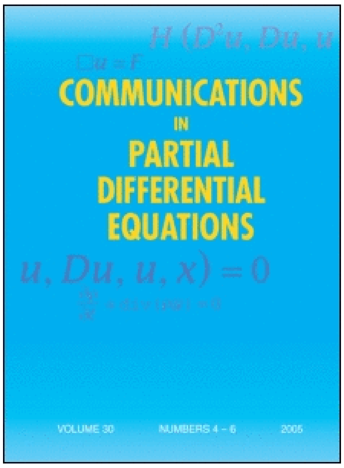

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Regularity Criteria for the Generalized MHD Equations

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This paper derives regularity criteria for the generalized magnetohydrodynamics (MHD) equations, a system of equations resulting from replacing the Laplacian $-\Delta$ in the usual MHD equations by a fractional Laplacian $(-\Delta)^\alpha$. These criteria impose assumptions on the velocity field u alone and sharpen a result of He and Xin (2005). In addition, these criteria apply to the incompressible Navier–Stokes equations and improve some existing results.

Keywords Besov space; Generalized MHD equations; Regularity criteria.

2000 Mathematics Subject Classification 76D03; 76W05; 35Q35.

1. Introduction

This paper focuses on the generalized magnetohydrodynamics (GMHD) equations

$$\begin{cases} u_t + u \cdot \nabla u + \nu(-\Delta)^\alpha u = -\nabla P + b \cdot \nabla b, & x \in \mathbb{R}^d, \ t > 0, \\ b_t + u \cdot \nabla b + \eta(-\Delta)^\beta b = b \cdot \nabla u, & x \in \mathbb{R}^d, \ t > 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, & x \in \mathbb{R}^d, \ t > 0, \end{cases} \tag{1.1}$$

where ν, η, α and β are positive parameters, u and b are d-dimensional divergence-free vector fields, and P is a scalar. The GMHD equations generalize the usual incompressible MHD equations by replacing the Laplacian $-\Delta$ in the MHD equations by a general fractional Laplacian $(-\Delta)^\alpha$ (see Wu, 2003). When $\alpha = \beta = 1$, (1.1) reduces to the MHD equations. Our goal here is to derive several regularity criteria for solutions to the initial-value problem for the GMHD equations with the initial data

$$u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \quad x \in \mathbb{R}^d.$$

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Aiming at the fundamental issue on the global existence of classical solutions to the incompressible Navier–Stokes equations

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad x \in \mathbb{R}^d, \ t > 0,$$

regularity criteria have been derived to reveal how far we are from resolving this difficult problem. Regularity criteria are sufficient conditions under which the corresponding solution becomes globally smooth. There are several types of regularity criteria. The Serrin type criterion states that a Leray weak solution u is smooth on $[0, T]$ if any one of the conditions

$$\begin{cases} u \in L^q([0,T]; L^p(\mathbb{R}^d)) \quad \text{with } \frac{2}{q} + \frac{d}{p} = 1 \quad \text{for } d < p \leq \infty, \\ u \in C([0,T]; L^3(\mathbb{R}^3)), \\ u \in L^\infty([0,T]; L^d(\mathbb{R}^d)) \quad \text{for } d \geq 4 \end{cases} \tag{1.2}$$

holds. See Furioli et al. (1997), Lions and Masmoudi (2001), Prodi (1959), Serrin (1963), Sohr and von Wahl (1984) and references therein. An alternative type of criterion requires that

$$\nabla u \in L^q([0,T]; L^p(\mathbb{R}^d)) \quad \text{with } \frac{2}{q} + \frac{d}{p} = 2 \quad \text{for } \frac{d}{2} < p < \infty. \tag{1.3}$$

See Beirao da Veiga (1987, 1995). Another type of criterion is expressed in terms of the vorticity $\omega = \nabla \times u$. The pioneering work of Beale et al. (1984) proves that if

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty} dt < \infty, \tag{1.4}$$

then the velocity u of the 3D Euler equations maintains its smoothness on $[0, T]$. For the 3D Navier–Stokes equations, (1.4) implies that u is smooth on $(0, T]$. The condition in (1.4) was later weakened by Kozono and Taniuchi (2000) to

$$\int_0^T \|\omega(\cdot,t)\|_{BMO} dt < \infty$$

and further reduced in Kozono et al. (2002) to

$$\int_0^T \|\omega(\cdot,t)\|^q_{\mathring{B}^0_{p,\infty}(\mathbb{R}^d)} dt < \infty \quad \text{with } \frac{2}{q} + \frac{d}{p} = 2 \quad \text{and} \quad d \leq p \leq \infty, \tag{1.5}$$

where $\mathring{B}^0_{p,\infty}$ represents a homogeneous Besov space to be defined in the next section. (1.5) currently represents the sharpest criterion in terms of vorticity.

As for the 3D Navier–Stokes equations, the global existence issue for the 3D MHD equations also remains open. Several regularity criteria are currently available. Early criteria impose conditions on both the velocity field u and the magnetic field b (Caflisch et al., 1997; Wu, 1997). Noticing the experimental and numerical results that appear to indicate the dominant role of the velocity field (Hasegawa, 1985; Politano et al., 1995), He and Xin were able to derive several regularity criteria for the 3D MHD equations that put no constraint on b

(He and Xin, 2005). One of them states that if a weak solution (u, b) of the 3D MHD equations satisfies

$$\nabla u \in L^q([0, T]; L^p(\mathbb{R}^3)) \quad \text{with } \frac{2}{q} + \frac{3}{p} = 2 \quad \text{and} \quad 3 \leq p < \infty, \tag{1.6}$$

then (u, b) is smooth on $[0, T]$. The goal of this paper is to weaken the assumption (1.6) and to extend the ranges of the indices. We consider a local solution (u, b) of the d-dimensional GMHD equations (1.1) with an initial data $(u_0, b_0) \in \mathring{B}^s_{p,r}$ for $1 < p < \infty$, $1 < r < \infty$ and $-\frac{d}{pr} < s < \frac{d}{p}(1 - \frac{1}{r})$. We prove that if

$$\int_0^T \|u(\cdot, t)\|^q_{\mathring{B}^1_{p,\infty}(\mathbb{R}^d)} dt < \infty \quad \text{with } \frac{2}{q} + \frac{d}{\alpha p} = 2 \quad \text{and} \quad \frac{d}{2\alpha} < p < \infty, \tag{1.7}$$

then (u, b) remains in $\mathring{B}^s_{p,r}$ on $[0, T]$. As a special consequence, (u, b) is smooth $[0, T]$ when $p \geq d$. In the borderline case $p = \frac{d}{2\alpha}$, the assumption is modified to

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{\mathring{B}^1_{p,\infty}} \leq C \max(\nu, \eta) \tag{1.8}$$

for some constant C independent of ν and η. Another borderline case is when $p = \infty$. The situation is more complex. We establish for this case that either

$$\int_0^T \|u(\cdot, t)\|^{1+\delta}_{B^1_{\infty,\infty}(\mathbb{R}^d)} dt < \infty \quad \text{for some } \delta > 0$$

or

$$\int_0^T \|u(\cdot, t)\|_{B^{1+\epsilon}_{\infty,\infty}(\mathbb{R}^d)} dt < \infty \quad \text{for some } \epsilon > 0$$

is sufficient for any solution (u, b) to be in $L^\infty([0, T]; H^s)$ for $s \in [-\rho, \rho]$, where ρ is related to δ and ϵ. Here $B^1_{\infty,\infty}$ denotes an inhomogeneous Besov space. Because of the genuine inclusion $\mathring{W}^{1,p} \subset \mathring{B}^1_{p,\infty}$, (1.7) does represent a weaker condition than (1.6). In addition, $p \in \left(\frac{d}{2\alpha}, \infty\right)$ in (1.7) extends $p \in [d, \infty)$ in (1.6).

Since the GMHD equations can be viewed as a generalization of the Navier–Stokes equations, these results also apply to the Navier–Stokes equations. Compared with (1.3), the assumption in (1.7) is weaker. In addition, the range of p in (1.7) includes $\left(\frac{d}{2\alpha}, d\right)$, which is excluded from (1.5).

This paper is organized as follows. Section 2 provides the definition of Besov spaces. Section 3 details the regularity criteria (1.7) and (1.8) while Section 4 states and proves the regularity criterion for the borderline case $p = \infty$.

