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STABILITY AND DECAY RATES FOR A VARIANT OF THE 2D BOUSSINESQ-BÉNARD SYSTEM*

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Abstract. This paper investigates the stability and large-time behavior of perturbations near the trivial solution to a variant of the 2D Boussinesq-Bénard system. This system does not involve thermal diffusion. Our research is partially motivated by a recent work [C.R. Doering, J. Wu, K. Zhao, and X. Zheng, Phys. D, 376/377:144–159, 2018] on the stability and large-time behavior of solutions near the hydrostatic balance concerning the 2D Boussinesq system. Due to the lack of thermal diffusion, these stability problems are difficult. The energy method and classical approaches are no longer effective in dealing with these partially dissipated systems. This paper presents a new approach that takes into account the special structure of the linearized system. The linearized parts of the vorticity equation and the temperature equation both obey a degenerate damped wave-type equation. By representing the nonlinear system in an integral form and carefully crafting the functional setting for the initial data and solution spaces, we are able to establish the long-term stability and global (in time) existence and uniqueness of smooth solutions. Simultaneously, we also obtain exact decay rates for various derivatives of the perturbations.

Keywords. Boussinesq-Bénard equations; global solution; large-time behavior; stability; velocity damping.

AMS subject classifications. 35Q35; 76D03; 76D05.

1. Introduction

This paper focuses on the following 2D Boussinesq-Bénard system

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u} + \nabla P = \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \Delta\mathbf{u}\cdot\mathbf{e}_2, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u}(t,x,y)\triangleq(u_1(t,x,y),u_2(t,x,y))$ denotes the 2D velocity field, $P=P(t,x,y)$ the pressure and $\theta=\theta(t,x,y)$ the temperature. Here $\mathbf{e}_2=(0,1)$ and the term $\theta\mathbf{e}_2$ represents the buoyancy forcing due to the gravity in the vertical direction. The first equation is the 2D damped Euler equation with a buoyancy forcing and the second equation simply states that the temperature is transported by the velocity field with a diffusion term of the second component of the velocity. Equation (1.1) is a variant of the standard Boussinesq-Bénard system. The standard Boussinesq-Bénard system contains $\Delta\mathbf{u}$ and the term $\mathbf{u}\cdot\mathbf{e}_2$ instead of the damping $\mathbf{u}$ and $\Delta\mathbf{u}\cdot\mathbf{e}_2$ in (1.1). Equation (1.1) may be physically relevant when the diffusive effect of u_2 plays a more dominant role in the equation of temperature.

The aim here is to understand the large-time stability property of perturbations near the trivial steady state $(\mathbf{u},\theta)=(\mathbf{0},0)$. Equivalently, we explore the global existence

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and uniqueness of small solutions and the precise large-time behavior of these solutions. This study is partially motivated by a recent work of Doering, Wu, Zhao and Zheng [18] and a followup paper [50] on the global stability and large-time behavior of perturbations near the hydrostatic balance concerning the Boussinesq equations. The 2D Boussinesq system

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{1.2}$$

admits a special class of solutions

$$\mathbf{u}_{he} = 0, \quad \theta_{he} = \beta y, \quad P_{he} = \frac{1}{2}\beta y^2, \tag{1.3}$$

where $\beta > 0$ is a parameter. The special solution in (1.3) is the so-called hydrostatic balance, which refers to the geophysical circumstance when the fluid is at rest and when the pressure gradient is balanced out by the buoyancy force in the direction of gravity. The perturbations

$$\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_{he}, \quad \widetilde{\theta} = \theta - \theta_{he}, \quad \widetilde{P} = P - P_{he}$$

satisfy

$$\begin{cases} \partial_t \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}\cdot\nabla\widetilde{\mathbf{u}} + \nabla\widetilde{P} = \Delta\widetilde{\mathbf{u}} + \widetilde{\theta}\mathbf{e}_2, \\ \partial_t\widetilde{\theta} + \widetilde{\mathbf{u}}\cdot\nabla\widetilde{\theta} + \beta\widetilde{u}_2 = 0, \\ \nabla\cdot\widetilde{\mathbf{u}} = 0. \end{cases} \tag{1.4}$$

The difference between (1.1) and (1.4) is that the velocity equation in (1.4) involves dissipation instead of damping, and the temperature equation contains $\widetilde{u}_2$ instead of Δu_2 as in (1.1). The recent papers [18] and [50] have significantly advanced our understanding of the stability problem involving (1.4). [18] obtains the global stability and large-time behavior of the perturbation $(\widetilde{\mathbf{u}}, \widetilde{P}, \widetilde{\theta})$. It is proven, for $\beta > 0$, that the L^2 norms of the velocity perturbation (not necessarily small) and its first-order spatial and temporal derivatives converge to zero as $t \to \infty$. Consequently it is found that the pressure and temperature functions stratify in the vertical direction in a weak topology. Remarkably, the second-order spatial derivatives of the velocity perturbation (not necessarily small) are shown to be bounded uniformly in time for all time. In addition, [18] contains extensive numerical simulations illustrating the analytic results and investigating unsolved problems. [50] furthers the studies of [18]. [50] obtains precise large-time decay rates for the velocity field of the linearized equations and explicit eventual profile for the temperature. Conditional decay rates are also obtained for the nonlinear system. There are still many interesting open issues concerning (1.4). One issue is whether or not $\|\nabla\widetilde{\theta}\|_{L^2}$ is bounded uniformly in time. The lack of thermal diffusion in (1.4) makes it extremely difficult to answer this question.

In contrast, (1.1) has a very special structure and solutions of (1.1) appear to behave differently from those of (1.4). We are able to prove the global stability of the solutions of (1.1) in a highly regular setting and extract precise decay rates for various derivatives of the solutions. We explain in some detail our main idea. We take advantage of the vorticity formulation. The vorticity $\omega = \nabla\times\mathbf{u} \triangleq \partial_x u_2 - \partial_y u_1$ measures how fast the fluid rotates and its control is very important in the study of fluid stability problem. Taking the curl of the velocity equation in (1.1) yields

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega + \omega = \partial_x \theta.$$

Due to the divergence-free condition, we have

$$\Delta \mathbf{u} \cdot \mathbf{e}_2 = (\partial_{xx} + \partial_{yy}) u_2 = \partial_{xx} u_2 - \partial_{xy} u_1 = \partial_x \omega.$$

Equation (1.1) is then reduced to the following system

$$\begin{cases} \partial_t \omega + \mathbf{u} \cdot \nabla \omega + \omega = \partial_x \theta, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = \partial_x \omega, \\ \mathbf{u} = \nabla^{\perp} \Delta^{-1} \omega. \end{cases} \tag{1.5}$$

Since (1.5) is translation invariant, we assume, without loss of generality, that the initial data start with $t=1$, namely

$$\mathbf{u}(1,x,y) = \mathbf{u}_0(x,y), \quad \omega(1,x,y) = \omega_0(x,y), \quad \theta(1,x,y) = \theta_0(x,y). \tag{1.6}$$

Differentiating (1.5) in t yields

$$\begin{cases} \partial_{tt} \omega + \partial_t \omega - \partial_{xx} \omega = F_1, \\ \partial_{tt} \theta + \partial_t \theta - \partial_{xx} \theta = F_2, \end{cases} \tag{1.7}$$

and

$$\begin{cases} F_1 = -\partial_x (\mathbf{u} \cdot \nabla \theta) - \partial_t (\mathbf{u} \cdot \nabla \omega), \\ F_2 = -\partial_x (\mathbf{u} \cdot \nabla \omega) - \partial_t (\mathbf{u} \cdot \nabla \theta) - \mathbf{u} \cdot \nabla \theta. \end{cases} \tag{1.8}$$

We observe that the linear parts ω and θ obey exactly the same degenerate damped wave equation. It follows from (1.5) and (1.6) that

$$\begin{aligned} \omega_1(1,x,y) &\triangleq (\partial_t \omega)(1,x,y) = -\mathbf{u}_0 \cdot \nabla \omega_0 - \omega_0 - \partial_x \theta_0, \\ \theta_1(1,x,y) &\triangleq (\partial_t \theta)(1,x,y) = -\mathbf{u}_0 \cdot \nabla \theta_0 + \partial_x \omega_0. \end{aligned} \tag{1.9}$$

We have derived an equivalent system of (1.1) and our attention will be focused on this equivalent initial-value problem

$$\begin{cases} \partial_{tt} \omega + \partial_t \omega - \partial_{xx} \omega = F_1, \\ \partial_{tt} \theta + \partial_t \theta - \partial_{xx} \theta = F_2, \\ \omega(1,x,y) = \omega_0(x,y), \ (\partial_t \omega)(1,x,y) = \omega_1(x,y), \\ \theta(1,x,y) = \theta_0(x,y), \ (\partial_t \theta)(1,x,y) = \theta_1(x,y). \end{cases} \tag{1.10}$$

Before we analyze the nonlinear system, we first understand the linearized system of (1.5), which is given by

$$\begin{cases} \partial_t \omega + \omega = \partial_x \theta, \\ \partial_t \theta = \partial_x \omega. \end{cases}$$

Clearly, for any integer $k\in\mathbb{N}$,

$$\|\Lambda^k\omega(t)\|_{L^2}^2+\|\Lambda^k\theta(t)\|_{L^2}^2+\int_0^t\|\Lambda^k\omega(\tau)\|_{L^2}^2\,d\tau=\|\Lambda^k\omega_0\|_{L^2}^2+\|\Lambda^k\theta_0\|_{L^2}^2, \tag{1.11}$$

which implies the linear stability in any Sobolev space $H^k(\mathbb{R}^2)$. Here $\Lambda=\sqrt{-\Delta}$ denotes the Zygmund operator. Λ and more general fractional Laplacian operators Λ^α with $\alpha\in\mathbb{R}$ are defined via the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi,\eta)=(\xi^2+\eta^2)^{\frac{\alpha}{2}}\widehat{f}(\xi,\eta),$$

where the Fourier transform is defined by

$$\widehat{f}(\xi,\eta)=\int_{\mathbb{R}^2}e^{-ix\xi-iy\eta}f(x,y)dxdy.$$

However, the energy estimate (1.11) does not give us any information on the decay rates of ω or θ, due to the lack of thermal diffusion. The Fourier-splitting method of Schonbek has been very effective on large-time decay problems (see, e.g., [9,47,48]), but it does not apply here when there is no thermal diffusion or damping in θ. Therefore, new ideas and different approaches appear to be necessary in order to handle the large-time behavior problem here.

To establish the long-term stability and large-time behavior of (ω,θ) of (1.10), we take advantage of the special structure of the linear portion in (1.10). Our first step is to represent the solution of the degenerate damped wave equation

$$\begin{cases}\partial_{tt}f+\partial_t f-\partial_{xx}f=F,\\ f(1,x,y)=f_0(x,y),\quad \partial_t f(1,x,y)=f_1(x,y)\end{cases} \tag{1.12}$$

to be the integral form

$$f(t,x,y)=K_0(t,\partial_x)f_0+K_1(t,\partial_x)\left(\frac{1}{2}f_0+f_1\right)+\int_1^t K_1(t-s,\partial_x)F(s,x,y)ds, \tag{1.13}$$

where K_0 and K_1 are Fourier multiplier operations,

$$K_0(t,\partial_x)=\frac{1}{2}\left(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+\partial_x^2}\right)(t-1)}+e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}+\partial_x^2}\right)(t-1)}\right),$$
$$K_1(t,\partial_x)=\frac{1}{2\sqrt{\frac{1}{4}+\partial_x^2}}\left(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+\partial_x^2}\right)(t-1)}-e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}+\partial_x^2}\right)(t-1)}\right).$$

A natural second step is to obtain the precise large-time behavior of $\widehat{K}_0$ and $\widehat{K}_1$ and sharp bounds for their action on L^1 and L^2 functions. Once these preparations are made, we then write (1.10) into an integral form and apply the continuity argument. We need a suitable functional setting for the initial data. Let $a\in\mathbb{N}$ be a sufficiently large positive integer, say $a>8$ and define X_0 to be the Sobolev space, equipped with the norm

$$\begin{aligned}\|(\omega_0,\theta_0)\|_{X_0}\triangleq\ &\|\langle\nabla\rangle^a(\omega_0,\theta_0)\|_{L^2}+\|\langle\nabla\rangle^4(\omega_0,\theta_0)\|_{L^1}+\|\langle\nabla\rangle^4(\omega_1,\theta_1)\|_{L^1}\\ &+\|\Lambda^{-1}(\omega_0,\theta_0)\|_{L^1}+\|\Lambda^{-1}(\omega_1,\theta_1)\|_{L^1}+\|\Lambda^{-2}\theta_0\|_{L^1}+\|\Lambda^{-2}\theta_1\|_{L^1},\end{aligned}$$

where (ω_1,θ_1) is given by (1.9) and $\langle\nabla\rangle$ denotes the inhomogeneous derivative,

$$\langle\nabla\rangle \triangleq (I-\Delta)^{\frac{1}{2}}.$$

We set the Banach space X as the working space for the solution (ω,θ) to system (1.5) equipped with the following norm

$$\begin{aligned}\|(\omega,\theta)\|_X = \sup_{t\geq 1}\{&t^{-\varepsilon}\|\langle\nabla\rangle^a\omega,\langle\nabla\rangle^a\theta\|_{L^2}+t^{\frac{1}{4}}\|\langle\nabla\rangle^2\theta\|_{L^2}+t^{\frac{1}{4}}\|\Lambda^{-1}\omega\|_{L^2}\\ &+t^{\frac{1}{4}}\|\langle\nabla\rangle^2\Lambda^{-1}\theta\|_{L^2}+t^{\frac{3}{4}}\|\partial_x\langle\nabla\rangle\omega\|_{L^2}+t^{\frac{3}{4}}\|\partial_x\langle\nabla\rangle^2\theta\|_{L^2}\\ &+t^{\frac{5}{4}}\|\partial_{xx}\langle\nabla\rangle\theta\|_{L^2}+t^{\frac{3}{4}}\|\partial_x\Lambda^{-1}\theta\|_{L^2}+t^{\frac{7}{8}}\|\partial_{xx}\Lambda^{-2}\theta\|_{L^2}\\ &+t^{\frac{5}{4}}\|\partial_t\langle\nabla\rangle^2\omega\|_{L^2}+t^{\frac{5}{4}}\|\partial_t\theta\|_{L^2}+t^{\frac{5}{4}}\|\partial_t\Lambda^{-1}\omega\|_{L^2}\}. \end{aligned}\tag{1.14}$$

The time weights in (1.14) except the first term are based on the decay properties of the kernels K_0 and K_1. With these preparations at our disposal, we can state our main result as follows. For notational convenience, we write $A\lesssim B$ for the statement that $A\leq CB$ for some absolute constant $C>0$.

