\tau
$$

for any $t \leq T$.

Let $s \geq 0$. We now estimate $\|\theta\|_s = |\Lambda^s\theta|_2$. If we take the inner product of $\Lambda^{2s}\theta$ with the first equation in Eq. (2.1), we obtain

$$
\frac{1}{2}\frac{d}{dt}|\Lambda^s\theta|_2^2 + \kappa|\Lambda^{s+\alpha}\theta|_2^2 = (\Lambda^{2s}\theta, f) - (\Lambda^{2s}\theta, u \cdot \nabla\theta)
\tag{2.2}
$$

The first term on the right hand side is bounded above by

$$
|(\Lambda^{2s}\theta, f)| \leq |\Lambda^{s+\alpha}\theta|_2 |\Lambda^{s-\alpha}f|_2 \leq \frac{\kappa}{2}|\Lambda^{s+\alpha}\theta|_2^2 + \frac{1}{2\kappa}|\Lambda^{s-\alpha}f|_2^2
\tag{2.3}
$$

For the second term, we have

$$
|(\Lambda^{2s}\theta, u \cdot \nabla\theta)| = |(\Lambda^{2s}\theta, \nabla(u\theta)| \leq |\Lambda^{s+\beta}\theta|_2 \, |\Lambda^{s+1-\beta}(u\theta)|_2
\tag{2.4}
$$

where $\beta \leq \alpha$ remains to be determined. To proceed, we need the calculus inequality

$$|\Lambda^{\gamma}(FG)|_r \leq C\Big[|\Lambda^{\gamma}F|_p|G|_q + |F|_q|\Lambda^{\gamma}G|_p\Big], \tag{2.5}$$

where $\gamma > 0$, $1 < r \leq p \leq \infty$ and $1/r = 1/p + 1/q$. The use of Eq. (2.5) gives

$$|(\Lambda^{2s}\theta, u \cdot \nabla\theta)| \leq C|\Lambda^{s+\beta}\theta|_2\big[|\Lambda^{s+1-\beta}u|_p|\theta|_q + |\Lambda^{s+1-\beta}\theta|_p|u|_q\big], \tag{2.6}$$

where $2 < q \leq \infty$ and $1/p + 1/q = 1/2$. Taking into account of the second equation in (2.1) and the inclusion $H^{s+2-(2/p)-\beta} \subset \mathcal{L}_{s+1-\beta}^p$, we have

$$\big|\Lambda^{s+1-\beta}u\big|_p \leq \big|\Lambda^{s+1-\beta}\theta\big|_p \leq \big|\Lambda^{s+2-2/p-\beta}\theta\big|_2.$$

It then follows from Eq. (2.6) that

$$|(\Lambda^{2s}\theta, u \cdot \nabla\theta)| \leq C(|\theta|_q + |u|_q)|\Lambda^{s+\beta}\theta|_2|\Lambda^{s+2-(2/p)-\beta}\theta|_2.$$

In the above, β is essentially arbitrary and we may choose

$$\beta = 1 - \frac{1}{p} = \frac{1}{2} + \frac{1}{q} \quad \text{so that } s + \beta = s + 2 - \frac{2}{p} - \beta.$$

Hence

$$|(\Lambda^{2s}\theta, u \cdot \nabla\theta)| \leq C(|\theta|_q + |u|_q)|\Lambda^{s+\beta}\theta|_2^2. \tag{2.7}$$

Combining Eqs. (2.2), (2.3) and (2.7), we have

$$\frac{d}{dt}|\Lambda^s\theta|_2^2 + \kappa|\Lambda^{s+\alpha}\theta|_2^2 \leq \frac{1}{\kappa}|\Lambda^{s-\alpha}f|_2^2 + C_0(|\theta|_q + |u|_q)|\Lambda^{s+\beta}\theta|_2^2 \tag{2.8}$$

for any $s \geq 0$, $2 < q \leq \infty$ and $\beta = 1/2 + 1/q \leq \alpha$.

We now prove that weak solutions of the IVP (2.1) with $\alpha > 1/2$ are actually regular. More precisely, we have the following theorem.

Theorem 2.1. *Let $\alpha > 1/2$ and assume for $T > 0$, $s > 0$ and $2 < q < 2/2\alpha - 1$*

$$\theta_0 \in H^s \cap L^q \quad \text{and} \quad f \in L^1([0, T]; L^2) \cap L^1([0, T]; L^q) \cap L^2([0, T]; H^{s-\alpha}).$$

Then any weak solution θ of the IVP (2.1) are regular in the sense that

$$\theta \in L^{\infty}([0, T]; H^s) \cap L^2([0, T]; H^{s+\alpha})$$

Proof. The idea of the proof is to obtain from Eq. (2.8) a closed differential inequality for $|\Lambda^s\theta|^2$. Since u is essentially the Riesz transform of θ from the second equation in (2.1), we have for $1 < q < \infty$

$$|u(\cdot, t)|_q \leq |\theta(\cdot, t)|_q \leq |\theta_0|_q + \int_0^t |f(\cdot, \tau)|_q\, d\tau. \tag{2.9}$$