2. Besov Spaces

This section provides the definition of Besov spaces and related facts. We denote by $\mathcal{S}(\mathbb{R}^d)$ the usual Schwarz class and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions. Let $\hat{f}$ denote the Fourier transform of f, defined by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx.$$

The fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in \mathbb{R}$ is defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha f} = |\xi|^{2\alpha} \hat{f}(\xi).$$

For notational convenience, we sometimes write Λ for $(-\Delta)^{\frac{1}{2}}$. We define $\mathcal{S}_0$ to be the following subspace of $\mathcal{S}$,

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbb{R}^d} \phi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \ldots \right\}.$$

Its dual $\mathcal{S}_0'$ is given by

$$\mathcal{S}_0' = \mathcal{S}' / \mathcal{S}_0^\perp = \mathcal{S}' / \mathcal{P},$$

where $\mathcal{P}$ is the space of polynomials. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}_0'$ if their difference is a polynomial.

For $j \in \mathbb{Z}$, we define

$$A_j = \{\xi \in \mathbb{R}^d : 2^{j-1} < |\xi| < 2^{j+1}\}.$$

Then there exists a sequence $\{\Phi_j\} \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\text{supp } \widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd} \Phi_0(2^j x).$$

and

$$\sum_{k=-\infty}^{\infty} \widehat{\Phi}_k(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^d \backslash \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

As a consequence, for any $f \in \mathcal{S}_0'$,

$$\sum_{k=-\infty}^{\infty} \Phi_k * f = f. \tag{2.1}$$

To define the homogeneous Besov space, we set

$$\Delta_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \ldots. \tag{2.2}$$

Definition 2.1. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ is defined by

$$\mathring{B}^s_{p,q} = \left\{ f \in \mathcal{S}_0' : \|f\|_{\mathring{B}^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\mathring{B}^s_{p,q}} = \begin{cases} \left(\sum_j \left(2^{js} \|\Delta_j f\|_{L^p} \right)^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_j 2^{js} \|\Delta_j f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

To define the inhomogeneous Besov space, we let $\Psi \in C_0^\infty(\mathbb{R}^d)$ be even and satisfy

$$\widehat{\Psi}(\xi) = 1 - \sum_{k=0}^{\infty} \widehat{\Phi}_k(\xi).$$

It is clearly that for any $f \in \mathcal{S}'$,

$$\Psi * f + \sum_{k=0}^{\infty} \Phi_k * f = f.$$

We further set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leq -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \ldots \end{cases} \tag{2.3}$$

Definition 2.2. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \{f \in \mathcal{S}' : \|f\|_{B^s_{p,q}} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}} \equiv \begin{cases} \left(\sum_{j=-1}^{\infty} \left(2^{js}\|\Delta_j f\|_{L^p}\right)^q \right)^{1/q}, & \text{if } q < \infty, \\ \sup_{-1 \leq j < \infty} 2^{js}\|\Delta_j f\|_{L^p}, & \text{if } q = \infty. \end{cases} \tag{2.4}$$

We caution that Δ_j with $j \leq -1$ associated with the homogeneous Besov space $\mathring{B}^s_{p,q}$ are defined differently from those associated with the inhomogeneous Besov space $B^s_{p,q}$. Therefore, it will be understood that Δ_j with $j \leq -1$ in the context of the homogeneous Besov space are given by (2.2) and by (2.3) in the context of the inhomogeneous Besov space. For Δ_j defined by either (2.2) or (2.3) and $S_j \equiv \sum_{k<j} \Delta_k$,

$$\Delta_j \Delta_k = 0 \quad \text{if } |j-k| \geq 2 \quad \text{and} \quad \Delta_j(S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j-k| \geq 4.$$

The Besov spaces and the standard Sobolev spaces defined by

$$\mathring{W}^{s,p} = \Lambda^{-s} L^p \quad \text{and} \quad W^{s,p} = (1-\Delta)^{-s/2} L^p$$

obey the simple facts stated in the following lemma (see Bergh and Löfström, 1976).

Lemma 2.3. *Assume that $s \in \mathbb{R}$ and $p, q \in [1, \infty]$.*

1) *If $s > 0$, then $B^s_{p,q} \subset \mathring{B}^s_{p,q}$.*
2) *If $s_1 \leq s_2$, then $B^{s_2}_{p,q} \subset B^{s_1}_{p,q}$. This inclusion relation is false for the homogeneous Besov spaces.*
3) *If $1 \leq q_1 \leq q_2 \leq \infty$, then $\mathring{B}^s_{p,q_1} \subset \mathring{B}^s_{p,q_2}$ and $B^s_{p,q_1} \subset B^s_{p,q_2}$.*

4) *If* $1 \le p_1 \le p_2 \le \infty$ *and* $s_1 = s_2 + d\big(\frac{1}{p_1} - \frac{1}{p_2}\big)$, *then* $\mathring{B}^{s_1}_{p_1,q}(\mathbb{R}^d) \subset \mathring{B}^{s_2}_{p_2,q}(\mathbb{R}^d)$.
5) *If* $1 \le p_1 \le p_2 \le \infty$, $1 \le q_1, q_2 \le \infty$, *and* $s_1 > s_2 + d\big(\frac{1}{p_1} - \frac{1}{p_2}\big)$, *then* $B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \subset B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$.
6) *If* $1 < p < \infty$, *then*

$$B^s_{p,\min(p,2)} \subset W^{s,p} \subset B^s_{p,\max(p,2)}, \quad \mathring{B}^s_{p,\min(p,2)} \subset \mathring{W}^{s,p} \subset \mathring{B}^s_{p,\max(p,2)}.$$

We will need a Bernstein type inequality for fractional derivatives.

Proposition 2.4. *Let* $\alpha \ge 0$. *Let* $1 \le p \le q \le \infty$.

1) *If* f *satisfies*

$$supp\ \hat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \le K2^j\},$$

for some integer j *and a constant* $K > 0$, *then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If* f *satisfies*

$$supp\ \hat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\} \tag{2.5}$$

for some integer j *and constants* $0 < K_1 \le K_2$, *then*

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 *and* C_2 *are constants depending on* α, p *and* q *only.*

The following proposition provides a lower bound for an integral originated from the dissipative term in the process of L^p estimates (see Chen et al., 2007; Wu, 2006).

Proposition 2.5. *Assume either* $\alpha \ge 0$ *and* $p = 2$ *or* $0 \le \alpha \le 1$ *and* $2 < p < \infty$. *Let* j *be an integer and* $f \in \mathcal{S}'$. *Then*

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \Lambda^{2\alpha} \Delta_j f dx \ge C 2^{2\alpha j} \|\Delta_j f\|^p_{L^p}$$

for some constant C *depending on* d, α *and* p.

3. Criteria in the Norm of $\mathring{B}^1_{p,\infty}$ with $p < \infty$

This section presents two regularity criteria that are expressed in terms of the norm of the homogeneous Besov space $\mathring{B}^1_{p,\infty}$ with $p < \infty$.

Theorem 3.1. *Consider the* d*-dimensional GMHD equations* (1.1) *with* $\nu > 0, \eta > 0$, *and* $0 < \alpha = \beta \le 1$. *Assume the initial data* $(u_0, b_0) \in \mathring{B}^s_{p,r}(\mathbb{R}^d)$ *with* s, p *and* r *satisfying*

$$1 < p < \infty, \quad 1 < r < \infty \quad and \quad -\frac{d}{pr} < s < \frac{d}{p}\left(1 - \frac{1}{r}\right).$$

Let $T_1 > 0$ and let (u, b) be a corresponding solution of the GMHD equations with

$$u, b \in L^\infty([0, T_1); \mathring{B}^s_{p,r}(\mathbb{R}^d)) \cap L^r([0, T_1); \mathring{B}^{s+\frac{2\alpha}{r}}_{p,r}(\mathbb{R}^d)). \tag{3.1}$$

Let $T > T_1$. If u satisfies

$$\int_0^T \|u(\cdot, t)\|^q_{\mathring{B}^1_{p,\infty}(\mathbb{R}^d)} dt < \infty \tag{3.2}$$

with p and q obeying

$$\max\left(1, \frac{d}{2\alpha}\right) < p < \infty, \quad \frac{2}{q} + \frac{d}{\alpha p} = 2, \tag{3.3}$$

then

$$(u, b) \in L^\infty\big([0, T]; \mathring{B}^s_{p,r}(\mathbb{R}^d)) \cap L^r([0, T]; \mathring{B}^{s+\frac{2\alpha}{r}}_{p,r}(\mathbb{R}^d)\big). \tag{3.4}$$

Remarks. We make two remarks.

i) The condition in (3.2) is solely imposed on u. Since $\mathring{W}^{1,p} \subset \mathring{B}^1_{p,\infty}$, (3.2) is weaker than $\nabla u \in L^q([0, T]; L^p)$;

ii) In the theorem, we used the fact that for $(u_0, b_0) \in \mathring{B}^s_{p,r}$, the GMHD equations have a local solution in the regularity class (3.1). This can be established by a priori estimates and standard methods such as successive approximations.

