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Global Regularity for Several Incompressible Fluid Models with Partial Dissipation

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Abstract. This paper examines the global regularity problem on several 2D incompressible fluid models with partial dissipation. They are the surface quasi-geostrophic (SQG) equation, the 2D Euler equation and the 2D Boussinesq equations. These are well-known models in fluid mechanics and geophysics. The fundamental issue of whether or not they are globally well-posed has attracted enormous attention. The corresponding models with partial dissipation may arise in physical circumstances when the dissipation varies in different directions. We show that the SQG equation with either horizontal or vertical dissipation always has global solutions. This is in sharp contrast with the inviscid SQG equation for which the global regularity problem remains outstandingly open. Although the 2D Euler is globally well-posed for sufficiently smooth data, the associated equations with partial dissipation no longer conserve the vorticity and the global regularity is not trivial. We are able to prove the global regularity for two partially dissipated Euler equations. Several global bounds are also obtained for a partially dissipated Boussinesq system.

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Keywords. Euler equation, surface quasi-geostrophic equation, Boussinesq equations, partial dissipation, global regularity.

1. Introduction

This paper studies the global regularity of several two-dimensional (2D) models with partial dissipation. The first is the surface quasi-geostrophic (SQG) equation with two different partial dissipation terms, which can be written as

$$\begin{cases} \partial_t\theta + u\partial_x\theta + v\partial_y\theta - \mu\partial_{xx}\theta = 0, \quad (x,y) \in \mathbb{R}^2,\ t>0, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t\theta + u\partial_x\theta + v\partial_y\theta - \mu\partial_{yy}\theta = 0, \quad (x,y) \in \mathbb{R}^2,\ t>0, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.2}$$

where θ is a scalar real-valued function, $\mu > 0$ is a constant, and the velocity $\vec{u} \equiv (u,\, v)$ is determined by θ through a stream function ψ, namely

$$\vec{u} = (u,\, v) = (-\partial_y\psi,\, \partial_x\psi), \quad \sqrt{-\Delta}\psi = \theta.$$

The above relations can be combined to

$$\vec{u} = (u,\, v) = \left(-\frac{\partial_y}{\sqrt{-\Delta}}\theta,\ \frac{\partial_x}{\sqrt{-\Delta}}\,\theta\right) = (-\mathcal{R}_2\theta,\ \mathcal{R}_1\theta),$$

where $\mathcal{R}_1, \mathcal{R}_2$ are the standard 2D Riesz transforms. Clearly, the velocity $\vec{u} = (u,\, v)$ is divergence free, namely $\partial_x u + \partial_y v = 0$. The SQG equation is an important model in geophysical fluid dynamics. In particular, it is the special case of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers, see [23,54] and the references cited there. Mathematically, as pointed out by Constantin, Majda and Tabak [23], the inviscid SQG equation ((1.1) with $\mu = 0$) shares many parallel properties with those of the 3D Euler equations such as the vortex-stretching mechanism and thus serves as a lower-dimensional model of the 3D Euler equations.

The inviscid SQG equation is among the simplest scalar partial differential equations for which the global well-posedness issue remains open. The global regularity problem on the SQG equation has recently been studied extensively and important progress has been made. Besides establishing the local well-posedness and several regularity criteria, Constantin, Majda and Tabak carefully examined the behavior of several special classes of solutions [23]. The perplexing behavior of solutions to the inviscid SQG equations was further investigated both theoretically and numerically and these studies have contributed substantially to our understanding of the global regularity problem (see, e.g., [22,24,29–32,36,52]).

The dissipative SQG equation with fractional Laplacian, namely (1.1) with $(-\Delta)^{\frac{\alpha}{2}}$ instead of $-\mu\partial_{xx}\theta$, has recently attracted enormous attention and significant progress has been made on the global well-posedness issue. The global regularity problem for the SQG equation with either subcritical ($\alpha > 1$) or critical ($\alpha = 1$) dissipation has been successfully resolved (see, e.g., [7,20,25,26,43,45,56]). Although the global regularity issue for the supercritical case $\alpha < 1$ remains outstandingly open, there are important recent developments (see, e.g., [11–13,15,17,27,28,33,34,37,42,50,51,57,60,63,64]).

We explore how partial dissipation would affect the regularity of solutions to the SQG equation. To the best of our knowledge, such systems of equations as in (1.1) and (1.2) have never been studied before. We are able to establosh the global regularity for both equations.

Theorem 1.1. *For any $\theta_0 \in H^2(\mathbb{R}^2)$, (1.1) admits a unique global solution θ such that for any given $T > 0$,*

$$\theta \in L^\infty([0,T]; H^2(\mathbb{R}^2)), \quad \partial_x\theta \in L^2([0,T]; H^2(\mathbb{R}^2)).$$

Theorem 1.2. *For any $\theta_0 \in H^2(\mathbb{R}^2)$, the system (1.2) admits a unique global solution θ such that for any given $T > 0$,*

$$\theta \in L^\infty([0,T]; H^2(\mathbb{R}^2)), \quad \partial_y\theta \in L^2([0,T]; H^2(\mathbb{R}^2)).$$

Remark 1.3. The proof of Theorem 1.1 is given in Sect. 2. Since the proof of Theorem 1.2 is largely parallel to that of Theorem 1.1, we shall omit the details.

We also investigate the global well-posedness issue on the following 2D Euler equations with partial dissipation.

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_{xx} u = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \end{cases} \tag{1.3}$$

and

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \partial_{yy} v = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y). \end{cases} \tag{1.4}$$

The Euler equations model the motion of ideal incompressible fluids and the global well-posedness problem on the 3D Euler equations is one of the most challenging problems in mathematical fluid dynamics (see [5,18,19,49] and references therein for a review of the subject). The 2D Euler equation has been extensively studied and the global regularity is known since the work of Wolibner [61] and Hölder [39] (see also [4,16,44,49]). The key observation is that the corresponding vorticity is simply transported by the velocity field due to the absence of the vortex stretching term in the 2D case. Consequently one easily obtains the boundedness of the vorticity if it is initially so. This is the key component in the proof of the global well-posedness for the 2D Euler equation. It appears that an alternative proof without resorting to the boundedness of the vorticity is currently lacking.

On the first look, it seems that we should be able to obtain the global regularity of (1.3) or (1.4) easily. However, when partial dissipation is added to the 2D Euler equation, the vorticity equation is more complex and it is then not easy to deduce the boundness of the vorticity. Then one has to rely on energy estimates to establish global bounds in consecutively more and more regular functional settings. We are able to prove the global regularity for (1.3) and (1.4).

Theorem 1.4. *For any* $(u_0,\ v_0) \in H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, (1.3) *admits a unique global solution* (u, v) *such that for any given* $T > 0$

$$\begin{aligned} &(u,\, v) \in L^\infty([0,T]; H^2(\mathbb{R}^2)) \times L^\infty([0,T]; H^2(\mathbb{R}^2)),\\ &\partial_x u \in L^2([0,T]; H^2(\mathbb{R}^2)). \end{aligned}$$

Theorem 1.5. *For* $(u_0,\ v_0) \in H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, (1.4) *admits a unique global solution* (u, v) *such that for any given* $T > 0$

$$\begin{aligned} &(u,\, v) \in L^\infty([0,T]; H^2(\mathbb{R}^2)) \times L^\infty([0,T]; H^2(\mathbb{R}^2)),\\ &\partial_y v \in L^2([0,T]; H^2(\mathbb{R}^2)). \end{aligned}$$

The proof of Theorem 1.4 is given in Sect. 3. We now explain that (1.4) can be converted into (1.3) and thus the proof of Theorem 1.5 follows. If we set

$$\begin{cases} \tilde{x} = y, \quad \tilde{y} = x, \\ \tilde{u}(\tilde{x},\tilde{y},t) = v(x,y,t), \quad \tilde{v}(\tilde{x},\tilde{y},t) = u(x,y,t), \quad \tilde{P}(\tilde{x},\tilde{y},t) = P(x,y,t), \end{cases}$$

then we can check that

$$u\partial_x u + v\partial_y u = \tilde{v}\partial_{\tilde{y}}\tilde{v} + \tilde{u}\partial_{\tilde{x}}\tilde{u}, \quad u\partial_x v + v\partial_y v = \tilde{v}\partial_{\tilde{y}}\tilde{u} + \tilde{u}\partial_{\tilde{x}}\tilde{u}.$$

Therefore, (1.4) is equivalent to

$$\begin{cases} \partial_t \tilde{v} + \tilde{u}\partial_{\tilde{x}}\tilde{v} + \tilde{v}\partial_{\tilde{y}}\tilde{v} + \partial_{\tilde{y}}\tilde{P} = 0, \\ \partial_t \tilde{u} + \tilde{u}\partial_{\tilde{x}}\tilde{u} + \tilde{v}\partial_{\tilde{y}}\tilde{u} + \partial_{\tilde{x}}\tilde{P} - \partial_{\tilde{x}\tilde{x}}\tilde{u} = 0, \\ \partial_{\tilde{x}}\tilde{u} + \partial_{\tilde{y}}\tilde{v} = 0, \\ \tilde{u}(\tilde{x},\tilde{y},0) = v_0(x,y), \quad \tilde{v}(\tilde{x},\tilde{y},0) = u_0(x,y). \end{cases} \tag{1.5}$$

It appears difficult to establish the global regularity of two other systems of the 2D Euler equations with partial dissipation,

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_{yy} u = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p = 0, \\ \partial_x u + \partial_y v = 0, \end{cases} \tag{1.6}$$

and

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \partial_{xx} v = 0, \\ \partial_x u + \partial_y v = 0. \end{cases} \tag{1.7}$$

It is easy to establish global bound for the vorticity ω in L^2, but our attempts for any global bound for ω in L^q with $q > 2$ have failed. Of course, if one is willing to add more partial dissipation, then the global regularity can be obtained. For example, the system

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_{yy} u = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \partial_{xx} v = 0, \\ \partial_x u + \partial_y v = 0. \end{cases} \tag{1.8}$$

does have a global classical solution.