To eliminate the occurrence of $|\Lambda^{s+\beta}\theta|_2$, we use one of the inequalities of Gagliardo and Nirenberg

$$\left|\Lambda^{s+\beta}\theta\right|_2 \leq C\left|\Lambda^{s+\alpha}\theta\right|_2^{\beta/\alpha}\left|\Lambda^s\theta\right|_2^{1-\beta/\alpha}.$$

We use Hölder's inequality to find

$$\left|\Lambda^{s+\beta}\theta\right|_2^2 \leq \frac{\kappa}{2}\left|\Lambda^{s+\alpha}\theta\right|_2^2 + \frac{C}{\kappa}\left|\Lambda^s\theta\right|_2^2. \tag{2.10}$$

Inserting Eqs. (2.9) and (2.10) into Eq. (2.8) and canceling a factor of $|\Lambda^{s+\alpha}\theta|_2^2$,

$$\frac{d}{dt}\left|\Lambda^s\theta\right|_2^2 + \frac{\kappa}{2}\left|\Lambda^{s+\alpha}\theta\right|_2^2 \leq \frac{1}{\kappa}|\Lambda^{s-\alpha}f|_2^2 + \frac{C}{\kappa}|\Lambda^s\theta|_2^2, \tag{2.11}$$

where C only depends on $|\theta_0|_q$ and $\int_0^T |f(\cdot, \tau)|_q d\tau$. The proof of Theorem 2.1 is then concluded after we apply Gronwall's lemma to Eq. (2.11).

Remark. The seemingly formal estimates in the proof of Theorem 2.1 can all be made rigorous through a well-known procedure of first proving them for a Galerkin-type approximating sequence $\{\theta^N\}_{N=1}^\infty$ of θ and then passing to the limit. Since the bounds for the sequence depend only on the initial data and forcing term (independent of N), the weak solution as the limit of the sequence obeys the same estimates as each θ^N does. This approach has been widely used in proving local or global regularity for weak solutions of the hydrodynamic equations (see e.g., Ref. [11] pp. 96–113], Ref. [23] pp. 278–308]) and only the a priori-type estimates are given here for the clarity of our presentation.

Now we turn our attention to the regularity issue of weak solutions of the IVP (2.1) with α equal to the critical index 1/2. The purpose of the next several theorems is to show how far one can go toward a regularity proof and what assumptions are needed to fill the gap.

Theorem 2.2. *Let* $\alpha = 1/2$ *and* $s > 0$. *Assume that* $\theta_0 \in H^s \cap L^\infty$ *and* $f \in L^1([0, T]; L^\infty) \cap L^2([0, T]; H^{s-1/2})$. *Consider a weak solution* θ *of the*

IVP (2.1) *and assume that*

$$|\theta(\cdot, t)|_\infty + |u(\cdot, t)|_\infty < \frac{\kappa}{C_0} \tag{2.12}$$

hols for all $t \in [0, T]$, *where* C_0 *is a constant as in Eq.* (2.8). *Then* θ *is regular in the sense that*

$$\theta \in L^\infty([0, T]; H^s) \cap L^2\left([0, T]; H^{s+(1/2)}\right) \tag{2.13}$$

In particular, if θ_0, f *and* u *satisfy*

$$|\theta_0|_\infty + \int_0^t |f(\cdot, \tau)|_\infty \, d\tau \leq \frac{\kappa}{2C_0} \quad \text{and} \quad |u(\cdot, t)|_\infty < \frac{\kappa}{2C_0}, \tag{2.14}$$

then Eq. (2.13) *holds.*

Remark. Assumption Eq. (2.12) is a smallness condition.

Proof. The situation is different when $\alpha = 1/2$. The differential inequality Eq. (2.8) only holds for $\beta = \alpha = 1/2$ and $q = \infty$. We have after replacing β with $1/2$ and q with ∞

$$\frac{d}{dt} |\Lambda^s \theta|_2^2 + \kappa |\Lambda^{s+(1/2)} \theta|_2^2 \leq \frac{1}{\kappa} |\Lambda^{s-(1/2)} f|_2^2 + C_0(|\theta|_\infty + |u|_\infty) |\Lambda^{s+(1/2)} \theta|_2^2$$

Using the assumption Eq. (2.12), we have

$$\frac{d}{dt} |\Lambda^s \theta|_2^2 \leq \frac{1}{\kappa} |\Lambda^{s-(1/2)} f|_2^2,$$

which gives Eq. (2.13). In view of the maximum principle

$$|\theta(\cdot, t)|_\infty \leq |\theta_0|_\infty + \int_0^t |f(\cdot, \tau)|_\infty \, d\tau,$$

it then follows that Eq. (2.14) implies Eq. (2.12) and thus Eq. (2.13).

For the critical index $\alpha = 1/2$, the terms $|\theta|_\infty$ and $|u|_\infty$ have so far stood in the way of finding a regularity proof. The problem of how to deal with $|\theta|_\infty$ and $|u|_\infty$ has to be solved. In order to estimate $|\theta|_\infty$ and $|u|_\infty$, we need the following lemma.[2]

Lemma 2.3. *Let $\Omega = [0, 2\pi]^2$ and $F \in H^\sigma(\Omega)$ ($\sigma > 1$) be periodic. Then*

$$|F|_\infty \le C\left[1 + \|F\|_1 \sqrt{log\left(1 + \|F\|_\sigma^{1/(\sigma-1)}\right)}\right] \qquad (2.15)$$

Now we can prove the following theorem, which can be reviewed as a ladder theorem for the QG equation.