We now exam the consequence of this theorem on the 3-D MHD equations. We take $d = 3$, $\alpha = \beta = 1$ and $r = 2$. Parallel to the regular class (1.2) for the Navier–Stokes equations, any solution (u, b) of the MHD equations satisfying one of the conditions

$$u, b \in C([0, T]; L^3(\mathbb{R}^3))$$
$$u, b \in L^\infty([0, T]; L^d(\mathbb{R}^d)) \quad \text{for } d > 3$$

is actually smooth. Combining this fact with the inclusion relation

$$\mathring{B}^s_{p,2} \subset \mathring{W}^{s,p} \quad \text{for } p \geq 2,$$

we have the following corollary.

Corollary 3.2. *Consider the 3D incompressible MHD equations, namely* (1.1) *with $d = 3$, and $\alpha = \beta = 1$. Let (u, b) be a local solution satisfying*

$$u, b \in L^\infty\big([0, T_1); \mathring{B}^s_{p,2}(\mathbb{R}^3)) \cap L^2([0, T_1); \mathring{B}^{s+1}_{p,2}(\mathbb{R}^3)\big) \tag{3.5}$$

for some $T_1 > 0$, where $-\frac{3}{2p} < s < \frac{3}{2p}$. If, for $T > T_1$, u obeys the assumption

$$\int_0^T \|u(\cdot, t)\|^q_{\mathring{B}^1_{p,\infty}(\mathbb{R}^3)} dt < \infty \quad \text{with } \frac{3}{2} < p < \infty, \quad \frac{2}{q} + \frac{3}{p} = 2,$$

then u and b remain in the regularity class (3.5) *up to T*,

$$u, b \in L^\infty\big([0, T]; \mathring{B}^s_{p,2}(\mathbb{R}^3)\big) \cap L^2\big([0, T]; \mathring{B}^{s+1}_{p,2}(\mathbb{R}^3)\big). \tag{3.6}$$

In particular, for $s \geq 0$ *and* $p \geq 3$, *u and b are smooth on* $[0, T]$.

The following theorem deals with the borderline case when $p = \frac{d}{2\alpha}$, which is not included in Theorem 3.1.

Theorem 3.2. *Let* $\nu > 0, \eta > 0$, *and* $0 < \alpha = \beta \leq 1$. *Assume that*

$$p = \frac{d}{2\alpha}, \quad 1 < r < \infty \quad \text{and} \quad -\frac{2\alpha}{r} < s < 2\alpha - \frac{2\alpha}{r}.$$

Let $u_0, b_0 \in \mathring{B}^s_{p,r}(\mathbb{R}^d)$ *be divergence free vector fields and let* (u, b) *be s solution of the GMHD equations* (1.1) *corresponding to* (u_0, b_0) *and satisfy*

$$u, b \in L^\infty\big([0, T_1]; \mathring{B}^s_{p,r}\big) \cap L^r\Big([0, T_1]; \mathring{B}^{s+\frac{2\alpha}{r}}_{p,r}\Big) \tag{3.7}$$

for some $T_1 > 0$. *If u satisfies*

$$\sup_{t\in[0,T]} \|u(\cdot, t)\|_{\mathring{B}^1_{p,\infty}} \leq C \max(\nu, \eta)$$

for some constant depending on α, *s*, *d and p, then* (u, b) *remains in the regularity class* (3.7) *over* $[0, T]$.

We now prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. For the sake of a concise presentation, we shall only consider the case when $r = p$. The general case of r can be treated in a very similar fashion. It suffices to derive an a priori estimate that implies (3.4) under (3.3).

Let j be an arbitrary integer. Projecting the velocity equation of (1.1) onto divergence-free vector fields and then applying Δ_j (defined in (2.2)), we have

$$\partial_t \Delta_j u + \nu(-\Delta)^\alpha \Delta_j u = -\mathbb{P}\Delta_j(u \cdot \nabla u) + \mathbb{P}\Delta_j(b \cdot \nabla b), \tag{3.8}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ is the projection operator onto divergence free vector fields. Similarly, applying Δ_j to the second equation in (1.1) yields

$$\partial_t \Delta_j b + +\eta(-\Delta)^\alpha \Delta_j b = -\Delta_j(u \cdot \nabla b) + \Delta_j(b \cdot \nabla u). \tag{3.9}$$

Multiplying (3.9) by $p|\Delta_j b|^{p-2}\Delta_j b$, integrating with respect to x and applying proposition 2.5, we obtain

$$\begin{aligned} &\frac{d}{dt}\|\Delta_j b\|^p_{L^p} + C\eta 2^{2\alpha j}\|\Delta_j b\|^p_{L^p} \\ &\quad \leq -p\int |\Delta_j b|^{p-2}\Delta_j b \Delta_j(u \cdot \nabla b)dx + p\int |\Delta_j b|^{p-2}\Delta_j b\Delta_j(b \cdot \nabla u)dx. \end{aligned}$$

Multiplying by 2^{psj} and summing over all j, we get

$$\frac{d}{dt}\|b\|^p_{\mathring{B}^s_{p,p}} + C_0\eta\|b\|^p_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}} = I_1 + I_2, \tag{3.10}$$

where

$$I_1 = -p\sum_j 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b\cdot\Delta_j(u\cdot\nabla b)dx, \tag{3.11}$$

$$I_2 = p\sum_j 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b\cdot\Delta_j(b\cdot\nabla u)dx. \tag{3.12}$$

To estimate I_1, we use Bony's notion of paraproduct to write

$$\begin{aligned}\Delta_j(u\cdot\nabla b) = &\sum_{|j-k|\le 2}\Delta_j(S_{k-1}u\cdot\nabla\Delta_k b) + \sum_{|j-k|\le 2}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}b)\\ &+\sum_{k\ge j-1}\sum_{|k-l|\le 1}\Delta_j(\Delta_k u\cdot\nabla\Delta_l b).\end{aligned} \tag{3.13}$$

Inserting this decomposition in (3.11), we splits I_1 into three parts,

$$I_1 \equiv I_{11} + I_{12} + I_{13} \tag{3.14}$$

with

$$I_{11} = -p\sum_j\sum_{|j-k|\le 2} 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b\cdot\Delta_j(S_{k-1}u\cdot\nabla\Delta_k b)dx,$$

$$I_{12} = -p\sum_j\sum_{|j-k|\le 2} 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b\cdot\Delta_j(\Delta_k u\cdot\nabla S_{k-1} b)dx,$$

$$I_{13} = -p\sum_j\sum_{k\ge j-1} 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b\cdot\sum_{|k-l|\le 1}\Delta_j(\Delta_k u\cdot\nabla\Delta_l b)dx.$$

We first bound I_{12}. Noticing that the second summation in I_{12} is over k satisfying $|j-k|\le 2$, we only need consider the term with $k=j$. By Hölder's inequality

$$I_{12} \le C\sum_j 2^{psj}\|\Delta_j b\|^{p-1}_{L^p}\|\Delta_j u\|_{L^p}\|\nabla S_{j-1}b\|_{L^\infty}.$$

Applying Bernstein's inequality yields

$$I_{12} \le C\sum_j 2^{psj}\|\Delta_j b\|^{p-1}_{L^p}\|\Delta_j u\|_{L^p}\sum_{m\le j-1}2^{(1+\frac{d}{p})m}\|\Delta_m b\|_{L^p}.$$

We write the right hand side as

$$\begin{aligned}I_{12} &\le C\sum_j 2^j\|\Delta_j u\|_{L^p}\left(2^{(ps+\frac{d}{p})j}\|\Delta_j b\|^p_{L^p}\right)^{1-\frac{1}{p}}\\ &\quad\times\sum_{m\le j-1}\left(2^{(ps+\frac{d}{p})m}\|\Delta_m b\|^p_{L^p}\right)^{\frac{1}{p}}2^{[\frac{d}{p}(1-\frac{1}{p})+1-s](m-j)}\\ &\equiv C\sum_j A_j B_j(C*D)_j,\end{aligned}$$

where $C * D$ denotes the convolution of two sequences, and

$$A_j = 2^j \|\Delta_j u\|_{L^p}, \quad B_j = \left(2^{(ps+\frac{d}{p})j}\|\Delta_j b\|_{L^p}^p\right)^{1-\frac{1}{p}},$$
$$C_j = \left(2^{(ps+\frac{d}{p})j}\|\Delta_j b\|_{L^p}^p\right)^{1/p} \quad \text{and} \quad D_j = \begin{cases} 2^{-[\frac{d}{p}(1-\frac{1}{p})+1-s]j}, & \text{if } j \geq 1, \\ 0, & \text{if } j < 1. \end{cases}$$