Our last result of this paper concerns the global regularity for a 2D Boussinesq system of equations with partial dissipation. The Boussinesq equations model many geophysical flows such as atmospheric fronts and ocean circulations (see, e.g., [48,54]). In addition, they are at the center of turbulence theories concerning turbulent thermal convection (see, e.g., [21,38]). Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism and the inviscid 2D Boussinesq equations can be identified as the Euler equations for the 3D axisymmetric swirling flows [49]. Extensive recent efforts have been devoted to obtain the global regularity for various partial dissipation cases involving the 2D Boussinesq equations (see, e.g., ([1–3,10,35,46]).

Our attention here focuses on the following 2D Boussinesq equations with horizontal dissipation in the vertical velocity equation and vertical dissipation in the temperature equation

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \partial_{xx} v = \theta, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta - \partial_{yy}\theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \quad \theta(x,y,0) = \theta_0(x,y), \end{cases} \tag{1.9}$$

we establish several global bounds, which may be useful in the eventual resolution of whether or not (1.9) is globally well-posed.

Theorem 1.6. *Assume that $(\vec{u}_0, \theta_0) \in H^\sigma(\mathbb{R}^2)$ with $\sigma > 2$ and $\nabla \cdot \vec{u}_0 = 0$. Let $(\vec{u}, \theta)$ be the corresponding solution of (1.9). Then, $(\vec{u}, \theta)$ admits the following global bounds, for any $T > 0$ and $t \le T$,*

$$\|\vec{u}(t)\|_{H^1}^2 + \int_0^t \|\partial_x \nabla v(\tau)\|_{L^2}^2 \, d\tau \le C, \tag{1.10}$$

where $C = C(T, \vec{u}_0, \theta_0)$;

$$\|\theta(t)\|_{H^1}^2 + \int_0^t \|\partial_y \nabla \theta(\tau)\|_{L^2}^2 \, d\tau \le C, \tag{1.11}$$

where $C = C(T, \vec{u}_0, \theta_0)$;

$$\big\||\partial_y|^{1+s}\theta(t)\big\|_{L^2}^2 + \int_0^t \big\||\partial_y|^{2+s}\theta(\tau)\big\|_{L^2}^2 \, d\tau \le C, \quad \int_0^t \|\partial_y \theta(\tau)\|_{L^\infty}^2 \, d\tau \le C, \tag{1.12}$$

where $0 < s < \frac{1}{2}$ and $C = C(T, \vec{u}_0, \theta_0)$;

$$\|\partial_x \theta(t)\|_{L^q} \le C, \quad 2 \le q < \infty \tag{1.13}$$

where $C = C(T, q, \vec{u}_0, \theta_0)$.

The rest of this paper is organized as follows. Section 2 proves Theorem 1.1 while Sects. 3 and 4 present the proof of Theorems 1.4 and 1.6, respectively. Throughout the rest of the paper, C denotes various positive and finite constants whose exact values are unimportant and may vary from line to line.

2. The Proof of Theorem 1.1

The existence and uniqueness of local smooth solutions can be established without difficulty. Thus, in order to complete the proof of Theorem 1.1, it is sufficient to establish *a priori* estimates that hold for any fixed $T > 0$.

We first recall the following logarithmic Sobolev inequality which will play an important role in the proof of Theorem 1.1.

Lemma 2.1. *The following logarithmic Sobolev embedding inequality holds for all vector fields f with $f \in H^s(\mathbb{R}^2)$ and $s > 1$*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\Big(1 + \|f\|_{L^2(\mathbb{R}^2)} + \|f\|_{\mathrm{BMO}(\mathbb{R}^2)} \sqrt{\log(e + \|f\|_{\dot{H}^s(\mathbb{R}^2)})}\Big), \tag{2.1}$$

where BMO *denotes the homogenous space of bounded mean oscillations associated with the norm (see [59] for more details)*

$$\|f\|_{\mathrm{BMO}} \triangleq \sup_{x\in\mathbb{R}^2,\, r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)}\left|f(y)-\frac{1}{|B_r(y)|}\int_{B_r(y)} f(z)\,dz\right|\,dy.$$

Here $B_r(x)$ denotes the disk of radius r and center x in $\mathbb{R}^2$.

Remark 2.2. It is worthy to emphasize that the power $\frac{1}{2}$ of $\log(e+\|f\|_{\dot{H}^s(\mathbb{R}^2)})$ plays a key role in proving our theorem.

Proof of Lemma 2.1. Although the approach in part available in the literature (see, e.g. [53, Corollary 2.4]), yet for the convenience of the reader, we give detailed proof via the Littlewood-Paley decomposition. By the Littlewood-Paley decomposition, we can rewrite

$$f=\sum_{j=-\infty}^{\infty}\dot{\Delta}_j f,$$

where $\dot{\Delta}_j$ denotes the homogeneous Fourier localization operator. By Bernstein inequality (see, e.g. [4,16]),

$$\begin{aligned}\|f\|_{L^\infty} &\le \sum_{j=-\infty}^{-1}\|\dot{\Delta}_j f\|_{L^\infty}+\left\|\sum_{j=0}^{N-1}\dot{\Delta}_j f\right\|_{L^\infty}+\sum_{j=N}^{\infty}\|\dot{\Delta}_j f\|_{L^\infty}\\ &\le C\left(\sum_{j=-\infty}^{-1}2^j\|f\|_{L^2}+N^{\frac{1}{2}}\left\|\Big\{\sum_{j=0}^{N-1}(\dot{\Delta}_j f)^2\Big\}^{\frac{1}{2}}\right\|_{L^\infty}+\sum_{j=N}^{\infty}2^{j(1-s)}2^{js}\|\dot{\Delta}_j f\|_{L^2}\right)\\ &\le C(\|f\|_{L^2}+N^{\frac{1}{2}}\|f\|_{\mathrm{BMO}}+2^{N(1-s)}\|f\|_{\dot{H}^s}),\end{aligned} \tag{2.2}$$

where we have used the following estimate from Lemma 3.2 of [40]

$$\left\|\Big\{\sum_{j=0}^{N-1}(\dot{\Delta}_j f)^2\Big\}^{\frac{1}{2}}\right\|_{L^\infty}\le C\|f\|_{\mathrm{BMO}}.$$

Now taking an integer N such that $2^{N(1-s)}\|f\|_{\dot{H}^s}\approx 1$, we thus get

$$N=\left[\frac{1}{s-1}\log(e+\|f\|_{\dot{H}^s})\right]+1.$$

Substituting this fixed N into (2.2), we obtain the desired inequality (2.1). Therefore, we complete the proof of Lemma 2.1. □

The following anisotropic Sobolev inequalities (see [9] and [8]) will be frequently used later

Lemma 2.3. *The following anisotropic Sobolev inequalities hold,*

$$\int_{\mathbb{R}^2}|fgh|\,dxdy\le C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_x g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_y h\|_{L^2}^{\frac{1}{2}}, \tag{2.3}$$

$$\int_{\mathbb{R}^2}|fgh|\,dxdy\le C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_x g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{2\gamma-1}{2\gamma}}\big\||\partial_y|^{\gamma}h\big\|_{L^2}^{\frac{1}{2\gamma}},\quad \frac{1}{2}<\gamma\le 1. \tag{2.4}$$

Now let us proceed to prove Theorem 1.1.