THEOREM 1.1. *There exists a small number $\epsilon_0>0$ such that, if the initial data (ω_0,θ_0) satisfies*

$$\|(\omega_0,\theta_0)\|_{X_0}\leq\epsilon_0,$$

then there exists a unique global solution (ω,θ) to system (1.5) or (1.10) with

$$(\omega,\theta)\in X, \qquad P\in C([1,\infty);H^a(\mathbb{R}^2)). \tag{1.15}$$

Moreover, the following decay estimates hold:

$$\|\omega(t)\|_{L^\infty_{xy}}\lesssim\epsilon_0 t^{-1}, \quad \|\theta(t)\|_{L^\infty_{xy}}\lesssim\epsilon_0 t^{-\frac{1}{2}}, \quad \|P(t)\|_{L^\infty_{xy}}\lesssim\epsilon_0 t^{-\frac{1}{4}}. \tag{1.16}$$

To prove Theorem 1.1, we make use of the following continuity argument.

LEMMA 1.1. *Assume the initial data $(\omega_0,\theta_0)\in X_0$ and the solution (ω,θ) given by (1.10) fulfills the following condition*

$$\|(\omega,\theta)\|_X\lesssim\|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\omega,\theta)\|_X), \tag{1.17}$$

where $Q(z)\geq Cz^\beta$ for $z\lesssim 1$ and $\beta>1$. Then there exists $\epsilon_0>0$ such that, if

$$\|(\omega_0,\theta_0)\|_{X_0}\lesssim\epsilon_0,$$

then (1.10) has a unique global solution $(\omega,\theta)\in X$ and satisfies

$$\|(\omega,\theta)\|_X\lesssim\epsilon_0.$$

To facilitate the proof of Lemma 1.1, we introduce a working space Y equipped with the norm

$$\begin{aligned}\|(\mathbf{u},\omega,\theta)\|_Y=\|(\omega,\theta)\|_X+\sup_{t\geq 1}\{&t^{\frac{3}{4}}\|\langle\nabla\rangle\omega\|_{L^2}+t^{\frac{3}{4}}\|u_1\|_{L^2}+t^{\frac{7}{8}}\|u_2\|_{L^2}\\ &+t\|\omega\|_{L^\infty}+t^{\frac{5}{4}}\|\partial_x\langle\nabla\rangle\omega\|_{L^2}\}\end{aligned}$$

and verify that the norm $\|(\mathbf{u},\omega,\theta)\|_Y$ is bounded by $\|(\omega,\theta)\|_X$ and $Q(\|(\omega,\theta)\|_X)$ with $Q(z)$ being of the form Cz^β with $\beta>1$, as stated in the following lemma.

LEMMA 1.2. *Let X and Y be the Banach spaces with their norms being defined as the above. Then*

$$\|(\mathbf{u},\omega,\theta)\|_Y \lesssim \|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X). \tag{1.18}$$

As a consequence of Lemma 1.2, to prove (1.17), it suffices to verify

$$\|(\omega,\theta)\|_X \lesssim \|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y). \tag{1.19}$$

So the proof of Theorem 1.1 is reduced to establishing (1.18) in Lemma 1.2 and (1.19).

The Boussinesq equations have attracted considerable interest recently and there are substantial developments. Physically the Boussinesq equations are important models in the study of geophysical fluids and the Rayleigh-Bénard convection (see, e.g., [14, 17, 22, 23, 27, 42, 43, 45, 51]). Mathematically they share many similar properties with the 3D Navier-Stokes and the 3D Euler equations such as the vortex stretching mechanism, which is believed to be the primary reason for any potential finite-time blowup [41]. The significance of the Boussinesq equations has made them the subject of numerous investigations. The global regularity problem and the issue of stability near physically relevant equilibria are among the most prominent topics on the Boussinesq equations, especially when there is only partial or fractional dissipation. Important progress has been made on both topics (see, e.g., [1–5, 8–21, 24–26, 28–40, 44, 46, 49, 50, 52–54, 56–63]). It is hoped that the results presented in this paper will help lead to a better understanding of the hydrostatic equilibrium.

We remark that Wu, Wu and Xu [55] studied the global solution of a magnetohydrodynamic (MHD) system near a background magnetic field and skillfully solved the small data global well-posedness problem there. We take advantage of some of the tools developed in [55]. Although there are resemblances between the linear part of (1.10) and that of the MHD system, there are a few differences between the Boussinesq-Bénard system and the MHD system in [55]. First, the equation of θ in (1.1) involves Δu_2 while the magnetic stream function in [55] depends on u_2. As a consequence, the functional setting for θ involves Sobolev spaces of negative indices, as in the definitions of X_0 and X. Second, ω and θ naturally form a self-contained system, as in (1.10) and it suffices to understand this system of two scalar functions. There are some other differences as well such as the nonlinear terms.

The rest of this paper is divided into five sections. The second section provides the derivation of the integral representation (1.13) for the convenience of the readers. In addition, decay rates of K_0 and K_1 and some estimates of K_0 and K_1 acting on L^1 and L^2 functions are also listed here. The third section verifies (1.18). The fourth section shows that

$$\sup_{t\geq 1} t^{-\varepsilon}\|\langle\nabla\rangle^a(\omega,\theta)\|_{L^2} \lesssim \|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y),$$

which fulfills part of the proof for (1.19). Section 5 serves as a preparation for Section 6. It estimates the nonlinear terms and provides bounds for various Sobolev norms of these terms. Section 6 completes the proof of (1.19) via the integral representation in (1.13) and thus finishes the proof of Theorem 1.1.

2. Integral representation and properties of kernel functions

This section serves as a preparation. We derive the integral representation in (1.13) and list several estimates on the kernel functions K_0 and K_1.

LEMMA 2.1. *The solution of the degenerate damped wave equation*

$$\begin{cases} \partial_{tt} f + \partial_t f - \partial_{xx} f = F, \\ f(1,x,y) = f_0(x,y),\ \partial_t f(1,x,y) = f_1(x,y) \end{cases}$$

can be written in the integral form

$$f(t,x,y) = K_0(t,\partial_x) f_0 + K_1(t,\partial_x)\left(\frac{1}{2} f_0 + f_1\right) + \int_1^t K_1(t-s,\partial_x) F(s,x,y)\,ds,$$

where K_0 and K_1 are Fourier multiplier operators given by

$$\widehat{K_0}(t,\xi) = \frac{1}{2}\left(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)(t-1)} + e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)(t-1)}\right), \tag{2.1}$$

$$\widehat{K_1}(t,\xi) = \frac{1}{2\sqrt{\frac{1}{4}-\xi^2}}\left(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)(t-1)} - e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)(t-1)}\right). \tag{2.2}$$

Proof. We decompose the second-order operator into first-order time operators,

$$\partial_{tt} f + \partial_t f - \partial_{xx} f = \left(\partial_t + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\partial_x^2}\right)\left(\partial_t + \frac{1}{2} + \frac{1}{2}\sqrt{1+4\partial_x^2}\right) f = 0.$$

Therefore $\partial_{tt} f + \partial_t f - \partial_{xx} f = 0$ can be written as

$$\begin{cases} \left(\partial_t + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\partial_x^2}\right) g = 0, \\ \left(\partial_t + \frac{1}{2} + \frac{1}{2}\sqrt{1+4\partial_x^2}\right) f = g \end{cases}$$

or

$$\begin{cases} \left(\partial_t + \frac{1}{2} + \frac{1}{2}\sqrt{1+4\partial_x^2}\right) h = 0, \\ \left(\partial_t + \frac{1}{2} - \frac{1}{2}\sqrt{1+4\partial_x^2}\right) f = h. \end{cases}$$

Clearly,

$$g(t,x,y) = e^{\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\partial_x^2}\right)(t-1)} g(1,x,y),$$
$$h(t,x,y) = e^{\left(-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\partial_x^2}\right)(t-1)} h(1,x,y).$$

Therefore,

$$\begin{aligned} f(t,x,y) &= \frac{1}{\sqrt{1+4\partial_x^2}}(g(t,x,y) - h(t,x,y)) \\ &= \frac{1}{\sqrt{1+4\partial_x^2}}\left(e^{\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\partial_x^2}\right)(t-1)} g(1,x,y) - e^{\left(-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\partial_x^2}\right)(t-1)} h(1,x,y)\right) \end{aligned}$$

with

$$g(1,x,y)=f_1+\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+4\partial_x^2}\right)f_0, \quad h(1,x,y)=f_1+\left(\frac{1}{2}-\frac{1}{2}\sqrt{1+4\partial_x^2}\right)f_0.$$

Regrouping the terms in the representation of f above yields the desired formula for f. This completes the proof of Lemma 2.1. ∎

The integral representation stated in Lemma 2.1 will be applied to (1.10) and also to the system for $(\Lambda^{-1}\omega,\Lambda^{-1}\theta)$, which satisfies

$$\begin{cases} \partial_{tt}\Lambda^{-1}\omega+\partial_t\Lambda^{-1}\omega-\partial_{xx}\Lambda^{-1}\omega=F_3, \\ \partial_{tt}\Lambda^{-1}\theta+\partial_t\Lambda^{-1}\theta-\partial_{xx}\Lambda^{-1}\theta=F_4, \end{cases} \tag{2.3}$$

where

$$\begin{cases} F_3=-\partial_x\Lambda^{-1}\nabla\cdot(\mathbf{u}\theta)-\partial_t\Lambda^{-1}\nabla\cdot(\mathbf{u}\omega), \\ F_4=-\partial_x\Lambda^{-1}\nabla\cdot(\mathbf{u}\omega)-\partial_t\Lambda^{-1}\nabla\cdot(\mathbf{u}\theta)-\Lambda^{-1}\nabla\cdot(\mathbf{u}\theta). \end{cases} \tag{2.4}$$

As a preparation for the estimates in the subsequent sections, the following lemma provides some decay estimates on K_0 and K_1. These estimates are derived in [55].

LEMMA 2.2. *Let K_0,K_1 be defined as above. Then for any $\alpha>0$, $1\le q\le\infty$, $i=0,1$,*

$$\begin{aligned} &(1) \quad \||\xi|^\alpha\widehat{K}_i(t,\cdot)\|_{L^q_\xi(|\xi|\le\frac{1}{2})}\lesssim\langle t\rangle^{-\frac{1}{2}(\frac{1}{q}+\alpha)}, \\ &(2) \quad \|\partial_t\widehat{K}_i(t,\cdot)\|_{L^q_\xi(|\xi|\le\frac{1}{2})}\lesssim\langle t\rangle^{-1-\frac{1}{2q}}, \\ &(3) \quad |\widehat{K}_i(t,\xi)|\lesssim e^{-\frac{1}{2}t}, \quad \textit{for any } |\xi|\ge\frac{1}{2}, \\ &(4) \quad |\langle\xi\rangle^{-1}\partial_t\widehat{K}_0(t,\xi)|,\ |\partial_t\widehat{K_1}(\xi)|\lesssim e^{-\frac{1}{2}t}, \quad \textit{for any } |\xi|\ge\frac{1}{2}. \end{aligned} \tag{2.5}$$

We list several bounds on K_0 and K_1 when acting on L^1 and L^2 functions. These bounds can be found in [55].

LEMMA 2.3. *Assume that $\|\widehat{K}(t,\cdot)\|_{L^\infty}$ is bounded. Then, for any Schwartz function f, we have*

$$\|K(t,\partial_x)f\|_{L^2_{xy}}\lesssim\|\widehat{K}(t,\xi)\|_{L^\infty_\xi}\|f\|_{L^2_{xy}}.$$

LEMMA 2.4. *Assume that $\|\widehat{K}(t,\cdot)\|_{L^2}$ is bounded. Then, for any Schwartz function f, we have*

$$\|K(t,\partial_x)f\|_{L^2_{xy}}\lesssim\|\widehat{K}(t,\xi)\|_{L^2_\xi}\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{2\epsilon}f\|_{L^1_{xy}}.$$

LEMMA 2.5. *Assume that $\|\widehat{K}(t,\cdot)\|_{L^\infty}$ is bounded. Then, for any Schwartz function f, we have*

$$\|K(t,\partial_x)f\|_{L^2_{xy}}\lesssim\|\widehat{K}(t,\xi)\|_{L^\infty_\xi}\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{1}{2}+2\epsilon}f\|_{L^1_{xy}}.$$

As a consequence of Lemma 2.4 and Lemma 2.5, we have the following corollary.