By Hölder's inequality and Young's inequality,

$$\begin{aligned} I_{12} &\leq C\|\{A_j\}\|_{l^\infty}\|\{B_j\}\|_{l^{p/(p-1)}}\|\{(C*D)_j\}\|_{l^p} \\ &\leq C\|\{A_j\}\|_{l^\infty}\|\{B_j\}\|_{l^{p/(p-1)}}\|\{C_j\}\|_{l^p}\|\{D_j\}\|_{l^1}. \end{aligned}$$

When $\frac{d}{p}\left(1-\frac{1}{p}\right)+1-s>0$, $\|\{D_j\}\|_{l^1} < \infty$ and thus

$$I_{12} \leq C \sup_j 2^j \|\Delta_j u\|_{L^p} \sum_j 2^{(ps+\frac{d}{p})j}\|\Delta_j b\|_{L^p}^p.$$

Setting

$$p_1 = p - \frac{d}{2\alpha} \quad \text{and} \quad p_2 = \frac{d}{2\alpha},$$

we can further bound I_{12} as

$$\begin{aligned} I_{12} &\leq C\|u\|_{\mathring{B}^1_{p,\infty}} \sum_j \left(2^{p_1 sj}\|\Delta_j b\|_{L^p}^{p_1}\right)\left(2^{(p_2 s+\frac{d}{p})j}\|\Delta_j b\|_{L^p}^{p_2}\right) \\ &\leq C\|u\|_{\mathring{B}^1_{p,\infty}}\|b\|_{\mathring{B}^s_{p,p}}^{p_1}\|b\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^{p_2} \\ &\leq C\|u\|_{\mathring{B}^1_{p,\infty}}^{\frac{p}{p_1}}\|b\|_{\mathring{B}^s_{p,p}}^p + \frac{C_0\eta}{16}\|b\|^p_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}. \end{aligned} \tag{3.15}$$

We now bound I_{11}. We rewrite I_{11} as

$$\begin{aligned} I_{11} = &-p\sum_j \sum_{|j-k|\leq 2} 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b \cdot [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k b\,dx \\ &- p\sum_j 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b\cdot (S_j u\cdot \Delta_j b)dx \\ &- p\sum_j\sum_{|j-k|\leq 2} 2^{psj}\int |\Delta_j b|^{p-2}\Delta_j b\cdot (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k b\,dx \\ = &\; I_{111}+I_{112}+I_{113}, \end{aligned}$$

where the brackets [ ] represent the commutator, namely

$$[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k b = \Delta_j(S_{k-1}u\cdot\nabla\Delta_k b) - S_{k-1}u\cdot\nabla\Delta_j\Delta_k b.$$

A key point of this decomposition is that the second term I_{112} becomes zero since u is divergence free. As we shall see in (3.18) below, the estimates based on this decomposition allow us to gain the good factor 2^m instead of 2^j. To bound I_{111} and I_{113}, we notice again that k in the summations satisfies $|j-k| \le 2$ and it suffices to consider the typical case when $k = j$. According to Hölder's inequality,

$$I_{111} \le p \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^{p-1} \, \|[\Delta_j, S_{j-1}u \cdot \nabla]\Delta_j b\|_{L^p},$$

$$I_{113} \le p \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^{p-1} \|\Delta_j u\|_{L^\infty} \, \|\nabla \Delta_j b\|_{L^p}.$$

To bound the commutator, we have by the definition of Δ_j

$$\begin{aligned}[][\Delta_j, S_{j-1}u \cdot \nabla]\Delta_j b &= \int_{\mathbb{R}^d} \Phi_j(x-y)(S_{j-1}u(y) - S_{j-1}u(x)) \cdot \nabla \Delta_j b(y) dy \\ &= \int_{\mathbb{R}^d} \nabla \Phi_j(x-y) \cdot (S_{j-1}u(y) - S_{j-1}u(x)) \cdot \Delta_j b(y) dy.\end{aligned}$$

where we have integrated by parts. By Young's inequality,

$$\begin{aligned}\|[\Delta_j, S_{j-1}u \cdot \nabla]\Delta_j b\|_{L^p} &\le \|\nabla S_{j-1}u\|_{L^\infty} \, \|\Delta_j b\|_{L^p} \int_{\mathbb{R}^d} |x||\nabla \Phi_j(x)| dx \\ &= C \, \|\nabla S_{j-1}u\|_{L^\infty} \, \|\Delta_j b\|_{L^p} \end{aligned} \tag{3.16}$$

Therefore,

$$I_{11} \le C \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^{p-1} \big(\|\Delta_j u\|_{L^\infty} \, \|\nabla \Delta_j b\|_{L^p} + \|\nabla S_{j-1}u\|_{L^\infty} \, \|\Delta_j b\|_{L^p} \big). \tag{3.17}$$

It then follows from Bernstein's inequality that

$$\begin{aligned} I_{11} &\le C \sum_j 2^{(ps+1+\frac{d}{p})j} \|\Delta_j u\|_{L^p} \, \|\Delta_j b\|_{L^p}^p \\ &\quad + C \sum_j 2^{psj} \, \|\Delta_j b\|_{L^p}^p \sum_{m \le j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p}. \end{aligned} \tag{3.18}$$

Regrouping the terms on the right side, we have

$$\begin{aligned} I_{11} &\le C \sup_j 2^j \|\Delta_j u\|_{L^p} \sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j b\|_{L^p}^p \\ &\quad + C \sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j b\|_{L^p}^p \sum_{m \le j-1} 2^m \|\Delta_m u\|_{L^p} 2^{\frac{d}{p}(m-j)}. \end{aligned}$$

As in the estimates for I_{12}, we can bound I_{11} by

$$\begin{aligned} I_{11} &\le C \sup_j 2^j \|\Delta_j u\|_{L^p} \sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j b\|_{L^p}^p \\ &\le C \|u\|_{\mathring{B}^1_{p,\infty}}^{\frac{p}{p_1}} \|b\|_{\mathring{B}^s_{p,p}}^p + \frac{C_0 \eta}{16} \|b\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^p. \end{aligned}$$

We now estimate I_{13}. Since k and l in the summation satisfies $|k-l| \leq 1$, we can take $l = k$ without loss of generality. By Hölder's inequality and Bernstein's inequality,

$$I_{13} \leq C \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^{p-1} 2^{(1+\frac{d}{p})j} \sum_{k \geq j-1} \|\Delta_k u\|_{L^p} \|\Delta_k b\|_{L^p} \tag{3.19}$$

To further bound I_{13}, we write the right side as

$$\sum_j \left(2^{(ps+\frac{d}{p})j} \|\Delta_j b\|_{L^p}^p\right)^{1-\frac{1}{p}} \sum_{k \geq j-1} (2^k \|\Delta_k u\|_{L^p}) \left(2^{(ps+\frac{d}{p})k} \|\Delta_k b\|_{L^p}^p\right)^{\frac{1}{p}} 2^{(s+1+\frac{d}{p^2})(j-k)}.$$

Therefore, for $s + 1 + \frac{d}{p^2} > 0$, I_{13} can be bounded by

$$\begin{aligned} I_{13} &\leq C \sup_j 2^j \|\Delta_j u\|_{L^p} \sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j b\|_{L^p}^p \\ &\leq C \|u\|_{\mathring{B}_{p,\infty}^1}^{\frac{p}{p_1}} \|b\|_{\mathring{B}_{p,p}^s}^p + \frac{C_0 \eta}{16} \|b\|_{\mathring{B}_{p,p}^{s+\frac{2\alpha}{p}}}^p. \end{aligned}$$

Collecting the estimates for I_{11}, I_{12} and I_{13}, we have

$$I_1 \leq C \|u\|_{\mathring{B}_{p,\infty}^1}^q \|b\|_{\mathring{B}_{p,p}^s}^p + \frac{C_0 \eta}{4} \|b\|_{\mathring{B}_{p,p}^{s+\frac{2\alpha}{p}}}^p,$$

where $q = p/p_1 = p/\left(p - \frac{d}{2\alpha}\right)$.