Theorem 2.4. *Let $\alpha = 1/2$, $T > 0$ and $s > 1$. Assume that $\theta_0 \in H^s$ and $f \in L^2([0, T]; H^{s-(1/2)})$. Then any solution θ of the IVP (2.1) satisfies for any $\sigma > 1$*

$$\frac{d}{dt}|\Lambda^s\theta|_2^2 + \kappa|\Lambda^{s+(1/2)}\theta|_2^2 \le \frac{1}{\kappa}|\Lambda^{s-(1/2)}f|_2^2$$
$$+ C\left[1 + \|\theta\|_1 \sqrt{log\left(1 + \|\theta\|_\sigma^{1/(\sigma-1)}\right)}\right]|\Lambda^{s+(1/2)}\theta|_2^2$$

If we further assume that $\|\theta(\cdot, t)\|_\sigma$ is bounded and small, say, for some constant C

$$1 + \|\theta\|_1 \sqrt{log\left(1 + \|\theta\|_\sigma^{1/(\sigma-1)}\right)} \le C\kappa,$$

then θ is regular in the sense that

$$\theta \in L^\infty([0, T]; H^s) \cap L^2([0, T]; H^{s+(1/2)}).$$

3 REGULARIZED QG EQUATION

In this section we are concerned with the IVP for the regularized QG equation

$$\begin{cases} \theta_t + u \cdot \nabla\theta + \mu(-\Delta)^\alpha\theta_t = f, & (x, t) \in \Omega \times [0, \infty), \\ u = (u_1, u_2) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), & (x, t) \in \Omega \times [0, \infty), \\ \theta(x, 0) = \theta_0(x), & x \in \Omega, \end{cases} \qquad (3.1)$$

where $0 \le \alpha \le 1$ and $\mu > 0$ are real numbers. The central issue is still whether or not weak solutions are regular for all time. Because of the insignificant role of f and for the sake of clarity of our presentation, we will set $f = 0$ in the rest of this section.

The exposition below is organized as follows. We first construct a local solution in H^s for $s > 1$ and then produce explicit bounds on the norms of all derivatives. From this we infer that for $\alpha > 1/2$ the solutions are smooth and unique for all time. For $\alpha = 1/2$, global regularity is established under a weak assumption.

We reformulate the problem as an integral equation and then apply the Banach contraction mapping principle to prove local existence. Now rewrite the first equation in (3.1) in the form

$$(1 + \mu \Lambda^{2\alpha}) \theta_t = -(f - u \cdot \nabla \theta)$$

and invert the operator $(1 + \mu \Lambda^{2\alpha})$ subject to periodic boundary condition to obtain

$$\theta(x, t) = \theta_0(x) + \int_0^t G * (u\theta)(\tau) \, d\tau, \tag{3.2}$$

where the convolution kernel $G(x)$ is defined through its Fourier transform

$$\hat{G}(k) = \hat{G}((k_1, k_2)) = \frac{i}{1 + \mu |k|^{2\alpha}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}. \tag{3.3}$$

Now notice that for $\alpha \geq 1/2$ and any $s \geq 0$

$$\|G * F\|_s = \left(\sum_k |k|^{2s} |\widehat{G * F}(k)|^2 \right)^{1/2}$$

$$\leq \left(\sum_k |\hat{G}(k)|^2 |k|^{2s} |\hat{F}(k)|^2 \right)^{\frac{1}{2}} \leq \sup_k |\hat{G}(k)| \, \|F\|_s \leq \frac{1}{\mu} \|F\|_s \tag{3.4}$$

We now state and prove a local existence result for smooth solutions of the IVP (3.1).

Theorem 3.1. *Let $\alpha \geq 1/2$ and assume that $\theta_0 \in H^s$ for some $s > 1$.*

(a) There exists a $T = T(\|\theta_0\|_s)$ such that the IVP (3.1) has a unique solution θ with $\theta \in L^\infty([0, T]; H^s)$.

(b) If T_* is the supremum of the set of all $T > 0$ such that (3.1) has a solution in $L^\infty([0, T]; H^s)$, then either $T_* = \infty$ or

$$\limsup_{t \uparrow T_*} \|\theta(\cdot, t)\|_s = \infty.$$

Proof. Write the Eq. (3.2) symbolically as $\theta = A\theta$. A is seen to be a mapping of the space $E = L^\infty([0, T]; H^s)$ into itself, where $T > 0$ is yet to be specified.