COROLLARY 2.1. *Assume the Fourier multiplier operator $K(t,\partial_x)$ satisfies, for some $\alpha \geq 0$,*

$$\|\widehat{\partial_x^\alpha K}(t,\xi)\|_{L^2(|\xi|\leq \frac{1}{2})} < \infty, \quad \|\widehat{K}(t,\xi)\|_{L^\infty(|\xi|\geq \frac{1}{2})} < \infty.$$

Then, for any Schwartz function f and any $\epsilon > 0$,

$$\|\partial_x^\alpha K(t,\partial_x) f\|_{L^2_{xy}} \lesssim (\|\widehat{\partial_x^\alpha K}(t,\xi)\|_{L^2_\xi(|\xi|\leq \frac{1}{2})} + \|\widehat{K}(t,\xi)\|_{L^\infty_\xi(|\xi|\geq \frac{1}{2})}) \times \|\langle\nabla\rangle^{\alpha+\frac{1}{2}+2\epsilon}\Lambda^{\frac{1}{2}-\epsilon} f\|_{L^1_{xy}}.$$

3. Verifying Lemma 1.2

This section and the subsequent three sections are devoted to the proof of Theorem 1.1. As aforementioned, the proof of Theorem 1.1 is reduced to the proof of (1.18) in Lemma 1.2 and (1.19). This section verifies (1.18), namely

$$\|(\mathbf{u},\omega,\theta)\|_Y \lesssim \|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X).$$

Due to the definitions of X and Y, it suffices to check that

$$\sup_{t\geq 1}\{t^{\frac{3}{4}}\|\langle\nabla\rangle\omega\|_{L^2} + t^{\frac{3}{4}}\|u_1\|_{L^2} + t^{\frac{7}{8}}\|u_2\|_{L^2} + t\|\omega\|_{L^\infty} + t^{\frac{5}{4}}\|\partial_x\langle\nabla\rangle\omega\|_{L^2}\} \lesssim \|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X). \tag{3.1}$$

We start with the first term in (3.1). Noticing that $\mathbf{u} = \nabla^\perp\Delta^{-1}\omega$, we have

$$\|\mathbf{u}\|_{L^2} = \|\nabla^\perp\Delta^{-1}\omega\|_{L^2} = \|\Lambda^{-1}\omega\|_{L^2}.$$

It follows from the basic inequality

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\|f'\|^{\frac{1}{2}}_{L^2(\mathbb{R})}$$

that

$$\begin{aligned}\|\mathbf{u}\|_{L^\infty(\mathbb{R}^2)} &\lesssim \|\mathbf{u}\|^{\frac{1}{4}}_{L^2}\|\partial_x\mathbf{u}\|^{\frac{1}{4}}_{L^2}\|\partial_y\mathbf{u}\|^{\frac{1}{4}}_{L^2}\|\partial_{xy}\mathbf{u}\|^{\frac{1}{4}}_{L^2}\\ &\lesssim \|\Lambda^{-1}\omega\|^{\frac{1}{4}}_{L^2}\|\omega\|^{\frac{1}{2}}_{L^2}\|\partial_x\omega\|^{\frac{1}{4}}_{L^2}.\end{aligned}$$

where L^∞ and L^2 are abbreviated for $L^\infty(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$, respectively. Unless otherwise stated, this convention is assumed throughout the rest of the paper. By an interpolation inequality and Young's inequality,

$$\begin{aligned}\|\omega\|_{L^2} &\leq \|\partial_t\omega\|_{L^2} + \|\mathbf{u}\cdot\nabla\omega\|_{L^2} + \|\partial_x\theta\|_{L^2}\\ &\leq \|\partial_t\omega\|_{L^2} + \|\mathbf{u}\|_{L^\infty}\|\nabla\omega\|_{L^2} + \|\partial_x\theta\|_{L^2}\\ &\leq \|\partial_t\omega\|_{L^2} + \|\Lambda^{-1}\omega\|^{\frac{1}{4}}_{L^2}\|\omega\|^{\frac{1}{2}}_{L^2}\|\partial_x\omega\|^{\frac{1}{4}}_{L^2}\|\omega\|^{1-\frac{1}{a}}_{L^2}\|\nabla^a\omega\|^{\frac{1}{a}}_{L^2} + \|\partial_x\theta\|_{L^2}\\ &\leq \|\partial_t\omega\|_{L^2} + \|\Lambda^{-1}\omega\|^{\frac{1}{4}}_{L^2}\|\omega\|^{\frac{7}{8}}_{L^2}\|\partial_x\omega\|^{\frac{1}{4}}_{L^2}\|\omega\|^{\frac{5}{8}-\frac{1}{a}}_{L^2}\|\nabla^a\omega\|^{\frac{1}{a}}_{L^2} + \|\partial_x\theta\|_{L^2}\\ &\leq \|\partial_t\omega\|_{L^2} + \frac{1}{2}\|\omega\|_{L^2} + \|\Lambda^{-1}\omega\|^2_{L^2}\|\partial_x\omega\|^2_{L^2}\|\langle\nabla\rangle^a\omega\|^5_{L^2} + \|\partial_x\theta\|_{L^2}.\end{aligned}$$

By the definition of X,

$$\begin{aligned}\|\omega\|_{L^2} &\lesssim \|\partial_t\omega\|_{L^2} + \|\Lambda^{-1}\omega\|^2_{L^2}\|\partial_x\omega\|^2_{L^2}\|\langle\nabla\rangle^a\omega\|^5_{L^2} + \|\partial_x\theta\|_{L^2}\\ &\lesssim s^{-\frac{5}{4}}\|(\omega,\theta)\|_X + s^{(-\frac{1}{4}\times 2-\frac{3}{4}\times 2+5\varepsilon)}\|(\omega,\theta)\|^7_X + s^{-\frac{3}{4}}\|(\omega,\theta)\|_X\\ &\lesssim s^{-\frac{3}{4}}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X)),\end{aligned}$$

where $-\frac{1}{2}-\frac{3}{2}+5\varepsilon\le-\frac{3}{4}$ and $Q(b)=b^7$. Therefore,

$$\begin{aligned}
\|\omega\|_{L^\infty} &\le \|\omega\|_{L^2}^{\frac14}\|\partial_x\omega\|_{L^2}^{\frac14}\|\partial_y\omega\|_{L^2}^{\frac14}\|\partial_{xy}\omega\|_{L^2}^{\frac14}\\
&\le \|\omega\|_{L^2}^{\frac14}\|\nabla\omega\|_{L^2}^{\frac12}\|\nabla^2\omega\|_{L^2}^{\frac14}\\
&\le \|\omega\|_{L^2}^{\frac14}\|\omega\|_{L^2}^{\frac12(1-\frac1a)}\|\nabla^a\omega\|_{L^2}^{\frac{1}{2a}}\|\omega\|_{L^2}^{\frac14(1-\frac2a)}\|\nabla^a\omega\|_{L^2}^{\frac{1}{2a}}\\
&\le \|\omega\|_{L^2}^{1-\frac1a}\|\nabla^a\omega\|_{L^2}^{\frac1a}\\
&\lesssim s^{-\frac34(1-\frac1a)}s^{\varepsilon\frac1a}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X))\\
&\lesssim s^{-\frac58}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)),
\end{aligned}$$

where a and ε satisfy $-\frac18+\frac1a(\frac34+\varepsilon)\le 0$. This is a preliminary estimate for $\|\omega\|_{L^\infty}$ and a better decay rate will be provided later. By an interpolation inequality and the Sobolev embedding,

$$\begin{aligned}
\|\nabla\omega\|_{L^\infty} &\lesssim \|\omega\|_{L^\infty}^{1-\frac{1}{a-2}}\|\nabla^{a-2}\omega\|_{L^\infty}^{\frac{1}{a-2}}\\
&\lesssim \|\omega\|_{L^\infty}^{1-\frac{1}{a-2}}\|\langle\nabla\rangle^{a}\omega\|_{L^\infty}^{\frac{1}{a-2}}\\
&\lesssim s^{-\frac58(1-\frac{1}{a-2})+\varepsilon\frac{1}{a-2}}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X))\\
&\lesssim s^{-\frac12}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)),
\end{aligned}\tag{3.2}$$

when $-\frac58(1-\frac{1}{a-2})+\varepsilon\frac{1}{a-2}\le-\frac12$. Similarly,

$$\|\nabla^2\omega\|_{L^\infty}\lesssim s^{-\frac12}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).$$

To bound $\|\partial_x\omega\|_{L^2}$, we apply ∂_x to the vorticity equation in (1.5) to get

$$\partial_t\partial_x\omega+\partial_x\mathbf{u}\cdot\nabla\omega+\mathbf{u}\cdot\nabla\partial_x\omega+\partial_x\omega=\partial_{xx}\theta.$$

Therefore,

$$\begin{aligned}
\|\partial_x\omega\|_{L^2} &\le \|\partial_t\partial_x\omega\|_{L^2}+\|\partial_x\mathbf{u}\cdot\nabla\omega\|_{L^2}+\|\mathbf{u}\cdot\nabla\partial_x\omega\|_{L^2}+\|\partial_{xx}\theta\|_{L^2}\\
&\le \|\partial_t\partial_x\omega\|_{L^2}+\|\partial_x\mathbf{u}\|_{L^2}\|\nabla\omega\|_{L^\infty}+\|\mathbf{u}\|_{L^\infty}\|\partial_x\nabla\omega\|_{L^2}+\|\partial_{xx}\theta\|_{L^2}\\
&\le \|\partial_t\partial_x\omega\|_{L^2}+\|\omega\|_{L^2}\|\nabla\omega\|_{L^\infty}+\|\Lambda^{-1}\omega\|_{L^2}^{\frac14}\|\omega\|_{L^2}^{\frac12}\|\partial_x\omega\|_{L^2}^{\frac14}\|\omega\|_{L^2}^{1-\frac2a}\|\nabla^a\omega\|_{L^2}^{\frac2a}\\
&\quad+\|\partial_{xx}\theta\|_{L^2}\\
&\le \|\partial_t\partial_x\omega\|_{L^2}+\|\omega\|_{L^2}\|\nabla\omega\|_{L^\infty}+\|\Lambda^{-1}\omega\|_{L^2}^{\frac13}\|\omega\|_{L^2}^{\frac23}\|\omega\|_{L^2}^{\frac43-\frac{8}{3a}}\|\langle\nabla\rangle^a\omega\|_{L^2}^{\frac{8}{3a}}\\
&\quad+\frac12\|\partial_x\omega\|_{L^2}+\|\partial_{xx}\theta\|_{L^2}.
\end{aligned}$$

Then, by the definition of $\|(\omega,\theta)\|_X$, we get

$$\begin{aligned}
&\|\partial_x\omega\|_{L^2}\\
&\lesssim \|\partial_t\nabla\omega\|_{L^2}+\|\omega\|_{L^2}\|\nabla\omega\|_{L^\infty}+\|\Lambda^{-1}\omega\|_{L^2}^{\frac13}\|\omega\|_{L^2}^{2-\frac{8}{3a}}\|\langle\nabla\rangle^a\omega\|_{L^2}^{\frac{8}{3a}}+\|\partial_{xx}\theta\|_{L^2}\\
&\lesssim (s^{-\frac54}+s^{-\frac34-\frac12}+s^{-\frac{1}{12}-\frac54}+s^{-\frac54})(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X))\\
&\lesssim s^{-\frac54}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).
\end{aligned}$$

Applying ∇ to the vorticity equation in (1.5), we obtain

$$\partial_t \nabla\omega + \nabla\mathbf{u}\cdot\nabla\omega + \mathbf{u}\cdot\nabla^2\omega + \nabla\omega = \partial_x\nabla\theta.$$

By Young's inequality,

$$\begin{aligned}
\|\nabla\omega\|_{L^2} &\le \|\partial_t\nabla\omega\|_{L^2} + \|\nabla\mathbf{u}\cdot\nabla\omega\|_{L^2} + \|\mathbf{u}\cdot\nabla^2\omega\|_{L^2} + \|\partial_x\nabla\theta\|_{L^2}\\
&\le \|\partial_t\nabla\omega\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}\|\nabla\omega\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty}\|\nabla^2\omega\|_{L^2} + \|\partial_x\nabla\theta\|_{L^2}\\
&\le \|\partial_t\nabla\omega\|_{L^2} + \|\omega\|_{L^2}\|\nabla\omega\|_{L^\infty} + \|\Lambda^{-1}\omega\|_{L^2}^{\frac14}\|\omega\|_{L^2}^{\frac12}\|\partial_x\omega\|_{L^2}^{\frac14}\|\omega\|_{L^2}^{1-\frac{2}{a}}\|\nabla^a\omega\|_{L^2}^{\frac{2}{a}}\\
&\quad + \|\partial_x\nabla\theta\|_{L^2}\\
&\le \|\partial_t\nabla\omega\|_{L^2} + \|\omega\|_{L^2}\|\nabla\omega\|_{L^\infty} + \|\Lambda^{-1}\omega\|_{L^2}^{\frac13}\|\omega\|_{L^2}^{\frac23}\|\omega\|_{L^2}^{\frac43-\frac{8}{3a}}\|\langle\nabla\rangle^a\omega\|_{L^2}^{\frac{8}{3a}}\\
&\quad + \frac12\|\nabla\omega\|_{L^2} + \|\partial_x\nabla\theta\|_{L^2}.
\end{aligned}$$