We now estimate I_2 defined in (3.12). By Höler's inequality, we get

$$I_2 \leq C \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^{p-1} \|\Delta_j (b \cdot \nabla u)\|_{L^p}.$$

Decomposing $\Delta_j(b \cdot \nabla u)$ into paraproducts as in (3.13), we can bound the L^p-norm of $\Delta_j(b \cdot \nabla u)$ as follows:

$$\begin{aligned} \|\Delta_j(b \cdot \nabla u)\|_{L^p} \leq\ & 2^j \|\Delta_j u\|_{L^p} \sum_{m \leq j-1} 2^{\frac{d}{p}m} \|\Delta_m b\|_{L^p} + \|\Delta_j b\|_{L^p} \sum_{m \leq j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p} \\ & + 2^{(1+\frac{d}{p})j} \sum_{k \geq j-1} \|\Delta_k u\|_{L^p} \|\Delta_k b\|_{L^p}, \end{aligned} \tag{3.20}$$

where we have applied Bernstein's inequality. Correspondingly, I_2 is bounded by

$$I_2 \leq I_{21} + I_{22} + I_{23},$$

where

$$\begin{aligned} I_{21} &= C \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^{p-1} 2^j \|\Delta_j u\|_{L^p} \sum_{m \leq j-1} 2^{\frac{d}{p}m} \|\Delta_m b\|_{L^p}, \\ I_{22} &= C \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^p \sum_{m \leq j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m u\|_{L^p}, \\ I_{23} &= C \sum_j 2^{psj} \|\Delta_j b\|_{L^p}^{p-1} 2^{(1+\frac{d}{p})j} \sum_{k \geq j-1} \|\Delta_k u\|_{L^p} \|\Delta_k b\|_{L^p}. \end{aligned}$$

I_{23} can be bounded in exactly the same way as I_{13} (see (3.19)), while I_{22} admits the same bound as I_{11} (see (3.18)). It remains to bound I_{21}. By writing it as

$$I_{21} = \sum_j (2^j \|\Delta_j u\|_{L^p})\big(2^{psj+\frac{d}{p}j}\|\Delta_j b\|_{L^p}^p\big)^{1-\frac{1}{p}} \times \sum_{m\le j-1} \big(2^{(ps+\frac{d}{p})m}\|\Delta_m b\|_{L^p}^p\big)^{1/p} 2^{(s-\frac{d}{p}(1-\frac{1}{p}))(j-m)},$$

we can show that for $s - \frac{d}{p}\big(1-\frac{1}{p}\big) < 0$,

$$I_{21} \le C \sup_j 2^j \|\Delta_j u\|_{L^p} \sum_j 2^{(ps+\frac{d}{p})j}\|\Delta_j b\|_{L^p}^p,$$

which admits the same bound as (3.15). Inserting the estimates for I_1 and I_2 in (3.10), we can conclude that

$$b \in L^\infty\big([0,T]; \mathring{B}^s_{p,p}(\mathbb{R}^d)\big) \cap L^p\Big([0,T]; \mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}(\mathbb{R}^d)\Big)$$

when (3.3) holds.

Multiplying (3.8) by $2^{psj}p|\Delta_j u|^{p-2}\Delta_j u$, integrating with respect to x and summing over j, we obtain

$$\frac{d}{dt}\|u\|^p_{\mathring{B}^s_{p,p}} + C_1\nu\|u\|^p_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}} = I_3 + I_4, \tag{3.21}$$

where

$$I_3 = -p\sum_j 2^{psj}\int |\Delta_j u|^{p-2}\Delta_j u \mathbb{P}\Delta_j(u\cdot\nabla u)dx, \tag{3.22}$$

$$I_4 = p\sum_j 2^{psj}\int |\Delta_j u|^{p-2}\Delta_j u \mathbb{P}\Delta_j(b\cdot\nabla b)dx. \tag{3.23}$$

The estimates for I_1 can be applied directly to I_3 and we have

$$I_3 \le C\|u\|^{\frac{p}{p_1}}_{\mathring{B}^1_{p,\infty}}\|u\|^p_{\mathring{B}^s_{p,p}} + \frac{C_1\nu}{8}\|u\|^p_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}.$$

To estimate I_4, we first apply Hölder's inequality to obtain

$$I_4 = p\sum_j 2^{psj}\|\Delta_j u\|^{p-1}_{L^p}\|\Delta_j(b\cdot\nabla b)\|_{L^p}.$$

As in (3.20), we have

$$\|\Delta_j(b\cdot\nabla b)\|_{L^p} \le 2^j\|\Delta_j b\|_{L^p}\sum_{m\le j-1} 2^{\frac{d}{p}m}\|\Delta_m b\|_{L^p} + \|\Delta_j b\|_{L^p}\sum_{m\le j-1}2^{(1+\frac{d}{p})m}\|\Delta_m b\|_{L^p} + 2^{(1+\frac{d}{p})j}\sum_{k\ge j-1}\|\Delta_k b\|_{L^p}\|\Delta_k b\|_{L^p}.$$

Inserting this bound in (3.23) naturally splits the right of (3.23) into three terms I_{41}, I_{42} and I_{43}. To bound

$$I_{41} \equiv p \sum_j 2^{psj} \|\Delta_j u\|_{L^p}^{p-1} 2^j \|\Delta_j b\|_{L^p} \sum_{m \le j-1} 2^{\frac{d}{p}m} \|\Delta_m b\|_{L^p},$$

we write it as

$$I_{41} = p \sum_j 2^j \|\Delta_j u\|_{L^p} \left(2^{psj+\frac{d}{p}j} \|\Delta_j u\|_{L^p}^p\right)^{1-\frac{2}{p}} \left(2^{psj+\frac{d}{p}j} \|\Delta_j b\|_{L^p}^p\right)^{\frac{1}{p}}$$
$$\times \sum_{m \le j-1} \left(2^{psm+\frac{d}{p}m} \|\Delta_m b\|_{L^p}^p\right)^{\frac{1}{p}} 2^{(\frac{d}{p}(1-\frac{1}{p})-s)(m-j)}.$$

As in the estimates for I_{12}, we have for $\frac{d}{p}\left(1-\frac{1}{p}\right) - s > 0$,

$$I_{41} \le \sup_j 2^j \|\Delta_j u\|_{L^p} \Big[\sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j u\|_{L^p}^p\Big]^{1-\frac{2}{p}} \Big[\sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j b\|_{L^p}^p\Big]^{\frac{2}{p}}.$$

We further write

$$2^{(ps+\frac{d}{p})j} \|\Delta_j u\|_{L^p}^p = 2^{p_1 sj} \|\Delta_j u\|_{L^p}^{p_1} 2^{(p_2 s+\frac{d}{p})j} \|\Delta_j u\|_{L^p}^{p_2}$$

and then apply Hölder's inequality to get

$$I_{41} \le C\|u\|_{\mathring{B}^1_{p,\infty}} \|u\|_{\mathring{B}^s_{p,p}}^{p_1(1-\frac{2}{p})} \|u\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^{p_2(1-\frac{2}{p})} \|b\|_{\mathring{B}^s_{p,p}}^{\frac{2p_1}{p}} \|b\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^{\frac{2p_2}{p}}.$$

By Young's inequality, we have

$$\begin{aligned} I_{41} &\le C\|u\|_{\mathring{B}^1_{p,\infty}} \|u\|_{\mathring{B}^s_{p,p}}^{p_1} \|u\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^{p_2} + C\|u\|_{\mathring{B}^1_{p,\infty}} \|b\|_{\mathring{B}^s_{p,p}}^{p_1} \|b\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^{p_2} \\ &\le C\|u\|_{\mathring{B}^1_{p,\infty}}^{\frac{p}{p_1}} \|u\|_{\mathring{B}^s_{p,p}}^p + C\|u\|_{\mathring{B}^1_{p,\infty}}^{\frac{p}{p_1}} \|b\|_{\mathring{B}^s_{p,p}}^p + \frac{C_1 \nu}{8} \|u\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^p + \frac{C_0 \eta}{8} \|b\|_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}^p. \end{aligned} \quad (3.24)$$

The bound for

$$I_{42} \equiv p \sum_j 2^{psj} \|\Delta_j u\|_{L^p}^{p-1} \|\Delta_j b\|_{L^p} \sum_{m \le j-1} 2^{(1+\frac{d}{p})m} \|\Delta_m b\|_{L^p}$$

is the same as the bound for I_{41} if $\frac{d}{p}\left(1-\frac{1}{p}\right) - 1 - s > 0$. We now bound I_{43},

$$I_{43} \equiv p \sum_j 2^{psj} \|\Delta_j u\|_{L^p}^{p-1} 2^{(1+\frac{d}{p})j} \sum_{k \ge j-1} \|\Delta_k b\|_{L^p} \|\Delta_k b\|_{L^p}.$$