Let $b = \|\theta_0\|_s$, and set $R = 2b$. Define B_R to be the ball with radius R centered at the origin in E. We now show that if T is sufficiently small, then A is a contraction map on B_R. Let θ and $\bar{\theta}$ be any two elements of B_R. Then we have

$$\|A\theta - A\bar{\theta}\|_E = \left\| \int_0^t G * (u\theta - \bar{u}\bar{\theta}) \, d\tau \right\|_E$$
$$\leq T\|G * (u\theta - \bar{u}\bar{\theta})\|_E \leq T\big[\|G * ((u - \bar{u})\theta)\|_E + \|G * (\bar{u}(\theta - \bar{\theta}))\|_E\big]$$

Using (3.4), we have

$$\|A\theta - A\bar{\theta}\|_E \leq \frac{T}{\mu} \sup_{t \in [0, T]} \big[|\Lambda^s((u - \bar{u})\theta)|_2 + |\Lambda^s(\bar{u}(\theta - \bar{\theta}))|_2\big] \tag{3.5}$$

Then applying the calculus inequality (2.5), we find that

$$|\Lambda^s((u - \bar{u})\theta)|_2 \leq C(|\Lambda^s(u - \bar{u})|_2|\theta|_\infty + |u - \bar{u}|_\infty|\Lambda^s\theta|_2) \tag{3.6}$$

and

$$|\Lambda^s(\bar{u}(\theta - \bar{\theta}))|_2 \leq C\big(|\Lambda^s(\theta - \bar{\theta})|_2|\bar{u}|_\infty + |\theta - \bar{\theta}|_\infty|\Lambda^s\bar{u}|_2\big) \tag{3.7}$$

Since $s > 1$, we have the Sobolev inequality

$$|F|_\infty \leq C\|F\|_s, \quad \text{for some constant } C. \tag{3.8}$$

Inserting Eqs. (3.6) and (3.7) into Eq. (3.5) after applying Eq. (3.8), we obtain

$$\|A\theta - A\bar{\theta}\|_E \leq \frac{T}{\mu} \sup_{t \in [0, T]} \big[\|\theta\|_s\|u - \bar{u}\|_s + \|\bar{u}\|_s\|\theta - \bar{\theta}\|_s\big] \tag{3.9}$$

Since u and θ, as well as $\bar{u}$ and $\bar{\theta}$, are related by the second equation in (3.1), one has

$$\|\bar{u}\|_s \leq \|\bar{\theta}\|_s, \quad \|u - \bar{u}\|_s \leq \|\theta - \bar{\theta}\|_s$$

It then follows from (3.9) that

$$\|A\theta - A\bar{\theta}\|_E \leq \frac{T}{\mu}(\|\theta\|_E + \|\bar{\theta}\|_E)\|\theta - \bar{\theta}\|_E \leq \frac{2TR}{\mu}\|\theta - \bar{\theta}\|_E.$$

Also, since $A0 = \theta_0$

$$\|A\theta\|_E = \|A\theta - A0 + \theta_0\|_E \leq \|A\theta - A0\|_E + \|\theta_0\|_E \leq \frac{2TR}{\mu}\|\theta\|_E + b$$

Now choose

$$T = \frac{\mu}{4R} \quad \text{so that} \quad \frac{2TR}{\mu} = \frac{1}{2}.$$

It is indeed that T depends only on $\|\theta_0\|_s$ (since $R = 2\|\theta_0\|_s$). For this choice of T, we have

$$\|A\theta - A\bar{\theta}\|_E \leq \frac{1}{2}\|\theta - \bar{\theta}\|_E, \quad \text{and} \quad \|A\theta\|_E \leq R$$

The conclusion of part (a) then follows from contraction mapping principle.

To prove part (b), suppose on the contrary that $T_* < \infty$ and that there exists a number $M > 0$ and a sequence t_n approaching T_* from below such that

$$\|\theta(\cdot, t_n)\|_s \leq M, \quad \text{for } n = 1, 2, 3, \ldots$$

By part (a), there exists some $T = T(M)$ such that the solution starting with any $\theta(x, t_n)$ is in $L^\infty([0, T]; H^s)$. Since t_n approaches T_*, we can choose t_n such that $t_n + T > T_*$. By extending θ to the interval $[0, t_n + T]$, we obtain a solution of the IVP (3.1) in $L^\infty([0, t_n + T]; H^s)$. But this contradicts the maximality of T_*.

Now we return to the central issue: can the solution obtained in Theorem 3.1 be extended for all time? This motivates us to explore the regularity properties of solutions. For $\alpha > 1/2$, it is indeed the case that the solution we constructed in Theorem 3.1 can be extended for all time.

Theorem 3.2. *Let $\alpha > 1/2$ and assume that $\theta_0 \in H^s$ for some $s > 1$. Then there exists a unique solution to the IVP (3.1) which lies in $L^\infty([0, \infty); H^s)$.*

Proof. Let $\theta \in L^\infty([0, T]; H^s)$ be the solution of the IVP (3.1) we constructed in Theorem 3.1. Take the inner product of $\Lambda^{2s-2\alpha}\theta$ with the first equation in (3.1)

$$\frac{1}{2}\frac{d}{dt}\left[|\Lambda^{s-\alpha}\theta|_2^2 + \mu\,|\Lambda^s\theta|_2^2\right] = -\left(\Lambda^{2s-2\alpha}\theta, u \cdot \nabla\theta\right).$$

