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Global regularity for the generalized incompressible Oldroyd-B model with only stress tensor dissipation in critical Besov spaces

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Abstract

This paper presents a new small data global well-posedness result on the incompressible Oldroyd-B model with only dissipation in the equation of stress tensor (without stress tensor damping or velocity dissipation). The dissipation is not necessarily given by the standard Laplacian operator and any fractional dissipation with fractional power equal to or greater than $1/2$ suffices. The functional setting is the hybrid homogeneous Besov spaces, which allow us to maximize the functional spaces of the initial data.

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1. Introduction

The Oldroyd-B model, derived by J.G. Oldroyd, reflects one of the most popular constitutive laws obeyed by viscoelastic fluids such as solvent with particles suspended in it (see, e.g., [3,12,15,34]). A general form of the d-dimensional incompressible Oldroyd-B model is given by

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$$\begin{cases} u_t + u \cdot \nabla u + \nu \Lambda^{2\alpha} u + \nabla p = \mu_1 \nabla \cdot \tau, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \tau_t + u \cdot \nabla \tau + a\tau + \eta \Lambda^{2\beta} \tau + Q(\tau, \nabla u) = \mu_2 D(u), \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x), \quad \tau(0,x) = \tau_0(x), \end{cases} \tag{1.1}$$

where $u(t,x)$ stands for the velocity, $p(t,x)$ the pressure and $\tau(t,x)$ the non-Newtonian part of the stress tensor (a d-by-d symmetric matrix), and $0 \le \alpha, \beta \le 1$ and ν, μ_1, a, μ_2 are nonnegative constants. Here $D(u)$ is the symmetric part of the velocity gradient, namely

$$D(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^\top\big)$$

and the bilinear term Q assumes the following form

$$Q(\tau, \nabla u) = \tau W(u) - W(u)\tau - b\big(D(u)\tau + \tau D(u)\big) \tag{1.2}$$

with $b \in [-1, 1]$ being a constant and $W(u)$ being the skew-symmetric part of the ∇u,

$$W(u) = \frac{1}{2}\big(\nabla u - (\nabla u)^\top\big).$$

In addition, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator and the general fractional Laplacian operator $(-\Delta)^\gamma$ is defined through the Fourier transform, namely

$$\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \widehat{f}(\xi).$$

When $\alpha = 1$ and $\beta = 1$, (1.1) reduces to the standard Oldroyd-B model. For general fractional powers $\alpha \ge 0$ and $\beta \ge 0$, (1.1) allows us to examine a family of equations simultaneously and helps us understand how the properties of the solutions vary with respect to the sizes of α and β.

Due to its special structure and features, the Oldroyd-B model has recently attracted considerable interests from the community of mathematical fluids. A rich array of results have been established on the well-posedness and closely related problems. To place our results into the context of existing research, we briefly describe some of related work. We start with the case $\alpha = 1$ and $\eta = 0$. When $\nu > 0$ and $a > 0$, the existence and uniqueness of local strong solutions have been established in Hilbert spaces H^s by Guillopé and Saut [24]. If the coupling parameters and the initial data are sufficiently small, these solutions are shown to be global [25]. Similar results in $L^s - L^r$ space were obtained by Fernandez-Cara, Guillén and Ortega [22]. The study of the existence and uniqueness in the critical Besov setting was initiated by Chemin and Masmoudi [7]. Their results were improved in the critical L^p framework for the case of the non-small coupling parameters by Zi, Fang and Zhang [41]. In the corotational case, namely $b = 0$ in (1.2), the global existence of weak solutions was established by Lions and Masmoudi [32].

Several more recent results dealt with the case when there is only kinematic dissipation (no damping or dissipation in τ), namely (1.1) with $\nu > 0$ and $a = \eta = 0$. Zhu [40] obtained small global smooth solutions of the 3D Oldroyd-B model with $\alpha = 1$ in time-weighted Sobolev spaces. Chen and Hao [8] extended this small data global well-posedness to the critical Besov setting, again for $\alpha = 1$. The work of Wu and Zhao [36] were able to establish the small data global well-posedness in critical Besov spaces for any α in the range $1/2 \le \alpha \le 1$.

We now turn to the case when there is no kinematic dissipation, namely (1.1) with $\nu = 0$. The well-posedness problem becomes extremely difficult. When both the damping mechanism and the Laplacian dissipation are present for τ, Elgindi and Rousset [18] were able to establish a small data global well-posedness result in the Sobolev space for the 2D Oldroyd-B. The 3D case was resolved by Elgindi and Liu [19]. We remark that the damping mechanism in τ plays a crucial role in [18,19]. A recent work of Constantin, Wu, Zhao and Zhu [15] were able to establish the small data global well-posedness for (1.1) with $\nu = 0$, $a = 0$ and $\frac{1}{2} \le \beta \le 1$, the case of no damping and general fractional dissipation in τ. This result is for general d-dimensional space in the Sobolev space $H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2}$. [15] offered a key observation that the non-Newtonian stress tensor can actually regularize the viscoelastic fluids. We remark that there is a very large literature on the Oldroyd-B model and interested readers may consult the references [2,9,11–15,20,21,26,27,29–31,33,35–39]. This list is by no means exhaustive.

This paper focuses on the following generalized Oldroyd-B model without dissipation or damping mechanism

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \mu_1 \nabla\cdot\tau, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d, \\ \tau_t + u\cdot\nabla\tau + \eta\Lambda^{2\beta}\tau + Q(\tau,\nabla u) = \mu_2 D(u), \\ \nabla\cdot u = 0, \\ u(0,x) = u_0(x), \quad \tau(0,x) = \tau_0(x). \end{cases} \tag{1.3}$$

The goal of this paper is to extend the work of Constantin, Wu, Zhao and Zhu [15] to critical Besov setting. The small data global well-posedness of [15] is in the Sobolev setting $H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2}$. The advantage of the critical Besov spaces is that they weaken the regularity requirements on the initial data and maximize the functional setting of the solutions.

Due to the lack of the kinematic dissipation, the global well-posedness and the stability problem on (1.3) is not trivial. The first equation in (1.3) is a forced incompressible Euler equation. As revealed in the work of Kiselev and Sverak [28], the gradient of the vorticity (the curl of the velocity) to the 2D Euler equation in a unit disk can grow double exponentially in time. These growth results on the Euler and forced Euler equations appear to suggest that we should not expect the stability of (1.3) near the trivial solution in any Sobolev or Besov settings. The results of this paper are possible due to a new observation. Let $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ denote the standard Leray projection onto divergence-free vector fields. It is easy to check from (1.3) that u and $\Lambda^{-1}\mathbb{P}\nabla\cdot\tau$ satisfy

$$\begin{cases} \partial_t u - \nabla^{-1}\mathbb{P}\nabla\cdot\tau = -\mathbb{P}(u\cdot\nabla u), \quad x\in\mathbb{R}^d, \quad t>0, \\ \partial_t\Lambda^{-1}\mathbb{P}\nabla\cdot\tau + \eta\Lambda^{2\beta}\Lambda^{-1}\mathbb{P}\nabla\cdot\tau + \frac{\mu_2}{2}\Lambda u \\ \qquad = -\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) - \Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u). \end{cases} \tag{1.4}$$

For the sake of clarity, we focus on the linearization of (1.4), which is given by

$$\begin{cases} \partial_t u - \nabla^{-1}\mathbb{P}\nabla\cdot\tau = 0, \\ \partial_t\Lambda^{-1}\mathbb{P}\nabla\cdot\tau + \eta\Lambda^{2\beta}\Lambda^{-1}\mathbb{P}\nabla\cdot\tau + \frac{\mu_2}{2}\Lambda u = 0. \end{cases} \tag{1.5}$$

By differentiating (1.5) in t and making suitable substitutions, we find that u and $\nabla^{-1}\mathbb{P}\nabla\cdot\tau$ satisfy exactly the same damped wave equation,

$$\begin{cases} \partial_{tt} u + \eta \Lambda^{2\beta} \partial_t u - \frac{\mu_2}{2} \Delta u = 0, \\ \partial_{tt} \nabla^{-1} \mathbb{P} \nabla \cdot \tau + \eta \Lambda^{2\beta} \partial_t \nabla^{-1} \mathbb{P} \nabla \cdot \tau - \frac{\mu_2}{2} \Delta \nabla^{-1} \mathbb{P} \nabla \cdot \tau = 0. \end{cases} \tag{1.6}$$

(1.6) reveals the hidden dissipation and dispersion regularization properties for u and $\nabla^{-1}\mathbb{P}\nabla \cdot \tau$. We exploit the regularization of (1.6) by constructing suitable energy functionals based on (1.6). This explains the prime reason why the small data global wellposedness and stability are possible even when the velocity equation involves no dissipation and the equation of τ has no damping.

We choose the critical Besov space as the functional setup for our global solutions. We explain how we select the precise regularity indices of the Besov spaces. For the d-dimensional incompressible Navier-Stokes equations with general fractional dissipation

$$\partial_t u + u \cdot \nabla u + \nabla p + \nu \Lambda^{2\alpha} u = 0, \quad x \in \mathbb{R}^d,\ t > 0, \tag{1.7}$$

one of the standard critical space is the homogeneous Besov space

$$\dot{B}_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d).$$

Critical spaces on the Navier-Stokes or generalized Navier-Stokes equations can be found in many papers and books (see, e.g., [1,4–6,23]). Any solution (u, p) of (1.7) and its naturally scaled counterpart $(u_\lambda, p_{\lambda)})$ with

$$u_\lambda(t,x) = \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x), \quad p_\lambda(t,x) = \lambda^{4\alpha-2} p(\lambda^{2\alpha} t, \lambda x)$$

share the equivalent norm

$$\|u(t,\cdot)\|_{\dot{B}_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)} \approx \|u_\lambda(\lambda^{-2\alpha} t, \cdot)\|_{\dot{B}_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)}.$$

As can be seen from (1.6), both u and $\Lambda^{-1}\mathbb{P}\nabla \cdot \tau$ are fractionally dissipated via the operator $(-\Delta)^\beta$ in addition to the dispersion effect. As a consequence, the natural setup for u and τ should involve the homogeneous critical Besov space

$$\dot{B}_{2,1}^{1+\frac{d}{2}-2\beta}(\mathbb{R}^d).$$

The situation here is more complex due to the nonlinear coupling and the partial dissipation in (1.3). Strictly speaking, there is no scaling invariance for (1.3). As we explain later in this introduction, the low frequencies and the high frequencies have different regularity setting and we employ the hybrid Besov spaces introduced by Danchin in [16,17] and used by Chen, Miao and Zhang [10] in their studies of the compressible Navier-Stokes equations.

After explaining some of the basic ingredients of our main result, we are ready to provide a precise statement.

Theorem 1.1. *Let $d \geq 2$ and $\mu_1, \mu_2, \eta > 0$. Assume*

$$\text{either} \quad \frac{1}{2} < \beta \leq 1 \quad \text{or} \quad \beta = \frac{1}{2} \quad \text{with} \quad \eta^2 \geq C\mu_1\mu_2,$$

where $C > 0$ is a pure constant. Then there exists a small constant ε such that if $\tau_0 \in \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}}$, $u_0 \in \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta-1}$ satisfy $\nabla \cdot u_0 = 0$ and

$$\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta-1}} + \|\tau_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}}} \le \varepsilon, \tag{1.8}$$

then (1.3) has a unique global solution (u, τ) satisfying

$$u \in C(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1});$$
$$\tau \in C(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}}), \quad \tau \in L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta}).$$

In fact, we have

$$\sup_t (\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta-1}} + \|\tau\|_{\dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}}}) + \int_0^\infty \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' + \int_0^\infty \|\tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}+2\beta}} \mathrm{d}t' \lesssim \varepsilon \tag{1.9}$$

and

$$\int_0^t \|\tau(t')\|^l_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' \lesssim \varepsilon + \varepsilon t. \tag{1.10}$$

Furthermore, if (u_0, τ_0) is more regular and sufficiently small, say $\tau_0 \in \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+s}$, $u_0 \in \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta+s-1}$ with $s > 0$ and their norms are smaller than ε depending on s, then we have

$$\begin{aligned} &u \in C(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta+s-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+s});\\ &\tau \in C(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1-2\beta} \cap \dot{B}_{2,1}^{\frac{d}{2}+s}), \quad \tau \in L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2\beta+s}). \end{aligned} \tag{1.11}$$

Our result establishes the small data global existence and regularity of (1.3) in critical Besov spaces. It is not clear if the upper bound (1.10) for the time integral of the lower frequency piece of τ can be improved to a time independent bound. In order to describe our proof, we explain how the low frequencies and the high frequencies are set in different regularity Besov spaces in order to suit the linearized system in (1.5) or equivalently in (1.6). (1.5) can be written as

$$\partial_t \begin{pmatrix} u \\ \nabla^{-1}\mathbb{P}\nabla \cdot \tau \end{pmatrix} = A \begin{pmatrix} u \\ \nabla^{-1}\mathbb{P}\nabla \cdot \tau \end{pmatrix}, \tag{1.12}$$

where

$$A(\Lambda) = \begin{pmatrix} 0 & \mu_1 \Lambda \\ -\frac{\mu_2}{2}\Lambda & -\eta \Lambda^{2\beta} \end{pmatrix}.$$

Let $\tilde{\tau} = \Lambda^{-1}\mathbb{P}\nabla \cdot \tau$, then the solution can be expressed as

$$(u(t), \tilde{\tau}(t))^\top = e^{A(\Lambda)t}(u(0), \tilde{\tau}(0))^\top.$$

In the frequency space, A becomes a multiplier and the eigenvalues of $A(\xi)$ are given by

$$\lambda_+ = -\frac{\eta|\xi|^{2\beta} + \sqrt{\eta^2|\xi|^{4\beta} - 2\mu_1\mu_2|\xi|^2}}{2},$$
$$\lambda_- = -\frac{\eta|\xi|^{2\beta} - \sqrt{\eta^2|\xi|^{4\beta} - 2\mu_1\mu_2|\xi|^2}}{2}.$$

The change of functions

$$\hat{v}^+(\xi) = \frac{\mu_2}{2}|\xi|\hat{u} + \lambda_+\hat{\tilde{\tau}},$$
$$\hat{v}^-(\xi) = \lambda_+\hat{u} + \mu_1|\xi|\hat{\tilde{\tau}}$$

diagonalize the system (1.12).

For $\frac{1}{2} \le \beta \le 1$ and as $\xi \to 0$,

$$\lambda_+ \sim -\frac{1}{2}\eta|\xi|^{2\beta} \quad \text{and} \quad \lambda_- \sim -\frac{1}{2}\eta|\xi|^{2\beta},$$

v^+ and v^- both behave like the heat kernel operator $e^{-\frac{1}{2}\eta t \Lambda^{2\beta}}$. Therefore, for low frequencies, u and $\tilde{\tau}$ have the parabolic behavior

$$u, \ \tilde{\tau} \sim e^{-\frac{1}{2}\eta t \Lambda^{2\beta}}$$

and this explains why we choose the critical Besov space $\dot{B}_{2,1}^{\frac{d}{2}+1-2\beta}$ for the low frequencies. Similarly, for $\frac{1}{2} \le \beta \le 1$ and as $\xi \to \infty$,

$$\lambda_+ \sim -\eta|\xi|^{2\beta} \quad \text{and} \quad \lambda_- \sim -\frac{\mu_1\mu_2}{2\eta}|\xi|^{2-2\beta},$$

and v^+ and v^- behave like the heat kernel operator $e^{-\eta t \Lambda^{2\beta}}$ and $e^{-\frac{\mu_1\mu_2}{2\eta} t \Lambda^{2-2\beta}}$, respectively, for the high frequencies. In the case when $\beta = 1$, $\lambda_+ \sim -\eta|\xi|^2$ and $\lambda_- \sim -\frac{\mu_1\mu_2}{2\eta}$ as $\xi \to \infty$, v^+ has the parabolic smoothing effect that behaves like heat kernel operator $e^{-\eta t \Lambda^{2\beta}}$, and v^- has the damping effect for the high frequencies. $\tilde{\tau}$ and u have similar behaviors as v^+ and v^-, respectively. This explains why we choose the critical Besov space $\dot{B}_{2,1}^{\frac{d}{2}+2\beta-1}$ for u and $\dot{B}_{2,1}^{\frac{d}{2}}$ for τ.