Proof of Theorem 1.1. Multiplying (1.1) by θ, using the divergence-free condition and integrating with respect to the space variable, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2+\|\partial_x\theta\|_{L^2}^2=0.$$

Integrating with respect to time yields

$$\|\theta(t)\|_{L^2}^2 + \int_0^T \|\partial_x\theta(\tau)\|_{L^2}^2\, d\tau \le \|\theta_0\|_{L^2}^2 < \infty. \tag{2.5}$$

Multiplying (1.1) by $|\theta|^{p-2}\theta$ and using the divergence-free condition, we have

$$\|\theta(t)\|_{L^p}^p + p(p-1)\int_0^t \|\partial_x\theta|\theta|^{\frac{p-2}{2}}(\tau)\|_{L^2}^2\, d\tau = \|\theta_0\|_{L^p}^p.$$

As a consequence, we obtain that, for any $t \ge 0$,

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \quad \forall\, p \in [2,\infty]. \tag{2.6}$$

Taking the inner product of (1.1) with $\Delta\theta$, we derive

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \|\partial_x\nabla\theta\|_{L^2}^2 = -\int_{\mathbb{R}^2} \nabla(u\partial_x\theta + v\partial_y\theta)\cdot\nabla\theta\, dxdy. \tag{2.7}$$

By the divergence-free condition $\partial_x u + \partial_y v = 0$, we rewrite the righthand side of (2.7) as

$$-\int_{\mathbb{R}^2} \partial_x u\partial_x\theta\partial_x\theta\, dxdy - \int_{\mathbb{R}^2} \partial_x v\partial_y\theta\partial_x\theta\, dxdy - \int_{\mathbb{R}^2} \partial_y u\partial_x\theta\partial_y\theta\, dxdy - \int_{\mathbb{R}^2} \partial_y v\partial_y\theta\partial_y\theta\, dxdy. \tag{2.8}$$

Now we start to estimate each term of (2.8). To estimate the first term, we integrate by parts and use Young inequality to obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_x u\partial_x\theta\partial_x\theta\, dxdy &= -\int_{\mathbb{R}^2} \theta(\partial_{xx}u\partial_x\theta + \partial_x u\partial_{xx}\theta)\, dxdy\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xx}u\|_{L^2}\|\partial_x\theta\|_{L^2} + \|\partial_x u\|_{L^2}\|\partial_{xx}\theta\|_{L^2})\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xx}\mathcal{R}_2\theta\|_{L^2}\|\partial_x\theta\|_{L^2} + \|\partial_x\mathcal{R}_2\theta\|_{L^2}\|\partial_{xx}\theta\|_{L^2})\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xx}\theta\|_{L^2}\|\partial_x\theta\|_{L^2} + \|\partial_x\theta\|_{L^2}\|\partial_{xx}\theta\|_{L^2})\\
&\le \varepsilon\|\partial_x\nabla\theta\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2,
\end{aligned} \tag{2.9}$$

where we have used the Boundedness of Riesz transform on L^q $(1 < q < \infty)$ spaces.

Applying similar arguments as above, we obtain the following bounds

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_x v\partial_y\theta\partial_x\theta\, dxdy &= -\int_{\mathbb{R}^2} \theta(\partial_{xx}v\partial_y\theta + \partial_x v\partial_{xy}\theta)\, dxdy\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xx}v\|_{L^2}\|\partial_y\theta\|_{L^2} + \|\partial_x v\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xx}\mathcal{R}_1\theta\|_{L^2}\|\partial_y\theta\|_{L^2} + \|\partial_x\mathcal{R}_1\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xx}\theta\|_{L^2}\|\partial_y\theta\|_{L^2} + \|\partial_x\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\
&\le \varepsilon\|\partial_x\nabla\theta\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\int_{\mathbb{R}^2} \partial_y u\partial_x\theta\partial_y\theta\, dxdy &= -\int_{\mathbb{R}^2} \theta(\partial_{xy}u\partial_y\theta + \partial_y u\partial_{xy}\theta)\, dxdy\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xy}u\|_{L^2}\|\partial_y\theta\|_{L^2} + \|\partial_y u\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xy}\mathcal{R}_2\theta\|_{L^2}\|\partial_y\theta\|_{L^2} + \|\partial_y\mathcal{R}_2\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\
&\le C\|\theta\|_{L^\infty}(\|\partial_{xy}\theta\|_{L^2}\|\partial_y\theta\|_{L^2} + \|\partial_y\theta\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\
&\le \varepsilon\|\partial_x\nabla\theta\|_{L^2}^2 + C_\varepsilon\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{2.11}$$

Unfortunately, it seems difficult to estimate the last term $\int_{\mathbb{R}^2} \partial_y v \partial_y \theta \partial_y \theta \, dxdy$. To handle this term, we first obtain

$$
\begin{aligned}
-\int_{\mathbb{R}^2} \partial_y v \partial_y \theta \partial_y \theta \, dxdy &= \int_{\mathbb{R}^2} \partial_x u \partial_y \theta \partial_y \theta \, dxdy \\
&= -2\int_{\mathbb{R}^2} u \partial_y \theta \partial_{xy} \theta \, dxdy \\
&\le C\|u\|_{L^\infty} \|\partial_y \theta\|_{L^2} \|\partial_{xy}\theta\|_{L^2} \\
&\le \varepsilon \|\partial_x \nabla \theta\|_{L^2}^2 + C_\varepsilon \|u\|_{L^\infty}^2 \|\nabla\theta\|_{L^2}^2. \qquad (2.12)
\end{aligned}
$$

In order to close the estimates, we need

$$\int_0^T \|u(t)\|_{L^\infty}^2 \, dt \le C < \infty.$$

Due to the unboundedness of the Riesz transforms on L^∞, it is not clear if $\|u\|_{L^\infty}$ is finite, even though $\|\theta\|_{L^\infty}$ is bounded. To circumvent this difficulty, we adapt the "weakly nonlinear" energy estimate approach introduced by Lei and Zhou [47], which enables us to get "almost a priori" bounds for L^2 norms of $\nabla\theta$ (see (2.15) below).

For any $T > 0$, we assume the solution is regular for $t < T$ and show that it remains regular at $t = T$. For any $t \in (T_0, T)$ (here $T_0 \in (0, T)$ to be specified later), we denote

$$M(t) \triangleq \max_{\tau\in[T_0,\, t]} \|\Delta\theta(\tau)\|_{L^2}^2.$$

Plugging the estimates (2.9)–(2.12) into (2.7), and choosing sufficiently small ε, we conclude that

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \|\partial_x \nabla\theta\|_{L^2}^2 \le C_\varepsilon \|\theta\|_{L^\infty}^2 \|\nabla\theta\|_{L^2}^2 + C_\varepsilon \|u\|_{L^\infty}^2 \|\nabla\theta\|_{L^2}^2. \qquad (2.13)$$

Gronwall inequality tells us that for any $0 \le s \le t$

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_s^t \|\partial_x \nabla\theta(\tau)\|_{L^2}^2 \, d\tau \le \|\nabla\theta(s)\|_{L^2}^2 \exp\Big[C\int_s^t (\|\theta\|_{L^\infty}^2 + \|u\|_{L^\infty}^2)(\tau)\, d\tau\Big].$$

Applying the logarithmic Sobolev inequality (2.1), we have, for any $T_0 \le t < T$,

$$
\begin{aligned}
&\|\nabla\theta(t)\|_{L^2}^2 + \int_{T_0}^t \|\partial_x \nabla\theta(\tau)\|_{L^2}^2 \, d\tau \\
&\quad \le \|\nabla\theta(T_0)\|_{L^2}^2 \exp\Big[C\int_{T_0}^t \Big(1 + \|\theta\|_{L^\infty} + \|u\|_{L^2} + \|u\|_{\mathrm{BMO}}\sqrt{\log(1+\|\Delta u\|_{L^2})}\Big)^2 d\tau\Big] \\
&\quad \le \|\nabla\theta(T_0)\|_{L^2}^2 \exp\Big[C\int_{T_0}^t \Big(1 + \|\theta\|_{L^\infty}^2 + \|u\|_{L^2}^2 + \|u\|_{\mathrm{BMO}}^2 \log(1+\|\Delta u\|_{L^2})\Big) d\tau\Big] \\
&\quad \le \|\nabla\theta(T_0)\|_{L^2}^2 \exp\Big[C\int_{T_0}^t \Big(1 + \|\theta\|_{L^\infty}^2 + \|\theta\|_{L^2}^2 + \|\theta\|_{L^\infty}^2 \log(1+\|\Delta \theta\|_{L^2})\Big) d\tau\Big] \\
&\quad \le \|\nabla\theta(T_0)\|_{L^2}^2 \exp\Big[C\int_{T_0}^t \Big(\|\theta(\tau)\|_{L^\infty}^2 \log(1+M(t))\Big) d\tau\Big] \\
&\quad \le \|\nabla\theta(T_0)\|_{L^2}^2 \exp\Big[C\int_{T_0}^t \|\theta(\tau)\|_{L^\infty}^2 \, d\tau \log(1+M(t))\Big], \qquad (2.14)
\end{aligned}
$$

where we have used the boundedness of Riesz transforms from L^∞ to BMO spaces (see, e.g., [58]), namely

$$\|\mathcal{R}f\|_{\mathrm{BMO}} \le C\|f\|_{L^\infty},$$

and the simple fact

$$\|\mathcal{R}f\|_{L^2} \le C\|f\|_{L^2}.$$

Note the following bound

$$\|\theta(t)\|_{L^\infty} \le C < \infty, \quad \text{for any } t \ge 0.$$

Thus we can choose T_0 close enough to T such that

$$C\int_{T_0}^{t} \|\theta(\tau)\|_{L^\infty}^2 \, d\tau \le \kappa$$

for sufficiently small $\kappa > 0$ to be specified later. Therefore,

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_{T_0}^{t} \|\partial_x \nabla\theta(\tau)\|_{L^2}^2 \, d\tau \le C(1 + M(t))^\kappa, \quad \text{for any } T_0 \le t < T. \tag{2.15}$$

Taking Δ to the first equation of (1.1) and testing by $\Delta\theta$, we see that

$$\frac{1}{2}\frac{d}{dt}\|\Delta\theta(t)\|_{L^2}^2 + \|\partial_x \Delta\theta\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta\{(u\partial_x\theta + v\partial_y\theta)\}\Delta\theta \, dxdy. \tag{2.16}$$