Then by the definition of $\|(\omega,\theta)\|_X$, we obtain

$$\begin{aligned}
&\|\nabla\omega\|_{L^2}\\
&\lesssim \|\partial_t\nabla\omega\|_{L^2} + \|\omega\|_{L^2}\|\nabla\omega\|_{L^\infty} + \|\Lambda^{-1}\omega\|_{L^2}^{\frac13}\|\omega\|_{L^2}^{2-\frac{8}{3a}}\|\langle\nabla\rangle^a\omega\|_{L^2}^{\frac{8}{3a}} + \|\partial_x\nabla\theta\|_{L^2}\\
&\lesssim (s^{-\frac54} + s^{-\frac34-\frac12} + s^{-\frac{1}{12}-\frac54} + s^{-\frac34})(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X))\\
&\lesssim s^{-\frac34}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X)).
\end{aligned} \tag{3.3}$$

Next we deal with the term $\|\partial_x\nabla\omega\|_{L^2}$. Applying $\partial_x\nabla$ to the vorticity equation in (1.5), we have

$$\partial_t\partial_x\nabla\omega + \partial_x\nabla\mathbf{u}\cdot\nabla\omega + \nabla\mathbf{u}\cdot\nabla\partial_x\omega + \partial_x\mathbf{u}\cdot\nabla^2\omega + \mathbf{u}\cdot\nabla^2\partial_x\omega + \partial_x\nabla\omega = \partial_{xx}\nabla\theta.$$

Taking the L^2-norm yields

$$\begin{aligned}
\|\partial_x\nabla\omega\|_{L^2} \le{}& \|\partial_t\partial_x\nabla\omega\|_{L^2} + \|\partial_x\nabla\mathbf{u}\cdot\nabla\omega\|_{L^2} + \|\nabla\mathbf{u}\cdot\nabla\partial_x\omega\|_{L^2} + \|\partial_x\mathbf{u}\cdot\nabla^2\omega\|_{L^2}\\
&+ \|\mathbf{u}\cdot\nabla^2\partial_x\omega\|_{L^2} + \|\partial_{xx}\nabla\theta\|_{L^2}.
\end{aligned} \tag{3.4}$$

By the definition of $\|(\omega,\theta)\|_X$, we have

$$\|\partial_t\partial_x\nabla\omega\|_{L^2} \lesssim \|\partial_t\nabla^2\omega\|_{L^2} \lesssim s^{-\frac54}\|(\omega,\theta)\|_X, \quad \|\partial_{xx}\nabla\theta\|_{L^2} \lesssim s^{-\frac54}\|(\omega,\theta)\|_X,$$

$$\begin{aligned}
\|\partial_x\nabla\mathbf{u}\cdot\nabla\omega\|_{L^2} &\lesssim \|\partial_x\omega\|_{L^2}\|\nabla\omega\|_{L^\infty}\\
&\lesssim s^{-\frac54-\frac12}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X))\\
&\lesssim s^{-\frac74}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X)),
\end{aligned}$$

$$\begin{aligned}
\|\nabla\mathbf{u}\cdot\nabla\partial_x\omega\|_{L^2} &\lesssim \|\omega\|_{L^2}\|\nabla^2\omega\|_{L^\infty}\\
&\lesssim s^{-\frac34-\frac12}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X))\\
&\lesssim s^{-\frac54}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X)),
\end{aligned}$$

$$\begin{aligned}
\|\partial_x\mathbf{u}\cdot\nabla^2\omega\|_{L^2} &\lesssim \|\omega\|_{L^2}\|\nabla^2\omega\|_{L^\infty}\\
&\lesssim s^{-\frac34-\frac12}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X))\\
&\lesssim s^{-\frac54}(\|(\omega,\theta)\|_X + Q(\|(\omega,\theta)\|_X))
\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{u}\cdot\nabla^2\partial_x\omega\|_{L^2} &\lesssim \|\mathbf{u}\|_{L^\infty}\|\partial_x\nabla^2\omega\|_{L^2}\\ &\lesssim \|\Lambda^{-1}\omega\|_{L^2}^{\frac14}\|\omega\|_{L^2}^{\frac12}\|\partial_x\omega\|_{L^2}^{\frac14}\|\partial_x\omega\|_{L^2}^{\frac{a-3}{a-1}}\|\partial_x\langle\nabla\rangle^{a-1}\omega\|_{L^2}^{\frac{a-3}{a-1}}\\ &\lesssim s^{-\frac54}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).\end{aligned}$$

Inserting these estimates in (3.4) leads to

$$\|\partial_x\nabla\omega\|_{L^2}\lesssim s^{-\frac54}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).$$

Using the estimate obtained above, we can calculate the decay estimate of $\|\omega\|_{L^\infty}$ again with a better decay rate,

$$\begin{aligned}\|\omega\|_{L^\infty} &\le \|\omega\|_{L^2}^{\frac14}\|\partial_x\omega\|_{L^2}^{\frac14}\|\partial_y\omega\|_{L^2}^{\frac14}\|\partial_{xy}\omega\|_{L^2}^{\frac14}\\ &\lesssim s^{\frac14\times(-\frac34-\frac54-\frac34-\frac54)}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X))\\ &\lesssim s^{-1}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).\end{aligned}\tag{3.5}$$

In order to obtain the decay estimate of $\|\mathbf{u}\|_{L^2}$, we make use of the structure of system (1.5) and the boundedness of Riesz transforms. Applying Λ^{-1} to the first equation of (1.5), we have

$$\partial_t\Lambda^{-1}\omega+\Lambda^{-1}\nabla\cdot(\mathbf{u}\omega)+\Lambda^{-1}\omega=\partial_x\Lambda^{-1}\theta.\tag{3.6}$$

By the boundedness of the Riesz transform in L^2, we have

$$\begin{aligned}\|\Lambda^{-1}\omega\|_{L^2} &\le \|\partial_t\Lambda^{-1}\omega\|_{L^2}+\|\Lambda^{-1}\nabla\cdot(\mathbf{u}\omega)\|_{L^2}+\|\partial_x\Lambda^{-1}\theta\|_{L^2}\\ &\le \|\partial_t\Lambda^{-1}\omega\|_{L^2}+C\|\mathbf{u}\omega\|_{L^2}+\|\partial_x\Lambda^{-1}\theta\|_{L^2}\\ &\le \|\partial_t\Lambda^{-1}\omega\|_{L^2}+C\|\Lambda^{-1}\omega\|_{L^2}\|\omega\|_{L^\infty}+\|\partial_x\Lambda^{-1}\theta\|_{L^2}\\ &\le \|\partial_t\Lambda^{-1}\omega\|_{L^2}+C\|\Lambda^{-1}\omega\|_{L^2}\|\omega\|_{L^\infty}+\|\partial_x\Lambda^{-1}\theta\|_{L^2}\\ &\lesssim (s^{-\frac54}+s^{-\frac14-1}+s^{-\frac34})(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X))\\ &\lesssim s^{-\frac34}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).\end{aligned}$$

Therefore,

$$\|\mathbf{u}\|_{L^2}=\|\nabla^\perp\Delta^{-1}\omega\|_{L^2}=\|\Lambda^{-1}\omega\|_{L^2}\lesssim s^{-\frac34}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).$$

Noticing that $u_2=\partial_x\Delta^{-1}\omega$ and using the first equation of (1.5), we have

$$u_2=-\partial_t\partial_x\Lambda^{-2}\omega+\partial_x\Lambda^{-2}(\mathbf{u}\cdot\nabla\omega)+\partial_{xx}\Lambda^{-2}\theta.$$

According to the boundedness of Riesz transform, divergence-free condition of $\mathbf{u}$ and Hölder inequality, we obtain

$$\begin{aligned}\|u_2\|_{L^2} &\lesssim \|\partial_t\Lambda^{-1}\omega\|_{L^2}+\|\mathbf{u}\omega\|_{L^2}+\|\partial_{xx}\Lambda^{-2}\theta\|_{L^2}\\ &\lesssim \|\partial_t\Lambda^{-1}\omega\|_{L^2}+\|\mathbf{u}\|_{L^\infty}\|\omega\|_{L^2}+\|\partial_{xx}\Lambda^{-2}\theta\|_{L^2}\\ &\lesssim s^{-\frac78}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).\end{aligned}$$

A special consequence of the estimates above is the following bound for $\|\mathbf{u}\|_{L^\infty}$, which will be used in a subsequent section.

$$\begin{aligned}\|\mathbf{u}\|_{L^\infty} &\lesssim \|\mathbf{u}\|_{L^2}^{\frac14}\|\partial_x\mathbf{u}\|_{L^2}^{\frac14}\|\partial_y\mathbf{u}\|_{L^2}^{\frac14}\|\partial_{xy}\mathbf{u}\|_{L^2}^{\frac14}\\ &\lesssim s^{-\frac14(\frac34+\frac34+\frac34+\frac54)}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X))\\ &= s^{-\frac78}(\|(\omega,\theta)\|_X+Q(\|(\omega,\theta)\|_X)).\end{aligned} \tag{3.7}$$

Thus we have verified (3.1).

4. Estimate for $\|\langle\nabla\rangle^a(\omega,\theta)\|_{L^2}$

This section and the rest two sections continue the proof of Theorem 1.1. The goal is to prove

$$\|(\omega,\theta)\|_X \lesssim \|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

This section verifies

$$\sup_{t\ge1} t^{-\varepsilon}\|\langle\nabla\rangle^a(\omega,\theta)\|_{L^2}\lesssim \|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

To do so, we first derive the L^2 bound of (ω,θ). Taking the L^2-inner product of the first two equations of (1.5) with $(\omega,\ \theta)$ yields

$$\frac12\frac{d}{dt}(\|\omega(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2)+\|\omega\|_{L^2}^2=\int_{\mathbb{R}^2}\partial_x\theta\omega+\partial_x\omega\theta\ dxdy=0.$$

Integrating from 1 to t with respect to the time variable leads to

$$\sup_{t\ge1}t^{-\varepsilon}\|(\omega,\theta)\|_{L^2}^2\lesssim\|(\omega_0,\theta_0)\|_{L^2}^2\lesssim\|(\omega_0,\theta_0)\|_{X_0}. \tag{4.1}$$

Next we establish the high-order energy estimate. Applying ∇^a to (1.5), we have

$$\begin{cases}\partial_t\nabla^a\omega+[\nabla^a,\mathbf{u}\cdot\nabla]\omega+\mathbf{u}\cdot\nabla\nabla^a\omega+\nabla^a\omega=\partial_x\nabla^a\theta,\\ \partial_t\nabla^a\theta+[\nabla^a,\mathbf{u}\cdot\nabla]\theta+\mathbf{u}\cdot\nabla\nabla^a\theta=\partial_x\nabla^a\omega,\end{cases} \tag{4.2}$$

where we have used the commutator notation

$$[f,g]=fg-gf.$$

Taking the L^2-inner product of (4.2) with $(\nabla^a\omega,\nabla^a\theta)$ yields

$$\begin{aligned}&\frac12\frac{d}{dt}(\|\nabla^a\omega\|_{L^2}^2+\|\nabla^a\theta\|_{L^2}^2)+\|\nabla^a\omega\|_{L^2}^2\\ &=-\int_{\mathbb{R}^2}[\nabla^a,\mathbf{u}\cdot\nabla]\omega\cdot\nabla^a\omega\ dxdy-[\nabla^a,\mathbf{u}\cdot\nabla]\theta\cdot\nabla^a\theta\ dxdy\\ &\lesssim\|[\nabla^a,\mathbf{u}\cdot\nabla]\omega\|_{L^2}\|\nabla^a\omega\|_{L^2}+\|[\nabla^a,\mathbf{u}\cdot\nabla]\theta\|_{L^2}\|\nabla^a\theta\|_{L^2}\\ &\lesssim(\|\nabla^a\mathbf{u}\|_{L^2}\|\nabla\omega\|_{L^\infty}+\|\nabla^a\omega\|_{L^2}\|\nabla\mathbf{u}\|_{L^\infty})\|\nabla^a\omega\|_{L^2}\\ &\quad+(\|\nabla^a\mathbf{u}\|_{L^2}\|\nabla\theta\|_{L^\infty}+\|\nabla^a\theta\|_{L^2}\|\nabla\mathbf{u}\|_{L^\infty})\|\nabla^a\theta\|_{L^2}.\\ &\lesssim\|\nabla^a\omega\|_{L^2}^{\frac{a-1}{a}}\|\omega\|_{L^2}^{\frac1a}(\|\nabla\omega\|_{L^\infty}\|\nabla^a\omega\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\nabla^a\theta\|_{L^2})\\ &\quad+\frac14\|\nabla^a\omega\|_{L^2}^2+(\|\nabla\mathbf{u}\|_{L^\infty}^2+\|\nabla\mathbf{u}\|_{L^\infty})(\|\nabla^a\omega\|_{L^2}^2+\|\nabla^a\theta\|_{L^2}^2)\\ &\lesssim\|\omega\|_{L^2}^{\frac{2}{a+1}}(\|\nabla\omega\|_{L^\infty}\|\nabla^a\omega\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\nabla^a\theta\|_{L^2})^{\frac{2a}{a+1}}\\ &\quad+\frac12\|\nabla^a\omega\|_{L^2}^2+(\|\nabla\mathbf{u}\|_{L^\infty}^2+\|\nabla\mathbf{u}\|_{L^\infty})(\|\nabla^a\omega\|_{L^2}^2+\|\nabla^a\theta\|_{L^2}^2).\end{aligned}$$