To accommodate the different behaviors of the solution (u, τ) at low and high frequencies, we adopt the hybrid Besov spaces (with different regularity indices for low and high frequencies) as our functional setting. This explains the selection of the Besov spaces for the initial data in Theorem 1.1. The proof of Theorem 1.1 focuses on establishing the global bound on the solution. The framework of the proof is the bootstrapping argument. This process starts with the definition of a suitable energy functional. As explained before, we need to make use of the stabilizing and smoothing effect of the wave structure in (1.6). In addition, we also incorporate the hybrid Besov setting in the energy functional. As detailed in the following section, we use $\|u\|^l_{\dot{B}^s_{p,q}}$ and $\|u\|^h_{\dot{B}^s_{p,q}}$ to represent the low and high frequency pieces of the Besov norm $\dot{B}^s_{p,q}$. Therefore, our energy functional E consists of four parts

$$E(t) = E_0^l(t) + E_0^h(t) + E^l(t) + E^h(t), \tag{1.13}$$

where $E_0^l(t)$ and $E_0^h(t)$ denote the low and high frequencies associated with the wave structure, more precisely,

$$\begin{aligned} E_0^l(t) &\triangleq \sup_t \|u\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \sup_t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} \\ &\quad + \int_0^t \|u\|^l_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' + \int_0^t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t', \\ E_0^h(t) &\triangleq \sup_t \|u\|^h_{\dot{B}^{\frac{d}{2}+2\beta-1}_{2,1}} + \sup_t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t', \end{aligned}$$

and $E^l(t)$ and $E^h(t)$ are the low and high frequency pieces for the original system,

$$\begin{aligned} E^l(t) &\triangleq \sup_t \|u\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \sup_t \|\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \int_0^t \|u\|^l_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' + \int_0^t \|\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^l_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t', \\ E^h(t) &\triangleq \sup_t \|u\|^h_{\dot{B}^{\frac{d}{2}+2\beta-1}_{2,1}} + \sup_t \|\tau\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' + \int_0^t \|\tau\|^h_{\dot{B}^{\frac{d}{2}+2\beta}_{2,1}} \mathrm{d}t'. \end{aligned}$$

It is clear from the definitions of $E_0^l(t)$, $E_0^h(t)$, $E^l(t)$ and $E^h(t)$ that

$$E(t) \approx E^l(t) + E^h(t).$$

In addition, we also estimate the time integral of the lower frequency piece of τ, namely

$$E_\tau^l(t) \triangleq \sup_t \|\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \int_0^t \|\tau\|^l_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t'.$$

To prove the global bound and the existence part of Theorem 1.1, our main efforts are devoted to establishing the inequality

$$E(t) \le C_1 E_0 + C_2 E^2(t), \tag{1.14}$$

where C_1 and C_2 are positive constants independent of t, and

$$E_0 = \|u_0\|_{\dot B^{\frac{d}{2}+1-2\beta, \frac{d}{2}+2\beta-1}} + \|\tau_0\|_{\dot B^{\frac{d}{2}+1-2\beta, \frac{d}{2}}}.$$

The bootstrapping argument applied to (1.14) implies the desired result. That is, there exists a small constant $\varepsilon > 0$ such that, if (1.8) holds or $E_0 \le \varepsilon^2$, then, for a pure constant $C > 0$,

$$E(t) \le C\,\varepsilon^2 \qquad \text{for all } t > 0.$$

This uniform upper bound, in particular, yields the global bound on the Besov norms of (u, τ). Together with the local well-posedness which follows from a standard procedure (see, e.g., [1,8, 16]), we obtain the global existence part of Theorem 1.1. The proof of (1.14) is very technical and takes advantage of the special wave structure. More details can be found in Sections 3 and 4.

The proof of the uniqueness part of Theorem 1.1 is not trivial. Due to the lack of the velocity dissipation in the original system (1.3), we also need to make use of the parabolic smoothing or damping effect of the wave structure as well. We establish a priori bounds on the difference of two solutions combining the wave equations and the original system. More technical details can be found in Subsection 4.2.

To establish the high regularity part of Theorem 1.1, we replace the energy pieces associated with the high frequencies in (1.13) by the following more regular pieces:

$$E_0'^h(t) \triangleq \sup_t \|u\|^h_{\dot B^{\frac{d}{2}+2\beta}_{2,1}} + \sup_t \|\mathbb{P}\nabla\cdot\tau\|^h_{\dot B^{\frac{d}{2}}_{2,1}} + \int_0^t \|u\|^h_{\dot B^{\frac{d}{2}+2}_{2,1}} \mathrm{d}t', \tag{1.15}$$

$$E'^h(t) \triangleq \sup_t \|u\|^h_{\dot B^{\frac{d}{2}+2\beta}_{2,1}} + \sup_t \|\tau\|^h_{\dot B^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|u\|^h_{\dot B^{\frac{d}{2}+2}_{2,1}} \mathrm{d}t' + \int_0^t \|\tau\|^h_{\dot B^{\frac{d}{2}+2\beta+1}_{2,1}} \mathrm{d}t'. \tag{1.16}$$

The key component of the proof is the following energy inequality

$$E_0'^h(t) + E'^h(t) \le C_1 E_0' + C_2((E'^h_0(t) + E'^h(t))E(t) + E^2(t)). \tag{1.17}$$

The proof of this energy inequality shares some similarities as the proof of (1.14), and is a consequence of a tedious process of controlling many terms. The high regularity part in (1.11) follows directly from (1.17). More details can be found in Section 5.

The rest of this paper is divided into four sections. Section 2 serves as a preparation. It provides the definitions of the homogeneous hybrid Besov spaces and supplies various inequalities such as bounds for products and triple products in Besov norms. Section 3 presents the proof of the key energy inequality, namely (1.14). The proof is long and involves many tedious estimates. Section 4 proves the existence part of Theorem 1.1 by applying the bootstrapping argument to (1.14). The proof of the uniqueness part is also detailed in this section. The last section, Section 5, establishes the higher regularity part of Theorem 1.1.

2. Littlewood-Paley theory and Besov spaces

We review several facts about the homogeneous Littlewood-Paley theory, Besov spaces, hybrid Besov spaces, and products and triple product estimates in these spaces.

2.1. Littlewood-Paley decomposition

The definition of the homogeneous Littlewood-Paley decomposition relies on the dyadic partition of unity (see, e.g., [1]). Let $\varphi \in C^\infty(\mathbb{R}^d)$ be a radial functions supported in $\mathcal{C} = \{\xi \in \mathbb{R}^d,\ \frac{5}{6} \le |\xi| \le \frac{12}{5}\}$ satisfying

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \ \text{ if } \ \xi \neq 0.$$

We use $\widehat{f}$ or $\mathcal{F}(f)$ to denotes the Fourier transform of f, and $\mathcal{F}^{-1}(f)$ to denote the inverse Fourier transform of f. We set

$$h(x) = \mathcal{F}^{-1}\big(\varphi(\xi)\big)$$

and define the dyadic blocks as follows

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^d} h(2^j y)u(x-y)\mathrm{d}y,$$

$$\dot{S}_j u = \sum_{j' \le j-1} \dot{\Delta}_{j'} u.$$

Definition 2.1. We denote by $\mathcal{S}'_h$ the space of tempered distributions u such that

$$\lim_{j\to-\infty} \dot{S}_j u = 0 \ \text{ in } \ \mathcal{S}'.$$

Then the homogeneous Littlewood-Paley decomposition is defined as

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u, \qquad \text{for} \quad u \in \mathcal{S}'_h.$$

With our choice of φ, we have

$$\dot{\Delta}_j \dot{\Delta}_k u = 0 \ \text{ if } \ |j-k| \ge 2, \quad \text{and} \quad \dot{\Delta}_j(\dot{S}_{k-1} u \dot{\Delta}_k u) = 0 \ \text{ if } \ |j-k| \ge 4.$$

The following lemma provides Bernstein-type inequalities for fractional derivatives.

Lemma 2.1. *Let $\beta \ge 0$. Let $1 \le p \le q \le +\infty$.*

(1) Let $j \in \mathbb{Z}$ and $m > 0$. If f satisfies

$$\operatorname{supp}\widehat{f} \subseteq \{\xi \in \mathbb{R}^d,\ |\xi| \le m\, 2^j\},$$

then, for some constant C independent of f and j,

$$\|\Lambda^\beta f\|_{L^q(\mathbb{R}^d)} \le C\, 2^{j|\beta|+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

(2) Let $j \in \mathbb{Z}$ and $m_1, m_2 > 0$. If f satisfies

$$\operatorname{supp}\widehat{f} \subseteq \{\xi \in \mathbb{R}^d,\ m_1 2^j \le |\xi| \le m_2 2^j\},$$

then, for two constants C_1 and C_2 independent of f and j,

$$C_1 2^{\beta j}\|f\|_{L^q(\mathbb{R}^d)} \le \|\Lambda^\beta f\|_{L^q(\mathbb{R}^d)} \le C_2 2^{\beta j+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

Especially, Lemma 2.1 holds for the dyadic blocks, namely for $f = \dot{\Delta}_j u$.

2.2. Homogeneous Besov spaces

Definition 2.2. For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as

$$\dot{B}^s_{p,r} \triangleq \{u \in \mathcal{S}'_h,\ \|u\|_{\dot{B}^s_{p,r}} < \infty\},$$

where the homogeneous Besov norm is given by

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \|\{2^{js}\|\dot{\Delta}_j u\|_{L^p}\}_j\|_{l^r}.$$

Clearly, the definition of the space $\dot{B}^s_{p,r}$ does not depend on the choice of φ.

2.3. Hybrid Besov spaces

The following hybrid Besov spaces allow different regularity indices for low and high frequencies (see [17]).

Definition 2.3. For $s, t \in \mathbb{R}$, the hybrid Besov space $\dot{B}^{s,t}$ is defined by

$$\dot{B}^{s,t} \triangleq \{u \in \mathcal{S}'_h,\ \|u\|_{\dot{B}^{s,t}} < \infty\}$$

with the norm given by

$$\|u\|_{\dot{B}^{s,t}} = \sum_{j\le 0} 2^{js}\|\dot{\Delta}_j u\|_{L^2} + \sum_{j>0} 2^{jt}\|\dot{\Delta}_j u\|_{L^2}.$$

We will use the notation

$$\|u\|^{l}_{\dot B^{s}_{2,1}} \triangleq \sum_{j\le 0} 2^{js}\|\dot\Delta_j u\|_2 \quad \text{and} \quad \|u\|^{h}_{\dot B^{s}_{2,1}} \triangleq \sum_{j> 0} 2^{js}\|\dot\Delta_j u\|_2.$$

For $s, t \in \mathbb{R}$ and $r \in [1, \infty]$, $L^r_T(\dot B^{s,t}) = L^r(0, T; \dot B^{s,t})$ denotes the standard space-time space with the norm

$$\|u\|_{L^r_T(\dot B^{s,t})} = \| \|u\|_{\dot B^{s,t}} \|_{L^r(0,T)}.$$

In contrast, the norm of the space-time Besov space $\widetilde L^r_T(\dot B^{s,t})$ is defined by

$$\|u\|_{\widetilde L^r_T(\dot B^{s,t})} = \sum_{j\le 0} 2^{js}\|\dot\Delta_j u\|_{L^r_T L^2} + \sum_{j>0} 2^{jt}\|\dot\Delta_j u\|_{L^r_T L^2}.$$

By the Minkowski inequality, we easily find that $\widetilde L^1_T(\dot B^{s,t}) = L^1_T(\dot B^{s,t})$ and $\widetilde L^r_T(\dot B^{s,t}) \subseteq L^r_T(\dot B^{s,t})$ for $r \ge 1$ (see, e.g., [1]).

The following lemma is a direct consequence of the definition of the hybrid Besov space. Please refer to [17] for more details.

Lemma 2.2.

(i) *We have* $\dot B^{s,s} = \dot B^{s}_{2,1}$.
(ii) *If* $s \le t$ *then* $\dot B^{s,t} = \dot B^{s}_{2,1} \cap \dot B^{t}_{2,1}$. *Otherwise,* $\dot B^{s,t} = \dot B^{s}_{2,1} + \dot B^{t}_{2,1}$.
(iii) *If* $s_1 \le s_2$ *and* $t_1 \ge t_2$*, then* $\dot B^{s_1,t_1} \hookrightarrow \dot B^{s_2,t_2}$.

2.4. Paraproducts and product estimates in hybrid Besov spaces

We continue to review more information on the Besov spaces and hybrid Besov spaces. Especially product and triple product estimates in these spaces are provided. We start by recalling the paraproduct decomposition

$$uv = \dot T_u v + \dot T_v u + \dot R(u, v),$$

where the homogeneous paraproduct of v by u is given by

$$\dot T_u v \triangleq \sum_q \dot S_{q-1} u \dot\Delta_q v,$$

and the homogeneous remainder of u and v by

$$\dot R(u, v) \triangleq \sum_q \dot\Delta_q u \tilde{\dot\Delta}_q v, \quad \text{and} \quad \tilde{\dot\Delta}_q = \dot\Delta_{q-1} + \dot\Delta_q + \dot\Delta_{q+1}.$$

One useful property of the homogeneous Besov spaces is the Besov embedding.

Proposition 2.3. *Assume $s, s_1, s_2 \in \mathbb{R}$ and $1 \le p, p_1, p_2, r, r_1, r_2 \le +\infty$. Then we have the following properties:*

(i) If $p_1 \le p_2$, $r_1 \le r_2$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-\frac{d}{p_1}+\frac{d}{p_2}}_{p_2,r_2}$.
(ii) If $s_1 \ne s_2$ and $\theta \in (0,1)$, then

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \le \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}.$$

(iii) $\dot{H}^s \approx \dot{B}^s_{2,2}$ and

$$\frac{1}{C^{|s|+1}}\|u\|_{\dot{B}^s_{2,2}} \le \|u\|_{\dot{H}^s} \le C^{|s|+1}\|u\|_{\dot{B}^s_{2,2}}.$$

(iv) If $s > 0$, then $\dot{B}^s_{2,1} \cap L^\infty$ (especially $\dot{B}^{\frac{d}{2}}_{2,1}$) is an algebra.