Taking the divergence free condition into account, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \Delta\{(u\partial_x\theta + v\partial_y\theta)\}\Delta\theta \, dxdy &= \int_{\mathbb{R}^2} (\Delta u\partial_x\theta + \Delta v\partial_y\theta + 2\partial_x u\partial_{xx}\theta + 2\partial_y u\partial_{xy}\theta + 2\partial_x v\partial_{xy}\theta + 2\partial_y v\partial_{yy}\theta)\Delta\theta \, dxdy\\ &\triangleq K_1 + K_2 + \cdots + K_6.\end{aligned} \tag{2.17}$$

Similar to the estimate (2.9), we can bound K_1 as

$$\begin{aligned}K_1 &= \int_{\mathbb{R}^2} \Delta u\partial_x\theta\Delta\theta \, dxdy\\ &= -\int_{\mathbb{R}^2} \theta(\partial_x\Delta u\Delta\theta + \Delta u\partial_x\Delta\theta) \, dxdy\\ &\le C\|\theta\|_{L^\infty}(\|\partial_x\Delta\mathcal{R}_2\theta\|_{L^2}\|\Delta\theta\|_{L^2} + \|\Delta\mathcal{R}_2\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2})\\ &\le C\|\theta\|_{L^\infty}(\|\partial_x\Delta\theta\|_{L^2}\|\Delta\theta\|_{L^2} + \|\Delta\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2})\\ &\le \epsilon\|\partial_x\Delta\theta\|_{L^2}^2 + C_\epsilon\|\theta\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2.\end{aligned} \tag{2.18}$$

Note the following fact

$$\begin{aligned}\|\Lambda f\|_{L^2}^2 &= \|\widehat{\Lambda f}(\xi)\|_{L^2}^2 = \int_{\mathbb{R}^2} |\xi|^2|\widehat{f}|^2(\xi) \, d\xi = \int_{\mathbb{R}^2} |\xi_1|^2|\widehat{f}|^2(\xi) \, d\xi + \int_{\mathbb{R}^2} |\xi_2|^2|\widehat{f}|^2(\xi) \, d\xi\\ &= \|\widehat{\partial_x f}(\xi)\|_{L^2}^2 + \|\widehat{\partial_y f}(\xi)\|_{L^2}^2\\ &= \|\partial_x f\|_{L^2}^2 + \|\partial_y f\|_{L^2}^2\\ &= \|\nabla f\|_{L^2}^2.\end{aligned}$$

The relation between $(u,\, v)$ and θ allows us to show

$$\|\Delta v\|_{L^2} = \|\Delta\frac{\partial_x}{(-\Delta)^{\frac{1}{2}}}\theta\|_{L^2} = \|\partial_x\Lambda\theta\|_{L^2} = \|\partial_x\nabla\theta\|_{L^2}$$

and

$$\|\partial_y\Delta v\|_{L^2} = \|\partial_x\Delta u\|_{L^2} \le C\|\partial_x\Delta\theta\|_{L^2}.$$

The combination of the above facts with the anisotropic Sobolev inequality (2.3) thus leads to

$$\begin{aligned}K_2 &= \int_{\mathbb{R}^2} \Delta v\partial_y\theta\Delta\theta \, dxdy\\ &\le C\|\Delta\theta\|_{L^2}\|\partial_y\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\frac{1}{2}}\|\partial_y\Delta v\|_{L^2}^{\frac{1}{2}}\\ &\le C\|\Delta\theta\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_x\nabla\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2}^{\frac{1}{2}}\\ &\le \epsilon\|\partial_x\Delta\theta\|_{L^2}^2 + C_\epsilon\|\nabla\theta\|_{L^2}^{\frac{2}{3}}\|\partial_x\nabla\theta\|_{L^2}^{\frac{4}{3}}\|\Delta\theta\|_{L^2}^{\frac{4}{3}}.\end{aligned} \tag{2.19}$$

We get by integrating by parts and applying the Young inequality

$$\begin{aligned} K_3 &= 2\int_{\mathbb{R}^2} \partial_x u \partial_{xx}\theta \Delta\theta \, dxdy \\ &= -2\int_{\mathbb{R}^2} u(\partial_x\partial_{xx}\theta\Delta\theta + \partial_{xx}\theta\partial_x\Delta\theta)\, dxdy \\ &\le C\|u\|_{L^\infty}\|\Delta\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2} \\ &\le \epsilon\|\partial_x\Delta\theta\|_{L^2}^2 + C_\epsilon\|u\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2. \end{aligned} \tag{2.20}$$

We again resort to the anisotropic Sobolev inequality (2.3) to obtain

$$\begin{aligned} K_4 &= 2\int_{\mathbb{R}^2} \partial_y u \partial_{xy}\theta \Delta\theta \, dxdy \\ &\le C\|\Delta\theta\|_{L^2}\|\partial_y u\|_{L^2}^{\frac{1}{2}}\|\partial_{xy} u\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xyy}\theta\|_{L^2}^{\frac{1}{2}} \\ &\le C\|\Delta\theta\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\partial_x\nabla\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2}^{\frac{1}{2}} \\ &\le \epsilon\|\partial_x\Delta\theta\|_{L^2}^2 + C_\epsilon\|\nabla\theta\|_{L^2}^{\frac{2}{3}}\|\partial_x\nabla\theta\|_{L^2}^{\frac{4}{3}}\|\Delta\theta\|_{L^2}^{\frac{4}{3}}. \end{aligned} \tag{2.21}$$

Similar to the bound (2.20),

$$\begin{aligned} K_5 &= 2\int_{\mathbb{R}^2} \partial_x v \partial_{xy}\theta \Delta\theta \, dxdy \\ &= -2\int_{\mathbb{R}^2} v(\partial_{xxy}\theta\Delta\theta + \partial_{xy}\theta\partial_x\Delta\theta)\, dxdy \\ &\le C\|v\|_{L^\infty}\|\Delta\theta\|_{L^2}\|\partial_x\Delta\theta\|_{L^2} \\ &\le \epsilon\|\partial_x\Delta\theta\|_{L^2}^2 + C_\epsilon\|v\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2. \end{aligned} \tag{2.22}$$

Utilizing the divergence free condition and the Young inequality, the last term can be estimated as

$$\begin{aligned} K_6 &= 2\int_{\mathbb{R}^2} \partial_y v \partial_{yy}\theta \Delta\theta \, dxdy \\ &= -2\int_{\mathbb{R}^2} \partial_x u \partial_{yy}\theta \Delta\theta \, dxdy \\ &= 2\int_{\mathbb{R}^2} u(\partial_{xyy}\theta\Delta\theta + \partial_{yy}\theta\partial_x\Delta\theta)\, dxdy \\ &\le C\|u\|_{L^\infty}\|\partial_x\Delta\theta\|_{L^2}\|\Delta\theta\|_{L^2} \\ &\le \epsilon\|\partial_x\Delta\theta\|_{L^2}^2 + C_\epsilon\|u\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2. \end{aligned} \tag{2.23}$$

Inserting the estimates for K_1 to K_6 in (2.16) and taking ϵ small enough, we obtain

$$\begin{aligned} \frac{d}{dt}\|\Delta\theta(t)\|_{L^2}^2 + \|\partial_x\Delta\theta\|_{L^2}^2 &\le C\|\theta\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2 + +C\|\nabla\theta\|_{L^2}^{\frac{2}{3}}\|\partial_x\nabla\theta\|_{L^2}^{\frac{4}{3}}\|\Delta\theta\|_{L^2}^{\frac{4}{3}} \\ &\quad + C\|u\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2 + C\|v\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2. \end{aligned} \tag{2.24}$$

Applying the logarithmic Sobolev inequality (2.1), we have

$$\begin{aligned}
\frac{d}{dt}\|\Delta\theta(t)\|_{L^2}^2+\|\partial_x\Delta\theta\|_{L^2}^2 &\le C\|\theta\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^{\frac{2}{3}}\|\partial_x\nabla\theta\|_{L^2}^{\frac{4}{3}}\|\Delta\theta\|_{L^2}^{\frac{4}{3}}\\
&\quad+C\Big(1+\|u\|_{L^2}+\|u\|_{\mathrm{BMO}}\sqrt{\log(1+\|\Delta u\|_{L^2})}\Big)^2\|\Delta\theta\|_{L^2}^2\\
&\quad+C\Big(1+\|v\|_{L^2}+\|v\|_{\mathrm{BMO}}\sqrt{\log(1+\|\Delta v\|_{L^2})}\Big)^2\|\Delta\theta\|_{L^2}^2\\
&\le C\|\theta\|_{L^\infty}^2\|\Delta\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^2}^{\frac{2}{3}}\|\partial_x\nabla\theta\|_{L^2}^{\frac{4}{3}}\|\Delta\theta\|_{L^2}^{\frac{4}{3}}\\
&\quad+C\Big(1+\|u\|_{L^2}^2+\|u\|_{\mathrm{BMO}}^2\log(1+\|\Delta u\|_{L^2})\Big)\|\Delta\theta\|_{L^2}^2\\
&\quad+C\Big(1+\|v\|_{L^2}^2+\|v\|_{\mathrm{BMO}}^2\log(1+\|\Delta v\|_{L^2})\Big)\|\Delta\theta\|_{L^2}^2\\
&\le C\|\theta\|_{L^\infty}^2 M(t)+C\|\partial_x\nabla\theta\|_{L^2}^{\frac{4}{3}}(1+M(t))^{\frac{2}{3}+\frac{\kappa}{3}}\\
&\quad+C\Big(1+\|\theta\|_{L^2}^2+\|\theta\|_{L^\infty}^2\log\big(1+M(t)\big)\Big)M(t)\\
&\le C\|\partial_x\nabla\theta\|_{L^2}^{\frac{4}{3}}(1+M(t))^{\frac{2}{3}+\frac{\kappa}{3}}+C\Big(1+\|\theta\|_{L^2}^2+C\|\theta\|_{L^\infty}^2\log\big(1+M(t)\big)\Big)M(t),
\end{aligned}$$

where we also have used

$$\|\mathcal{R}f\|_{\mathrm{BMO}}\le C\|f\|_{L^\infty}\quad\text{and}\quad\|\mathcal{R}f\|_{L^2}=\|f\|_{L^2}.$$