Here we have used the standard commutator estimate

$$\|[\nabla^a, f\cdot\nabla]g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty}\|\nabla^a g\|_{L^2} + \|\nabla g\|_{L^\infty}\|\nabla^a f\|_{L^2}$$

and the interpolation formula

$$\|\nabla^a \mathbf{u}\|_{L^2} \approx \|\nabla^{a-1}\omega\|_{L^2} \lesssim \|\nabla^a\omega\|_{L^2}^{1-\frac{1}{a}}\|\omega\|_{L^2}^{\frac{1}{a}}.$$

Thus

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla^a\omega\|_{L^2}^2 + \|\nabla^a\theta\|_{L^2}^2) \lesssim{}& \|\omega\|_{L^2}^{\frac{2}{a+1}}(\|\nabla\omega\|_{L^\infty}\|\nabla^a\omega\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\nabla^a\theta\|_{L^2})^{\frac{2a}{a+1}}\\ &+ (\|\nabla\mathbf{u}\|_{L^\infty}^2 + \|\nabla\mathbf{u}\|_{L^\infty})(\|\nabla^a\omega\|_{L^2}^2 + \|\nabla^a\theta\|_{L^2}^2).\end{aligned}$$

Integrating from 1 to t with respect to the time variable, we have

$$\begin{aligned}&\|\nabla^a\omega\|_{L^2}^2 + \|\nabla^a\theta\|_{L^2}^2\\ \lesssim{}& \|\nabla^a\omega_0\|_{L^2}^2 + \|\nabla^a\theta_0\|_{L^2}^2 + \int_1^t (\|\nabla\mathbf{u}\|_{L^\infty}^2 + \|\nabla\mathbf{u}\|_{L^\infty})(\|\nabla^a\omega\|_{L^2}^2 + \|\nabla^a\theta\|_{L^2}^2)\, ds\\ &+ \int_1^t \|\omega\|_{L^2}^{\frac{2}{a+1}}(\|\nabla\omega\|_{L^\infty}\|\nabla^a\omega\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\nabla^a\theta\|_{L^2})^{\frac{2a}{a+1}}\, ds.\end{aligned}$$

By the definition of $\|(\mathbf{u},\omega,\theta)\|_Y$,

$$\begin{aligned}\|\nabla\theta\|_{L^\infty} &\lesssim \|\nabla\theta\|_{L^2}^{\frac{1}{4}}\|\partial_x\nabla\theta\|_{L^2}^{\frac{1}{4}}\|\partial_y\nabla\theta\|_{L^2}^{\frac{1}{4}}\|\partial_{xy}\nabla\theta\|_{L^2}^{\frac{1}{4}}\\ &\lesssim \|\langle\nabla\rangle^2\theta\|_{L^2}^{\frac{1}{2}}\|\partial_x\langle\nabla\rangle^2\theta\|_{L^2}^{\frac{1}{2}}\\ &\lesssim s^{-\frac{1}{4}\times\frac{1}{2}-\frac{3}{4}\times\frac{1}{2}}\|(u,\omega,\theta)\|_Y\\ &\lesssim s^{-\frac{1}{2}}\|(\mathbf{u},\omega,\theta)\|_Y.\end{aligned}\tag{4.3}$$

Invoking the estimates in (3.2), (3.3) and (4.3), we find

$$\begin{aligned}&\int_1^t \|\omega\|_{L^2}^{\frac{2}{a+1}}(\|\nabla\omega\|_{L^\infty}\|\nabla^a\omega\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\nabla^a\theta\|_{L^2})^{\frac{2a}{a+1}}\, ds\\ \lesssim{}& \int_1^t s^{-\frac{3}{2(a+1)}}(s^{-1+\varepsilon} + s^{-\frac{1}{2}+\varepsilon})^{\frac{2a}{a+1}} ds(\|(\mathbf{u},\omega,\theta)\|_Y^{4-\frac{2}{a+1}} + Q(\|(\mathbf{u},\omega,\theta)\|_Y))\\ \lesssim{}& \int_1^t s^{-1+2\varepsilon} ds(\|(\mathbf{u},\omega,\theta)\|_Y^{4-\frac{2}{a+1}} + Q(\|(\mathbf{u},\omega,\theta)\|_Y))\\ \lesssim{}& t^{2\varepsilon}(\|(\mathbf{u},\omega,\theta)\|_Y^{4-\frac{2}{a+1}} + Q(\|(\mathbf{u},\omega,\theta)\|_Y)).\end{aligned}\tag{4.4}$$

As for the term $\|\nabla\mathbf{u}\|_{L^\infty}$, the definition of $\|(\mathbf{u},\omega,\theta)\|_Y$ yields that

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^\infty} &\lesssim \|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{4}}\|\partial_x\nabla\mathbf{u}\|_{L^2}^{\frac{1}{4}}\|\partial_y\nabla\mathbf{u}\|_{L^2}^{\frac{1}{4}}\|\partial_{xy}\nabla\mathbf{u}\|_{L^2}^{\frac{1}{4}}\\ &\lesssim \|\omega\|_{L^2}^{\frac{1}{4}}\|\partial_x\omega\|_{L^2}^{\frac{1}{4}}\|\nabla\omega\|_{L^2}^{\frac{1}{4}}\|\partial_x\nabla\omega\|_{L^2}^{\frac{1}{4}}\\ &\lesssim s^{\frac{1}{4}\times(-\frac{3}{4}-\frac{5}{4}-\frac{3}{4}-\frac{5}{4})}\|(\mathbf{u},\omega,\theta)\|_Y\\ &\lesssim s^{-1}\|(\mathbf{u},\omega,\theta)\|_Y.\end{aligned}$$

Thus

$$\int_1^t (\|\nabla \mathbf{u}\|_{L^\infty}^2 + \|\nabla \mathbf{u}\|_{L^\infty})(\|\nabla^a \omega\|_{L^2}^2 + \|\nabla^a \theta\|_{L^2}^2)\, ds$$
$$\lesssim t^{2\varepsilon}(\|(\mathbf{u},\omega,\theta)\|_Y^3 + \|(\mathbf{u},\omega,\theta)\|_Y^4). \tag{4.5}$$

Therefore

$$\|\nabla^a \omega\|_{L^2}^2 + \|\nabla^a \theta\|_{L^2}^2 \lesssim \|\nabla^a \omega_0\|_{L^2}^2 + \|\nabla^a \theta_0\|_{L^2}^2$$
$$+ t^{2\varepsilon}(\|(\mathbf{u},\omega,\theta)\|_Y^3 + \|(\mathbf{u},\omega,\theta)\|_Y^{4-\frac{2}{a+1}} + \|(\mathbf{u},\omega,\theta)\|_Y^4 + Q(\|(\mathbf{u},\omega,\theta)\|_Y)). \tag{4.6}$$

Then we deduce that

$$\sup_{t\geq 1} t^{-\varepsilon} \|\nabla^a(\omega,\theta)\|_{L^2} \lesssim \|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y). \tag{4.7}$$

Combining (4.1) and (4.7), we obtain

$$\sup_{t\geq 1} t^{-\varepsilon} \|\langle\nabla\rangle^a(\omega,\theta)\|_{L^2} \lesssim \|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y). \tag{4.8}$$

5. Estimates for the nonlinear terms

This section serves as a preparation for the estimates to be presented in the subsequent section. We bound various Sobolev norms of the nonlinear terms F_i for $i=1,2,3,4$ defined in (1.8) and (2.4). The Littlewood-Paley techniques will be employed. We give a brief introduction to Littlewood-Paley theory. Let $\phi(\xi)$ be a smooth bump function supported in the ball $|\xi|\leq 2$ and equal to 1 on the ball $|\xi|\leq 1$. For any real number $N>0$ and $f\in\mathcal{S}'$ (tempered distributions), projection operators are defined as follows:

$$\widehat{P_{\leq N} f}(\xi) \triangleq \phi\left(\frac{\xi}{N}\right)\widehat{f}(\xi),$$
$$\widehat{P_{>N} f}(\xi) \triangleq \left(1-\phi\left(\frac{\xi}{N}\right)\right)\widehat{f}(\xi),$$
$$\widehat{P_N f}(\xi) \triangleq \left(\phi\left(\frac{\xi}{N}\right) - \phi\left(2\frac{\xi}{N}\right)\right)\widehat{f}(\xi).$$

$P_{<N}$ and $P_{\geq N}$ are similarly defined. The following Bernstein inequalities play an important role in the process of estimating the nonlinear terms (see, e.g., [6, 7]).

Lemma 5.1 (Bernstein Inequalities). *Let $\alpha\geq 0$ and $N>0$ be real numbers and let $1\leq p\leq q\leq\infty$. Then there exist three constants C_1, C_2, C_3 such that*

$$\|\Lambda^\alpha P_{\leq N} f\|_{L^q(\mathbb{R}^d)} \leq C_1 N^{\alpha + d(\frac{1}{p}-\frac{1}{q})} \|P_{\leq N} f\|_{L^p(\mathbb{R}^d)},$$
$$C_2 N^\alpha \|P_N f\|_{L^q(\mathbb{R}^d)} \leq \|\Lambda^\alpha P_N f\|_{L^q(\mathbb{R}^d)} \leq C_3 N^\alpha \|P_N f\|_{L^q(\mathbb{R}^d)}.$$

Our first lemma bounds F_1.

Lemma 5.2. *For any $s\geq 1$,*

$$\|\langle\nabla\rangle^5 \Lambda^{\frac{1}{2}-\epsilon} F_1(s,\cdot)\|_{L^1_{xy}} \lesssim s^{-1-\varepsilon} Q(\|(\mathbf{u},\omega,\theta)\|_Y), \tag{5.1}$$

where ϵ is same as in Corollary 2.1 and ε is same as in the definition of X-norm defined in (1.14).

Proof. We recall that F_1 is given by

$$\begin{aligned} F_1 &= -\partial_x(\mathbf{u}\cdot\nabla\theta) - \partial_t(\mathbf{u}\cdot\nabla\omega) \\ &= -\partial_x\mathbf{u}\cdot\nabla\theta - \mathbf{u}\cdot\nabla\partial_x\theta - \partial_t\mathbf{u}\cdot\nabla\omega - \mathbf{u}\cdot\nabla\partial_t\omega \\ &\triangleq F_{11}+F_{12}+F_{13}+F_{14}. \end{aligned}$$

For $F_{11}(\mathbf{u},\theta)$, we divide it into three parts via the frequency decomposition defined above

$$F_{11} = -\partial_x(1-P_{\le s^\delta})\mathbf{u}\cdot\nabla\theta - \partial_x P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\theta - \partial_x P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\theta. \tag{5.2}$$

For the first high-frequency part, we have

$$\begin{aligned} &\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(\partial_x(1-P_{\le s^\delta})\mathbf{u}\cdot\nabla\theta)\|_{L^1_{xy}} \\ &\lesssim \|\langle\nabla\rangle^6(\partial_x P_{\gtrsim s^\delta}\mathbf{u}\cdot\nabla\theta)\|_{L^1_{xy}} \\ &\lesssim \|\langle\nabla\rangle^6\partial_x P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6\nabla\theta\|_{L^2_{xy}} \\ &\lesssim \|\langle\nabla\rangle^7 P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^7\theta\|_{L^2_{xy}} \\ &\lesssim s^{-(a-7)\delta}\|\langle\nabla\rangle^a P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^a\theta\|_{L^2_{xy}} \\ &\lesssim s^{-\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y) \end{aligned}$$

for large a and small δ. For the second high-frequency part in (5.2), we can bound it in the same way. For sufficiently large a,

$$\begin{aligned} &\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(\partial_x P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\theta)\|_{L^1_{xy}} \\ &\lesssim \|\langle\nabla\rangle^6(\partial_x P_{\lesssim s^\delta}\mathbf{u}\cdot\nabla P_{\gtrsim s^\delta}\theta)\|_{L^1_{xy}} \\ &\lesssim \|\langle\nabla\rangle^6\partial_x P_{\lesssim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6\nabla P_{\gtrsim s^\delta}\theta\|_{L^2_{xy}} \\ &\lesssim \|\langle\nabla\rangle^7 P_{\lesssim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^7 P_{\gtrsim s^\delta}\theta\|_{L^2_{xy}} \\ &\lesssim s^{-(a-7)\delta}\|\langle\nabla\rangle^a P_{\lesssim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^a P_{\gtrsim s^\delta}\theta\|_{L^2_{xy}} \\ &\lesssim s^{-\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y). \end{aligned}$$