The estimates for the term on the right hand side are similarly to those in the previous section, so we only give the most important lines here. Instead of Eq. (2.8), we have

$$\frac{1}{2}\frac{d}{dt}\big[|\Lambda^{s-\alpha}\theta|_2^2 + \mu\,|\Lambda^s\theta|_2^2\big] \le C_0(|\theta|_q + |u|_q)|\Lambda^{s-\alpha+\beta}\theta|_2^2 \qquad (3.10)$$

for any $2 < q \le \infty$ and $\beta = 1/2 + 1/q \le \alpha$. Inserting Eq. (2.9) and the following modified version of Eq. (2.10) in Theorem 2.1

$$|\Lambda^{s-\alpha+\beta}\theta|_2^2 \le \sqrt{\mu}|\Lambda^s\theta|_2^2 + \frac{C}{\sqrt{\mu}}|\Lambda^{s-\alpha}\theta|_2^2$$

into Eq. (3.10), we obtain

$$\frac{1}{2}\frac{d}{dt}\big[|\Lambda^{s-\alpha}\theta|_2^2 + \mu\,|\Lambda^s\theta|_2^2\big] \le \frac{C}{\sqrt{\mu}}\big[|\Lambda^{s-\alpha}\theta|_2^2 + \mu\,|\Lambda^s\theta|_2^2\big]$$

Gronwall's lemma then implies that for any $t > 0$

$$|\Lambda^{s-\alpha}\theta(\cdot,t)|_2^2 + \mu\,|\Lambda^s\theta(\cdot,t)|_2^2 \le \big[|\Lambda^{s-\alpha}\theta_0|_2^2 + \mu\,|\Lambda^s\theta_0|_2^2\big]\,e^{(C/\sqrt{\mu})t}.$$

This estimate indicates that the solution of the IVP (3.1) with $\alpha > 1/2$ remains bounded at later time. Therefore, by part (b) of Theorem 3.1, $\theta \in L^\infty([0,\infty); H^s)$.

Now we turn our attention to $\alpha = 1/2$. The following theorem asserts that if the solution θ loses its regularity at a later time, then the maximum of θ or the maximum of u necessarily grows without a bound. The conclusion is in the spirit of the well-known Beale–Kato–Majda result for the 3D Euler equations.[3]

Theorem 3.3. *Let* $\alpha = 1/2$ *and* θ *be the solution constructed in Theorem* 3.1. *If* T_* *is the supremum of the set of all* $T > 0$ *such that* θ *in the class* $L^\infty([0,T]; H^s)$ *and* $T_* < \infty$, *then one of the following*

$$\int_0^{T_*} |\theta(\cdot,\tau)|_\infty\,d\tau = \infty, \quad \int_0^{T_*} |u(\cdot,\tau)|_\infty\,d\tau = \infty$$

holds. In other words, θ *can be extended beyond* T_* *to* T_1 *if for some constant* M *and all* $T < T_1$

$$\int_0^T |\theta(\cdot,\tau)|_\infty\,d\tau < M, \quad \text{and} \quad \int_0^T |u(\cdot,\tau)|_\infty\,d\tau < M.$$

Proof. The differential inequality (3.10) is valid for all $\alpha > 0$, therefore we have by choosing $\alpha = 1/2$, $q = \infty$ and $\beta = 1/2$

$$\frac{1}{2}\frac{d}{dt}\Big[|\Lambda^{s-(1/2)}\theta|_2^2 + \mu\,|\Lambda^s\theta|_2^2\Big] \le C_0(|\theta|_\infty + |u|_\infty)|\Lambda^s\theta|_2^2, \qquad (3.11)$$

which implies

$$|\Lambda^{s-(1/2)}\theta(\cdot,t)|_2^2 + \mu\,|\Lambda^s\theta(\cdot,t)|_2^2$$
$$\le \Big[|\Lambda^{s-(1/2)}\theta_0|_2^2 + \mu\,|\Lambda^s\theta_0|_2^2\Big]\exp\left[\frac{C}{\mu}\int_0^t [|\theta(\cdot,\tau)|_\infty + |u(\cdot,\tau)|_\infty]\,d\tau\right].$$

The conclusion of the theorem is then inferred from this inequality.

Because of the Sobolev inequality Eq. (3.8), the following result is an easy consequence of Theorem 3.3.

Corollary 3.4. *Let $\alpha = 1/2$ and $\theta \in L^\infty([0,T];H^s)$ $(s > 1)$ be the solution constructed in Theorem 3.1. Assume that θ satisfies for $T_1 > T$*

$$\int_0^{T_1} \|\theta(\cdot,\tau)\|_\rho\,d\tau < \infty,$$

where $\rho > 1$. Then we can extend θ to be a solution of the IVP (3.1) in the class $L^\infty([0,T_1];H^s)$.

The following theorem concludes that no singularities in $\|\theta\|_s$ $(s > 1)$ are possible before $\|\theta\|_1$ becomes unbounded.