Integrating above inequality over interval (T_0,t) and observing that $M(t)$ is a monotonically increasing function, we thus obtain

$$\begin{aligned}
M(t)-M(T_0)&\le C\int_{T_0}^t\Big(1+\|\theta\|_{L^\infty}^2\log(1+M(s))\Big)\big(1+M(s)\big)\,ds\\
&\quad+C\int_{T_0}^t\Big(1+M(s)\Big)^{\frac{2+\kappa}{3}}\|\partial_x\nabla\theta(s)\|_{L^2}^{\frac{4}{3}}\,ds\\
&\le C\int_{T_0}^t\Big(1+\|\theta\|_{L^\infty}^2\log(1+M(s))\Big)\big(1+M(s)\big)\,ds\\
&\quad+C\Big(1+M(t)\Big)^{\frac{2+\kappa}{3}}\int_{T_0}^t\|\partial_x\nabla\theta(s)\|_{L^2}^{\frac{4}{3}}\,ds\\
&\le C\int_{T_0}^t\Big(1+\|\theta\|_{L^\infty}^2\log(1+M(s))\Big)\big(1+M(s)\big)\,ds\\
&\quad+C(T-T_0)^{\frac{1}{3}}\Big(1+M(t)\Big)^{\frac{2+3\kappa}{3}}.
\end{aligned}\tag{2.25}$$

Now we take $0<\kappa<\frac{1}{3}$ and then apply Young's inequality to obtain

$$C(T-T_0)^{\frac{1}{3}}\Big(1+M(t)\Big)^{\frac{2+3\kappa}{3}}\le C+\frac{1}{2}\big(1+M(t)\big).$$

As a consequence, the following inequality holds

$$1+M(t)\le C+C\int_{T_0}^t\Big(1+\|\theta\|_{L^\infty}^2\log(1+M(s)\Big)\big(1+M(s)\big)\,ds.$$

Letting

$$H(t)\triangleq C+C\int_{T_0}^t\Big(1+\|\theta\|_{L^\infty}^2\log(1+M(s)\Big)\big(1+M(s)\big)\,ds$$

and noting that

$$1+M(t)\le H(t),$$

we have

$$\begin{aligned}\frac{d}{dt}H(t) &= C\Big(1+\|\theta\|_{L^\infty}^2\log(1+M(t)\Big)\big(1+M(t)\big)\\ &\le C(1+\|\theta\|_{L^\infty}^2)\log\Big(1+M(t)\Big)\big(1+M(t)\big)\\ &\le C(1+\|\theta\|_{L^\infty}^2)H(t)\log H(t).\end{aligned}$$

Therefore, for all $t\in[T_0,\,T)$

$$H(t)\le H(T_0)\exp\exp\Big(C\int_{T_0}^t\big(1+\|\theta(\tau)\|_{L^\infty}^2\big)\,d\tau\Big)\le H(T_0)\exp\exp(CT+\kappa),$$

or

$$M(T)\le\big(1+M(T_0)\big)\exp\exp(CT+\kappa)-1<\infty,$$

which gives rise to

$$\max_{0\le t\le T}\|\Delta\theta(t)\|_{L^2}^2\le C<\infty.$$

Recalling the inequality (2.24), we can deduce from above bound that

$$\|\Delta\theta(t)\|_{L^2}^2+\int_0^T\|\partial_x\Delta\theta(\tau)\|_{L^2}^2\,d\tau\le C(T,\,T_0,\,\theta(T_0),\,\theta_0)<\infty. \tag{2.26}$$

It follows from the above bound (2.26) and the following inequality (see [35])

$$\|h\|_{L^\infty}\le C(\|h\|_{L^2}+\|\partial_y h\|_{L^2}+\|\partial_{xx}h\|_{L^2})$$

that

$$\begin{aligned}\int_0^T\|\nabla\theta(t)\|_{L^\infty}\,dt &\le C\int_0^T\big(\|\nabla\theta(t)\|_{L^2}+\|\partial_y\nabla\theta(t)\|_{L^2}+\|\partial_{xx}\nabla\theta(t)\|_{L^2}\big)\,dt\\ &\le C\int_0^T\big(\|\nabla\theta(t)\|_{L^2}+\|\nabla^2\theta(t)\|_{L^2}+\|\partial_x\Delta\theta(t)\|_{L^2}\big)\,dt\\ &\le C(T,\,\theta_0)<\infty,\end{aligned}$$

which is enough for high regularity as shown in [23,62]. This fact implies that the solution is regular at $t=T$. Moreover, the uniqueness is clear. Thus, we have completed the proof of Theorem 1.1. □

We end up this section with the following remark.

Remark 2.4. The method adopted in proving Theorem 1.1 may also be adapted with almost no change to the study of the following 2D incompressible porous medium equation with partial dissipation:

$$\begin{cases}\partial_t\theta+(\vec{u}\cdot\nabla)\theta-\partial_{xx}\theta=0, \quad (x,y)\in\mathbb{R}^2, \quad t>0,\\ \vec{u}=-\nabla p-\theta e_2,\\ \nabla\cdot\vec{u}=0,\\ \theta(x,0)=\theta_0(x).\end{cases} \tag{2.27}$$

Actually, combining the equation $\vec{u}=-\nabla p-\theta e_2$ and the incompressible condition $\nabla\cdot\vec{u}=0$, one can easily deduce

$$\vec{u}=(-\mathcal{R}_1\mathcal{R}_2\theta,\ \mathcal{R}_1\mathcal{R}_1\theta).$$

Consequently, by the method adopted in proving Theorem 1.1, we can show that the system (2.27) admits a unique global smooth solution.

3. The Proof of Theorem 1.4

This section proves Theorem 1.4. The focus is on how to obtain the global *a priori* bounds for the solution on any time interval $[0, T]$.

Proof of Theorem 1.4. The basic energy estimate entails that

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}(t)\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 \le 0.$$

Integrating in time yields

$$\|\vec{u}(t)\|_{L^2}^2 + \int_0^T \|\partial_x u(\tau)\|_{L^2}^2\, d\tau \le C(u_0,\, v_0) < \infty. \tag{3.1}$$

Taking the inner product of the first equation in (1.3) with Δu and the second equation in (1.3) with Δv, integrating over $\mathbb{R}^2$ in variable x and then adding them up, we deduce

$$\frac{1}{2}\frac{d}{dt}\|\nabla\vec{u}(t)\|_{L^2}^2 + \|\partial_x \nabla u\|_{L^2}^2 = 0, \tag{3.2}$$

where we have used the identity

$$\int_{\mathbb{R}^2} (\vec{u}\cdot\nabla\vec{u})\cdot\Delta\vec{u}\, dxdy = 0.$$

The above identity can be proved as follows. In the case dimension is two, we have

$$\Delta\vec{u} = \nabla^{\perp}\omega, \quad \nabla^{\perp} = (-\partial_y,\ \partial_x)$$

and

$$\nabla^{\perp}\cdot(\vec{u}\cdot\nabla\vec{u}) = \vec{u}\cdot\nabla\omega,$$

which leads to

$$\begin{aligned}
\int_{\mathbb{R}^2} (\vec{u}\cdot\nabla\vec{u})\cdot\Delta\vec{u}\, dxdy &= \int_{\mathbb{R}^2} (\vec{u}\cdot\nabla\vec{u})\cdot\nabla^{\perp}\omega\, dxdy \\
&= -\int_{\mathbb{R}^2} \nabla^{\perp}\cdot(\vec{u}\cdot\nabla\vec{u})\omega\, dxdy \\
&= -\int_{\mathbb{R}^2} (\vec{u}\cdot\nabla\omega)\omega\, dxdy \\
&= 0.
\end{aligned} \tag{3.3}$$

Integrating over $(0,\, t)$ with respect to the time variable leads to

$$\|\nabla\vec{u}(t)\|_{L^2}^2 + \int_0^T \|\partial_x\nabla u(\tau)\|_{L^2}^2\, d\tau \le C(u_0,\, v_0) < \infty. \tag{3.4}$$

In order to obtain the a priori global H^2 bound, we will apply Δ to the Eqs. $(1.3)_1$ and $(1.3)_2$, respectively, then we get

$$\begin{cases} \partial_t\Delta u + \Delta\partial_x p - \Delta\partial_{xx} u = -\Delta\{(\vec{u}\cdot\nabla)u\}, \\ \partial_t\Delta v + \Delta\partial_y p = -\Delta\{(\vec{u}\cdot\nabla)v\}. \end{cases} \tag{3.5}$$

Taking the inner products of $(3.5)_1$ with Δu and $(3.5)_2$ with Δv, adding the results and integrating by parts, it is easy to show

$$\frac{1}{2}\frac{d}{dt}\|\Delta\vec{u}(t)\|_{L^2}^2 + \|\partial_x\Delta u\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta\{(\vec{u}\cdot\nabla)u\}\Delta u\, dxdy - \int_{\mathbb{R}^2} \Delta\{(\vec{u}\cdot\nabla)v\}\Delta v\, dxdy. \tag{3.6}$$

In what follows, we will deal with each term on the right-hand side of (3.6) separately.