For the low-frequency part in (5.2), we can bound it by

$$\begin{aligned} &\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(\partial_x P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\theta)\|_{L^1_{xy}} \\ &\lesssim \|\langle\nabla\rangle^6 P_{\le 4s^\delta}\partial_x P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\theta\|_{L^1_{xy}} \\ &\lesssim s^{6\delta}(\|(P_{\le s^\delta}\partial_x u_1\partial_x P_{\le s^\delta}\theta)\|_{L^1_{xy}} + \|(P_{\le s^\delta}\partial_x u_2\partial_y P_{\le s^\delta}\theta)\|_{L^1_{xy}}) \\ &\lesssim s^{6\delta}\|\omega\|_{L^2_{xy}}\|\partial_x\theta\|_{L^2_{xy}} + s^{7\delta}\|u_2\|_{L^2_{xy}}\|\partial_y\theta\|_{L^2_{xy}} \\ &\lesssim (s^{6\delta}s^{-\frac34-\frac34} + s^{7\delta}s^{-\frac78-\frac14})Q(\|(\mathbf{u},\omega,\theta)\|_Y) \\ &\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y) \end{aligned}$$

for δ small. Thus we proved

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}F_{11}\|_{L^1_{xy}} \lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y). \tag{5.3}$$

Along the same lines, we can also prove that

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}F_{1i}\|_{L^1_{xy}} \lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y), \qquad i=2,3,4. \tag{5.4}$$

For $F_{12}(\mathbf{u},\theta)$, we again divide it into three parts

$$F_{12} = -(1-P_{\le s^\delta})\mathbf{u}\cdot\nabla\partial_x\theta - P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\partial_x\theta - P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_x\theta.$$

Then, for the first high-frequency part, we have

$$\begin{aligned}
&\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}((1-P_{\le s^\delta})\mathbf{u}\cdot\nabla\partial_x\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6(P_{\gtrsim s^\delta}\mathbf{u}\cdot\nabla\partial_x\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6\nabla\partial_x\theta\|_{L^2_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^8\theta\|_{L^2_{xy}}\\
\lesssim& s^{-(a-6)\delta}\|\langle\nabla\rangle^aP_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^a\theta\|_{L^2_{xy}}\\
\lesssim& s^{-\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y),
\end{aligned}$$

for large enough a and small δ. For the second high-frequency part, we can bound it by

$$\begin{aligned}
&\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\partial_x\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6(P_{\lesssim s^\delta}\mathbf{u}\cdot\nabla P_{\gtrsim s^\delta}\partial_x\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6P_{\lesssim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6\nabla P_{\gtrsim s^\delta}\partial_x\theta\|_{L^2_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6P_{\lesssim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6P_{\gtrsim s^\delta}\theta\|_{L^2_{xy}}\\
\lesssim& s^{-(a-8)\delta}\|\langle\nabla\rangle^aP_{\lesssim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^aP_{\gtrsim s^\delta}\theta\|_{L^2_{xy}}\\
\lesssim& s^{-\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).
\end{aligned}$$

For the low-frequency part, we can bound it by

$$\begin{aligned}
&\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_x\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6P_{\le 4s^\delta}(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_x\theta)\|_{L^1_{xy}}\\
\lesssim& s^{6\delta}\|(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_x\theta)\|_{L^1_{xy}}\\
\lesssim& s^{6\delta}\|P_{\le s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\nabla P_{\le s^\delta}\partial_x\theta\|_{L^2_{xy}}\\
\lesssim& s^{7\delta}\|\Lambda^{-1}\omega\|_{L^2_{xy}}\|\partial_x\theta\|_{L^2_{xy}}\\
\lesssim& s^{7\delta}s^{-\frac34-\frac34}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\
\lesssim& s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).
\end{aligned}$$

Thus we have shown

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}F_{12}\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y). \tag{5.5}$$

For $F_{13}(\mathbf{u},\omega)$, we also divide it into three parts

$$F_{13} = -(1-P_{\le s^\delta})\partial_t\mathbf{u}\cdot\nabla\omega - P_{\le s^\delta}\partial_t\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\omega - P_{\le s^\delta}\partial_t\mathbf{u}\cdot\nabla P_{\le s^\delta}\omega.$$

For the first high-frequency part, we have

$$\begin{aligned}
&\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}((1-P_{\le s^\delta})\partial_t\mathbf{u}\cdot\nabla\omega)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6(P_{\gtrsim s^\delta}\partial_t\mathbf{u}\cdot\nabla\omega)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6P_{\gtrsim s^\delta}\partial_t\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6\nabla\omega\|_{L^2_{xy}}.
\end{aligned}$$

By Lemma 5.1 and (1.5), we obtain

$$\begin{aligned}\|\langle\nabla\rangle^6 P_{\gtrsim s^\delta}\partial_t\mathbf{u}\|_{L^2_{xy}} &\lesssim s^{-(a-3)\delta}\|\langle\nabla\rangle^{a-2}\partial_t\omega\|_{L^2_{xy}}\\ &\lesssim s^{-(a-3)\delta}\|\langle\nabla\rangle^{a-2}(\partial_x\theta-\omega-\mathbf{u}\cdot\nabla\omega)\|_{L^2_{xy}}\\ &\lesssim s^{-(a-3)\delta}s^{C\varepsilon}(\|(\mathbf{u},\omega,\theta)\|_Y+Q(\|(\mathbf{u},\omega,\theta)\|_Y)).\end{aligned}$$

Therefore, combining two estimates above, we obtain

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}((1-P_{\le s^\delta})\partial_t\mathbf{u}\cdot\nabla\omega)\|_{L^1_{xy}}\lesssim s^{\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

for large enough a and small δ. Similarly, for the second high-frequency part,

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(P_{\le s^\delta}\partial_t\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\omega)\|_{L^1_{xy}}\lesssim s^{-\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

The low-frequency part is bounded by

$$\begin{aligned}&\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(P_{\le s^\delta}\partial_t\mathbf{u}\cdot\nabla P_{\le s^\delta}\omega)\|_{L^1_{xy}}\\ &\lesssim\|\langle\nabla\rangle^6 P_{\le 4s^\delta}(P_{\le s^\delta}\partial_t\mathbf{u}\cdot\nabla P_{\le s^\delta}\omega)\|_{L^1_{xy}}\\ &\lesssim s^{6\delta}\|(P_{\le s^\delta}\partial_t\mathbf{u}\cdot\nabla P_{\le s^\delta}\omega)\|_{L^1_{xy}}\\ &\lesssim s^{6\delta}\|P_{\le s^\delta}\partial_t\mathbf{u}\|_{L^2_{xy}}\|\nabla P_{\le s^\delta}\omega\|_{L^2_{xy}}\\ &\lesssim s^{7\delta}\|\partial_t\Lambda^{-1}\omega\|_{L^2_{xy}}\|\omega\|_{L^2_{xy}}\\ &\lesssim s^{7\delta}s^{-\frac54-\frac34}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\ &\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).\end{aligned}$$

Thus we have proven

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}F_{13}\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y). \tag{5.6}$$

For $F_{14}(\mathbf{u},\theta)$, we also divide it into three parts,

$$F_{14}=-(1-P_{\le s^\delta})\mathbf{u}\cdot\nabla\partial_t\omega-P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\partial_t\omega-P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_t\omega.$$

Then, for the first high-frequency part, we have

$$\begin{aligned}&\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}((1-P_{\le s^\delta})\mathbf{u}\cdot\nabla\partial_t\omega)\|_{L^1_{xy}}\\ &\lesssim\|\langle\nabla\rangle^6(P_{\gtrsim s^\delta}\mathbf{u}\cdot\nabla\partial_t\omega)\|_{L^1_{xy}}\\ &\lesssim\|\langle\nabla\rangle^6 P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6\nabla\partial_t\omega\|_{L^2_{xy}}\\ &\lesssim\|\langle\nabla\rangle^6 P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^7(\partial_x\theta-\omega-\mathbf{u}\cdot\nabla\omega)\|_{L^2_{xy}}\\ &\lesssim s^{-(a-6)\delta}\|\langle\nabla\rangle^a P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}s^{C\varepsilon}(\|(\mathbf{u},\omega,\theta)\|_Y+Q(\|(\mathbf{u},\omega,\theta)\|_Y))\\ &\lesssim s^{-\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y),\end{aligned}$$

for large enough a and small δ. Similarly, for the second high-frequency part,

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\partial_t\omega)\|_{L^1_{xy}}\lesssim s^{-\frac32-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

For the low-frequency part, we can bound it by

$$\begin{aligned}
&\|\langle\nabla\rangle^5\Lambda^{\frac{1}{2}-\epsilon}(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_t\omega)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6 P_{\le 4s^\delta}(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_t\omega)\|_{L^1_{xy}}\\
\lesssim& s^{6\delta}\|(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\partial_t\omega)\|_{L^1_{xy}}\\
\lesssim& s^{6\delta}\|P_{\le s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\nabla P_{\le s^\delta}\partial_t\omega\|_{L^2_{xy}}\\
\lesssim& s^{7\delta}\|\Lambda^{-1}\omega\|_{L^2_{xy}}\|\partial_t\omega\|_{L^2_{xy}}\\
\lesssim& s^{7\delta}s^{-\frac{3}{4}-\frac{5}{4}}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\
\lesssim& s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).
\end{aligned}$$

Thus we have proven

$$\|\langle\nabla\rangle^5\Lambda^{\frac{1}{2}-\epsilon}F_{14}\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y). \tag{5.7}$$

This completes the proof of Lemma 5.2. □

F_2 admits a similar bound.

LEMMA 5.3. *For any $s\ge 1$,*

$$\|\langle\nabla\rangle^5\Lambda^{\frac{1}{2}-\epsilon}F_2(s,\cdot)\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y), \tag{5.8}$$

where ϵ is same as in Corollary 2.1 and ε is same as in the definition of X-norm in (1.14).

Proof. We can rewrite F_2 explicitly as

$$\begin{aligned}
F_2&=-\partial_x(\mathbf{u}\cdot\nabla\omega)-\partial_t(\mathbf{u}\cdot\nabla\theta)-\mathbf{u}\cdot\nabla\theta\\
&=-\partial_x\mathbf{u}\cdot\nabla\omega-\mathbf{u}\cdot\nabla\partial_x\omega-\partial_t\mathbf{u}\cdot\nabla\theta-\mathbf{u}\cdot\nabla\partial_t\theta-\mathbf{u}\cdot\nabla\theta\\
&\triangleq F_{21}+F_{22}+F_{23}+F_{24}+F_{25}.
\end{aligned}$$

It suffices to analyze $\mathbf{u}\cdot\nabla\theta$, which is the worst term in F_2. Other terms can be treated in the same way as in the proof of Lemma 5.2. We again use the frequency-decomposition technique to handle this term. Firstly we can divide it into three parts

$$\mathbf{u}\cdot\nabla\theta=-(1-P_{\le s^\delta})u\cdot\nabla\theta-P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\theta-P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\theta.$$

So for the first high-frequency part, we have

$$\begin{aligned}
&\|\langle\nabla\rangle^5\Lambda^{\frac{1}{2}-\epsilon}((1-P_{\le s^\delta})\mathbf{u}\cdot\nabla\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6(P_{\gtrsim s^\delta}\mathbf{u}\cdot\nabla\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6 P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^6\nabla\theta\|_{L^2_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6 P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^7\theta\|_{L^2_{xy}}\\
\lesssim& s^{-(a-6)\delta}\|\langle\nabla\rangle^a P_{\gtrsim s^\delta}\mathbf{u}\|_{L^2_{xy}}\|\langle\nabla\rangle^a\theta\|_{L^2_{xy}}\\
\lesssim& s^{-\frac{3}{2}-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y)
\end{aligned}$$

for large enough a and small δ, say $(a-6)\delta\ge\frac{3}{2}+\varepsilon$. Using the same method, one can obtain the estimate for the second high-frequency part,

$$\|\langle\nabla\rangle^5\Lambda^{\frac{1}{2}-\epsilon}(P_{\le s^\delta}\mathbf{u}\cdot\nabla(1-P_{\le s^\delta})\theta)\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

For the low-frequency part,

$$
\begin{aligned}
&\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\theta)\|_{L^1_{xy}}\\
\lesssim&\|\langle\nabla\rangle^6 P_{\le 4s^\delta}(P_{\le s^\delta}\mathbf{u}\cdot\nabla P_{\le s^\delta}\theta)\|_{L^1_{xy}}\\
\lesssim& s^{6\delta}(\|P_{\le s^\delta}u_1\partial_x P_{\le s^\delta}\theta\|_{L^1_{xy}}+\|P_{\le s^\delta}u_2\partial_y P_{\le s^\delta}\theta\|_{L^1_{xy}})\\
\lesssim& s^{6\delta}(\|u_1\|_{L^2_{xy}}\|\partial_x\theta\|_{L^2_{xy}}+\|u_2\|_{L^2_{xy}}\|\partial_y\theta\|_{L^2_{xy}})\\
\lesssim& s^{6\delta}(s^{-\frac34-\frac34}+s^{-\frac78-\frac14})Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\
\lesssim& s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).
\end{aligned}
$$

This finishes the proof of Lemma 5.3. □

According to the boundness of Riesz transforms, we can also bound F_3 and F_4 in the same way. The main result can be stated as follows.

LEMMA 5.4. *For any* $s\ge 1$,

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}F_3\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y), \tag{5.9}$$

$$\|\langle\nabla\rangle^5\Lambda^{\frac12-\epsilon}F_4\|_{L^1_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y), \tag{5.10}$$

where ϵ *is same as in Corollary 2.1 and* ε *is same as in the definition of* X*-norm in (1.14).*

In order to bound $\Lambda^{-2}\theta$, we need a decay estimate for the nonlinear term $\Lambda^{-1}F_4$, which can be stated as follows.