Theorem 3.5. *Let $\alpha = 1/2$ and $\theta \in L^\infty([0,T];H^s)$ $(s > 1)$ be the solution constructed in Theorem 3.1. For any $T_1 > T$, if $\theta \in L^\infty([0,T_1];H^1)$, then θ can be extended to $L^\infty([0,T_1];H^s)$.*

Proof. After applying Lemma 2.3 to control $|\theta|_\infty$ and $|u|_\infty$, we have from Eq. (3.11) that

$$\frac{1}{2}\frac{d}{dt}\Big[|\Lambda^{s-(1/2)}\theta|_2^2 + \mu\,|\Lambda^s\theta|_2^2\Big] \le C\left[1 + \|\theta\|_1\sqrt{\log\Big(1 + \|\theta\|_s^{1/(s-1)}\Big)}\right]|\Lambda^s\theta|_2^2$$

If $\theta \in L^\infty([0,T_1];H^1)$, i.e., $\|\theta\|_1 < C$, then the above inequality becomes

$$\frac{dz}{dt} \le Cz\sqrt{1 + \log(z)} \qquad (3.12)$$

where we have set

$$z(t) = |\Lambda^{s-(1/2)}\theta(\cdot, t)|_2^2 + \mu |\Lambda^s\theta(\cdot, t)|_2^2.$$

It follows from applying Gronwall's lemma to Eq. (3.12) that $z \in L^\infty([0, T_1])$, which implies $\theta \in L^\infty([0, T_1]; H^s)$.

4 WEAK SOLUTIONS OF THE QG EQUATION

Onsager conjectured[19] that weak solutions of the 3D Euler equations in a Hölder space C^γ with exponent $\gamma > 1/3$ should conserve energy. In Ref. [13] Eyink proved energy conservation for weak solutions in a strong form of Hölder space C_*^γ ($\gamma > 1/3$), in which the norm is defined in terms of absolute Fourier coefficients. Constantin et al.[4] proved a sharp version of Onsager's conjecture in Besov space $B_3^{s,\infty}$ with $s > \frac{1}{3}$. In Ref. [12] Duchon and Robert explored possible sources for energy losses and proved the existence of the so-called dissipative weak solutions for the 3D Euler equations. Their notion of dissipative weak solutions can be regarded as a special type of dissipative solutions proposed by Lions.[17] For the 2D Euler equations, DiPerna and Majda constructed weak solutions of the vorticity–velocity formulation with data in $L^p (p > 4/3)$ and Eyink[14] validated such weak solutions as dissipative weak solutions in the sense of Duchon and Robert and argued for their relevance to the entropy cascade of 2D turbulence.

The goal of this section is to prove Onsager's conjecture and extend the notion of dissipative weak solutions to the IVP for the QG equation

$$\begin{cases} \theta_t + u \cdot \nabla\theta = 0, & (x, t) \in \Omega \times [0, \infty), \\ u = (u_1, u_2) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), & (x, t) \in \Omega \times [0, \infty), \\ \theta(x, 0) = \theta_0(x), & x \in \Omega. \end{cases} \quad (4.1)$$

Understanding the zero-dissipation limits of the Navier-Stokes equations[7,8] is crucial in hydrodynamics turbulence theory and we believe that similar limits for the regularizations of the QG equation will be equally important in the turbulence theory for quasi-geostrophic flows. At the end of this section we show that the smooth solution of the regularized QG equation converges to that of the QG equation as $\mu \to 0$.

For $\theta_0 \in L^2$ and any $T > 0$, the IVP (4.1) has been shown to possess global weak solutions (in the distributional sense) in $L^\infty([0, T]; L^2)$.[21] In the rest of this section we will use ϕ to denote the standard mollifier in $\mathbb{R}^2$, $\phi^\epsilon(x) = 1/\epsilon^2\phi(x/\epsilon)$ and $F^\epsilon = \phi^\epsilon * F$ for any tempered distribution F.

First we show that if the weak solution θ is also in the Besov space $B_3^{s,\infty}$ with $s > 1/3$, then θ conserves the L^2-norm. More details on Besov spaces can be found in Ref. [22], but here we only list some of the basic facts we will use. If $F \in B_p^{s,\infty}$ for $p \geq 1$, then

$$|F(\cdot + y) - F(\cdot)|_p \leq C|y|^s \|F\|_{B_p^{s,\infty}};$$
$$|F - F^\epsilon|_p \leq C\epsilon^s \|F\|_{B_p^{s,\infty}};$$
$$|\nabla F|_p \leq C\epsilon^{s-1} \|F\|_{B_p^{s,\infty}}. \tag{4.2}$$

Theorem 4.1. *Let* $\theta \in L^\infty([0,T]; L^2)$ *be a weak solution of the IVP* (4.1) *corresponding to* $\theta_0 \in L^2$ *and arbitrary* $T > 0$. *If further* $\theta \in L^3([0,T];$ $B_3^{s,\infty})$ *with* $s > 1/3$, *then for any* $t \leq T$

$$|\theta(\cdot, t)|_2 = |\theta_0|_2. \tag{4.3}$$

Proof. The idea of the proof is similar to the one in Ref. [4]. If θ solves the IVP (4.1), then θ^ϵ satisfies the following equation

$$\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon = \nabla \cdot \sigma^\epsilon, \tag{4.4}$$

where $\sigma^\epsilon = u^\epsilon \theta^\epsilon - (u\theta)^\epsilon$. It is easy to check that σ^ϵ can be represented by the formula

$$\sigma^\epsilon = (u - u^\epsilon)(\theta - \theta^\epsilon) - r^\epsilon(u, \theta),$$

with

$$r^\epsilon(u, \theta) = \int \phi(y)[(u(x - \epsilon y) - u(x))(\theta(x - \epsilon y) - \theta(x))] \, dy.$$