Thanks to the divergence-free condition, the first term can be rewritten as follows

$$\int_{\mathbb{R}^2} \Delta\{(\vec{u}\cdot\nabla)u\}\Delta u\,dxdy = \int_{\mathbb{R}^2} (\Delta u\partial_x u + \Delta v\partial_y u + 2\partial_x u\partial_{xx}u + 2\partial_y u\partial_{xy}u + 2\partial_x v\partial_{xy}u + 2\partial_y v\partial_{yy}u)\Delta u\,dxdy \triangleq N_1 + N_2 + \cdots + N_6. \tag{3.7}$$

With the aid of the inequality (2.3), the above six terms can be estimated as follows.

$$\begin{aligned} N_1 &= \int_{\mathbb{R}^2} \Delta u\partial_x u\Delta u\,dxdy \\ &\le C\|\Delta u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_x u\|_{L^2}^{\frac12}\|\partial_y\partial_x u\|_{L^2}^{\frac12} \\ &\le \varepsilon\|\partial_x\Delta u\|_{L^2}^2 + C_\varepsilon\|\partial_x u\|_{L^2}^{\frac23}\|\partial_{xy}u\|_{L^2}^{\frac23}\|\Delta u\|_{L^2}^2, \end{aligned} \tag{3.8}$$

$$\begin{aligned} N_2 &= \int_{\mathbb{R}^2} \Delta v\partial_y u\Delta u\,dxdy \\ &\le C\|\partial_y u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\Delta v\|_{L^2}^{\frac12}\|\partial_y\Delta v\|_{L^2}^{\frac12} \\ &= C\|\partial_y u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}\|\Delta v\|_{L^2}^{\frac12} \\ &\le \varepsilon\|\partial_x\Delta u\|_{L^2}^2 + C_\varepsilon\|\partial_y u\|_{L^2}^2(\|\Delta u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2), \end{aligned} \tag{3.9}$$

$$\begin{aligned} N_3 &= 2\int_{\mathbb{R}^2} \partial_x u\partial_{xx}u\Delta u\,dxdy \\ &\le C\|\partial_x u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_{xx}u\|_{L^2}^{\frac12}\|\partial_{yxx}u\|_{L^2}^{\frac12} \\ &\le C\|\partial_x u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_{xx}u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12} \\ &\le \varepsilon\|\partial_x\Delta u\|_{L^2}^2 + C_\varepsilon\|\partial_x u\|_{L^2}^2\|\partial_{xx}u\|_{L^2}\|\Delta u\|_{L^2}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} N_4 &= 2\int_{\mathbb{R}^2} \partial_y u\partial_{xy}u\Delta u\,dxdy \\ &\le C\|\partial_y u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_{xy}u\|_{L^2}^{\frac12}\|\partial_{xyy}u\|_{L^2}^{\frac12} \\ &\le C\|\partial_y u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_{xy}u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12} \\ &\le \varepsilon\|\partial_x\Delta u\|_{L^2}^2 + C_\varepsilon\|\partial_y u\|_{L^2}^2\|\partial_{xy}u\|_{L^2}\|\Delta u\|_{L^2}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} N_5 &= 2\int_{\mathbb{R}^2} \partial_x v\partial_{xy}u\Delta u\,dxdy \\ &\le C\|\partial_x v\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_{xy}u\|_{L^2}^{\frac12}\|\partial_{xyy}u\|_{L^2}^{\frac12} \\ &\le C\|\partial_y v\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_{xy}u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12} \\ &\le \varepsilon\|\partial_x\Delta u\|_{L^2}^2 + C_\varepsilon\|\partial_y v\|_{L^2}^2\|\partial_{xy}u\|_{L^2}\|\Delta u\|_{L^2}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} N_6 &= 2\int_{\mathbb{R}^2} \partial_y v\partial_{yy}u\Delta u\,dxdy \\ &\le C\|\partial_{yy}u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_y v\|_{L^2}^{\frac12}\|\partial_{yy}v\|_{L^2}^{\frac12} \\ &\le C\|\Delta u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x\Delta u\|_{L^2}^{\frac12}\|\partial_y v\|_{L^2}^{\frac12}\|\partial_{xy}u\|_{L^2}^{\frac12} \\ &\le \varepsilon\|\partial_x\Delta u\|_{L^2}^2 + C_\varepsilon\|\partial_y v\|_{L^2}^{\frac23}\|\partial_{xy}u\|_{L^2}^{\frac23}\|\Delta u\|_{L^2}^2. \end{aligned} \tag{3.13}$$

Similarly, the second term can be rewritten as

$$\int_{\mathbb{R}^2} \Delta\{(\vec{u}\cdot\nabla)v\}\Delta v\,dxdy = \int_{\mathbb{R}^2} (\Delta u\partial_x v + \Delta v\partial_y v + 2\partial_x u\partial_{xx}v + 2\partial_y u\partial_{xy}v + 2\partial_x v\partial_{xy}v + 2\partial_y v\partial_{yy}v)\Delta v\,dxdy \triangleq L_1 + L_2 + \cdots + L_6. \tag{3.14}$$

We again resort to (2.3) to obtain

$$
\begin{aligned}
L_1 &= \int_{\mathbb{R}^2} \Delta u \partial_x v \Delta v \, dxdy \\
&\le C\|\partial_x v\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\|\partial_x \Delta u\|_{L^2}^{\frac12}\|\Delta v\|_{L^2}^{\frac12}\|\partial_y \Delta v\|_{L^2}^{\frac12} \\
&\le \varepsilon\|\partial_x \Delta u\|_{L^2}^2 + C_\varepsilon \|\partial_x v\|_{L^2}^2(\|\Delta u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2), \qquad (3.15)\\
L_2 &= \int_{\mathbb{R}^2} \Delta v \partial_y v \Delta v \, dxdy \\
&\le C\|\Delta v\|_{L^2}\|\Delta v\|_{L^2}^{\frac12}\|\partial_y \Delta v\|_{L^2}^{\frac12}\|\partial_y v\|_{L^2}^{\frac12}\|\partial_{xy} v\|_{L^2}^{\frac12} \\
&= C\|\Delta v\|_{L^2}^{\frac32}\|\partial_x \Delta u\|_{L^2}^{\frac12}\|\partial_y v\|_{L^2}^{\frac12}\|\partial_{xx} u\|_{L^2}^{\frac12} \\
&\le \varepsilon\|\partial_x \Delta u\|_{L^2}^2 + C_\varepsilon \|\partial_y v\|_{L^2}^{\frac23}\|\partial_{xx} u\|_{L^2}^{\frac23}\|\Delta v\|_{L^2}^2, \qquad (3.16)\\
L_3 &= 2\int_{\mathbb{R}^2} \partial_x u \partial_{xx} v \Delta v \, dxdy \\
&\le C\|\Delta v\|_{L^2}\|\partial_{xx} v\|_{L^2}^{\frac12}\|\partial_{xxy} v\|_{L^2}^{\frac12}\|\partial_x u\|_{L^2}^{\frac12}\|\partial_{xx} u\|_{L^2}^{\frac12} \\
&\le C\|\Delta v\|_{L^2}^{\frac32}\|\partial_x \Delta u\|_{L^2}^{\frac12}\|\partial_x u\|_{L^2}^{\frac12}\|\partial_{xx} u\|_{L^2}^{\frac12} \\
&\le \varepsilon\|\partial_x \Delta u\|_{L^2}^2 + C_\varepsilon \|\partial_x u\|_{L^2}^{\frac23}\|\partial_{xx} u\|_{L^2}^{\frac23}\|\Delta v\|_{L^2}^2, \qquad (3.17)\\
L_4 &= 2\int_{\mathbb{R}^2} \partial_y u \partial_{xy} v \Delta v \, dxdy \\
&\le C\|\Delta v\|_{L^2}\|\partial_{xy} v\|_{L^2}^{\frac12}\|\partial_{xyy} v\|_{L^2}^{\frac12}\|\partial_y u\|_{L^2}^{\frac12}\|\partial_{xy} u\|_{L^2}^{\frac12} \\
&\le C\|\Delta v\|_{L^2}^{\frac32}\|\partial_x \Delta u\|_{L^2}^{\frac12}\|\partial_y u\|_{L^2}^{\frac12}\|\partial_{xy} u\|_{L^2}^{\frac12} \\
&\le \varepsilon\|\partial_x \Delta u\|_{L^2}^2 + C_\varepsilon \|\partial_y u\|_{L^2}^{\frac23}\|\partial_{xy} u\|_{L^2}^{\frac23}\|\Delta v\|_{L^2}^2, \qquad (3.18)\\
L_5 &= 2\int_{\mathbb{R}^2} \partial_x v \partial_{xy} v \Delta v \, dxdy \\
&\le C\|\Delta v\|_{L^2}\|\partial_{xy} v\|_{L^2}^{\frac12}\|\partial_{xxy} v\|_{L^2}^{\frac12}\|\partial_x v\|_{L^2}^{\frac12}\|\partial_{xy} v\|_{L^2}^{\frac12} \\
&\le C\|\Delta v\|_{L^2}\|\partial_x \Delta u\|_{L^2}^{\frac12}\|\partial_x v\|_{L^2}^{\frac12}\|\partial_{xx} u\|_{L^2} \\
&\le \varepsilon\|\partial_x \Delta u\|_{L^2}^2 + C_\varepsilon \|\partial_x v\|_{L^2}^{\frac23}\|\partial_{xx} u\|_{L^2}^{\frac43}\|\Delta v\|_{L^2}^{\frac43}, \qquad (3.19)\\
L_6 &= 2\int_{\mathbb{R}^2} \partial_y v \partial_{yy} v \Delta v \, dxdy \\
&\le C\|\Delta v\|_{L^2}\|\partial_{yy} v\|_{L^2}^{\frac12}\|\partial_{xyy} v\|_{L^2}^{\frac12}\|\partial_y v\|_{L^2}^{\frac12}\|\partial_{yy} v\|_{L^2}^{\frac12} \\
&\le C\|\Delta v\|_{L^2}^{\frac32}\|\partial_x \Delta u\|_{L^2}^{\frac12}\|\partial_y v\|_{L^2}^{\frac12}\|\partial_{xy} u\|_{L^2}^{\frac12} \\
&\le \varepsilon\|\partial_x \Delta u\|_{L^2}^2 + C_\varepsilon \|\partial_y v\|_{L^2}^{\frac23}\|\partial_{xy} u\|_{L^2}^{\frac23}\|\Delta v\|_{L^2}^2. \qquad (3.20)
\end{aligned}
$$