By writing the right side as

$$I_{43} = p \sum_j (2^j \|\Delta_j u\|_{L^p}) \left(2^{(ps+\frac{d}{p})j} \|\Delta_j u\|_{L^p}^p\right)^{1-\frac{2}{p}}$$
$$\times \sum_{k \ge j-1} \left(2^{(ps+\frac{d}{p})k} \|\Delta_k b\|_{L^p}^p\right)^{\frac{2}{p}} 2^{2(s+\frac{d}{p^2})(j-k)}$$

we have for $s + \frac{d}{p^2} > 0$,

$$I_{43} \le \sup_j 2^j \|\Delta_j u\|_{L^p} \Big[\sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j u\|_{L^p}^p\Big]^{1-\frac{2}{p}} \Big[\sum_j 2^{(ps+\frac{d}{p})j} \|\Delta_j b\|_{L^p}^p\Big]^{\frac{2}{p}}.$$

Thus I_{43} is bounded by (3.24). Inserting the estimates for I_3 and I_4 in (3.21) leads to (3.4). This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. We establish Theorem 3.2 by modifying the proof of Theorem 3.1. We shall not provide the details for all the terms, but focus on how we deal with I_{11} differently. We start with (3.18). When $p = \frac{d}{2\alpha}$, we have

$$\begin{aligned} I_{11} &\le C \sup_j 2^j \|\Delta_j u\|_{L^p} \sum_j 2^{(ps+2\alpha)j} \|\Delta_j b\|_{L^p}^p \\ &\quad + C \sum_j 2^{(ps+2\alpha)j} \|\Delta_j b\|_{L^p}^p \sum_{m\le j-1} 2^{2m} \|\Delta_m u\|_{L^p} 2^{2\alpha(j-m)} \\ &\le C \|u\|_{\mathring{B}^1_{p,\infty}} \|b\|^p_{\mathring{B}^{s+\frac{2\alpha}{p}}_{p,p}}. \end{aligned}$$

Other terms can be similarly modified. When these estimates replace the corresponding estimates in the proof of Theorem 3.1, we are led to the conclusion of Theorem 3.2.

4. Criterion in the Norm of $B^1_{\infty,\infty}$

This section derives a regularity criterion expressed in terms of the norm in $B^1_{\infty,\infty}$. This result complements the theorems of Section 3.

Theorem 4.1. *Consider the GMHD equations* (1.1) *with* $\nu > 0$, $\eta > 0$, *and* $\alpha = \beta > 0$. *Let* u_0, $b_0 \in H^s(\mathbb{R}^d)$ *with s satisfying* $-\rho < s < \rho$ *for some* $0 < \rho < \alpha$. *Let* (u, b) *be a solution of* (1.1) *corresponding to the initial data* (u_0, b_0) *satisfying*

$$(u, b) \in L^\infty([0, T_1]; H^s(\mathbb{R}^d)) \cap L^2([0, T_1]; H^{s+\alpha}(\mathbb{R}^d))$$

for some $T_1 > 0$. *Let* $T > T_1$. *If u satisfies either*

$$\int_0^T \|u(\cdot, t)\|^{1+\delta}_{B^1_{\infty,\infty}(\mathbb{R}^d)} dt < \infty \quad \textit{for some } \delta \ge \frac{\rho}{\alpha - \rho}$$

or

$$\int_0^T \|u(\cdot, t)\|_{B^{1+\epsilon}_{\infty,\infty}(\mathbb{R}^d)} dt < \infty \quad \textit{for some } \epsilon \ge \rho,$$

then

$$(u, b) \in L^\infty([0, T]; H^s(\mathbb{R}^d)) \cap L^2([0, T]; H^{s+\alpha}(\mathbb{R}^d)). \tag{4.1}$$

Remark. As we shall point out in the proof of this theorem, a single term originated in $b \cdot \nabla b$ prevented us from extending s to the better range $-\rho < s < 1 + \rho$.

Proof. Let $j \geq -1$ be an integer. Let Δ_j be defined as in (2.3). Consider the equations

$$\partial_t \Delta_j u + \nu(-\Delta)^\alpha \Delta_j u = -\nabla \Delta_j P - \Delta_j(u \cdot \nabla u) + \Delta_j(b \cdot \nabla b),$$
$$\partial_t \Delta_j b + \eta(-\Delta)^\alpha \Delta_j b = -\Delta_j(u \cdot \nabla b) + \Delta_j(b \cdot \nabla u).$$

Dotting the first equation by $2^{2sj}\Delta_j u$ and the second by $2^{2sj}\Delta_j b$, integrating with respect to x, and summing over all $j \geq -1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right) + \nu\|u\|_{H^{s+\alpha}}^2 + \eta\|b\|_{H^{s+\alpha}}^2 = J_1 + J_2 + J_3 + J_4, \tag{4.2}$$

where

$$J_1 = -\sum_j 2^{2sj} \int \Delta_j b \cdot \Delta_j(u \cdot \nabla b)\, dx, \tag{4.3}$$

$$J_2 = \sum_j 2^{2sj} \int \Delta_j b \cdot \Delta_j(b \cdot \nabla u)\, dx, \tag{4.4}$$

$$J_3 = -\sum_j 2^{2sj} \int \Delta_j u \cdot \Delta_j(u \cdot \nabla u)\, dx, \tag{4.5}$$

$$J_4 = \sum_j 2^{2sj} \int \Delta_j u \cdot \Delta_j(b \cdot \nabla b)\, dx. \tag{4.6}$$

As in (3.14) of Section 3, we write J_1 as $J_1 = J_{11} + J_{12} + J_{13}$ with

$$J_{11} = -\sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int \Delta_j b \cdot \Delta_j(S_{k-1}u \cdot \nabla \Delta_k b)\, dx,$$

$$J_{12} = -\sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int \Delta_j b \cdot \Delta_j(\Delta_k u \cdot \nabla S_{k-1} b)\, dx,$$

$$J_{13} = -\sum_j \sum_{k\geq j-1} 2^{2sj} \int \Delta_j b \cdot \sum_{|k-l|\leq 1} \Delta_j(\Delta_k u \cdot \nabla \Delta_l b)\, dx.$$

We can bound J_{11} in a similar fashion as we did to I_{11} in Section 3. Corresponding to (3.17), we have

$$J_{11} \leq C \sum_j 2^{(2s+1)j} \|\Delta_j b\|_{L^2}^2 \|\Delta_j u\|_{L^\infty} + C \sum_j 2^{2sj} \|\Delta_j b\|_{L^2}^2 \sum_{m\leq j-1} 2^m \|\Delta_m u\|_{L^\infty}.$$

Noticing that $\Delta_m = 0$ for $m < -1$ (see 2.3)), we have

$$J_{11} \leq C \sup_j 2^j \|\Delta_j u\|_{L^\infty} \|b\|_{B_{2,2}^s}^2 + C \sup_m 2^m \|\Delta_m u\|_{L^\infty} \sum_j 2^{(2s+\sigma)j} \|\Delta_j b\|_{L^2}^2 2^{-\sigma j}(j+1),$$

where $\sigma = \frac{2\alpha\delta}{(1+\delta)} > 0$. Letting $q_1 = 2 - \frac{\sigma}{\alpha}$ and $q_2 = \frac{\sigma}{\alpha}$, we have

$$\sum_j 2^{(2s+\sigma)j} \|\Delta_j b\|_{L^2}^2 = \sum_j \left(2^{q_1 sj} \|\Delta_j b\|_{L^2}^{q_1}\right)\left(2^{(q_2 s+\sigma)j} \|\Delta_j b\|_{L^2}^{q_2}\right) \leq \|b\|_{H^s}^{q_1} \|b\|_{H^{s+\alpha}}^{q_2}.$$

Therefore,

$$J_{11} \leq C\|u\|_{B^1_{\infty,\infty}}\|b\|^2_{H^s} + C\|u\|_{B^1_{\infty,\infty}}\|b\|^{q_1}_{H^s}\,\|b\|^{q_2}_{H^{s+\alpha}}$$
$$\leq C\|u\|_{B^1_{\infty,\infty}}\|b\|^2_{H^s} + C\|u\|^{1+\delta}_{B^1_{\infty,\infty}}\|b\|^2_{H^s} + \frac{\eta}{8}\|b\|^2_{H^{s+\alpha}}. \tag{4.7}$$

Alternatively, we can also bound J_{11} as follows.

$$J_{11} \leq C\sup_j 2^j\|\Delta_j u\|_{L^\infty}\|b\|^2_{H^s} + C\sup_j 2^{(1+\epsilon)j}\|\Delta_j u\|_{L^\infty}\sum_j 2^{2sj}\|\Delta_j b\|^2_{L^2}\sum_{m\leq j-1}2^{-\epsilon m}$$
$$\leq C\|u\|_{B^{1+\epsilon}_{\infty,\infty}}\|b\|^2_{H^s}.$$

J_{12} is bounded by

$$J_{12} \leq C\sum_j 2^{2sj}\|\Delta_j b\|_{L^2}\|\Delta_j u\|_{L^\infty}\sum_{m\leq j-1}2^m\|\Delta_m b\|_{L^2} \tag{4.8}$$