Proposition 2.4. *Assume $s > 0$, $u \in L^\infty \cap \dot{B}^s_{2,1}$ and $v \in L^\infty \cap \dot{B}^s_{2,1}$. Then $uv \in L^\infty \cap \dot{B}^s_{2,1}$ and*

$$\|uv\|_{\dot{B}^s_{2,1}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^s_{2,1}} + \|v\|_{L^\infty}\|u\|_{\dot{B}^s_{2,1}}.$$

Let $s_1, s_2 \le \frac{d}{2}$ such that $s_1 + s_2 > 0$, $u \in \dot{B}^{s_1}_{2,1}$ and $v \in \dot{B}^{s_2}_{2,1}$. Then $uv \in \dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}$ and

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}}\|v\|_{\dot{B}^{s_2}_{2,1}}.$$

The two Propositions above can be found in [1]. The following estimates in hybrid Besov spaces are very useful and their proofs can be found in [17]. For reader's convenience, we provide the proof.

Proposition 2.5. *Let $s_1, s_2, t_1, t_2 \in \mathbb{R}$ and $s_1 \le \frac{d}{2}$ and $s_2 \le \frac{d}{2}$. Then following estimate holds*

$$\|\dot{T}_u v\|_{\dot{B}^{s_1+t_1-\frac{d}{2}, s_2+t_2-\frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{s_1,s_2}}\|v\|_{\dot{B}^{t_1,t_2}}.$$

If $\min(s_1+t_1, s_2+t_2) > 0$, then

$$\|\dot{R}(u,v)\|_{\dot{B}^{s_1+t_1-\frac{d}{2}, s_2+t_2-\frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{s_1,s_2}}\|v\|_{\dot{B}^{t_1,t_2}}.$$

If $u \in L^\infty$,

$$\|\dot{T}_u v\|_{\dot{B}^{t_1,t_2}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{t_1,t_2}},$$

and, if $\min(t_1, t_2) > 0$, then

$$\|\dot{R}(u,v)\|_{\dot{B}^{t_1,t_2}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{t_1,t_2}}.$$

Remark 2.6. When $d \geq 2$, we have $\|uv\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|v\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}$ $(\frac{1}{2} \leq \beta \leq 1)$.

Proof. Clearly,

$$\dot{\Delta}_p \dot{T}_u v \triangleq \sum_{|q-p|\leq 3} \dot{\Delta}_p(\dot{S}_{q-1} u \dot{\Delta}_q v).$$

When $p \leq 0, q \approx p$ and $s_1 \leq \frac{d}{2}$, then

$$\begin{aligned}
\|\dot{S}_{q-1}u\|_{L^\infty} &\lesssim \sum_{q'\leq q-2} 2^{q'\frac{d}{2}} \|\dot{\Delta}_{q'}u\|_{L^2} \\
&\lesssim \sum_{q'\leq q-2} 2^{q'(\frac{d}{2}-s_1)} 2^{q's_1} \|\dot{\Delta}_{q'}u\|_{L^2} \\
&\lesssim 2^{q(\frac{d}{2}-s_1)} \|u\|_{\dot{B}^{s_1,s_2}}.
\end{aligned}$$

When $p > 0, q \approx p$ and $s_1, s_2 \leq \frac{d}{2}$, then

$$\begin{aligned}
\|\dot{S}_{q-1}u\|_{L^\infty} &\lesssim \sum_{q'\leq q-2} 2^{q'\frac{d}{2}} \|\dot{\Delta}_{q'}u\|_{L^2} \\
&\lesssim \sum_{q'\leq 0} 2^{q'(\frac{d}{2}-s_1)} 2^{q's_1} \|\dot{\Delta}_{q'}u\|_{L^2} + \sum_{0<q'\leq q-2} 2^{q'(\frac{d}{2}-s_2)} 2^{q's_2} \|\dot{\Delta}_{q'}u\|_{L^2} \\
&\lesssim (\sum_{q'\leq 0} 2^{q'(\frac{d}{2}-s_1)} c_{q'} + \sum_{0<q'\leq q-2} 2^{q'(\frac{d}{2}-s_2)} c_{q'}) \|u\|_{\dot{B}^{s_1,s_2}} \\
&\lesssim 2^{q(\frac{d}{2}-s_2)} \|u\|_{\dot{B}^{s_1,s_2}},
\end{aligned}$$

where the sequence c_j satisfies $\sum_{j\in\mathbb{Z}} c_j \leq 1$. Thus,

$$\begin{aligned}
\|\dot{T}_u v\|_{\dot{B}^{s_1+t_1-\frac{d}{2},s_2+t_2-\frac{d}{2}}} &\lesssim \sum_{p\leq 0} 2^{p(s_1+t_1-\frac{d}{2})} \|\dot{\Delta}_p \dot{T}_u v\|_{L^2} + \sum_{p>0} 2^{p(s_2+t_2-\frac{d}{2})} \|\dot{\Delta}_p \dot{T}_u v\|_{L^2} \\
&\lesssim \sum_{p\leq 0} 2^{p(s_1+t_1-\frac{d}{2})} \sum_{|q-p|\leq 3} \|\dot{S}_{q-1}u\|_{L^\infty} \|\dot{\Delta}_q v\|_{L^2} \\
&\quad + \sum_{p>0} 2^{p(s_2+t_2-\frac{d}{2})} \sum_{|q-p|\leq 3} \|\dot{S}_{q-1}u\|_{L^\infty} \|\dot{\Delta}_q v\|_{L^2} \\
&\lesssim \sum_{p\leq 0} 2^{p(s_1+t_1-\frac{d}{2})} \sum_{|q-p|\leq 3} 2^{q(\frac{d}{2}-s_1)} \|u\|^l_{\dot{B}^{s_1,s_2}} \|\dot{\Delta}_q v\|_{L^2} \\
&\quad + \sum_{p>0} 2^{p(s_2+t_2-\frac{d}{2})} \sum_{|q-p|\leq 3} 2^{q(\frac{d}{2}-s_2)} \|u\|_{\dot{B}^{s_1,s_2}} \|\dot{\Delta}_q v\|_{L^2} \\
&\lesssim \|u\|_{\dot{B}^{s_1,s_2}} \|v\|_{\dot{B}^{t_1,t_2}}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\|\dot{T}_u v\|_{\dot{B}^{t_1,t_2}} &\lesssim \sum_{p\le 0} 2^{pt_1}\|\dot{\Delta}_p \dot{T}_u v\|_{L^2} + \sum_{p>0} 2^{pt_2}\|\dot{\Delta}_p \dot{T}_u v\|_{L^2}\\
&\lesssim \sum_{p\le 0} 2^{pt_1} \sum_{|q-p|\le 3} \|\dot{S}_{q-1}u\|_{L^\infty}\|\dot{\Delta}_q v\|_{L^2}\\
&\quad + \sum_{p>0} 2^{pt_2} \sum_{|q-p|\le 3} \|\dot{S}_{q-1}u\|_{L^\infty}\|\dot{\Delta}_q v\|_{L^2}\\
&\lesssim \sum_{p\le 0} 2^{pt_1}\|u\|_{L^\infty}\|\dot{\Delta}_p v\|_{L^2} + \sum_{p>0} 2^{pt_2}\|u\|_{L^\infty}\|\dot{\Delta}_p v\|_{L^2}\\
&\lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{t_1,t_2}}.
\end{aligned}$$

Recall that

$$\dot{\Delta}_p \dot{R}(u,v) = \sum_{q\ge p-2} \dot{\Delta}_p(\dot{\Delta}_q u \tilde{\dot{\Delta}}_q v).$$

If $\min(s_1+t_1, s_2+t_2) > 0$, we have

$$\begin{aligned}
&\|\dot{R}(u,v)\|_{\dot{B}^{s_1+t_1-\frac{d}{2},s_2+t_2-\frac{d}{2}}} \lesssim \sum_{p\le 0} 2^{p(s_1+t_1-\frac{d}{2})}\|\dot{\Delta}_p \dot{R}(u,v)\|_{L^2}\\
&\quad + \sum_{p>0} 2^{p(s_2+t_2-\frac{d}{2})}\|\dot{\Delta}_p \dot{R}(u,v)\|_{L^2}\\
&\lesssim \sum_{p\le 0} 2^{p(s_1+t_1-\frac{d}{2})} 2^{p\frac{d}{2}} \sum_{q\ge p-2} \|\dot{\Delta}_q u\|_{L^2}\|\tilde{\dot{\Delta}}_q v\|_{L^2}\\
&\quad + \sum_{p>0} 2^{p(s_2+t_2-\frac{d}{2})} 2^{p\frac{d}{2}} \sum_{q\ge p-2} \|\dot{\Delta}_q u\|_{L^2}\|\tilde{\dot{\Delta}}_q v\|_{L^2}\\
&\lesssim \sum_{p\le 0} 2^{p(s_1+t_1)}\Big(\sum_{p-2\le q\le 0} + \sum_{q>0}\Big)\|\dot{\Delta}_q u\|_{L^2}\|\tilde{\dot{\Delta}}_q v\|_{L^2}\\
&\quad + \sum_{p>0} 2^{p(s_2+t_2)} \sum_{q\ge p-2} \|\dot{\Delta}_q u\|_{L^2}\|\tilde{\dot{\Delta}}_q v\|_{L^2}\\
&\lesssim \sum_{p\le 0} 2^{p(s_1+t_1)}\Big(\sum_{p-2\le q\le 0} 2^{-q(s_1+t_1)}c_q + \sum_{q>0} 2^{-q(s_2+t_2)}c_q\Big)\|u\|_{\dot{B}^{s_1,s_2}}\|v\|_{\dot{B}^{t_1,t_2}}\\
&\quad + \sum_{p>0} 2^{p(s_2+t_2)} \sum_{q\ge p-2} 2^{-q(s_2+t_2)}c_q\|u\|_{\dot{B}^{s_1,s_2}}\|v\|_{\dot{B}^{t_1,t_2}}\\
&\lesssim \|u\|_{\dot{B}^{s_1,s_2}}\|v\|_{\dot{B}^{t_1,t_2}}.
\end{aligned}$$

If $t_1, t_2 > 0$, we could get directly that

$$\|\dot R(u,v)\|_{\dot B^{t_1,t_2}} \lesssim \sum_{p\le 0} 2^{pt_1}\|\dot\Delta_p \dot R(u,v)\|_{L^2} + \sum_{p>0} 2^{pt_2}\|\dot\Delta_p \dot R(u,v)\|_{L^2}$$

$$\lesssim \sum_{p\le 0} 2^{pt_1} \sum_{q\ge p-2} \|u\|_{L^\infty}\|\tilde{\dot\Delta}_q v\|_{L^2} + \sum_{p>0} 2^{pt_2} \sum_{q\ge p-2} \|u\|_{L^\infty}\|\tilde{\dot\Delta}_q v\|_{L^2}$$

$$\lesssim \sum_{p\le 0} 2^{pt_1}\Big(\sum_{p-2\le q\le 0} + \sum_{q>0}\Big)\|\tilde{\dot\Delta}_q v\|_{L^2}\|u\|_{L^\infty} + \sum_{p>0} 2^{pt_2} \sum_{q\ge p-2} \|\tilde{\dot\Delta}_q v\|_{L^2}\|u\|_{L^\infty}$$

$$\lesssim \sum_{p\le 0} 2^{pt_1}\Big(\sum_{p-2\le q\le 0} 2^{-qt_1}c_q + \sum_{q>0} 2^{-qt_2}c_q\Big)\|u\|_{L^\infty}\|v\|_{\dot B^{t_1,t_2}}$$

$$+ \sum_{p>0} 2^{pt_2} \sum_{q\ge p-2} 2^{-qt_2}c_q\|u\|_{L^\infty}\|v\|_{\dot B^{t_1,t_2}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot B^{t_1,t_2}}.$$

This completes the proof of Proposition 2.5. □

2.5. *Triple product estimates in hybrid Besov spaces*

The following triple product estimates will be used frequently.

Proposition 2.7. *Let u be a vector with $\nabla\cdot u = 0$ and F be a homogeneous smooth function of degree m. Suppose that $-1-\frac{d}{2} < s_1, t_1, s_2, t_2 \le 1+\frac{d}{2}$ and $r_1, r_2 > -1-\frac{d}{2}$. The following estimates hold*

$$\big|(F(D)\dot\Delta_p(u\cdot\nabla v), F(D)\dot\Delta_p v)\big| \lesssim 2^{(m-s_1)p}c_p\|u\|_{\dot B^{\frac d2+1}_{2,1}}\|v\|_{\dot B^{s_1}_{2,1}}\|\dot\Delta_p F(D)v\|_{L^2},$$

$$\big|(F(D)\dot\Delta_p(u\cdot\nabla v), F(D)\dot\Delta_p v)\big| \lesssim c_p 2^{pm}2^{-p\psi^{s_1,s_2}(p)}\|u\|_{\dot B^{\frac d2+1}_{2,1}}\|v\|_{\dot B^{s_1,s_2}}\|F(D)\dot\Delta_p v\|_{L^2},$$

$$\big|(F(D)\dot\Delta_p(u\cdot\nabla v), F(D)\dot\Delta_p v)\big| \lesssim c_p 2^{p(m-r_2)}(\|u\|_{\dot B^{\frac d2+1}_{2,1}}\|v\|_{\dot B^{r_1,r_2}} + \|v\|_{\dot B^{\frac d2+1}_{2,1}}\|u\|_{\dot B^{r_1,r_2}}) \times\|F(D)\dot\Delta_p v\|_{L^2} \quad \text{for } p>0,$$

$$\big|(F(D)\dot\Delta_p(u\cdot\nabla v), \dot\Delta_p w) + (\dot\Delta_p(u\cdot\nabla w), F(D)\dot\Delta_p v)\big| \lesssim c_p\|u\|_{\dot B^{\frac d2+1}_{2,1}}(2^{pm}2^{-p\psi^{s_1,s_2}(p)}\|v\|_{\dot B^{s_1,s_2}}\|\dot\Delta_p w\|_{L^2} + 2^{-p\psi^{t_1,t_2}(p)}\|w\|_{\dot B^{t_1,t_2}}\|F(D)\dot\Delta_p v\|_{L^2}),$$

where the function $\psi^{\alpha,\beta}(p)$ defines as $\psi^{\alpha,\beta}(p) = \alpha$ if $p\le 0$, $\psi^{\alpha,\beta}(p) = \beta$, if $p>0$, and $\sum_{p\in\mathbb{Z}} c_p \le 1$.