Putting all the above estimates into (3.6) and taking ε small enough yields

$$
\frac{d}{dt}\|\Delta \vec{u}(t)\|_{L^2}^2 + \|\partial_x \Delta u\|_{L^2}^2 \le CH(t)\|\Delta \vec{u}\|_{L^2}^2,
$$

where

$$
\begin{aligned}
H(t) = C(&\|\partial_x u\|_{L^2}^2 + \|\partial_y u\|_{L^2}^2 + \|\partial_x v\|_{L^2}^2 + \|\partial_y v\|_{L^2}^{\frac23}\|\partial_{xx} u\|_{L^2}^{\frac23} + \|\partial_y u\|_{L^2}^{\frac23}\|\partial_{xy} u\|_{L^2}^{\frac23} \\
&+ \|\partial_x v\|_{L^2}^{\frac23}\|\partial_{xx} u\|_{L^2}^{\frac43} + \|\partial_x u\|_{L^2}^{\frac23}\|\partial_{xy} u\|_{L^2}^{\frac23})
\end{aligned}
$$

is an integrable function, namely

$$\int_0^T H(t)\,dt \le C(T,\,u_0,\,v_0) < \infty.$$

Gronwall's inequality gives, for any $0 \le t \le T$,

$$\|\Delta\vec{u}(t)\|_{L^2}^2 + \int_0^t \|\partial_x \Delta u(\tau)\|_{L^2}^2\,d\tau \le C(T,\,u_0,\,v_0) < \infty.$$

Thus, we obtain the desired bound of Theorem 1.4. Invoking the following inequalities (see [35])

$$\|h\|_{L^\infty} \le C(\|h\|_{L^2} + \|\partial_x h\|_{L^2} + \|\partial_{yy} h\|_{L^2}),$$
$$\|h\|_{L^\infty} \le C(\|h\|_{L^2} + \|\partial_y h\|_{L^2} + \|\partial_{xx} h\|_{L^2}),$$

we have

$$\begin{aligned}
\int_0^T \|\partial_y v(t)\|_{L^\infty}\,dt &= \int_0^T \|\partial_x u(t)\|_{L^\infty}\,dt \\
&\le C\int_0^T \big(\|\partial_x u(t)\|_{L^2} + \|\partial_y\partial_x u(t)\|_{L^2} + \|\partial_{xx}\partial_x u(t)\|_{L^2}\big)\,dt \\
&\le C\int_0^T \big(\|\nabla u(t)\|_{L^2} + \|\Delta u(t)\|_{L^2} + \|\partial_x \Delta u(t)\|_{L^2}\big)\,dt \\
&\le C(T,\,u_0,\,v_0) < \infty, \\
\int_0^T \|\partial_y u(t)\|_{L^\infty}\,dt &\le C\int_0^T \big(\|\partial_y u(t)\|_{L^2} + \|\partial_x\partial_y u(t)\|_{L^2} + \|\partial_{xx}\partial_y u(t)\|_{L^2}\big)\,dt \\
&\le C\int_0^T \big(\|\nabla u(t)\|_{L^2} + \|\Delta u(t)\|_{L^2} + \|\partial_x \Delta u(t)\|_{L^2}\big)\,dt \\
&\le C(T,\,u_0,\,v_0) < \infty, \\
\int_0^T \|\partial_x v(t)\|_{L^\infty}\,dt &\le C\int_0^T \big(\|\partial_x v(t)\|_{L^2} + \|\partial_x\partial_x v(t)\|_{L^2} + \|\partial_{yy}\partial_x v(t)\|_{L^2}\big)\,dt \\
&\le C\int_0^T \big(\|\partial_x v(t)\|_{L^2} + \|\partial_x\partial_x v(t)\|_{L^2} + \|\partial_{xy}\partial_x u(t)\|_{L^2}\big)\,dt \\
&\le C\int_0^T \big(\|\nabla u(t)\|_{L^2} + \|\Delta u(t)\|_{L^2} + \|\partial_x \Delta u(t)\|_{L^2}\big)\,dt \\
&\le C(T,\,u_0,\,v_0) < \infty.
\end{aligned}$$

Therefore, it is easily obtained that

$$\int_0^T \|\nabla\vec{u}(t)\|_{L^\infty}\,dt \le C(T,\,u_0,\,v_0) < \infty,$$

which implies the uniqueness of the solutions. This completes the proof of Theorem 1.4. □

4. The Proof of Theorem 1.6

This section proves Theorem 1.6. Again our focus is on how to obtain the global *a priori* bounds for the solution on any time interval $[0, T]$.

Proof of Theorem 1.6. Our approach in this section is partially inspired by [35]. To start with, let us apply the basic energy estimate to the system (1.9) to obtain

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\partial_y\theta\|_{L^2}^2\,d\tau \le \|\theta_0\|_{L^2}^2,$$
$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q}, \quad q\in[2,\infty],$$

and

$$\|\vec{u}(t)\|_{L^2} + \int_0^t \|\partial_x v\|_{L^2}^2\,d\tau \le C(t,\vec{u}_0,\theta_0).$$

Taking the inner product of the first two equations in (1.9) with $\Delta\vec{u}$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\vec{u}\|_{L^2}^2 + \|\partial_x\nabla v\|_{L^2}^2 &= \int_{\mathbb{R}^2}\theta\Delta v\,dxdy\\ &= \int_{\mathbb{R}^2}\theta\partial_{xx}v\,dxdy + \int_{\mathbb{R}^2}\theta\partial_{yy}v\,dxdy\\ &\le \|\theta\|_{L^2}\|\partial_x\nabla v\|_{L^2} + \|\partial_y\theta\|_{L^2}\|\nabla v\|_{L^2}\\ &\le \frac{1}{2}\|\partial_x\nabla v\|_{L^2}^2 + C\|\theta\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}\|\nabla\vec{u}\|_{L^2}.\end{aligned}$$

Applying the Gronwall inequality yields

$$\|\nabla\vec{u}(t)\|_{L^2}^2 + \int_0^t \|\partial_x\nabla v\|_{L^2}^2\,d\tau \le C < \infty. \tag{4.1}$$

Taking the inner product of the third equation in (1.9) with $\Delta\theta$ leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \|\partial_y\nabla\theta\|_{L^2}^2 &= -\int_{\mathbb{R}^2}\nabla(u\partial_x\theta + v\partial_y\theta)\cdot\nabla\theta\,dxdy\\ &= -\int_{\mathbb{R}^2}\partial_x u\partial_x\theta\partial_x\theta\,dxdy - \int_{\mathbb{R}^2}\partial_x v\partial_y\theta\partial_x\theta\,dxdy\\ &\quad -\int_{\mathbb{R}^2}\partial_y u\partial_x\theta\partial_y\theta\,dxdy - \int_{\mathbb{R}^2}\partial_y v\partial_y\theta\partial_y\theta\,dxdy.\end{aligned} \tag{4.2}$$