Writing the right side as

$$\sum_j 2^j\|\Delta_j u\|_{L^\infty}\left(2^{2sj}\|\Delta_j b\|^2_{L^2}\right)^{\frac{1}{2}}\sum_{m\leq j-1}\left(2^{2sm}\|\Delta_m b\|^2_{L^2}\right)^{\frac{1}{2}}2^{(s-1)(j-m)},$$

we have for $s-1<0$

$$J_{12} \leq C\sup_j 2^j\|\Delta_j u\|_{L^\infty}\sum_j 2^{2sj}\|\Delta_j b\|^2_{L^2} = C\|u\|_{B^1_{\infty,\infty}}\|b\|^2_{H^s}.$$

We can also modify the estimates to give s more freedom. In fact, if we rewrite the right-hand side of (4.8) as

$$J_{12} \leq \sum_j 2^j\|\Delta_j u\|_{L^\infty}\left(2^{(2s+\sigma)j}\|\Delta_j b\|^2_{L^2}\right)^{\frac{1}{2}}\sum_{m\leq j-1}\left(2^{(2s+\sigma)m}\|\Delta_m b\|^2_{L^2}\right)^{\frac{1}{2}},\,2^{(s-1-\frac{\sigma}{2})(j-m)}2^{-\sigma m}, \tag{4.9}$$

then the following bound holds for $s-1-\frac{\sigma}{2}<0$

$$J_{12} \leq C\sup_j 2^j\|\Delta_j u\|_{L^\infty}\sum_j 2^{(2s+\sigma)j}\|\Delta_j b\|^2_{L^2} \leq C\|u\|^{1+\delta}_{B^1_{\infty,\infty}}\|b\|^2_{H^s} + \frac{\eta}{8}\|b\|^2_{H^{s+\alpha}}.$$

Alternatively, one can also bound J_{12} by

$$J_{12} \leq C\|u\|_{B^{1+\epsilon}_{\infty,\infty}}\|b\|^2_{H^s}$$

valid for any $s-1-\epsilon<0$. J_{13} can be estimated as follows.

$$J_{13} \leq C\sum_j 2^{2sj}\|\Delta_j b\|_{L^2}2^j\sum_{k\geq j-1}\|\Delta_k u\|_{L^\infty}\|\Delta_k b\|_{L^2}$$
$$\leq C\sum_j\left(2^{2sj}\|\Delta_j b\|^2_{L^2}\right)^{\frac{1}{2}}\sum_{k\geq j-1}2^k\|\Delta_k u\|_{L^\infty}\left(2^{2sk}\|\Delta_k b\|^2_{L^2}\right)^{\frac{1}{2}}2^{(s+1)(j-k)}$$

When $s + 1 > 0$,

$$J_{13} \leq C\|u\|_{B^1_{\infty,\infty}} \|b\|^2_{B^s_{2,2}}.$$

We will combine J_2 and J_4 to take advantage of some cancellations between them. Using the notion of paraproducts, we write

$$\Delta_j(b \cdot \nabla u) = \sum_{|j-k|\leq 2} \Delta_j(S_{k-1}b \cdot \nabla\Delta_k u) + \sum_{|j-k|\leq 2} \Delta_j(\Delta_k b \cdot \nabla S_{k-1} u) + \sum_{k\geq j-1} \sum_{|k-l|\leq 1} \Delta_j(\Delta_k b \cdot \nabla \Delta_l u).$$

$\Delta_j(b \cdot \nabla b)$ can be decomposed similarly. Inserting these decompositions in J_2 and J_4, we have

$$J_2 + J_4 = J_{21} + J_{22} + J_{23} + J_{42} + J_{43},$$

where

$$J_{21} = \sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int \Delta_j b \cdot \Delta_j(S_{k-1}b \cdot \nabla\Delta_k u) dx + \sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int \Delta_j u \cdot \Delta_j(S_{k-1}b \cdot \nabla\Delta_k b) dx,$$

$$J_{22} = \sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int \Delta_j b \cdot \Delta_j(\Delta_k b \cdot \nabla S_{k-1} u) dx,$$

$$J_{23} = \sum_j \sum_{k\geq j-1} 2^{2sj} \int \Delta_j b \cdot \sum_{|k-l|\leq 1} \Delta_j(\Delta_k b \cdot \nabla\Delta_l u) dx,$$

$$J_{42} = \sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int \Delta_j u \cdot \Delta_j(\Delta_k b \cdot \nabla S_{k-1} b) dx,$$

$$J_{23} = \sum_j \sum_{k\geq j-1} 2^{2sj} \int \Delta_j u \cdot \sum_{|k-l|\leq 1} \Delta_j(\Delta_k b \cdot \nabla\Delta_l b) dx.$$

The cancellation occurs in J_{21}. To see this, we use the notation of commutators to rewrite it as

$$\begin{aligned} J_{21} &= \sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int (\Delta_j b \cdot [\Delta_j, S_{k-1}b \cdot \nabla]\Delta_k u + \Delta_j u \cdot [\Delta_j, S_{k-1} b \cdot \nabla]\Delta_k b) dx \\ &\quad + \sum_j \sum_{|j-k|\leq 2} 2^{2sj} \int (\Delta_j b \cdot S_{k-1} b \cdot \nabla\Delta_j\Delta_k u + \Delta_j u \cdot S_{k-1} b \cdot \nabla\Delta_j\Delta_k b) dx \\ &\equiv J_{211} + J_{212}. \end{aligned}$$

The second part J_{212} is more or less zero and this is where the cancellation is. More precisely, we have

$$J_{212} = \sum_j 2^{2sj} \int \Delta_j b \cdot S_j b \cdot \nabla\Delta_j u \, dx + \sum_j 2^{2sj} \sum_{|j-k|\leq 2} \int \Delta_j b \cdot (S_{k-1}b - S_j b) \cdot \nabla\Delta_j\Delta_k u \, dx$$

$$+\sum_j 2^{2sj}\int \Delta_j u\cdot S_j b\cdot \nabla\Delta_j b\,dx$$
$$+\sum_j 2^{2sj}\sum_{|j-k|\le 2}\int \Delta_j u\cdot (S_{k-1}b - S_j b)\cdot \nabla\Delta_j\Delta_k b\,dx.$$

If we integrate by parts, we have

$$\int \Delta_j b\cdot (S_j b\cdot \nabla\Delta_j u)dx + \int \Delta_j u\cdot (S_j b\cdot \nabla\Delta_j b)dx = 0.$$

Noting that $S_{k-1}b - S_j b$ is the summation of a few $\Delta_l b$, we obtain

$$J_{212} = \sum_j \sum_{|j-k|\le 2}\sum_{|l-j|\le 3}\int (\Delta_j b\cdot \Delta_l b\cdot \nabla\Delta_j\Delta_k u + \Delta_j u\cdot \Delta_l b\cdot \nabla\Delta_j\Delta_k b)dx.$$

To bound the commutators, we state and prove a lemma.

Lemma 4.2. *Let $1\le q\le\infty$. Let U be a divergence free vector.*

$$\|[\Delta_j, S_{k-1}U\cdot\nabla]\Delta_k V\|_{L^q} \le C\|\nabla S_{k-1}U\|_{L^\infty}\|\Delta_k V\|_{L^q}, \tag{4.10}$$
$$\|[\Delta_j, S_{k-1}U\cdot\nabla]\Delta_k V\|_{L^q} \le C\|\nabla S_{k-1}U\|_{L^q}\|\Delta_k V\|_{L^\infty}. \tag{4.11}$$

Proof. (4.10) is essentially the same as (3.16). It suffices to prove (4.11). We write

$$[\Delta_j, S_{k-1}U\cdot\nabla]\Delta_k V = \int_{\mathbb{R}^d}\Phi_j(x-y)(S_{k-1}U(y) - S_{k-1}U(x))\cdot\nabla\Delta_k V(y)dy$$
$$= \int_{\mathbb{R}^d}\nabla\Phi_j(x-y)\cdot(S_{k-1}U(y) - S_{k-1}U(x))\cdot\Delta_k V(y)dy.$$

Since

$$S_{k-1}U(y) - S_{k-1}U(x) = \int_0^1 \frac{d}{dt}S_{k-1}U(ty+(1-t)x)dt$$
$$= \int_0^1 (y-x)\cdot\nabla S_{k-1}U(ty+(1-t)x)dt,$$

we obtain

$$|[\Delta_j, S_{k-1}U\cdot\nabla]\Delta_k V| \le \|\Delta_k V\|_{L^\infty}\int_0^1\int_{\mathbb{R}^d} h(x-y)|\nabla S_{k-1}U(ty+(1-t)x)|dy\,dt,$$

where $h(x) = |x||\nabla\Phi_j(x)|$. Making the substitution $z = ty+(1-t)x$ yields

$$\int_{\mathbb{R}^d} h(x-y)|\nabla S_{k-1}U(ty+(1-t)x)|dy = \int_{\mathbb{R}^d} t^d h(t^{-1}(x-z))|\nabla S_{k-1}U(z)|dz.$$

It then follows from Young's inequality that

$$\|[\Delta_j, S_{k-1}U\cdot\nabla]\Delta_k V\|_{L^q} \le C\|\Delta_k V\|_{L^\infty}\,\|\nabla S_{k-1}U\|_{L^q}$$

where the constant C comes from the integral

$$C = \int_0^1 \int_{\mathbb{R}^d} t^d h(t^{-1}x)\,dx\,dt.$$

This completes the proof of Lemma 4.2.