Proof. We start with the following Bony decomposition

$$\begin{aligned}F(D)\dot{\Delta}_p(u\cdot\nabla v) &= F(D)\dot{\Delta}_p\sum_{|q-p|\le 3}\dot{S}_{q-1}u\cdot\nabla\dot{\Delta}_q v\\&\quad+F(D)\dot{\Delta}_p\sum_{|q-p|\le 3}\dot{\Delta}_q u\cdot\nabla\dot{S}_{q-1}v\\&\quad+F(D)\dot{\Delta}_p\sum_{q\ge p-2}\dot{\Delta}_q u\cdot\nabla\dot{\tilde{\Delta}}_q v\\&=I_1+I_2+I_3.\end{aligned}$$

First, we give estimates for I_2 for $s_1, s_2 \le \frac{d}{2}+1$:

$$\begin{aligned}\|I_2\|_{L^2} &\lesssim 2^{pm}\sum_{|q-p|\le 3}\|\dot{\Delta}_q u\|_{L^2}\|\dot{S}_{q-1}\nabla v\|_{L^\infty}\\&\lesssim 2^{pm}\sum_{|q-p|\le 3}c_q 2^{-q(\frac{d}{2}+1)}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}2^{q(\frac{d}{2}+1-s_1)}\|v\|_{\dot{B}^{s_1}_{2,1}}\\&\lesssim 2^{p(m-s_1)}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1}_{2,1}}.\end{aligned}$$

When $p\le 0$, we have

$$\begin{aligned}\|I_2\|_{L^2} &\lesssim 2^{pm}\sum_{|q-p|\le 3}\|\dot{\Delta}_q u\|_{L^2}\|\dot{S}_{q-1}\nabla v\|_{L^\infty}\\&\lesssim 2^{pm}\sum_{|q-p|\le 3}c_q 2^{-q(\frac{d}{2}+1)}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}2^{q(\frac{d}{2}+1-s_1)}\|v\|_{\dot{B}^{s_1,s_2}}\\&\lesssim 2^{p(m-s_1)}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1,s_2}}.\end{aligned}$$

When $p>0$, then

$$\begin{aligned}\|I_2\|_{L^2} &\lesssim 2^{pm}\sum_{|q-p|\le 3}\|\dot{\Delta}_q u\|_{L^2}\|\dot{S}_{q-1}\nabla v\|_{L^\infty}\\&\lesssim 2^{pm}\sum_{|q-p|\le 3}c_q 2^{-q(\frac{d}{2}+1)}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}2^{q(\frac{d}{2}+1-s_2)}\|v\|_{\dot{B}^{s_1,s_2}}\\&\lesssim 2^{p(m-s_2)}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1,s_2}}.\end{aligned}$$

Next, we bound I_3 for $s_1, s_2 > -\frac{d}{2}-1$:

$$\|I_3\|_{L^2}\lesssim 2^{p(m+1+\frac{d}{2})}\sum_{q\ge p-2}\|\dot{\Delta}_q u\|_{L^2}\|\dot{\Delta}_q v\|_{L^2}$$

$$\lesssim 2^{p(m+1+\frac{d}{2})}\sum_{q\geq p-2}c_q 2^{-q(\frac{d}{2}+1)}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}2^{-qs_1}\|v\|_{\dot{B}^{s_1}_{2,1}}$$
$$\lesssim 2^{p(m-s_1)}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1}_{2,1}}.$$

When $p\leq 0$, then

$$\begin{aligned}\|I_3\|_{L^2}&\lesssim 2^{p(m+1+\frac{d}{2})}\sum_{q\geq p-2}\|\dot{\Delta}_q u\|_{L^2}\|\dot{\Delta}_q v\|_{L^2}\\&\lesssim 2^{p(m+1+\frac{d}{2})}(\sum_{p-2\leq q\leq 0}+\sum_{q>0})\|\dot{\Delta}_q u\|_{L^2}\|\dot{\Delta}_q v\|_{L^2}\\&\lesssim 2^{p(m+1+\frac{d}{2})}(\sum_{p-2\leq q\leq 0}2^{-q(\frac{d}{2}+1)}c_q 2^{-qs_1}+\sum_{q>0}2^{-q(\frac{d}{2}+1)}c_q 2^{-qs_2})\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1,s_2}}\\&\lesssim 2^{p(m-s_1)}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1,s_2}}.\end{aligned}$$

When $p>0$, then

$$\begin{aligned}\|I_3\|_{L^2}&\lesssim 2^{p(m+1+\frac{d}{2})}\sum_{q\geq p-2}\|\dot{\Delta}_q u\|_{L^2}\|\dot{\Delta}_q v\|_{L^2}\\&\lesssim 2^{p(m+1+\frac{d}{2})}\sum_{q\geq p-2}c_q 2^{-q(\frac{d}{2}+1)}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}2^{-qs_2}\|v\|_{\dot{B}^{s_1,s_2}}\\&\lesssim 2^{p(m-s_2)}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1,s_2}}.\end{aligned}$$

Notice that

$$\begin{aligned}I_1=\sum_{|q-p|\leq 3}&[F(D)\dot{\Delta}_p,\dot{S}_{q-1}u]\cdot\nabla\dot{\Delta}_q v\\&+(\dot{S}_{q-1}-\dot{S}_{p-1})u\cdot\nabla\dot{\Delta}_q\dot{\Delta}_p F(D)v\\&+\dot{S}_{p-1}u\cdot\nabla\dot{\Delta}_p F(D)v.\end{aligned}$$

Thanks to $\nabla\cdot u=0$,

$$\begin{aligned}&\left|(I_1,\dot{\Delta}_p F(D)v\right|\\&\lesssim(2^{(m-1)p}\|\nabla u\|_{L^\infty}\|\nabla\dot{\Delta}_q v\|_{L^2}+2^{p\frac{d}{2}}\|\dot{\Delta}_p u\|_{L^2}\|\dot{\Delta}_p\nabla F(D)v\|_{L^2})\|\dot{\Delta}_p F(D)v\|_{L^2}\\&\lesssim\Big(2^{(m-1)p}\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}c_p 2^{-p(s_1-1)}\|v\|_{\dot{B}^{s_1}_{2,1}}\\&\quad+2^{-p}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}2^{p(m+1-s_1)}\|v\|_{\dot{B}^{s_1}_{2,1}}\Big)\|\dot{\Delta}_p F(D)v\|_{L^2}\\&\lesssim 2^{(m-s_1)p}c_p\|u\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot{B}^{s_1}_{2,1}}\|\dot{\Delta}_p F(D)v\|_{L^2}.\end{aligned}$$

When $p \le 0$, we get

$$\begin{aligned}
&\big|(I_1, \dot\Delta_p F(D)v\big| \\
&\lesssim (2^{(m-1)p}\|\nabla u\|_{L^\infty}\|\nabla \dot\Delta_q v\|_{L^2} + 2^{p\frac{d}{2}}\|\dot\Delta_p u\|_{L^2}\|\dot\Delta_p \nabla F(D)v\|_{L^2})\|\dot\Delta_p F(D)v\|_{L^2} \\
&\lesssim \Big(2^{(m-1)p}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}} c_p 2^{-p(s_1-1)}\|v\|_{\dot B^{s_1,s_2}} \\
&\quad + 2^{-p} c_p \|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}} 2^{p(m+1-s_1)}\|v\|_{\dot B^{s_1,s_2}}\Big)\|\dot\Delta_p F(D)v\|_{L^2} \\
&\lesssim 2^{(m-s_1)p} c_p \|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot B^{s_1,s_2}}\|\dot\Delta_p F(D)v\|_{L^2}.
\end{aligned}$$

When $p > 0$, we have

$$\big|(I_1, \dot\Delta_p F(D)v\big| \lesssim 2^{(m-s_2)p} c_p \|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|v\|_{\dot B^{s_1,s_2}}\|\dot\Delta_p F(D)v\|_{L^2}.$$

Combining the estimates, we obtain the first two inequalities of the proposition. Similarly, we can show the other two. This completes the proof. □

3. A priori estimates

This section presents the proof of the key energy estimate, namely (1.14). To achieve this, we need to overcome two main difficulties. The firs one is the lack of dissipation in the velocity equation. This is dealt with by taking advantage of the wave structure described in the introduction and involving suitable Lyapunov functional with inner product terms in the energy estimates (see Lemmas 3.2 and 3.3). The second main complication is that the proof of (1.14) estimates numerous terms and the dissipation is given by a general fractional operator. To handle this issue, we make suitable combinations and make full use of the fractional Laplacian.

More precisely, this section proves the proposition.

Proposition 3.1. *Assume that (τ, u) is a solution to the system (1.3) on $[0, T)$. Then, there exist two positive constants C_1, C_2 independent of T such that*

$$E(t) \le C_1 E_0 + C_2 E^2(t) \ \ \text{and} \ \ E^l_\tau(t) \lesssim E_0 + (t + E(t))E(t), \tag{3.1}$$

where $E_0 = \|u_0\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}} + \|\tau_0\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}$.

In order to prove Proposition 3.1, we need the following two important lemmas. The first lemma sets up the estimates for $E^l_0(t) + E^h_0(t)$, the low-frequency and the high-frequency energy pieces associated with $(u, \mathbb{P}\nabla\cdot\tau)$. As we have explained in the introduction, $(u, \mathbb{P}\nabla\cdot\tau)$ satisfies a system of wave equations, who exhibits extra smoothing and stabilizing properties. This lemma exploits this extra regularization to gain time integrability of u.

Lemma 3.2. *Let (u, τ) be the solution of system (1.3) on $[0, T)$. Then there exist two positive constants C_1, C_2 independent of T such that*

$$E_0^l(t)+E_0^h(t)\le C_1E_0+C_2\int\limits_0^t\sum_{j\le 0}2^{j(\frac{d}{2}+1-2\beta)}|G_j|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'$$
$$+C_2\int\limits_0^t\sum_{j>0}2^{j\frac{d}{2}}\Big(|H_j|/(\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\Big)\mathrm{d}t',$$

where

$$\begin{aligned}G_j=&-\frac{\mu_2}{2}(\dot\Delta_j\mathbb{P}(u\cdot\nabla u),\dot\Delta_j u)\\&-\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau+Q(\tau,\nabla u)),\dot\Delta_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau))\\&-K_1(\Lambda^{2\beta-1}\dot\Delta_j\mathbb{P}(u\cdot\nabla u),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\\&-K_1(\Lambda^{2\beta-1}\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau+Q(\tau,\nabla u)),\dot\Delta_j u)\end{aligned}$$

and

$$\begin{aligned}H_j=&-\eta(\Lambda^{2\beta-1}\dot\Delta_j\mathbb{P}(u\cdot\nabla u),\Lambda^{2\beta-1}\dot\Delta_j u)\\&-\frac{4\mu_1^2}{\eta}(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau+Q(\tau,\nabla u)),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\\&-\mu_1(\Lambda^{2\beta-1}\dot\Delta_j\mathbb{P}(u\cdot\nabla u),\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\\&-\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau+Q(\tau,\nabla u)),\Lambda^{2\beta-1}\dot\Delta_j u).\end{aligned}$$

The second lemma completes the first-stage estimates on $E^l(t)+E^h(t)$. We bound it in terms of the initial data and the nonlinear terms.

Lemma 3.3. *Let (u,τ) be the solution of system (1.3) on $[0,T)$. There exist two constants C_1, C_2 independent of T such that*

$$E^l(t)+E^h(t)\le C_1E_0+C_2\int\limits_0^t\sum_{j\le 0}2^{j(\frac{d}{2}+1-2\beta)}\Big(|G_j|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})$$
$$+|V_j|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\tau\|_{L^2})\Big)\mathrm{d}t'$$
$$+C_2\int\limits_0^t\sum_{j>0}2^{j\frac{d}{2}}\Big(|H_j|/(\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})$$
$$+|Y_j|/\|\dot\Delta_j\tau\|_{L^2}\Big)\mathrm{d}t',$$

where G_j, H_j are defined as in Lemma 3.2 and

$$V_j=-\mu_2(\dot\Delta_j(u\cdot\nabla u),\dot\Delta_j u)-\mu_1(\dot\Delta_j(u\cdot\nabla\tau),\dot\Delta_j\tau)-\mu_1(\dot\Delta_jQ(\tau,\nabla u),\dot\Delta_j\tau),$$
$$Y_j=-(\dot\Delta_j(u\cdot\nabla\tau),\dot\Delta_j\tau)-(\dot\Delta_jQ(\tau,\nabla u),\dot\Delta_j\tau).$$

The rest of this section proves Lemmas 3.2 and 3.3, and then Proposition 3.1. For the sake of the clarity, we divide the rest of this section into three subsections.

3.1. Proof of Lemma 3.2

Proof of Lemma 3.2. Naturally we divide the proof into two major steps. The first step focuses on the estimate of $E_0^l(t)$ while the second step provides the estimate for $E_0^h(t)$.

Step 1: Estimate of $E_0^l(t)$.

Let $\widetilde{C}_0 = 2^{j_0}$, where $j_0 \in \mathbb{Z}$ is a fixed constant which will be chosen in Step 2. By Lemma 2.1, we can deduce that, for any function f, there exist two constants $\widetilde{C}_1$, $\widetilde{C}_2$ such that

$$\widetilde{C}_1 2^{\kappa j}\|\dot{\Delta}_j f\|_{L^p} \le \|\Lambda^\kappa \dot{\Delta}_j f\|_{L^p} \le \widetilde{C}_2 2^{\kappa j}\|\dot{\Delta}_j f\|_{L^p}, \tag{3.2}$$

where $\kappa \in [0, m]$ (m is a large positive integer). Applying the operator $\dot{\Delta}_j\mathbb{P}$ and $\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot$ to the first equation and the second equation of the system (1.3) respectively, we obtain

$$\begin{cases} (\dot{\Delta}_j u)_t - \mu_1\Lambda\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau = -\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \\ (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)_t + \eta\Lambda^{2\beta}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) + \frac{\mu_2}{2}\Lambda\dot{\Delta}_j u \\ \qquad\qquad = -\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)). \end{cases} \tag{3.3}$$

Taking the L^2-inner product of the first equation of (3.3) with $\dot{\Delta}_j u$ and of the second with $\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau$, we obtain the following two identities:

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|_{L^2}^2 - \mu_1(\Lambda\dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau), \dot{\Delta}_j u) = -(\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j u), \tag{3.4}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\|_{L^2}^2 + \eta\|\Lambda^\beta\dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\|_{L^2}^2 + \frac{\mu_2}{2}(\Lambda\dot{\Delta}_j u, \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) \\ &= -(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \dot{\Delta}_j(\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)). \end{aligned} \tag{3.5}$$

Applying $\Lambda^{2\beta-1}$ to the first equation of (3.3) and taking the inner product with $\Lambda^{-1}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau$, taking the L^2-inner product of the second equation of (3.3) with $\Lambda^{2\beta-1}\dot{\Delta}_j u$, and then summing them up, we have

$$\begin{aligned} &\frac{d}{dt}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau, \Lambda^{2\beta-1}\dot{\Delta}_j u) - \mu_1\|\Lambda^\beta\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2 \\ &\quad + \eta(\Lambda^{2\beta-1}\dot{\Delta}_j u, \Lambda^{2\beta}\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) + \frac{\mu_2}{2}\|\Lambda^\beta\dot{\Delta}_j u\|_{L^2}^2 \\ &= -(\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) \\ &\quad - (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau + Q(\tau,\nabla u)), \Lambda^{2\beta-1}\dot{\Delta}_j u). \end{aligned} \tag{3.6}$$

A linear combination of (3.4), (3.5) and (3.6) with K_1 to be determined later leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\frac{\mu_2}{2}\|\dot\Delta_j u\|_{L^2}^2+\mu_1\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2+2K_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau,\Lambda^{2\beta-1}\dot\Delta_j u)\Big)\\
&+\frac{\mu_2K_1}{2}\|\Lambda^{\beta}\dot\Delta_j u\|_{L^2}^2+(\mu_1\eta-\mu_1K_1)\|\Lambda^{\beta-1}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\\
&+\eta K_1(\Lambda^{2\beta-1}\dot\Delta_j u,\Lambda^{2\beta}\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)=G_j.
\end{aligned}
\tag{3.7}
$$

With $\frac{1}{2}\le\beta\le 1$ and (3.2), we have, for $j\le j_0$ and for any $\epsilon_0,\epsilon_1>0$,

$$
\begin{aligned}
&2K_1|(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau,\Lambda^{2\beta-1}\dot\Delta_j u)|\\
&\qquad\le\frac{1}{\epsilon_0}\|\dot\Delta_j u\|_2^2+\epsilon_0\widetilde{C}_2^2\widetilde{C}_0^{2(2\beta-1)}K_1^2\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2,\\
&\eta K_1|(\Lambda^{2\beta-1}\dot\Delta_j u,\Lambda^{2\beta}\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)|\\
&\qquad\le\frac{\epsilon_1\eta K_1}{2}\|\Lambda^{\beta}\dot\Delta_j u\|_{L^2}^2+\frac{\widetilde{C}_0^{2(2\beta-1)}\widetilde{C}_2^2\eta K_1}{2\epsilon_1}\|\Lambda^{\beta-1}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}^2.
\end{aligned}
$$