It then follows (4.4) that

$$\int_\Omega |\theta^\epsilon(\cdot, t)|^2 \, dx - \int_\Omega |\theta_0^\epsilon|^2 \, dx = \int_0^t \int_\Omega \sigma^\epsilon \cdot \nabla \theta^\epsilon \, dx d\tau \tag{4.5}$$

We now estimate the term on the right hand side.

$$\left| \int_0^t \int_\Omega \sigma^\epsilon \cdot \nabla \theta^\epsilon \, dx d\tau \right| \leq C \int_0^t |\nabla \theta^\epsilon(\cdot, \tau)|_3 \cdot |\sigma^\epsilon(\cdot, \tau)|_{3/2} \, d\tau$$
$$\leq C \int_0^t |\nabla \theta^\epsilon(\cdot, \tau)|_3 \cdot \left[|u - u^\epsilon|_3 |\theta - \theta^\epsilon|_3 + |r^\epsilon(u, \theta)|_{3/2} \right] d\tau$$

Since θ and u are related by the second equation in (4.1), we have from Eq. (4.2)

$$|u - u^\epsilon|_3 \le |\theta - \theta^\epsilon|_3 \le C\epsilon^s \|\theta\|_{B_3^{s,\infty}}, \quad \text{and} \quad |r^\epsilon(u,\theta)|_{3/2} \le C\epsilon^{2s} \|\theta\|_{B_3^{s,\infty}}.$$

Therefore

$$\left| \int_0^t \int_\Omega \sigma^\epsilon \cdot \nabla\theta^\epsilon \, dx d\tau \right| \le C\epsilon^{3s-1} \int_0^t \|\theta\|_{B_3^{s,\infty}}^3 \, d\tau$$

and it approaches zero as $\epsilon \to 0$. The proof is then completed after letting $\epsilon \to 0$ in (4.5).

The following theorem extends the notion of dissipative weak solutions of the inviscid hydrodynamics equations[12,14] to the QG equation

Theorem 4.2. *Let $\theta \in L^\infty([0,T]; L^2)$ be a weak solution of the IVP (4.1) corresponding to $\theta_0 \in L^2$ and arbitrary $T > 0$. If a function $G : \mathbb{R} \to \mathbb{R}$ is C^2, strictly convex and has bounded derivative, then the equation*

$$\partial_t G(\theta) + u \cdot \nabla G(\theta) = -F(G, \theta), \tag{4.6}$$

holds in the sense of distribution, where $F(G, \theta)$ is the limit of

$$G''(\theta_\epsilon) \nabla\theta_\epsilon \cdot ((u\theta)_\epsilon - u_\epsilon \theta_\epsilon)$$

in the sense of distribution.

Proof. Let $G \in C^2(\mathbb{R}, \mathbb{R})$. Multiplying Eq. (4.4) by $G'(\theta^\epsilon)$, we have

$$\partial_t G(\theta^\epsilon) + u^\epsilon \cdot \nabla G(\theta^\epsilon) - \nabla \cdot (G'(\theta^\epsilon)\sigma^\epsilon) = -G''(\theta^\epsilon)\sigma^\epsilon \cdot \nabla\theta^\epsilon. \tag{4.7}$$

Now we start showing that Eq. (4.7) converges to Eq. (4.6) in the distributional sense. Since G has bounded derivative

$$|G'(x)| \le C$$

uniformly for all $x \in \mathbb{R}^2$, we have

$$|G(\theta^\epsilon)(\cdot, t) - G(\theta)(\cdot, t)|_2 \le |G'|_\infty |\theta^\epsilon(\cdot, t) - \theta(\cdot, t)|_2 \quad \to 0$$

as $\epsilon \to 0$. Now we show that $u^\epsilon G(\theta^\epsilon)$ converges to $uG(\theta)$ in L^1, which can be deduced from the following estimate

$$\begin{aligned}
|u^\epsilon G(\theta^\epsilon) - uG(\theta)|_1 &\le |(u^\epsilon - u)G(\theta^\epsilon)|_1 + |u(G(\theta^\epsilon) - G(\theta))|_1 \\
&\le |u^\epsilon - u|_2 |G(\theta^\epsilon)|_2 + |u|_2 |G(\theta^\epsilon) - G(\theta)|_2 \\
&\le C|\theta^\epsilon - \theta|_2 |G(\theta^\epsilon)|_2 + |u|_2 |G'|_\infty |\theta^\epsilon - \theta|_2
\end{aligned}$$

and the fact that $\theta^\epsilon \to \theta$ in L^2. It now remains to show that $G'(\theta^\epsilon)\sigma^\epsilon \to 0$ in the distributional sense.

$$
\begin{aligned}
|G'(\theta^\epsilon)\sigma^\epsilon|_1 &\leq |G'|_\infty |(u\theta)^\epsilon - u^\epsilon\theta^\epsilon|_1 \\
&\leq C|(u\theta)^\epsilon - u\theta|_1 + |u^\epsilon - u|_2|\theta|_2 + |u^\epsilon|_2|\theta^\epsilon - \theta|_2
\end{aligned}
$$

which approaches zero as $\epsilon \to 0$. Therefore the limit of the term on the right hand side should also exist in the distributional sense. This completes the proof of the theorem.