We now bound J_{211}. It suffices to consider the term with $k = j$. Applying Lemma 4.2 to bound the first commutator and Hölder's inequality to bound the second term in J_{211}, we have

$$J_{211} \le \sum_j 2^{2sj} \|\Delta_j b\|_{L^2} \|\nabla S_{j-1} b\|_{L^2} \|\Delta_j u\|_{L^\infty} + \sum_j 2^{2sj} \|\Delta_j u\|_{L^\infty} \|[\Delta_j, S_{j-1} b \cdot \nabla]\Delta_j b\|_{L^1}.$$

The term $\|[\Delta_j, S_{j-1} b \cdot \nabla]\Delta_j b\|_{L^1}$ can not be bounded suitably by Lemma 4.2, so we bound it directly by Hölder's inequality

$$\|[\Delta_j, S_{j-1} b \cdot \nabla]\Delta_j b\|_{L^1} \le \|S_{j-1} b\|_{L^2} \|\nabla \Delta_j b\|_{L^2}.$$

By Bernstein's inequality,

$$\begin{aligned} J_{211} \le & \sum_j 2^{2sj} \|\Delta_j b\|_{L^2} \|\Delta_j u\|_{L^\infty} \sum_{m \le j-1} 2^m \|\Delta_m b\|_{L^2} \\ & + \sum_j 2^{2sj} \|\Delta_j u\|_{L^\infty} 2^j \|\Delta_j b\|_{L^2} \sum_{m \le j-1} \|\Delta_m b\|_{L^2}. \end{aligned}$$

Hölder's inequality again yields

$$J_{212} \le \sum_j 2^{2sj} 2^j \|\Delta_j u\|_{L^\infty} \|\Delta_j b\|_{L^2}^2.$$

To further bound $J_{21} = J_{211} + J_{212}$, we write

$$\begin{aligned} & \sum_j 2^{2sj} \|\Delta_j u\|_{L^\infty} 2^j \|\Delta_j b\|_{L^2} \sum_{m \le j-1} \|\Delta_m b\|_{L^2} \\ & = \sum_j 2^j \|\Delta_j u\|_{L^\infty} \left(2^{(2s+\sigma)j} \|\Delta_j b\|_{L^2}^2\right)^{\frac{1}{2}} \sum_{m \le j-1} \left(2^{(2s+\sigma)m} \|\Delta_m b\|_{L^2}^2\right)^{\frac{1}{2}} 2^{(s-\frac{\sigma}{2})(j-m)} 2^{-\sigma m}. \end{aligned}$$

When $s - \frac{\sigma}{2} < 0$, this term is bounded by

$$C\|u\|_{B^1_{\infty,\infty}}^{1+\delta} \|b\|_{H^s}^2 + \frac{\eta}{8} \|b\|_{H^{s+\alpha}}^2$$

Therefore,

$$J_{21} \le C\|u\|_{B^1_{\infty,\infty}}^1 \|b\|_{H^s}^2 + C\|u\|_{B^1_{\infty,\infty}}^{1+\delta} \|b\|_{H^s}^2 + \frac{\eta}{8} \|b\|_{H^{s+\alpha}}^2.$$

Alternatively, J_{21} also obeys

$$J_{21} \le C\|u\|_{B^{1+\epsilon}_{\infty,\infty}} \|b\|_{H^s}^2$$

valid for $s - \epsilon < 0$. We remark that the range for s could have been enlarged if we had a better estimate for the term $\|[\Delta_j, S_{j-1} b \cdot \nabla]\Delta_j b\|_{L^1}$. To estimate J_{22}, we first bound it by

$$J_{22} \le \sum_j 2^{2sj} \|\Delta_j b\|_{L^2} \, \|\Delta_j b\|_{L^2} \sum_{m\le j-1} 2^m \|\Delta_m u\|_{L^\infty},$$

which is the same as the second part of J_{11}. For J_{23}, we have

$$\begin{aligned} J_{23} &\le \sum_j 2^{2sj} \|\Delta_j b\|_{L^2} 2^j \sum_{k\ge j-1} \|\Delta_k u\|_{L^\infty} \|\Delta_k b\|_{L^2} \\ &\le \sum_j \left(2^{2sj} \|\Delta_j b\|_{L^2}^2\right)^{\frac{1}{2}} \sum_{k\ge j-1} 2^k \|\Delta_k u\|_{L^\infty} (2^{2sk} \|\Delta_k b\|_{L^2})^{\frac{1}{2}} 2^{(s+1)(j-k)}. \end{aligned}$$

For $s+1>0$,

$$J_{23} \le C\|u\|_{B^1_{\infty,\infty}}^1 \|b\|_{H^s}^2.$$

We now estimate J_{42} and J_{43}. First, we bound them by

$$\begin{aligned} J_{42} &= \sum_j 2^{2sj} \|\Delta_j u\|_{L^\infty} \|\Delta_j b\|_{L^2} \sum_{m\le j-1} 2^m \|\Delta_m b\|_{L^2}, \\ J_{43} &= \sum_j 2^{2sj} \|\Delta_j u\|_{L^\infty} 2^j \sum_{k\ge j-1} \|\Delta_k b\|_{L^2}^2. \end{aligned}$$

As in (4.8), we have for $s-1<0$,

$$J_{42} \le C\|u\|_{B^1_{\infty,\infty}} \|b\|_{H^s}^2.$$

We write J_{43} as

$$J_{43} = \sum_j 2^j \|\Delta_j u\|_{L^\infty} \sum_{k\ge j-1} 2^{2sm} \|\Delta_m b\|_{L^2}^2 2^{2s(j-m)} \tag{4.12}$$

and for $s>0$,

$$J_{43} \le C\|u\|_{B^1_{\infty,\infty}} \|b\|_{H^s}^2.$$

To slightly relax the range of s, we can alternatively bound J_{43} as follows. Instead of (4.12), we write

$$J_{43} = \sum_j 2^j \|\Delta_j u\|_{L^\infty} \sum_{k\ge j-1} 2^{(2s+\sigma)m} \|\Delta_m b\|_{L^2}^2 2^{(2s+\sigma)(j-m)} 2^{-\sigma j}.$$

Then for $s > -\frac{\sigma}{2}$,

$$J_{43} \le C\|u\|_{B^1_{\infty,\infty}}^{1+\delta} \|b\|_{H^s}^2 + \frac{\eta}{8}\|b\|_{H^{s+\alpha}}^2.$$

J_3 can be similarly estimated as the previous terms. It can be bounded either by

$$J_3 \le C\|u\|_{B^1_{\infty,\infty}}^{1+\delta} \|u\|_{H^s}^2 + \frac{\nu}{8}\|u\|_{H^{s+\alpha}}^2 \quad \text{valid for } -1<s<1+\sigma/2$$

or by

$$J_3 \le C\|u\|_{B^{1+\epsilon}_{\infty,\infty}} \|u\|_{H^s}^2 + \frac{\nu}{8}\|u\|_{H^{s+\alpha}}^2 \quad \text{valid for } -1<s<1+\epsilon.$$

Inserting all the bounds in (4.2), we find that either

$$\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right) + \nu\|u\|_{H^{s+\alpha}}^2 + \eta\|b\|_{H^{s+\alpha}}^2 \le C\|u\|_{B^1_{\infty,\infty}}^{1+\delta}\left(\|b\|_{H^s}^2 + \|u\|_{H^s}^2\right)$$

or

$$\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right) + \nu\|u\|_{H^{s+\alpha}}^2 + \eta\|b\|_{H^{s+\alpha}}^2 \le C\|u\|_{B^{1+\epsilon}_{\infty,\infty}}\left(\|b\|_{H^s}^2 + \|u\|_{H^s}^2\right).$$

(4.1) follows as a special consequence. This completes the proof of Theorem 4.1.

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