Thanks to (3.2), choosing $\epsilon_0=\frac{4}{\mu_2}$, $\epsilon_1=\frac{\mu_2}{2\eta}$, K_1 small enough and inserting the two inequalities above in (3.7), we obtain

$$
\begin{aligned}
&\frac{\mu_2}{2}\|\dot\Delta_j u\|_{L^2}^2+\mu_1\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2+2K_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau,\Lambda^{2\beta-1}\dot\Delta_j u)\\
&\qquad\approx\|\dot\Delta_j u\|_{L^2}^2+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\frac{\mu_2}{2}\|\dot\Delta_j u\|_{L^2}^2+\mu_1\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2+2K_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau,\Lambda^{2\beta-1}\dot\Delta_j u)\Big)\\
&+\frac{\mu_2K_1}{4}\widetilde{C}_1^2 2^{2\beta j}\|\dot\Delta_j u\|_{L^2}^2+(\mu_1\eta-\mu_1K_1-\frac{\widetilde{C}_0^{2(2\beta-1)}\widetilde{C}_2^2\eta^2K_1}{\mu_2})\widetilde{C}_1^2 2^{2\beta j}\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\\
&\le|G_j|.
\end{aligned}
\tag{3.8}
$$

We choose $K_1>0$ sufficiently small such that

$$
(\mu_1\eta-\mu_1K_1-\frac{\widetilde{C}_0^{2(2\beta-1)}\widetilde{C}_2^2\eta^2K_1}{\mu_2})>0,\qquad\Big(\mu_1-\frac{4\widetilde{C}_0^{2(2\beta-1)}\widetilde{C}_2^2K_1^2}{\mu_2}\Big)>0.
$$

Dividing by $\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}$, (3.8) can be written as

$$
\begin{aligned}
&\frac{d}{dt}\sqrt{\frac{\mu_2}{2}\|\dot\Delta_j u\|_{L^2}^2+\mu_1\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2+2K_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau,\Lambda^{2\beta-1}\dot\Delta_j u)}\\
&+2^{2\beta j}\Big(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}\Big)\le C_2|G_j|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}).
\end{aligned}
\tag{3.9}
$$

Multiplying both sides of (3.9) by $2^{j(\frac{d}{2}+1-2\beta)}$, summing over $j\le j_0$ (we can choose $j_0=0$, see Step 2), and performing a time integration, we obtain

$$
\begin{aligned}
E_0^l(t) \le & C_1(\|u_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|\tau_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}}) \\
& + C_2 \int_0^t \sum_{j\le j_0} 2^{j(\frac{d}{2}+1-2\beta)} |G_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t'.
\end{aligned} \tag{3.10}
$$

Step 2: Estimate of $E_0^h(t)$.

Applying $\Lambda^{2\beta-1}$ to the first equation of (3.3) and taking the L^2 inner product with $\Lambda^{2\beta-1}\dot{\Delta}_j u$, we have

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|^2_{L^2} - \mu_1(\Lambda^{2\beta-1}\dot{\Delta}_j u, \Lambda^{2\beta}\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) \\
= & -(\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(u\cdot\nabla u), \Lambda^{2\beta-1}\dot{\Delta}_j u).
\end{aligned} \tag{3.11}
$$

A linear combination of (3.11), (3.6) and (3.5) (for some positive constant K_2 to be determined later) gives

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\Big(\eta\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|^2_{L^2} + K_2\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^2_{L^2} + 2\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau, \Lambda^{2\beta-1}\dot{\Delta}_j u)\Big) \\
& + \frac{\mu_1\mu_2}{2}\|\Lambda^\beta\dot{\Delta}_j u\|^2_{L^2} + (\eta K_2 - \mu_1^2)\|\Lambda^{\beta-1}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|^2_{L^2} \\
& + \frac{\mu_2}{2}K_2(\Lambda\dot{\Delta}_j u, \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) = H_j.
\end{aligned} \tag{3.12}
$$

It is easy to check that for any $\epsilon_0, \epsilon_1 > 0$, we have

$$
\begin{aligned}
& 2\mu_1|(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau, \Lambda^{2\beta-1}\dot{\Delta}_j u)| \le \frac{2\mu_1^2}{\epsilon_0}\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|^2_2 + \frac{\epsilon_0}{2}\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^2_{L^2}, \\
& \frac{\mu_2}{2}K_2|(\Lambda\dot{\Delta}_j u, \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)| \le \frac{\epsilon_1\mu_2K_2}{4}\|\Lambda^\beta\dot{\Delta}_j u\|^2_{L^2} + \frac{\mu_2K_2}{4\epsilon_1}\|\Lambda^{-\beta}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|^2_{L^2}.
\end{aligned} \tag{3.13}
$$

Using $j \ge j_0 + 1$ and $\frac{1}{2} \le \beta \le 1$, we have

$$
\frac{\mu_2K_2}{4\epsilon_1}\|\Lambda^{-\beta}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|^2_{L^2} \le \frac{\mu_2K_2}{4\epsilon_1}\widetilde{C}_2^2\widetilde{C}_0^{2(1-2\beta)}\|\Lambda^{\beta-1}\dot{\Delta}_j\mathbb{P}\nabla\cdot\tau\|^2_{L^2}. \tag{3.14}
$$

Combining (3.12), (3.13) and (3.14), and choosing

$$
K_2 = \epsilon_0 = \frac{4\mu_1^2}{\eta}, \quad \epsilon_1 = \frac{\mu_1}{K_2},
$$

we obtain

$$
\begin{aligned}
& \eta\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|^2_{L^2} + K_2\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^2_{L^2} + 2\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau, \Lambda^{2\beta-1}\dot{\Delta}_j u) \\
\approx & \|\Lambda^{2\beta-1}\dot{\Delta}_j u\|^2_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|^2_{L^2}
\end{aligned}
$$

and

$$\frac{1}{2}\frac{d}{dt}\Big(\eta\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2}^2+K_2\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2+2\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau,\Lambda^{2\beta-1}\dot{\Delta}_j u)\Big)$$
$$+\frac{\mu_1\mu_2}{4}\|\Lambda^{\beta}\dot{\Delta}_j u\|_{L^2}^2+(3\mu_1^2-\frac{4\mu_2\mu_1^3}{\eta^2}\widetilde{C}_2^2\widetilde{C}_0^{2(1-2\beta)})\|\Lambda^{\beta}\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2\leqslant|H_j|.$$

We then choose j_0 or $\widetilde{C}_0$ to ensure that $(3\mu_1^2-\frac{4\mu_2\mu_1^3}{\eta^2}\widetilde{C}_2^2\widetilde{C}_0^{2(1-2\beta)})>0$ for $\frac{1}{2}<\beta\le 1$. However, in the case when $\beta=\frac{1}{2}$, we need

$$\eta^2\ge C\mu_1\mu_2 \tag{3.15}$$

for a suitable constant $C>0$ in order to have

$$(3\mu_1^2-\frac{4\mu_2\mu_1^3}{\eta^2}\widetilde{C}_2^2\widetilde{C}_0^{2(1-2\beta)})>0.$$

This explains why we need (3.15) in the case when $\beta=\frac{1}{2}$ in Theorem 1.1. Without loss of generality, we set $j_0=0$. Thanks to $\frac{1}{2}\le\beta\le 1$, dividing by $\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}$, we have

$$\begin{aligned}&\frac{d}{dt}\sqrt{\Big(\eta\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2}^2+K_2\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}^2+2\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau,\Lambda^{2\beta-1}\dot{\Delta}_j u)\Big)}\\&+2^{2(1-\beta)j}(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\\&\le C_2|H_j|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}).\end{aligned}\tag{3.16}$$

Multiplying both sides of (3.16) by $2^{j\frac{d}{2}}$, summing over $j>0$, and integrating in time, we have

$$\begin{aligned}E_0^h(t)&=\sup_t\|u\|^h_{\dot{B}^{\frac{d}{2}+2\beta-1}_{2,1}}+\sup_t\|\mathbb{P}\nabla\cdot\tau\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\int_0^t\|u\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\&\lesssim\|u_0\|^h_{\dot{B}^{\frac{d}{2}+2\beta-1}_{2,1}}+\|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}+\int_0^t\sum_{j>0}2^{j\frac{d}{2}}|H_j|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'.\end{aligned}\tag{3.17}$$

Combining (3.10) and (3.17), we establish the proof of Lemma 3.2. □

3.2. Proof of Lemma 3.3

Proof of Lemma 3.3. We again divide the proof into two steps.

Step 1: Estimate of $E^l(t)$.

Thanks to

$$(\dot{\Delta}_j\nabla p,\dot{\Delta}_j u)=0,$$

we obtain from (1.3) that

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|_{L^2}^2 = \mu_1(\dot{\Delta}_j \nabla\cdot\tau, \dot{\Delta}_j u) - (\dot{\Delta}_j(u\cdot\nabla u), \dot{\Delta}_j u) \tag{3.18}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j \tau\|_{L^2}^2 + \eta\|\Lambda^\beta \dot{\Delta}_j\tau\|_{L^2}^2 \\ &= \mu_2(\dot{\Delta}_j D(u), \dot{\Delta}_j \tau) - (\dot{\Delta}_j(u\cdot\nabla\tau), \dot{\Delta}_j\tau) - (\dot{\Delta}_j Q(\tau,\nabla u), \dot{\Delta}_j\tau).\end{aligned} \tag{3.19}$$

Adding (3.18) and (3.19) and making use of

$$(\dot{\Delta}_j\nabla\cdot\tau, \dot{\Delta}_j u) + (\dot{\Delta}_j D(u), \dot{\Delta}_j\tau) = 0,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\tau\|_{L^2}\Big) \le C_2|V_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\tau\|_{L^2}).$$

Multiplying by $2^{(\frac{d}{2}+1-2\beta)j}$ and summing over $j \le 0$ lead to the following estimates for the low frequencies,

$$\begin{aligned}\sup_t \|u\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \sup_t \|\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} &\le C_1(\|u_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|\tau_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}}) \\ &+ C_2\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)}|V_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\tau\|_{L^2})\mathrm{d}t',\end{aligned}$$

which, together with (3.10), yields

$$\begin{aligned}E^l(t) \le& C_1(\|u_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|\tau_0\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}}) + C_2\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)} \\ &\times \Big(|G_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) + |V_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j\tau\|_{L^2})\Big)\mathrm{d}t'.\end{aligned} \tag{3.20}$$

Step 2: Estimate of $E^h(t)$.

By (3.19),

$$\frac{d}{dt}\|\dot{\Delta}_j\tau\|_2 + 2^{2\beta j}\|\dot{\Delta}_j\tau\|_2 \le C2^j\|\dot{\Delta}_j u\|_2 + C|Y_j|/\|\dot{\Delta}_j\tau\|_{L^2}. \tag{3.21}$$

This leads to the following estimate for the high frequencies

$$\sup_t \|\tau\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + \int_0^t \|\tau\|^h_{\dot{B}^{\frac{d}{2}+2\beta}_{2,1}} \mathrm{d}t'$$
$$\leq \|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} + C\int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t' + C\int_0^t \sum_{j>0} 2^{\frac{d}{2}j} |Y_j|/\|\dot{\Delta}_j\tau\|_{L^2} \mathrm{d}t'. \tag{3.22}$$

To eliminate the term $C\int_0^t \|u\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t'$ on the right side of the inequality above, we calculate (3.17) + η_2(3.21) with η_2 small enough such that $\eta_2 C \leq \frac{1}{2}$ to get

$$\begin{aligned} E^h(t) \leq & C_1(\|u_0\|^h_{\dot{B}^{\frac{d}{2}+2\beta-1}_{2,1}} + \|\tau_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}) + C_2\int_0^t \sum_{j>0} 2^{j\frac{d}{2}} \Big(|H_j|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} \\ & + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) + |Y_j|/\|\dot{\Delta}_j\tau\|_{L^2}\Big)\mathrm{d}t'. \end{aligned} \tag{3.23}$$

Combining (3.23) with (3.20) finishes the proof of Lemma 3.3. □

3.3. *Proof of Proposition 3.1*

With the two lemmas at our disposal, this subsection proves Proposition 3.1. We need an identity stated in the following lemma. A proof of this lemma can be found in [40].

Lemma 3.4. *For any smooth tensor $[\tau^{i,j}]_{d\times d}$ and d dimensional vector u, it always holds that*

$$\mathbb{P}\nabla\cdot(u\cdot\nabla\tau) = \mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau) + \mathbb{P}(\nabla u\cdot\nabla\tau) - \mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau),$$

where the ith component of $\nabla u\cdot\nabla\tau$ is

$$[\nabla u\cdot\nabla\tau]^i = \sum_j \partial_j u\cdot\nabla\tau^{i,j},$$

and

$$[\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau]^i = \partial_i u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau.$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. The proof makes use of Lemmas 3.2 and 3.3. The main efforts are devoted to bounding G_j, H_j, V_j and Y_j. To bound G_j and H_j suitably, we divide G_j and H_j each into three parts,

$$G_j = G_j^1 + G_j^2 + G_j^3, \quad H_j = H_j^1 + H_j^2 + H_j^3$$

with

$$
\begin{aligned}
G_j^1 =& -\frac{\mu_2}{2}(\dot\Delta_j \mathbb{P}(u\cdot\nabla u), \dot\Delta_j u) - \mu_1(\dot\Delta_j \Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\\
&- K_1(\Lambda^{2\beta-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot\Delta_j u),\\
G_j^2 =& -\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau),\\
G_j^3 =& -K_1(\Lambda^{2\beta-1}\dot\Delta_j\mathbb{P}(u\cdot\nabla u), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) - K_1(\Lambda^{2\beta-2}\dot\Delta_j\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \dot\Delta_j u),\\
H_j^1 =& -\frac{4\mu_1^2}{\eta}(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)\\
&- \mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \Lambda^{2\beta-1}\dot\Delta_j u)\\
&- \eta(\Lambda^{2\beta-1}\dot\Delta_j\mathbb{P}(u\cdot\nabla u), \Lambda^{2\beta-1}\dot\Delta_j u),\\
H_j^2 =& -\frac{4\mu_1^2}{\eta}(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau),\\
H_j^3 =& -\mu_1(\Lambda^{2\beta-1}\dot\Delta_j\mathbb{P}(u\cdot\nabla u), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) - \mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), \Lambda^{2\beta-1}\dot\Delta_j u).
\end{aligned}
$$

For the sake of clarity, we divide the rest of the proof into several steps.

Step 1: Estimate for $G_j^1,\ H_j^1,\ V_j,\ Y_j$.