LEMMA 5.5. *For any* $s\ge 1$,

$$\||\partial_x|^{\frac14}\Lambda^{-1}F_4(s,\cdot)\|_{L^2_{xy}}\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y), \tag{5.11}$$

where $|\partial_x|^\gamma$ *with a fractional power* $\gamma\ge 0$ *is defined via the Fourier transform*

$$\widehat{|\partial_x|^\gamma f}(\xi,\eta)=|\xi|^\gamma\hat f(\xi,\eta)$$

and ϵ *is same as in Corollary 2.1 and* ε *is same as in the definition of* X*-norm in (1.14).*

Proof. F_4 is given by

$$
\begin{aligned}
F_4&=-\partial_x\Lambda^{-1}\nabla\cdot(\mathbf{u}\omega)-\partial_t\Lambda^{-1}\nabla\cdot(\mathbf{u}\theta)-\Lambda^{-1}\nabla\cdot(\mathbf{u}\theta)\\
&=-\Lambda^{-1}\nabla\cdot(\partial_x\mathbf{u}\omega)-\Lambda^{-1}\nabla\cdot(\mathbf{u}\partial_x\omega)-\Lambda^{-1}\nabla\cdot(\partial_t\mathbf{u}\theta)\\
&\quad-\Lambda^{-1}\nabla\cdot(\mathbf{u}\partial_t\theta)-\Lambda^{-1}\nabla\cdot(\mathbf{u}\theta)\\
&\triangleq F_{41}+F_{42}+F_{43}+F_{44}+F_{45}.
\end{aligned}\tag{5.12}
$$

By the standard bounds for the Riesz transforms, Hardy-Littlewood-Sobolev inequality, Hölder inequality and interpolation inequality,

$$
\begin{aligned}
\||\partial_x|^{\frac14}\Lambda^{-1}F_{41}\|_{L^2_{xy}}&\lesssim\|\Lambda^{-\frac34}(\partial_x\mathbf{u}\omega)\|_{L^2_{xy}}\lesssim\|\partial_x\mathbf{u}\omega\|_{L^{\frac87}_{xy}}\lesssim\|\partial_x\mathbf{u}\|_{L^2_{xy}}\|\omega\|_{L^{\frac83}_{xy}}\\
&\lesssim\|\partial_x\mathbf{u}\|_{L^2_{xy}}\|\omega\|^{\frac34}_{L^2_{xy}}\|\omega\|^{\frac14}_{L^\infty_{xy}}\lesssim s^{-\frac34-\frac34\times\frac34-\frac14}Q(\|(\mathbf{u},\omega,\theta)\|_Y)
\end{aligned}
$$

$$\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

Invoking the bound for $\|\mathbf{u}\|_{L^\infty}$ in (3.7), we have

$$\begin{aligned}\||\partial_x|^{\frac14}\Lambda^{-1}F_{42}\|_{L^2_{xy}} &\lesssim \|\Lambda^{-\frac34}(\mathbf{u}\partial_x\omega)\|_{L^2_{xy}} \lesssim \|\mathbf{u}\partial_x\omega\|_{L^{\frac87}_{xy}} \lesssim \|\partial_x\omega\|_{L^2_{xy}}\|\mathbf{u}\|_{L^{\frac83}_{xy}}\\ &\lesssim \|\partial_x\omega\|_{L^2_{xy}}\|\mathbf{u}\|^{\frac34}_{L^2_{xy}}\|\mathbf{u}\|^{\frac14}_{L^\infty_{xy}} \lesssim s^{-\frac34-\frac34\times\frac34-\frac14}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\ &\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).\end{aligned}$$

To bound F_{43}, we invoke the bound

$$\begin{aligned}\|\theta\|_{L^\infty} &\lesssim \|\theta\|^{\frac14}_{L^2}\|\partial_x\theta\|^{\frac14}_{L^2}\|\partial_y\theta\|^{\frac14}_{L^2}\|\partial_{xy}\theta\|^{\frac14}_{L^2}\\ &\lesssim s^{-\frac14(\frac14+\frac34+\frac14+\frac34)}Q(\|(\mathbf{u},\omega,\theta)\|_Y) = s^{-\frac12}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\end{aligned}$$

to obtain

$$\begin{aligned}\||\partial_x|^{\frac14}\Lambda^{-1}F_{43}\|_{L^2_{xy}} &\lesssim \|\Lambda^{-\frac34}(\partial_t\mathbf{u}\theta)\|_{L^2_{xy}} \lesssim \|\partial_t\mathbf{u}\theta\|_{L^{\frac87}_{xy}} \lesssim \|\partial_t\mathbf{u}\|_{L^2_{xy}}\|\theta\|_{L^{\frac83}_{xy}}\\ &\lesssim \|\partial_t\mathbf{u}\|_{L^2_{xy}}\|\theta\|^{\frac34}_{L^2_{xy}}\|\theta\|^{\frac14}_{L^\infty_{xy}} \lesssim s^{-\frac54-\frac14\times\frac34-\frac12\times\frac14}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\ &\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).\end{aligned}$$

$$\begin{aligned}\||\partial_x|^{\frac14}\Lambda^{-1}F_{44}\|_{L^2_{xy}} &\lesssim \|\Lambda^{-\frac34}(\mathbf{u}\partial_t\theta)\|_{L^2_{xy}} \lesssim \|\mathbf{u}\partial_t\theta\|_{L^{\frac87}_{xy}} \lesssim \|\partial_t\theta\|_{L^2_{xy}}\|\mathbf{u}\|_{L^{\frac83}_{xy}}\\ &\lesssim \|\partial_t\theta\|_{L^2_{xy}}\|\mathbf{u}\|^{\frac34}_{L^2_{xy}}\|\mathbf{u}\|^{\frac14}_{L^\infty_{xy}} \lesssim s^{-\frac54-\frac34\times\frac34-\frac14}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\ &\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).\end{aligned}$$

$$\begin{aligned}\||\partial_x|^{\frac14}\Lambda^{-1}F_{45}\|_{L^2_{xy}} &\lesssim \|\Lambda^{-\frac34}(\mathbf{u}\theta)\|_{L^2_{xy}} \lesssim \|\mathbf{u}\theta\|_{L^{\frac87}_{xy}} \lesssim \|\mathbf{u}\|_{L^2_{xy}}\|\theta\|_{L^{\frac83}_{xy}}\\ &\lesssim \|\mathbf{u}\|_{L^2_{xy}}\|\mathbf{u}\|^{\frac34}_{L^2_{xy}}\|\theta\|^{\frac14}_{L^\infty_{xy}} \lesssim s^{-\frac34-\frac34\times\frac34-\frac12\times\frac14}Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\ &\lesssim s^{-1-\varepsilon}Q(\|(\mathbf{u},\omega,\theta)\|_Y).\end{aligned}$$

Collecting these estimates yield the desirable bound (5.11), which finishes the proof of Lemma 5.5. □

6. Proof of Equation (1.19)

This section completes the proof of Theorem 1.1. We continue to verify

$$\|(\omega,\theta)\|_X \lesssim \|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y).$$

For the sake of clarity, we divide this section into several subsections, each dealing with some of the terms in $\|(\omega,\theta)\|_X$.

6.1. Estimates of $\|\langle\nabla\rangle^2\theta\|_{L^2}$, $\|\Lambda^{-1}\omega\|_{L^2}$ and $\|\langle\nabla\rangle^2\Lambda^{-1}\theta\|_{L^2}$.

Using Duhamel's formula,

$$\theta(t,x,y) = K_0(t,\partial_x)\theta_0 + K_1(t,\partial_x)\left(\frac12\theta_0+\theta_1\right) + \int_1^t K_1(t-s,\partial_x)F_2(s)ds. \tag{6.1}$$

For notational convenience, we may sometimes write $K_0(t)$ for $K_0(t,\partial_x)$ and $K_1(t)$ for $K_1(t,\partial_x)$. Then by Lemma 2.2, Corollary 2.1 and Lemma 5.3, we have

$$\begin{aligned}
\|\langle\nabla\rangle^2\theta\|_{L^2} \lesssim& \|K_0(t)\langle\nabla\rangle^2\theta_0\|_{L^2} + \|K_1(t)\langle\nabla\rangle^2\left(\frac{1}{2}\theta_0+\theta_1\right)\|_{L^2} \\
&+ \|\int_1^t K_1(t-s)\langle\nabla\rangle^2 F_2(s)ds\|_{L^2} \\
\lesssim& t^{-\frac{1}{4}}(\|\langle\nabla\rangle^{3+\epsilon}\theta_0\|_{L^1_{xy}} + \|\langle\nabla\rangle^{3+\epsilon}\theta_1\|_{L^1_{xy}}) + \int_1^t (\|\widehat{K}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})} \\
&+ \|\widehat{K}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{5}{2}+2\epsilon}F_2(s)\|_{L^1_{xy}}ds \\
\lesssim& t^{-\frac{1}{4}}\|(\omega_0,\theta_0)\|_{X_0} + \int_1^t \langle t-s\rangle^{-\frac{1}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y) \\
\lesssim& t^{-\frac{1}{4}}(\|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y)).
\end{aligned}$$

Again by Duhamel's formula,

$$\begin{aligned}
\Lambda^{-1}\omega(t,x,y) =& K_0(t,\partial_x)\Lambda^{-1}\omega_0 + K_1(t,\partial_x)\left(\frac{1}{2}\Lambda^{-1}\omega_0+\Lambda^{-1}\omega_1\right) \\
&+ \int_1^t K_1(t-s,\partial_x)F_3(s)ds, \\
\Lambda^{-1}\theta(t,x,y) =& K_0(t,\partial_x)\Lambda^{-1}\theta_0 + K_1(t,\partial_x)\left(\frac{1}{2}\Lambda^{-1}\theta_0+\Lambda^{-1}\theta_1\right) \\
&+ \int_1^t K_1(t-s,\partial_x)F_4(s)ds.
\end{aligned}$$

Thus, by Lemma 2.2, Corollary 2.1 and Lemma 5.4,

$$\begin{aligned}
\|\Lambda^{-1}\omega\|_{L^2} \lesssim& \|K_0(t)\Lambda^{-1}\omega_0\|_{L^2} + \|K_1(t)\left(\frac{1}{2}\Lambda^{-1}\omega_0+\Lambda^{-1}\omega_1\right)\|_{L^2} \\
&+ \|\int_1^t K_1(t-s)F_3(s)ds\|_{L^2} \\
\lesssim& t^{-\frac{1}{4}}(\|\langle\nabla\rangle^{1+\epsilon}\Lambda^{-1}\omega_0\|_{L^1_{xy}} + \|\langle\nabla\rangle^{1+\epsilon}\Lambda^{-1}\omega_1\|_{L^1_{xy}}) \\
&+ \int_1^t (\|\widehat{K}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})} \\
&+ \|\widehat{K}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\langle\nabla\rangle^{\frac{1}{2}+2\epsilon}\Lambda^{\frac{1}{2}-\epsilon}F_3(s)\|_{L^1_{xy}}ds \\
\lesssim& t^{-\frac{1}{4}}\|(\omega_0,\theta_0)\|_{X_0} + \int_1^t \langle t-s\rangle^{-\frac{1}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y) \\
\lesssim& t^{-\frac{1}{4}}(\|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y)).
\end{aligned}$$

Similarly, by Lemma 2.2, Corollary 2.1 and Lemma 5.4,

$$\|\langle\nabla\rangle^2\Lambda^{-1}\theta\|_{L^2} \lesssim t^{-\frac{1}{4}}(\|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y)).$$

6.2. Estimates of $\|\partial_x\langle\nabla\rangle\omega\|_{L^2}$, $\|\partial_x\langle\nabla\rangle^2\theta\|_{L^2}$, $\|\partial_{xx}\langle\nabla\rangle^2\theta\|_{L^2}$ and $\|\partial_x\Lambda^{-1}\theta\|_{L^2}$. To estimate $\|\partial_x\langle\nabla\rangle\omega\|_{L^2}$, we start with Duhamel's formula

$$\omega(t,x,y)=K_0(t,\partial_x)\omega_0+K_1(t,\partial_x)\left(\frac{1}{2}\omega_0+\omega_1\right)+\int_1^t K_1(t-s,\partial_x)F_1(s)ds. \tag{6.2}$$

By Lemma 2.2, Corollary 2.1 and Lemma 5.3,

$$\begin{aligned}
\|\partial_x\langle\nabla\rangle\omega\|_{L^2}\lesssim&\|\partial_xK_0(t)\langle\nabla\rangle\omega_0\|_{L^2}+\|\partial_xK_1(t)\langle\nabla\rangle\left(\frac{1}{2}\omega_0+\omega_1\right)\|_{L^2}\\
&+\|\int_1^t\partial_xK_1(t-s)\langle\nabla\rangle F_1(s)ds\|_{L^2}\\
\lesssim& t^{-\frac{3}{4}}(\|\langle\nabla\rangle^{2+\epsilon}\omega_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{2+\epsilon}\omega_1\|_{L^1_{xy}})\\
&+\int_1^t(\|\widehat{\partial_xK}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})}\\
&+\|\widehat{\partial_xK}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{3}{2}+2\epsilon}F_1(s)\|_{L^1_{xy}}ds\\
\lesssim& t^{-\frac{3}{4}}\|(\omega_0,\theta_0)\|_{X_0}+\int_1^t\langle t-s\rangle^{-\frac{3}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\
\lesssim& t^{-\frac{3}{4}}(\|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y)).
\end{aligned}$$