We now show that the two models Eqs. (1.1) and (1.3) are close by examining the limit of Eq. (1.3) as $\mu \to 0$. First we recall that if $\theta_0 \in H^s$ for $s \geq 3$, then the IVP (4.1) is known to have a unique smooth solution θ on a finite time interval satisfying $\theta \in L^\infty([0, T]; H^s)$.[5]

Theorem 4.3. *Assume that $\{\theta_0^\mu\}_{\mu>0}$ and θ_0 lie in H^s with $s \geq 3$. Then the difference $\theta^\mu - \theta$ between θ^μ of the IVP (3.1) with initial data θ_0^μ and the solution θ of the IVP (4.1) with initial data θ_0 has the property*

$$
\begin{aligned}
|\theta^\mu(\cdot, t) &- \theta(\cdot, t)|_2^2 + \mu\|\theta^\mu(\cdot, t) - \theta(\cdot, t)\|_\alpha^2 \\
&\leq \left(|\theta_0^\mu - \theta_0|_2^2 + \mu\|\theta_0^\mu - \theta_0\|_\alpha^2\right) \exp\left[C\int_0^t (1 + \|\theta(\cdot, \tau)\|_s)\, d\tau\right] \\
&\quad + C\mu^2 \int_0^t \exp\left[C\int_0^{t-\zeta} (1 + \|\theta(\cdot, \tau)\|_s)\, d\tau\right] \|\theta(\cdot, \zeta)\|_{2\alpha+1}^4\, d\zeta.
\end{aligned}
$$

uniformly for $0 \leq t \leq T$, where C is a pure constant and T is any fixed time less than the existence time for θ.

In particular, if there is a constant C such that

$$
|\theta_0^\mu - \theta_0|_2^2 + \mu\|\theta_0^\mu - \theta_0\|_\alpha^2 \leq C\mu^2 \quad \text{as } \mu \to 0,
$$

then

$$
|\theta^\mu(\cdot, t) - \theta(\cdot, t)|_2^2 + \mu\|\theta^\mu(\cdot, t) - \theta(\cdot, t)\|_\alpha^2 \leq C\mu^2
$$

uniformly for $0 \leq t \leq T$, where C is a constant depending only on T and $\|\theta_0\|_s$.

Proof. The difference $w(x, t) = \theta^\mu(x, t) - \theta(x, t)$ solves the equation

$$
w_t + u^\mu \cdot \nabla w + v \cdot \nabla\theta + \mu\Lambda^{2\alpha}w_t + \mu\Lambda^{2\alpha}\theta_t = 0, \tag{4.8}
$$

where $v = u^\mu - u$. Since θ^μ and θ are both in H^s with $s \geq 3$, θ_t^μ and θ_t are at least in L^2. This validify the steps that follow. Multiplying Eq. (4.8) by w and integrate over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\left[w^2 + \mu|\Lambda^\alpha w|^2\right] = -\int v \cdot \nabla\theta \cdot w - \mu\int w\Lambda^{2\alpha}\theta_t,$$

where the two terms on the right-hand side may be estimated as follows.

$$-\int v \cdot \nabla\theta \cdot w \leq |\nabla\theta(\cdot, t)|_\infty |v|_2 \, |w|_2$$

Since $|\nabla\theta(\cdot, t)|_\infty \leq C\|\theta(\cdot, t)\|_s$ and $|v|_2 \leq C|w|_2$, it follows that for $s > 2$

$$-\int v \cdot \nabla\theta \cdot w \leq C\|\theta(\cdot, t)\|_s|w|_2^2.$$

Noticing that $\theta_t = -u \cdot \nabla\theta$ and applying the calculus inequality Eq. (2.5), we can bound the second term by

$$
\begin{aligned}
\mu\left|\int w\Lambda^{2\alpha+1}(u\theta)\right| &\leq |w|_2^2 + \frac{\mu^2}{4}\left|\Lambda^{2\alpha+1}(u\theta)\right|_2^2 \\
&\leq |w|_2^2 + C\mu^2\left[\left|\Lambda^{2\alpha+1}u\right|_2|\theta|_\infty + |u|_\infty\left|\Lambda^{2\alpha+1}\theta\right|_2\right]^2 \\
&\leq |w|_2^2 + C\mu^2\left|\Lambda^{2\alpha+1}\theta\right|_2^2 \|\theta\|_{2\alpha+1}^2 \\
&\leq |w|_2^2 + C\mu^2\|\theta\|_{2\alpha+1}^4
\end{aligned}
$$

Collecting the above estimates, there appears

$$\frac{d}{dt}\int\left[w^2 + \mu|\Lambda^\alpha w|^2\right] \leq C(1 + \|\theta(\cdot, t)\|_s)\int w^2 + C\mu^2\|\theta\|_{2\alpha+1}^4, \qquad (4.9)$$

where the pure constant C does not depend on μ. The desired result then follows from applying Gronwall's lemma to Eq. (4.9).

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