We first deal with the terms in $G_j^1,\ H_j^1,\ V_j,\ Y_j$ that do not involve $Q(\tau,\nabla u)$. Due to $\nabla\cdot u=0$,

$$
(\dot\Delta_j\mathbb{P}(u\cdot\nabla u), \dot\Delta_j u) = (\dot\Delta_j(u\cdot\nabla u), \dot\Delta_j u).
$$

By Propositions 2.5 and 2.7, for $j\le 0$,

$$
|(\dot\Delta_j(u\cdot\nabla u), \dot\Delta_j u)| \lesssim c_j 2^{-j(\frac{d}{2}+1-2\beta)}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|u\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}\|\dot\Delta_j u\|_{L^2}
$$

and, for $j>0$,

$$
|\Lambda^{2\beta-1}(\dot\Delta_j(u\cdot\nabla u), \Lambda^{2\beta-1}\dot\Delta_j u)| \lesssim c_j 2^{-j\frac{d}{2}}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|u\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2}.
$$

In addition,

$$
|(\dot\Delta_j(u\cdot\nabla\tau), \dot\Delta_j\tau)| \lesssim c_j 2^{-j\psi^{\frac{d}{2}+1-2\beta,\frac{d}{2}}(j)}\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\|\dot\Delta_j\tau\|_{L^2}.
$$

We now turn to the terms that contain $Q(\tau,\nabla u)$. By Lemma 2.1 and Hölder's inequality, for $j\le 0$,

$$
\begin{aligned}
&\big|-\mu_1(\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) - K_1(\Lambda^{2\beta-2}\dot\Delta_j\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot\Delta_j u)\big|\\
&\lesssim (1+2^{(2\beta-1)j})\|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\\
&\lesssim \|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}(\|\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}),\\
&\big|-\mu_1(\dot\Delta_j Q(\tau,\nabla u), \dot\Delta_j\tau)\big| \lesssim \|\dot\Delta_j Q(\tau,\nabla u)\|_{L^2}\|\dot\Delta_j\tau\|_{L^2}.
\end{aligned}
$$

For $j \geq 0$, we have

$$
\begin{aligned}
&\big| -\frac{4\mu_1^2}{\eta}(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau) - \mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau,\nabla u), \Lambda^{2\beta-1}\dot{\Delta}_j u)\big| \\
&\lesssim \|\dot{\Delta}_j Q(\tau,\nabla u)\|_{L^2}(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}), \\
&\big| -(\dot{\Delta}_j Q(\tau,\nabla u), \dot{\Delta}_j \tau)\big| \lesssim \|\dot{\Delta}_j Q(\tau,\nabla u)\|_{L^2}\|\dot{\Delta}_j \tau\|_{L^2}.
\end{aligned}
$$

Combining estimates above, we conclude that

$$
\begin{aligned}
&\int_0^t \sum_{j\leq 0} 2^{j(\frac{d}{2}+1-2\beta)}\Big(|G_j^1|/(\|\dot{\Delta}_j u\|_2 + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) + |V_j|/(\|\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \tau\|_{L^2}\Big)\mathrm{d}t' \\
&+ \int_0^t \sum_{j>0} 2^{j\frac{d}{2}}\Big(|H_j^1|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) + |Y_j|/\|\dot{\Delta}_j \tau\|_{L^2}\Big)\mathrm{d}t' \\
&\lesssim \big(\sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} + \sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}\big)\int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t' \\
&\qquad + \int_0^t \|Q(\tau,\nabla u)\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\mathrm{d}t' \\
&\lesssim \big(\sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} + \sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}\big)\int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t',
\end{aligned}
\tag{3.24}
$$

where we have used Remark 2.6 in the second inequality.

Step 2: Estimate for G_j^2, G_j^3.

To bound the difficult term G_j^2, we divide it into three terms according to Lemma 3.4,

$$
G_j^2 = G_j^{2,1} + G_j^{2,2} + G_j^{2,3},
$$

where

$$
\begin{aligned}
G_j^{2,1} &= -\mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau), \dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau), \\
G_j^{2,2} &= -\mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}(\nabla u\cdot\nabla\tau), \dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau), \\
G_j^{2,3} &= \mu_1(\dot{\Delta}_j \Lambda^{-1}\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau), \dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau).
\end{aligned}
$$

By Proposition 2.7,

$$\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)}|G_j^{2,1}|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'$$
$$\lesssim \int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)}c_j 2^{-j}2^{-j(\frac{d}{2}-2\beta)}\|u\|_{\dot B_{2,1}^{\frac{d}{2}+1}}\|\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}-2\beta,\frac{d}{2}-1}}\mathrm{d}t'$$
$$\lesssim \sup_t \|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\int_0^t \|u\|_{\dot B_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t'. \tag{3.25}$$

Next we estimate the terms $G_j^{2,2}$ and $G_j^{2,3}$. Thanks to $\nabla\cdot u=0$,

$$[\nabla u\cdot\nabla\tau]^i \triangleq \sum_{j,k}\partial_j u^k\partial_k\tau^{i,j} = \sum_{j,k}\partial_k(\partial_j u^k\tau^{i,j}),$$
$$[\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau]^i \triangleq \sum_k \partial_i u^k\partial_k\Delta^{-1}\nabla\cdot\nabla\cdot\tau = \sum_k\partial_k(\partial_i u^k\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau). \tag{3.26}$$

Then we have

$$\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)}\left|G_j^{2,2}+G_j^{2,3}\right|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'$$
$$\lesssim \int_0^t \|\nabla u\cdot\nabla\tau\|^l_{\dot B_{2,1}^{\frac{d}{2}-2\beta}} + \|\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|^l_{\dot B_{2,1}^{\frac{d}{2}-2\beta}}\mathrm{d}t'$$
$$\lesssim \int_0^t \|\nabla u\otimes\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} + \|\nabla u\otimes\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\mathrm{d}t'$$
$$\lesssim \sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\int_0^t\|u\|_{\dot B_{2,1}^{\frac{d}{2}+1}}\mathrm{d}t' \tag{3.27}$$

by Remark 2.6. (3.25) and (3.27) imply

$$\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)}|G_j^2|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \lesssim E^2(t). \tag{3.28}$$

To deal with G_j^3, we decompose it into three terms according to Lemma 3.4

$$G_j^3 = G_j^{3,1}+G_j^{3,2}+G_j^{3,3},$$

where

$$G_j^{3,1} = -K_1\big((\Lambda^{2\beta-1}\dot\Delta_j\mathbb{P}(u\cdot\nabla u), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) + (\Lambda^{2\beta-2}\dot\Delta_j\mathbb{P}(u\cdot\nabla\mathbb{P}\nabla\cdot\tau), \dot\Delta_j u)\big),$$
$$G_j^{3,2} = -K_1(\Lambda^{2\beta-2}\dot\Delta_j\mathbb{P}(\nabla u\cdot\nabla\tau), \dot\Delta_j u),$$
$$G_j^{3,3} = K_1(\Lambda^{2\beta-2}\dot\Delta_j\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau), \dot\Delta_j u).$$

Observing that $j \le 0$ and using Proposition 2.7, we have

$$\begin{aligned}|G_j^{3,1}| &\lesssim c_j\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\Big(2^{-j(\frac{d}{2}+1-2\beta)}\|u\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}\|\Lambda^{2\beta-2}\dot\Delta_j\mathbb{P}\nabla\cdot\tau\|_{L^2}\\
&\quad +2^{(2\beta-2)j}2^{-(\frac{d}{2}-2\beta)j}\|\mathbb{P}\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}-2\beta,\frac{d}{2}-1}}\|\dot\Delta_j u\|_{L^2}\Big)\\
&\lesssim c_j\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}2^{-j(\frac{d}{2}+2-4\beta)}\Big(\|u\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}\\
&\qquad\qquad +\|\mathbb{P}\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}-2\beta,\frac{d}{2}-1}}\|\dot\Delta_j u\|_{L^2}\Big).\end{aligned}$$

Thanks to $\frac{1}{2}\le\beta\le 1$,

$$\begin{aligned}&\int_0^t\sum_{j\le 0}2^{j(\frac{d}{2}+1-2\beta)}|G_j^{3,1}|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\
&\lesssim\int_0^t\sum_{j\le 0}c_j\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}2^{(2\beta-1)j}\Big(\|u\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}+\|\mathbb{P}\nabla\cdot\tau\|_{\dot B^{\frac{d}{2}-2\alpha,\frac{d}{2}-1}}\Big)\\
&\lesssim\big(\sup_t\|u\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}}+\sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\big)\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'.\end{aligned}\tag{3.29}$$

As in the estimates of $G_j^{2,2}$ and $G_j^{2,3}$, we have

$$\begin{aligned}&\int_0^t\sum_{j\le 0}2^{j(\frac{d}{2}+1-2\beta)}\big|G_j^{3,2}+G_j^{3,3}\big|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\\
&\lesssim\sup_t\|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\int_0^t\|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'.\end{aligned}$$

Combining this with (3.29), we obtain

$$\int_0^t\sum_{j\le 0}2^{j(\frac{d}{2}+1-2\beta)}|G_j^3|/(\|\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'\lesssim E^2(t).\tag{3.30}$$

Step 3: Estimate for H_j^2, H_j^3.

In order to estimate H_j^2, we apply Lemma 3.4 to write

$$H_j^2 = H_j^{2,1} + H_j^{2,2} + H_j^{2,3},$$

where

$$\begin{aligned}
H_j^{2,1} &= -\frac{4\mu_1^2}{\eta}(\Lambda^{-1}\dot{\Delta}_j \mathbb{P}(u \cdot \nabla \mathbb{P}\nabla \cdot \tau), \dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla \cdot \tau),\\
H_j^{2,2} &= -\frac{4\mu_1^2}{\eta}(\Lambda^{-1}\dot{\Delta}_j \mathbb{P}(\nabla u \cdot \nabla \tau), \dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla \cdot \tau),\\
H_j^{2,3} &= \frac{4\mu_1^2}{\eta}(\Lambda^{-1}\dot{\Delta}_j \mathbb{P}(\nabla u \cdot \nabla \Delta^{-1}\nabla \cdot \nabla \cdot \tau), \dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla \cdot \tau).
\end{aligned}$$

By Proposition 2.7,

$$|H_j^{2,1}| \lesssim c_j 2^{-j} 2^{-j(\frac{d}{2}-1)} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\mathbb{P}\nabla \cdot \tau\|_{\dot{B}^{\frac{d}{2}-2\beta, \frac{d}{2}-1}} \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla \cdot \tau\|_{L^2},$$

which implies

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j\frac{d}{2}} |H_j^{2,1}|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla \cdot \tau\|_{L^2}) dt'\\
&\lesssim \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta, \frac{d}{2}}} \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} dt'.
\end{aligned} \tag{3.31}$$

Then we deal with $H_j^{2,2}$ and $H_j^{2,3}$. Using Remark 2.6 and (3.26), we obtain

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j\frac{d}{2}} \left| H_j^{2,2} + H_j^{2,3} \right| /(\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla \cdot \tau\|_{L^2}) dt'\\
&\lesssim \int_0^t \|\nabla u \otimes \tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla u \otimes \Delta^{-1}\nabla \cdot \nabla \cdot \tau\|^h_{\dot{B}_{2,1}^{\frac{d}{2}}} dt'\\
&\lesssim \int_0^t \|\nabla u \otimes \tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta, \frac{d}{2}}} + \|\nabla u \otimes \Delta^{-1}\nabla \cdot \nabla \cdot \tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta, \frac{d}{2}}} dt'\\
&\lesssim \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta, \frac{d}{2}}} \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} dt'.
\end{aligned}$$

Combining this with (3.31), we obtain

$$\int_0^t \sum_{j>0} 2^{j\frac{d}{2}} |H_j^2| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t' \lesssim E^2(t). \tag{3.32}$$

By Lemma 3.4, H_j^3 can be rewritten as

$$H_j^3 = H_j^{3,1} + H_j^{3,2} + H_j^{3,3},$$

where

$$\begin{aligned} H_j^{3,1} &= -\mu_1\big(\Lambda^{2\beta-1}(\dot{\Delta}_j(u\cdot\nabla u), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau) \\ &\quad + (\Lambda^{-1}\dot{\Delta}_j(u\cdot\nabla\mathbb{P}\nabla\cdot\tau), \Lambda^{2\beta-1}\dot{\Delta}_j u)\big), \\ H_j^{3,2} &= -\mu_1(\Lambda^{-1}\dot{\Delta}_j\mathbb{P}(\nabla u\cdot\nabla\tau), \Lambda^{2\beta-1}\dot{\Delta}_j u), \\ H_j^{3,3} &= \mu_1(\Lambda^{-1}\dot{\Delta}_j\mathbb{P}(\nabla u\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\tau), \Lambda^{2\beta-1}\dot{\Delta}_j u). \end{aligned}$$

By Proposition 2.7,

$$\begin{aligned} \big|H_j^{3,1}\big| &\lesssim c_j \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \big(2^{-j(\frac{d}{2}+2\beta-1)} \|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}} \|\dot{\Delta}_j\Lambda^{2\beta-2}\mathbb{P}\nabla\cdot\tau\|_{L^2} \\ &\quad + 2^{j(2\beta-2)} 2^{-j(\frac{d}{2}-1)} \|\mathbb{P}\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\beta,\frac{d}{2}-1}} \|\dot{\Delta}_j u\|_{L^2}\big) \\ &\lesssim c_j 2^{-j\frac{d}{2}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \big(\|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}} \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2} \\ &\quad + \|\mathbb{P}\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\beta,\frac{d}{2}-1}} \|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2}\big), \end{aligned}$$

which implies

$$\begin{aligned} &\int_0^t \sum_{j>0} 2^{j\frac{d}{2}} |H_j^{3,1}| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t' \\ &\lesssim \big(\sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-1}} + \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\big) \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t'. \end{aligned} \tag{3.33}$$

As in the estimates of $H_j^{2,2}$ and $H_j^{2,3}$, we derive

$$\int_0^t \sum_{j>0} 2^{j\frac{d}{2}} \big|H_j^{3,2} + H_j^{3,3}\big| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t'$$

$$\lesssim \sup_t \|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} \int_0^t \|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}} dt'.$$

Combining this with (3.33), we conclude

$$\int_0^t \sum_{j>0} 2^{j\frac{d}{2}} |H_j^3| / (\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) dt' \lesssim E^2(t). \tag{3.34}$$

Gathering (3.24), (3.28), (3.30), (3.32) and (3.34), we obtain (3.1). From (3.22), we have the following estimates for low frequencies of τ,

$$E^l_\tau(t) = \sup_t \|\tau\|^l_{\dot B^{\frac{d}{2}+1-2\beta}_{2,1}} + \int_0^t \|\tau\|^l_{\dot B^{\frac{d}{2}+1}_{2,1}} dt' \le \|\tau_0\|^l_{\dot B^{\frac{d}{2}+1-2\beta}_{2,1}} + C\int_0^t \|u\|^l_{\dot B^{\frac{d}{2}+2-2\beta}_{2,1}} dt'$$
$$+ C\int_0^t \sum_{j\le 0} 2^{(\frac{d}{2}+1-2\beta)j} |Y_j| / \|\dot\Delta_j \tau\|_{L^2} dt'.$$

Thanks to Proposition 2.5 and Proposition 2.7, we have

$$\begin{aligned}
E^l_\tau(t) &\lesssim \|\tau_0\|^l_{\dot B^{\frac{d}{2}+1-2\beta}_{2,1}} + t\sup_t \|u\|^l_{\dot B^{\frac{d}{2}+1-2\beta}_{2,1}} \\
&\quad + \int_0^t \Big(\sum_{j\le 0} c_j \|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}} \|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} + \|Q(\tau,\nabla u)\|^l_{\dot B^{\frac{d}{2}+1-2\beta}_{2,1}}\Big) dt' \\
&\lesssim \|\tau_0\|^l_{\dot B^{\frac{d}{2}+1-2\beta}_{2,1}} + t\sup_t \|u\|^l_{\dot B^{\frac{d}{2}+1-2\beta}_{2,1}} + \sup_t \|\tau\|_{\dot B^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} \int_0^t \|u\|_{\dot B^{\frac{d}{2}+1}_{2,1}} dt' \\
&\lesssim E_0 + (t + E(t))E(t).
\end{aligned}$$

We complete the proof of Proposition 3.1. □

4. The global existence and the uniqueness

This section proves the existence and uniqueness part of Theorem 1.1. The high regularity part will be established in Section 5. The existence part applied the bootstrapping argument to the *a priori* energy inequality obtained in Proposition 3.1. Due to the lack of velocity dissipation and the general fractional dissipation, the proof of the uniqueness part is not trivial. We need to make use of the parabolic smoothing or damping effect of the wave structure in order to establish the uniqueness. The rest of this section is naturally divided into two sections.