Now we consider $\|\partial_x\langle\nabla\rangle^2\theta\|_{L^2}$. As in the estimates above,

$$\begin{aligned}
\|\partial_x\langle\nabla\rangle^2\theta\|_{L^2}\lesssim&\|\partial_xK_0(t)\langle\nabla\rangle^2\theta_0\|_{L^2}+\|\partial_xK_1(t)\langle\nabla\rangle^2\left(\frac{1}{2}\theta_0+\theta_1\right)\|_{L^2}\\
&+\|\int_1^t\partial_xK_1(t-s)\langle\nabla\rangle^2F_2(s)ds\|_{L^2}\\
\lesssim& t^{-\frac{3}{4}}(\|\langle\nabla\rangle^{3+\epsilon}\theta_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{3+\epsilon}\theta_1\|_{L^1_{xy}})+\int_1^t(\|\widehat{\partial_xK}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})}\\
&+\|\widehat{\partial_xK}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{5}{2}+2\epsilon}F_2(s)\|_{L^1_{xy}}ds\\
\lesssim& t^{-\frac{3}{4}}\|(\omega_0,\theta_0)\|_{X_0}+\int_1^t\langle t-s\rangle^{-\frac{3}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\
\lesssim& t^{-\frac{3}{4}}(\|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y)).
\end{aligned}$$

As for $\|\partial_{xx}\langle\nabla\rangle^2\theta\|_{L^2}$, using Lemma 2.2, Corollary 2.1 and Lemma 5.3 again, we obtain

$$\begin{aligned}
\|\partial_{xx}\langle\nabla\rangle^2\theta\|_{L^2}\lesssim&\|\partial_{xx}K_0(t)\langle\nabla\rangle^2\theta_0\|_{L^2}+\|\partial_{xx}K_1(t)\langle\nabla\rangle^2\left(\frac{1}{2}\theta_0+\theta_1\right)\|_{L^2}\\
&+\|\int_1^t\partial_{xx}K_1(t-s)\langle\nabla\rangle^2F_2(s)ds\|_{L^2}\\
\lesssim& t^{-\frac{5}{4}}(\|\langle\nabla\rangle^{3+\epsilon}\theta_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{3+\epsilon}\theta_1\|_{L^1_{xy}})+\int_1^t(\|\widehat{\partial_{xx}K}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})}\\
&+\|\widehat{\partial_{xx}K}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{5}{2}+2\epsilon}F_2(s)\|_{L^1_{xy}}ds\\
\lesssim& t^{-\frac{5}{4}}\|(\omega_0,\theta_0)\|_{X_0}+\int_1^t\langle t-s\rangle^{-\frac{5}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y)
\end{aligned}$$

$$\lesssim t^{-\frac{5}{4}}(\|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y)).$$

Then we deal with the term $\|\partial_x\Lambda^{-1}\theta\|_{L^2}$. We proceed in the same way as in the previous estimate,

$$\begin{aligned}\|\partial_x\Lambda^{-1}\theta\|_{L^2}\lesssim&\|\partial_xK_0(t)\Lambda^{-1}\theta_0\|_{L^2}+\|\partial_xK_1(t)\left(\frac{1}{2}\Lambda^{-1}\theta_0+\Lambda^{-1}\theta_1\right)\|_{L^2}\\&+\|\int_1^t\partial_xK_1(t-s)F_4(s)ds\|_{L^2}\\\lesssim&t^{-\frac{3}{4}}(\|\langle\nabla\rangle^{1+\epsilon}\Lambda^{-1}\theta_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{1+\epsilon}\Lambda^{-1}\theta_1\|_{L^1_{xy}})\\&+\int_1^t(\|\widehat{\partial_xK}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})}\\&+\|\widehat{\partial_xK}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\langle\nabla\rangle^{\frac{1}{2}+2\epsilon}\Lambda^{\frac{1}{2}-\epsilon}F_4(s)\|_{L^1_{xy}}ds\\\lesssim&t^{-\frac{3}{4}}\|(\omega_0,\theta_0)\|_{X_0}+\int_1^t\langle t-s\rangle^{-\frac{3}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\\lesssim&t^{-\frac{3}{4}}(\|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y)).\end{aligned}$$

6.3. Estimates of $\|\partial_t\langle\nabla\rangle^2\omega\|_{L^2}$, $\|\partial_t\Lambda^{-1}\omega\|_{L^2}$, $\|\partial_t\theta\|_{L^2}$ and $\|\partial_{xx}\Lambda^{-2}\theta\|_{L^2}$. By Lemma 2.2, Corollary 2.1 and Lemma 5.2, we have

$$\begin{aligned}\|\partial_t\langle\nabla\rangle^2\omega\|_{L^2}\lesssim&\|\partial_tK_0(t)\langle\nabla\rangle^2\omega_0\|_{L^2}+\|\partial_tK_1(t)\langle\nabla\rangle^2\left(\frac{1}{2}\omega_0+\omega_1\right)\|_{L^2}\\&+\|\int_1^t\partial_tK_1(t-s)\langle\nabla\rangle^2F_1(s)ds\|_{L^2}\\\lesssim&t^{-\frac{5}{4}}(\|\langle\nabla\rangle^{3+\epsilon}\omega_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{3+\epsilon}\omega_1\|_{L^1_{xy}})+\int_1^t(\|\partial_t\widehat{K}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})}\\&+\|\partial_t\widehat{K}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{5}{2}+2\epsilon}F_1(s)\|_{L^1_{xy}}ds\\\lesssim&t^{-\frac{5}{4}}\|(\omega_0,\theta_0)\|_{X_0}+\int_1^t\langle t-s\rangle^{-\frac{5}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\\lesssim&t^{-\frac{5}{4}}(\|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y)).\end{aligned}$$

Next we bound $\|\partial_t\Lambda^{-1}\omega\|_{L^2}$. By Lemma 2.2, Corollary 2.1 and Lemma 5.4, we have

$$\begin{aligned}\|\partial_t\Lambda^{-1}\omega\|_{L^2}\lesssim&\|\partial_tK_0(t)\Lambda^{-1}\omega_0\|_{L^2}+\|\partial_tK_1(t)\Lambda^{-1}\left(\frac{1}{2}\omega_0+\omega_1\right)\|_{L^2}\\&+\|\int_1^t\partial_tK_1(t-s)\Lambda^{-1}F_3(s)ds\|_{L^2}\\\lesssim&t^{-\frac{5}{4}}(\|\langle\nabla\rangle^{1+\epsilon}\Lambda^{-1}\omega_0\|_{L^1_{xy}}+\|\langle\nabla\rangle^{1+\epsilon}\Lambda^{-1}\omega_1\|_{L^1_{xy}})\\&+\int_1^t(\|\partial_t\widehat{K}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})}\\&+\|\partial_t\widehat{K}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{1}{2}+2\epsilon}F_3(s)\|_{L^1_{xy}}ds\\\lesssim&t^{-\frac{5}{4}}\|(\omega_0,\theta_0)\|_{X_0}+\int_1^t\langle t-s\rangle^{-\frac{5}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y)\\\lesssim&t^{-\frac{5}{4}}(\|(\omega_0,\theta_0)\|_{X_0}+Q(\|(\mathbf{u},\omega,\theta)\|_Y)).\end{aligned}$$

For $\|\partial_t\theta\|_{L^2}$, by (6.1), Lemma 2.2, Corollary 2.1 and Lemma 5.3, we have

$$\begin{aligned}\|\partial_t\theta\|_{L^2} \lesssim & \|\partial_t K_0(t)\theta_0\|_{L^2} + \|\partial_t K_1(t)\Big(\frac{1}{2}\theta_0+\theta_1\Big)\|_{L^2} \\ & + \|\int_1^t \partial_t K_1(t-s)F_2(s)ds\|_{L^2} \\ & \lesssim t^{-\frac{5}{4}}(\|\langle\nabla\rangle^{1+\epsilon}\theta_0\|_{L^1_{xy}} + \|\langle\nabla\rangle^{1+\epsilon}\theta_1\|_{L^1_{xy}}) + \int_1^t (\|\partial_t\widehat{K}_1(t-s,\xi)\|_{L^2_\xi(|\xi|\le\frac{1}{2})} \\ & + \|\partial_t\widehat{K}_1(t-s,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac{1}{2})})\|\Lambda^{\frac{1}{2}-\epsilon}\langle\nabla\rangle^{\frac{1}{2}+2\epsilon}F_2(s)\|_{L^1_{xy}}ds \\ & \lesssim t^{-\frac{5}{4}}\|(\omega_0,\theta_0)\|_{X_0} + \int_1^t \langle t-s\rangle^{-\frac{5}{4}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y) \\ & \lesssim t^{-\frac{5}{4}}(\|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y)).\end{aligned}$$

Now we consider the last term $\|\partial_{xx}\Lambda^{-2}\theta\|_{L^2}$. By Duhamel's formula,

$$\begin{aligned}\Lambda^{-2}\theta(t,x,y) = & K_0(t,\partial_x)\Lambda^{-2}\theta_0 + K_1(t,\partial_x)\Big(\frac{1}{2}\Lambda^{-2}\theta_0+\Lambda^{-2}\theta_1\Big) \\ & + \int_1^t K_1(t-s,\partial_x)\Lambda^{-1}F_4(s)ds.\end{aligned}$$

Using Lemma 2.2, Lemma 2.3 and Lemma 5.5, we deduce that

$$\begin{aligned}\|\partial_{xx}\Lambda^{-2}\theta\|_{L^2} \lesssim & \|\partial_{xx}K_0(t)\Lambda^{-2}\theta_0\|_{L^2} + \|\partial_{xx}K_1(t)\Lambda^{-2}\Big(\frac{1}{2}\theta_0+\theta_1\Big)\|_{L^2} \\ & + \|\int_1^t |\partial_x|^{\frac{7}{4}}K_1(t-s)|\partial_x|^{\frac{1}{4}}\Lambda^{-1}F_4(s)ds\|_{L^2} \\ & \lesssim \|\partial_{xx}\widehat{K}_0(t)\|_{L^2}\|\Lambda^{-2}\theta_0\|_{L^1_{xy}} + \|\partial_{xx}\widehat{K}_1(t)\|_{L^2}\|\Lambda^{-2}\Big(\frac{1}{2}\theta_0+\theta_1\Big)\|_{L^1_{xy}} \\ & + \int_1^t \|\widehat{|\partial_x|^{\frac{7}{4}}K}_1(t-s)\|_{L^\infty}\||\partial_x|^{\frac{1}{4}}\Lambda^{-1}F_4(s)\|_{L^2_{xy}}ds \\ & \lesssim t^{-\frac{5}{4}}(\|\Lambda^{-2}\theta_0\|_{L^1_{xy}} + \|\Lambda^{-2}\theta_1\|_{L^1_{xy}}) + \int_1^t \langle t-s\rangle^{-\frac{7}{8}}s^{-1-\varepsilon}ds\cdot Q(\|(\mathbf{u},\omega,\theta)\|_Y) \\ & \lesssim t^{-\frac{7}{8}}(\|(\omega_0,\theta_0)\|_{X_0} + Q(\|(\mathbf{u},\omega,\theta)\|_Y)).\end{aligned}$$

Finally we estimate the pressure P. According to the divergence-free condition, we have

$$P = \Delta^{-1}\partial_y\theta - \Delta^{-1}\nabla\cdot(\nabla\cdot(\mathbf{u}\otimes\mathbf{u})).$$

Therefore, for any $t\ge 1$, by the boundedness of Riesz transforms and the definition of Y-norm,

$$\begin{aligned}\|P(t)\|_{H^a} & \lesssim \|\Delta^{-1}\partial_y\theta\|_{H^a} + \|\Delta^{-1}\nabla\cdot(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}))\|_{H^a} \\ & \lesssim \|\Delta^{-1}\partial_y\theta\|_{L^2} + \|\Lambda^a\Delta^{-1}\partial_y\theta\|_{L^2} + \|\Delta^{-1}\nabla\cdot(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}))\|_{H^a} \\ & \lesssim \|\Lambda^{-1}\theta\|_{L^2} + \|\Lambda^{a-1}\|_{L^2} + \|\mathbf{u}\|_{H^a}\|\mathbf{u}\|_{L^\infty} \\ & \lesssim t^\varepsilon(\|(\mathbf{u},\omega,\theta)\|_Y + Q(\|(\mathbf{u},\omega,\theta)\|_Y)) \\ & \lesssim t^\varepsilon\epsilon_0.\end{aligned}$$

This implies that $P\in C([1,+\infty);H^a(\mathbb{R}^2))$. Moreover,

$$\begin{aligned}\|P(t)\|_{L^\infty} &\lesssim \|\Delta^{-1}\partial_y\theta\|_{L^\infty}+\|\Delta^{-1}\nabla\cdot(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}))\|_{L^\infty}\\ &\lesssim \|\langle\nabla\rangle^2\Delta^{-1}\partial_y\theta\|_{L^2}+\|\langle\nabla\rangle^2\Delta^{-1}\nabla\cdot(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}))\|_{L^2}\\ &\lesssim \|\langle\nabla\rangle^2\Lambda^{-1}\theta\|_{L^2}+\|\langle\nabla\rangle^2(\mathbf{u}\otimes\mathbf{u})\|_{L^2}\\ &\lesssim \|\langle\nabla\rangle^2\Lambda^{-1}\theta\|_{L^2}+\|\langle\nabla\rangle^2\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^\infty}\\ &\lesssim t^{-\frac{1}{4}}\epsilon_0.\end{aligned}$$

This finishes the proof of Theorem 1.1.

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