4.1. The global existence

The local existence can be established via a standard procedure. In fact, we could modify the methods in [16] or [8] to achieve the local existence. It then suffices to establish the global bound on the Besov norm of (u, τ).

Proof of the existence part of Theorem 1.1. By Proposition 3.1, the energy functional defined in (1.13) satisfies

$$E(t) \le C_1 E_0 + C_2 E^2(t), \quad t > 0 \tag{4.1}$$

for some positive constants C_1 and C_2. An application of the bootstrapping argument to (4.1) implies that, if the initial norm

$$\|u_0\|_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1} \cap \dot{B}^{\frac{d}{2}+2\beta-1}_{2,1}} + \|\tau_0\|_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1} \cap \dot{B}^{\frac{d}{2}}_{2,1}} \le \varepsilon$$

for sufficiently small $\varepsilon > 0$ or $E_0 \le \varepsilon^2$, then, for any $t > 0$,

$$E(t) \le 2C_1\varepsilon^2,$$

which, especially, yields the desired global upper bound on the norms of (u, τ). In fact, if we take

$$\varepsilon \le \frac{1}{\sqrt{8C_1C_2}}$$

and make the ansatz that

$$E(t) \le \frac{1}{2C_2}, \tag{4.2}$$

then (4.1) implies that

$$E(t) \le C_1 E_0 + C_2 \frac{1}{2C_2} E(t)$$

or

$$E(t) \le 2C_1\, E_0 \le 2C_1\varepsilon^2 \le \frac{1}{4C_2} \tag{4.3}$$

The bound in (4.3) is only half of the one in the ansatz (4.2). The bootstrapping argument then implies (4.3) indeed holds for any $t > 0$. Especially, (1.9) holds. The upper bound in (1.10) is a consequence of the following inequality from Proposition 3.1,

$$E_\tau^l(t) \lesssim E_0 + (t + \varepsilon)\varepsilon. \tag{4.4}$$

This completes the proof of the existence part of Theorem 1.1. □

4.2. The uniqueness

Due to the lack of velocity dissipation and the inclusion of a range of fractional dissipation in τ, the proof of the uniqueness is not direct. We need the extra smoothing and damping effect of the wave structure. We use some of the ideas implemented in Section 3.

Proof of the uniqueness part of Theorem 1.1. Assume (u_1, τ_1) and (u_2, τ_2) are two solutions of (1.3) with the same initial data. Denote $\delta u = u_1 - u_2, \delta\tau = \tau_1 - \tau_2, \delta p = p_1 - p_2$. Then $(\delta u, \delta\tau)$ satisfies

$$\begin{cases} (\delta u)_t + \nabla\delta p = \mu_1\nabla\cdot\delta\tau - u_1\cdot\nabla\delta u - \delta u\cdot\nabla u_2, \\ (\delta\tau)_t + u_1\cdot\nabla\delta\tau + \eta\Lambda^{2\beta}\delta\tau = \mu_2 D(\delta u) - \delta u\cdot\nabla\tau_2 - Q(\tau_1,\nabla\delta u) - Q(\delta\tau,\nabla u_2), \\ \nabla\cdot\delta u = 0, \\ \delta u(0,x) = 0, \ \delta\tau(0,x) = 0. \end{cases}$$

Similar to Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} &\sup_t \|\delta u\|_{\dot B^{\frac d2+1-2\beta,\frac d2+2\beta-2}} + \sup_t \|\delta\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2-1}} + \int_0^t \|\delta u\|_{\dot B^{\frac d2+1,\frac d2}}\,\mathrm{d}t' \\ &\quad + \int_0^t (\|\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|^l_{\dot B^{\frac d2+1}_{2,1}} + \|\tau\|^h_{\dot B^{\frac d2+2\beta-1}_{2,1}})\mathrm{d}t' \\ &\lesssim C_2\int_0^t \sum_{j\le 0} 2^{j(\frac d2+1-2\beta)}\Big(|G_j'|/(\|\dot\Delta_j\delta u\|_{L^2} + \|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}) \\ &\qquad\qquad + |V_j'|/(\|\dot\Delta_j\delta u\|_{L^2} + \|\dot\Delta_j\delta\tau\|_{L^2})\Big)\mathrm{d}t' \\ &\quad + C_2\int_0^t \sum_{j>0} 2^{j(\frac d2-1)}\Big(|H_j'|/(\|\Lambda^{2\beta-1}\dot\Delta_j\delta u\|_{L^2} \\ &\qquad\qquad + \|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}) + |Y_j'|/\|\dot\Delta_j\delta\tau\|_{L^2}\Big)\mathrm{d}t', \end{aligned}$$

where

$$G_j' = G_j'^1 + G_j'^2 + G_j'^3, \qquad H_j' = H_j'^1 + H_j'^2 + H_j'^3$$

with

$$\begin{aligned} G_j'^1 = &-\frac{\mu_2}{2}((\dot\Delta_j\mathbb{P}(u_1\cdot\nabla\delta u), \dot\Delta_j\delta u) + (\dot\Delta_j\mathbb{P}(\delta u\cdot\nabla u_2), \dot\Delta_j\delta u)) \\ &- \mu_1((\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau) \\ &\quad + (\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2), \dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)) \end{aligned}$$

$$
\begin{aligned}
&- K_1((\Lambda^{2\beta-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot{\Delta}_j\delta u)\\
&\qquad + (\Lambda^{2\beta-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot{\Delta}_j\delta u)),\\
G_j'^2 =& - \mu_1((\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\qquad + (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau)),\\
G_j'^3 =& - K_1(\Lambda^{2\beta-1}(\dot{\Delta}_j\mathbb{P}(u_1\cdot\nabla\delta u),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\qquad + (\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(\delta u\cdot\nabla u_2),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau))\\
&- K_1((\Lambda^{2\beta-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\dot{\Delta}_j\delta u)\\
&\qquad + (\Lambda^{2\beta-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\dot{\Delta}_j\delta u)),\\
H_j'^1 =& - \frac{4\mu_1^2}{\eta}((\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\qquad + (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau))\\
&- \mu_1((\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)\\
&\qquad + (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u))\\
&- \eta((\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(u_1\cdot\nabla\delta u),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)\\
&\qquad + (\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(\delta u\cdot\nabla u_2),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)),\\
H_j'^2 =& - \frac{4\mu_1^2}{\eta}((\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\qquad + (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)),\\
H_j'^3 =& - \mu_1(\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(u_1\cdot\nabla\delta u),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\qquad - \mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)\\
&- \mu_1(\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(\delta u\cdot\nabla u_2),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)\\
&\qquad - \mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u),
\end{aligned}
$$

and

$$
\begin{aligned}
V_j' =& -\mu_2((\dot{\Delta}_j(u_1\cdot\nabla\delta u),\dot{\Delta}_j\delta u)+(\dot{\Delta}_j(\delta u\cdot\nabla u_2),\dot{\Delta}_j\delta u))\\
&-\mu_1((\dot{\Delta}_j(u_1\cdot\nabla\delta\tau),\dot{\Delta}_j\delta\tau)+(\dot{\Delta}_j(\delta u\cdot\nabla\tau_2),\dot{\Delta}_j\delta\tau))\\
&-\mu_1((\dot{\Delta}_jQ(\tau_1,\nabla\delta u),\dot{\Delta}_j\delta\tau)+(\dot{\Delta}_jQ(\delta\tau,\nabla u_2),\dot{\Delta}_j\delta\tau)),\\
Y_j' =& -((\dot{\Delta}_j(u_1\cdot\nabla\delta\tau),\dot{\Delta}_j\delta\tau)+(\dot{\Delta}_j(\delta u\cdot\nabla\tau_2),\dot{\Delta}_j\delta\tau))\\
&-((\dot{\Delta}_jQ(\tau_1,\nabla\delta u),\dot{\Delta}_j\delta\tau)+(\dot{\Delta}_jQ(\delta\tau,\nabla u_2),\dot{\Delta}_j\delta\tau)).
\end{aligned}
$$

According to Proposition 2.5 and Proposition 2.7, for $j\le 0$,

$$|(\dot{\Delta}_j(u_1\cdot\nabla\delta u),\dot{\Delta}_j\delta u)|\lesssim c_j 2^{-j(\frac{d}{2}+1-2\beta)}\|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\|\dot{\Delta}_j\delta u\|_{L^2},$$
$$|(\dot{\Delta}_j(\delta u\cdot\nabla u_2),\dot{\Delta}_j\delta u)|\lesssim c_j 2^{-j(\frac{d}{2}+1-2\beta)}\|u_2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\|\dot{\Delta}_j\delta u\|_{L^2};$$

and for $j>0$,

$$\begin{aligned}&|(\Lambda^{2\beta-1}\dot{\Delta}_j(u_1\cdot\nabla\delta u),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)|\\&\quad\lesssim c_j2^{-j(\frac{d}{2}-1)}\|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2},\\&|(\Lambda^{2\beta-1}\dot{\Delta}_j(\delta u\cdot\nabla u_2),\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)|\\&\lesssim c_j2^{-j(\frac{d}{2}-1)}\|u_2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2};\end{aligned}$$

and for all j,

$$|(\dot{\Delta}_j(u_1\cdot\nabla\delta\tau),\dot{\Delta}_j\delta\tau)|\lesssim c_j2^{-j\psi^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}(j)}\|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}\|\dot{\Delta}_j\delta\tau\|_{L^2},$$
$$|(\dot{\Delta}_j(\delta u\cdot\nabla\tau_2),\dot{\Delta}_j\delta\tau)|\lesssim c_j2^{-j\psi^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}(j)}\|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}\|\dot{\Delta}_j\delta\tau\|_{L^2}.$$

For $j\le 0$, we have

$$\begin{aligned}&\big|-\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)-K_1(\Lambda^{2\beta-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot{\Delta}_j\delta u)\big|\\&\lesssim\|\dot{\Delta}_jQ(\tau_1,\nabla\delta u)\|_{L^2}(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}),\\&\big|-\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)-K_1(\Lambda^{2\beta-2}\dot{\Delta}_j\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot{\Delta}_j\delta u)\big|\\&\lesssim\|\dot{\Delta}_jQ(\delta\tau,\nabla u_2)\|_{L^2}(\|\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}),\end{aligned}$$

and

$$\begin{aligned}&\big|-\mu_1(\dot{\Delta}_jQ(\tau_1,\nabla\delta u),\dot{\Delta}_j\delta\tau)\big|\lesssim\|\dot{\Delta}_jQ(\tau_1,\nabla\delta u)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2},\\&\big|-\mu_1(\dot{\Delta}_jQ(\delta\tau,\nabla u),\dot{\Delta}_j\delta\tau)\big|\lesssim\|\dot{\Delta}_jQ(\delta\tau,\nabla u_2)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2}.\end{aligned}$$

And for $j\ge 0$, we have

$$\begin{aligned}&\Big|-\frac{4\mu_1^2}{\eta}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)-\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\tau_1,\nabla\delta u),\\&\qquad\qquad\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)\Big|\\&\lesssim\|\dot{\Delta}_jQ(\tau_1,\nabla\delta u)\|_{L^2}(\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}),\\&\Big|-\frac{4\mu_1^2}{\eta}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau)-\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot Q(\delta\tau,\nabla u_2),\\&\qquad\qquad\Lambda^{2\beta-1}\dot{\Delta}_j\delta u)\Big|\end{aligned}$$

$$\lesssim \|\dot{\Delta}_j Q(\delta\tau, \nabla u_2)\|_{L^2}(\|\Lambda^{2\beta-1}\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}),$$

and

$$\begin{aligned}
\big| -(\dot{\Delta}_j Q(\tau_1, \nabla\delta u), \dot{\Delta}_j\delta\tau)\big| &\lesssim \|\dot{\Delta}_j Q(\tau_1, \nabla\delta u)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2},\\
\big| -(\dot{\Delta}_j Q(\delta\tau, \nabla u_2), \dot{\Delta}_j\delta\tau)\big| &\lesssim \|\dot{\Delta}_j Q(\delta\tau, \nabla u_2)\|_{L^2}\|\dot{\Delta}_j\delta\tau\|_{L^2}.
\end{aligned}$$

Combining the estimates above, we obtain

$$\begin{aligned}
&\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)}\Big(|G_j'^1|/(\|\dot{\Delta}_j\delta u\|_2 + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\\
&\qquad\qquad + |V_j'|/(\|\dot{\Delta}_j\delta u\|_{L^2} + \|\dot{\Delta}_j\delta\tau\|_{L^2}\Big)\mathrm{d}t'\\
&+\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)}\Big(|H_j'^1|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}) + |Y_j'|/\|\dot{\Delta}_j\delta\tau\|_{L^2}\Big)\mathrm{d}t'\\
&\lesssim \sup_t\big(\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}} + \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\big)\int_0^t \|(u_1,u_2)\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\
&\quad+\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}\mathrm{d}t' + \int_0^t \|(Q(\tau_1,\nabla\delta u), Q(\delta\tau,\nabla u_2))\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}\mathrm{d}t'\\
&\lesssim \sup_t\big(\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}} + \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\big)\int_0^t \|(u_1,u_2)\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'\\
&\quad+\sup_t(\|\tau_1\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} + \|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}})\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\mathrm{d}t'.
\end{aligned}$$

$G_j'^2$ and $G_j'^3$ can be bounded directly via Proposition 2.5,

$$\begin{aligned}
&\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)}|G_j'^2|/(\|\dot{\Delta}_j\delta u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'\\
&\lesssim \int_0^t (\|u_1\cdot\nabla\delta\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|\delta u\cdot\nabla\tau_2\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}})\mathrm{d}t'\\
&\lesssim \int_0^t (\|u_1\delta\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|\delta u\tau_2\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}})\mathrm{d}t'
\end{aligned}$$

$$\lesssim \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}} \int_0^t \|u_1\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathrm{d}t' + t\sup_t(\|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}),$$

and

$$\int_0^t \sum_{j\le 0} 2^{j(\frac{d}{2}+1-2\beta)} |G_j'^{3}|/(\|\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$

$$\lesssim \int_0^t (\|u_1\delta u\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|\delta u\nabla u_2\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|u_1\delta\tau\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}} + \|\delta u\tau_2\|^l_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}})\mathrm{d}t'$$

$$\lesssim \sup_t(\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}} + \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}) \int_0^t (\|u_1\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|u_2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}})\mathrm{d}t'$$
$$+ t\sup_t(\|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}).$$

To estimate $H_j'^{2}$, we rewrite it as $H_j'^{2} = H_j'^{2,1} + H_j'^{2,2} + H_j'^{2,3} + H_j'^{2,4}$ according to Lemma 3.4, where

$$H_j'^{2,1} = -\frac{4\mu_1^2}{\eta}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}(u_1\cdot\nabla\mathbb{P}\nabla\cdot\delta\tau), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau),$$
$$H_j'^{2,2} = -\frac{4\mu_1^2}{\eta}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}(\nabla u_1\cdot\nabla\delta\tau), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau),$$
$$H_j'^{2,3} = \frac{4\mu_1^2}{\eta}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}(\nabla u_1\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\delta\tau), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau),$$
$$H_j'^{2,4} = -\frac{4\mu_1^2}{\eta}(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau).$$

By Proposition 2.7, we obtain

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)} |H_j'^{2,1}|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$

$$\lesssim \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}} \int_0^t \|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t'.$$

It follows from Proposition 2.5 that

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)} \big|H_j'^{2,2} + H_j'^{2,3}\big|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2} + \|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$

$$\lesssim \int_0^t \|\nabla u_1 \otimes \delta\tau\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\nabla u_1 \otimes \Delta^{-1}\nabla\cdot\nabla\cdot\delta\tau\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \mathrm{d}t'$$

$$\lesssim \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}} \int_0^t \|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t',$$

and

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)} \big|H'^{2,4}_j\big| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$

$$\lesssim \int_0^t \|\delta u \otimes \nabla\tau_2\|^h_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \mathrm{d}t' \lesssim \sup_t \|\tau_2\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \mathrm{d}t'.$$

To estimate H'^3_j, we rewrite it as $H'^3_j = H'^{3,1}_j + H'^{3,2}_j + H'^{3,3}_j + H'^{3,4}_j + H'^{3,5}_j$ by Lemma 3.4, where

$$\begin{aligned}
H'^{3,1}_j &= -\mu_1\big((\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(u_1\cdot\nabla\delta u), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau) \\
&\qquad + (\dot{\Delta}_j\Lambda^{-1}\mathbb{P}(u_1\cdot\nabla\mathbb{P}\nabla\cdot\delta\tau), \Lambda^{2\beta-1}\dot{\Delta}_j\delta u)\big), \\
H'^{3,2}_j &= -\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}(\nabla u_1\cdot\nabla\delta\tau), \Lambda^{2\beta-1}\dot{\Delta}_j\delta u), \\
H'^{3,3}_j &= \mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}(\nabla u_1\cdot\nabla\Delta^{-1}\nabla\cdot\nabla\cdot\delta\tau), \Lambda^{2\beta-1}\dot{\Delta}_j\delta u), \\
H'^{3,4}_j &= -\mu_1(\Lambda^{2\beta-1}\dot{\Delta}_j\mathbb{P}(\delta u\cdot\nabla u_2), \dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau), \\
H'^{3,5}_j &= -\mu_1(\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2), \Lambda^{2\beta-1}\dot{\Delta}_j\delta u).
\end{aligned}$$

By Proposition 2.7, we obtain

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)} |H'^{3,1}_j| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$

$$\lesssim \big(\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}} + \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}\big) \int_0^t \|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \mathrm{d}t'.$$

Similar to the estimates for $H'^{2,2}_j$, $H'^{2,3}_j$ and $H'^{2,4}_j$, we have

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)}|H_j'^{3,2}+H_j'^{3,3}|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$
$$\lesssim \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}\int_0^t \|u_1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t',$$

and

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)}|H_j'^{3,5}|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$
$$\lesssim \sup_t \|\tau_2\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\mathrm{d}t'.$$

By Proposition 2.5, we have

$$\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}-1)}|H_j'^{3,4}|/(\|\Lambda^{2\beta-1}\dot{\Delta}_j\delta u\|_{L^2}+\|\dot{\Delta}_j\Lambda^{-1}\mathbb{P}\nabla\cdot\delta\tau\|_{L^2})\mathrm{d}t'$$
$$\lesssim \int_0^t \|\delta u\cdot\nabla u_2\|^h_{\dot{B}^{\frac{d}{2}+2\beta-2}_{2,1}}\mathrm{d}t' \lesssim \sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\int_0^t \|u_2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t'.$$

Combining the estimates above, we obtain

$$\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}+\sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}+\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}}}\mathrm{d}t'$$
$$\lesssim \sup_t\big(\|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}}+\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}\big)\int_0^t(\|(u_1,u_2)\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}+\|u_1\|_{\dot{B}^{\frac{d}{2}}_{2,1}})\mathrm{d}t'$$
$$+\sup_t \|(\tau_1,\tau_2)\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\mathrm{d}t' + t\sup_t(\|\tau_2\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}}\|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}).$$

Notice that

$$\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\mathrm{d}t' \lesssim t\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}}+\int_0^t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}}}\mathrm{d}t'.$$

Thanks to the uniform a priori estimates, we have $\sup_t \|(\tau_1,\tau_2)\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}}} \lesssim \varepsilon$. Then we can choose ε and t small such that

$$\sup_t \|\delta u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta-2}} + \sup_t \|\delta\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}-1}} \leqslant 0.$$

This completes the proof of the uniqueness part of Theorem 1.1. □

5. High regularity properties for more regular data

This section is devoted to proving the higher regularity part of Theorem 1.1. More precisely, we show that, if the initial datum (u_0, τ_0) is in a more regular Besov space and sufficiently small,

$$\|u_0\|_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}\cap\dot{B}^{\frac{d}{2}+2\beta+s-1}_{2,1}} + \|\tau_0\|_{\dot{B}^{\frac{d}{2}+1-2\beta}_{2,1}\cap\dot{B}^{\frac{d}{2}+s}_{2,1}} \le \varepsilon,$$

then the corresponding solution (u, τ) of (1.3) is in a more regular space,

$$\begin{aligned} &u \in C(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1-2\beta}_{2,1} \cap \dot{B}^{\frac{d}{2}+2\beta+s-1}_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1} \cap \dot{B}^{\frac{d}{2}+1+s}_{2,1});\\ &\tau \in C(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1-2\beta}_{2,1} \cap \dot{B}^{\frac{d}{2}+s}_{2,1}), \quad \tau \in L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1} \cap \dot{B}^{\frac{d}{2}+2\beta+s}_{2,1}). \end{aligned} \tag{5.1}$$

The proof shares some similarities with the proof of the existence part. The key component of the proof is the energy inequality stated in the following proposition.

Proposition 5.1. *Assume that (u, τ) solves (1.3). Then, there exist two positive constants C_1 and C_2 such that, for $t > 0$,*

$$E_0'^h(t) + E'^h(t) \le C_1 E_0' + C_2((E_0'^h(t) + E'^h(t))E(t) + E^2(t)),$$

where $E_0'^h(t)$ is defined in (1.15) and $E'^h(t)$ in (1.16), and $E(t)$ is defined in (1.13) as before.

In the rest of this section, we first assume Proposition 5.1 and provide the proof for the higher regularity part of Theorem 1.1, and then prove Proposition 5.1.

Proof for the higher regularity part of Theorem 1.1. By Proposition 5.1, we have

$$E_0'^h(t) + E'^h(t) \le C_1 E_0' + C_2((E_0'^h(t) + E'^h(t))E(t) + E^2(t)).$$

Therefore, thanks to the results in Subsection 4.1, we can choose ε small such that $C_2E(t) \le \frac{1}{2}$ such that

$$E_0'^h(t) + E'^h(t) \lesssim E_0' + \varepsilon.$$

Combining this inequality with the results in Subsection 4.1 and inequality (4.4) yields

$$\begin{aligned} &\sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta.\frac{d}{2}+2\beta}} + \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+1}} + \int_0^t \|u\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}+2}} dt' + \int_0^t \|\tau\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}+2\beta+1}} dt' \\ &\lesssim E_0' + \varepsilon(\varepsilon + t). \end{aligned}$$

Similarly, we can choose ε depending on s sufficiently small to get further regularity,

$$\begin{aligned}&\sup_t \|u\|_{\dot B^{\frac d2+1-2\beta,\frac d2+2\beta+s-1}} + \sup_t \|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2+s}}\\&+\int_0^t \|u\|_{\dot B^{\frac d2+1,\frac d2+1+s}}\mathrm{d}t' + \int_0^t \|\tau\|_{\dot B^{\frac d2+1,\frac d2+2\beta+s}}\mathrm{d}t'\\&\lesssim \|u\|^h_{\dot B^{\frac d2+2\beta+s-1}_{2,1}} + \|\tau\|^h_{\dot B^{\frac d2+s}_{2,1}} + \varepsilon(\varepsilon+t).\end{aligned}$$

This completes the proof of the higher regularity in (5.1) or in (1.9). □

We now turn to the proof of Proposition 5.1. The proof of Proposition 5.1 relies on the following lemma, which is the higher regularity version of Lemma 3.2 and Lemma 3.3. Due to its similarity with Lemmas 3.2 and 3.3, we omit its proof.

Lemma 5.2. *Let (u,τ) be the solution to the system (1.3) on $[0,T)$, we have the following estimates*

$$\begin{aligned}E_0'^h(t) + E'^h(t) \le C_1 E_0' + C_2 \int_0^t \sum_{j>0} 2^{j(\frac d2+1)}\Big(&|H_j|/(\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2}\\&+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) + |Y_j|/\|\dot\Delta_j\tau\|_{L^2}\Big)\mathrm{d}t',\end{aligned}$$

where C_1 and C_2 are independent of T and $E_0' = \|u_0\|_{\dot B^{\frac d2+1-2\beta,\frac d2+2\beta}} + \|\tau_0\|_{\dot B^{\frac d2+1-2\beta,\frac d2+1}}$.

Proof of Proposition 5.1. According to Proposition 2.5 and Proposition 2.7, for $j>0$,

$$|(\Lambda^{2\beta-1}\dot\Delta_j(u\cdot\nabla u),\Lambda^{2\beta-1}\dot\Delta_j u)| \lesssim c_j 2^{-j(\frac d2+1)}\|u\|_{\dot B^{\frac d2+1}_{2,1}}\|u\|_{\dot B^{\frac d2+1-2\beta,\frac d2+2\beta}}\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2},$$

$$|(\dot\Delta_j(u\cdot\nabla\tau),\dot\Delta_j\tau)| \lesssim c_j 2^{-(\frac d2+1)j}\|u\|_{\dot B^{\frac d2+1}_{2,1}}\|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2+1}}\|\dot\Delta_j\tau\|_{L^2}.$$

Similar to the inequality (3.24), we have

$$\begin{aligned}&\int_0^t \sum_{j>0} 2^{j(\frac d2+1)}\Big(|H_j^1|/(\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2} + \|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) + |Y_j|/\|\dot\Delta_j\tau\|_{L^2}\Big)\mathrm{d}t'\\&\lesssim \big(\sup_t\|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2+1}} + \sup_t\|u\|_{\dot B^{\frac d2+1-2\beta,\frac d2+2\beta}}\big)\int_0^t \|u\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t'\\&\quad+\int_0^t \|Q(\tau,\nabla u)\|^h_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t'\end{aligned}$$

$$
\begin{aligned}
&\lesssim \big(\sup_t \|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2+1}} + \sup_t \|u\|_{\dot B^{\frac d2+1-2\beta,\frac d2+2\beta}}\big)\int_0^t \|u\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t' \\
&\quad + \sup_t \|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2}} \int_0^t \|u\|^h_{\dot B^{\frac d2+2}_{2,1}}\mathrm{d}t' \\
&\lesssim (E_0'^h(t)+E'^h(t))E(t)+E^2(t). \qquad (5.2)
\end{aligned}
$$

By Proposition 2.7, we obtain

$$
|H_j^{2,1}| \lesssim c_j 2^{-j}2^{-j\frac d2}\|u\|_{\dot B^{\frac d2+1}_{2,1}}\|\mathbb{P}\nabla\cdot\tau\|_{\dot B^{\frac d2-2\beta,\frac d2}}\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2},
$$

which implies

$$
\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac d2+1)}|H_j^{2,1}|/(\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \\
&\lesssim \sup_t \|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2+1}}\int_0^t \|u\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t'.
\end{aligned} \qquad (5.3)
$$

By Proposition 2.5,

$$
\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac d2+1)}\big|H_j^{2,2}+H_j^{2,3}\big|/(\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t' \\
&\lesssim \int_0^t \|\nabla u\otimes\tau\|^h_{\dot B^{\frac d2+1}_{2,1}} + \|\nabla u\otimes\Delta^{-1}\nabla\cdot\nabla\cdot\tau\|^h_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t' \\
&\lesssim \big(\sup_t \|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2+1}} + \sup_t \|u\|_{\dot B^{\frac d2+1-2\beta,\frac d2+2\beta-1}}\big)\int_0^t \|u\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t' \\
&\quad + \sup_t \|\tau\|_{\dot B^{\frac d2+1-2\beta,\frac d2}} \int_0^t \|u\|^h_{\dot B^{\frac d2+2}_{2,1}}\mathrm{d}t' \\
&\lesssim (E_0'^h(t)+E'^h(t))E(t)+E^2(t).
\end{aligned}
$$

Combining this with (5.3), we obtain

$$
\int_0^t \sum_{j>0} 2^{j(\frac d2+1)}|H_j^{2}|/(\|\Lambda^{2\beta-1}\dot\Delta_j u\|_{L^2}+\|\dot\Delta_j\Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2})\mathrm{d}t'
$$

$$\lesssim (E_0'^h(t) + E'^h(t))E(t) + E^2(t). \tag{5.4}$$

By Proposition 2.7,

$$\begin{aligned}
\big|H_j^{3,1}\big| &\lesssim c_j \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \big(2^{-j(\frac{d}{2}+2\beta)} \|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta}} \|\dot{\Delta}_j \Lambda^{2\beta-2}\mathbb{P}\nabla\cdot\tau\|_{L^2} \\
&+ 2^{2\beta-2}2^{-j\frac{d}{2}} \|\mathbb{P}\nabla\cdot\tau\|_{\dot{B}^{\frac{d}{2}-2\beta,\frac{d}{2}}} \|\dot{\Delta}_j u\|_{L^2}\big).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1)} |H_j^{3,1}| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t' \\
&\lesssim \big(\sup_t \|u\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+2\beta}} + \sup_t \|\tau\|_{\dot{B}^{\frac{d}{2}+1-2\beta,\frac{d}{2}+1}}\big) \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \mathrm{d}t' \\
&\lesssim (E_0'^h(t) + E'^h(t))E(t) + E^2(t).
\end{aligned} \tag{5.5}$$

As in the estimates of the terms $H_j^{2,2}$ and $H_j^{2,3}$, we have

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1)} \big|H_j^{3,2} + H_j^{3,3}\big| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t' \\
&\lesssim (E_0'^h(t) + E'^h(t))E(t) + E^2(t).
\end{aligned}$$

Combining this with (5.5), we obtain

$$\begin{aligned}
&\int_0^t \sum_{j>0} 2^{j(\frac{d}{2}+1)} |H_j^{3}| / (\|\Lambda^{2\beta-1}\dot{\Delta}_j u\|_{L^2} + \|\dot{\Delta}_j \Lambda^{-1}\mathbb{P}\nabla\cdot\tau\|_{L^2}) \mathrm{d}t' \\
&\lesssim (E_0'^h(t) + E'^h(t))E(t) + E^2(t).
\end{aligned} \tag{5.6}$$

Putting (5.2), (5.4) and (5.6) together, we complete the proof of Proposition 5.1. □

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