By (2.3) and $\partial_x u + \partial_y v = 0$,

$$\begin{aligned}\int_{\mathbb{R}^2}\partial_x u\partial_x\theta\partial_x\theta\,dxdy &= -\int_{\mathbb{R}^2}\partial_y v\partial_x\theta\partial_x\theta\,dxdy = 2\int_{\mathbb{R}^2} v\partial_x\theta\partial_{xy}\theta\,dxdy\\ &\le C\|\partial_{xy}\theta\|_{L^2}\|\partial_x\theta\|_{L^2}^{\frac{1}{2}}\|\partial_{xy}\theta\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}\|\partial_x v\|_{L^2}^{\frac{1}{2}}\\ &\le \frac{1}{8}\|\partial_y\nabla\theta\|_{L^2}^2 + C\|v\|_{L^2}^2\|\nabla\vec{u}\|_{L^2}^2\|\nabla\theta\|_{L^2}^2.\end{aligned} \tag{4.3}$$

The other three terms can be bounded as follows,

$$\begin{aligned}\int_{\mathbb{R}^2}\partial_x v\partial_y\theta\partial_x\theta\,dxdy &= -\int_{\mathbb{R}^2}\theta(\partial_{xy}v\partial_x\theta + \partial_x v\partial_{xy}\theta)\,dxdy\\ &\le C\|\theta\|_{L^\infty}(\|\partial_{xy}v\|_{L^2}\|\partial_x\theta\|_{L^2} + \|\partial_x v\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\ &\le C\|\theta\|_{L^\infty}\|\partial_y\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^2}\\ &\le \frac{1}{8}\|\partial_y\nabla\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\int_{\mathbb{R}^2}\partial_y u\partial_x\theta\partial_y\theta\,dxdy &= -\int_{\mathbb{R}^2}\theta(\partial_{xy}u\partial_y\theta + \partial_y u\partial_{xy}\theta)\,dxdy\\ &\le C\|\theta\|_{L^\infty}(\|\partial_{xy}u\|_{L^2}\|\partial_y\theta\|_{L^2} + \|\partial_y u\|_{L^2}\|\partial_{xy}\theta\|_{L^2})\\ &\le C\|\theta\|_{L^\infty}\|\partial_y\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^2}\end{aligned}$$

$$\leq \frac{1}{8}\|\partial_y\nabla\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2, \tag{4.5}$$

$$\begin{aligned}
-\int_{\mathbb{R}^2}\partial_y v\partial_y\theta\partial_y\theta\,dxdy &= \int_{\mathbb{R}^2}\theta(\partial_{yy}v\partial_y\theta+\partial_y v\partial_{yy}\theta)\,dxdy\\
&\leq C\|\theta\|_{L^\infty}(\|\partial_{yy}v\|_{L^2}\|\partial_y\theta\|_{L^2}+\|\partial_y v\|_{L^2}\|\partial_{yy}\theta\|_{L^2})\\
&\leq C\|\theta\|_{L^\infty}\|\partial_y\theta\|_{L^2}\|\partial_y\nabla\theta\|_{L^2}\\
&\leq \frac{1}{8}\|\partial_y\nabla\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{4.6}$$

Plugging above estimates (4.3)–(4.6) into (4.2), we obtain

$$\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \|\partial_y\nabla\theta\|_{L^2}^2 \leq C\|\theta\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2 + C\|v\|_{L^2}^2\|\nabla\vec{u}\|_{L^2}^2\|\nabla\theta\|_{L^2}^2.$$

An easy application of Gronwall's inequality gives

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t\|\partial_y\nabla\theta\|_{L^2}^2\,d\tau < \infty. \tag{4.7}$$

Combining above estimates, one gets

$$\|\vec{u}(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + \int_0^t(\|\partial_x v\|_{H^1}^2 + \|\partial_y\theta\|_{H^1}^2)\,d\tau \leq C < \infty. \tag{4.8}$$

Applying $|\partial_y|^{1+s}$ $(0<s<\frac{1}{2})$ to the temperature equation and multiplying the resulting equation by $|\partial_y|^{1+s}\theta$, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\||\partial_y|^{1+s}\theta(t)\|_{L^2}^2 + \||\partial_y|^{2+s}\theta\|_{L^2}^2 &= \int_{\mathbb{R}^2}|\partial_y|^{1+s}(\vec{u}\cdot\nabla\theta)|\partial_y|^{1+s}\theta\,dxdy\\
&= \int_{\mathbb{R}^2}|\partial_y|(\vec{u}\cdot\nabla\theta)|\partial_y|^{1+2s}\theta\,dxdy\\
&= \int_{\mathbb{R}^2}\frac{\partial_y}{|\partial_y|}(\partial_y\vec{u}\cdot\nabla\theta+\vec{u}\cdot\nabla\partial_y\theta)|\partial_y|^{1+2s}\theta\,dxdy\\
&:= I + J.
\end{aligned} \tag{4.9}$$

By (2.4), the term J can be bounded as

$$\begin{aligned}
J &= \int_{\mathbb{R}^2}\frac{\partial_y}{|\partial_y|}(\vec{u}\cdot\nabla\partial_y\theta)|\partial_y|^{1+2s}\theta\,dxdy\\
&= \int_{\mathbb{R}^2}(\vec{u}\cdot\nabla\partial_y\theta)\frac{\partial_y}{|\partial_y|}|\partial_y|^{1+2s}\theta\,dxdy\\
&\leq C\|\partial_y\nabla\theta\|_{L^2}\|\vec{u}\|_{L^2}^{\frac{1}{2}}\|\partial_x\vec{u}\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial_y}{|\partial_y|}|\partial_y|^{1+2s}\theta\right\|_{L^2}^{\frac{1-2s}{2-2s}}\left\|\frac{\partial_y}{|\partial_y|}|\partial_y|^{1+2s+(1-s)}\theta\right\|_{L^2}^{\frac{1}{2-2s}} \quad \left(0<s<\frac{1}{2}\right)\\
&\leq C\|\vec{u}\|_{H^1}\|\partial_y\nabla\theta\|_{L^2}^{\frac{3-4s}{2-2s}}\||\partial_y|^{2+s}\theta\|_{L^2}^{\frac{1}{2-2s}} \qquad \left(0<s<\frac{1}{2}\right)\\
&\leq \frac{1}{8}\||\partial_y|^{2+s}\theta\|_{L^2}^2 + C\|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}}\|\partial_y\nabla\theta\|_{L^2}^2.
\end{aligned} \tag{4.10}$$

Now we turn to the term I, which can be written as

$$I = \int_{\mathbb{R}^2}\frac{\partial_y}{|\partial_y|}(\partial_y u\partial_x\theta + \partial_y v\partial_y\theta)|\partial_y|^{1+2s}\theta\,dxdy := I_1 + I_2. \tag{4.11}$$

Integrating by parts, we get

$$\begin{aligned} I_1 \ &= \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|}(\partial_y u \partial_x \theta)|\partial_y|^{1+2s}\theta \, dxdy \\ &= -\int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|}(u \partial_y \partial_x \theta)|\partial_y|^{1+2s}\theta \, dxdy - \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|}(u \partial_x \theta)\partial_y|\partial_y|^{1+2s}\theta \, dxdy \\ &= -\int_{\mathbb{R}^2} (u \partial_y \partial_x \theta)\frac{\partial_y}{|\partial_y|}|\partial_y|^{1+2s}\theta \, dxdy - \int_{\mathbb{R}^2} |\partial_y|^s(u \partial_x \theta)\frac{\partial_y}{|\partial_y|}\partial_y|\partial_y|^{1+s}\theta \, dxdy \\ &:= I_{11} + I_{12}. \end{aligned}$$

Obviously, the term I_{11} admits the same bound as the term J, that is,

$$I_{11} \le \frac{1}{8}\||\partial_y|^{2+s}\theta\|_{L^2}^2 + C\|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}}\|\partial_y \nabla\theta\|_{L^2}^2.$$

By virtue of the Sobolev embedding, the term I_{12} can be estimated as follows

$$\begin{aligned} I_{12} &\le C\||\partial_y|^s(u\partial_x\theta)\|_{L^2}\||\partial_y|^{2+s}\theta\|_{L^2} \\ &\le C\|u\|_{H^1}(\|\partial_x\theta\|_{L^2} + \|\partial_y\nabla\theta\|_{L^2})\||\partial_y|^{2+s}\theta\|_{L^2} \\ &\le \frac{1}{8}\||\partial_y|^{2+s}\theta\|_{L^2}^2 + C\|u\|_{H^1}^2(\|\partial_x\theta\|_{L^2} + \|\partial_y\nabla\theta\|_{L^2})^2, \end{aligned}$$

where we have applied the following estimate

$$\begin{aligned} \||\partial_y|^s(u\partial_x\theta)\|_{L^2} &= \|(u\partial_x\theta)(x,y)\|_{L_x^2 H_y^s} \\ &\le C\Big\|\|(u\partial_x\theta)(x,y)\|_{H_y^s}\Big\|_{L_x^2} \\ &\le C\Big\|\|u(x,y)\|_{H_y^{s_1}}\|\partial_x\theta(x,y)\|_{H_y^{s_2}}\Big\|_{L_x^2} \\ &\quad \times \left(s_1,\ s_2 < \frac{1}{2},\ s + \frac{1}{2} = s_1 + s_1 > 0\right) \\ &\le C\Big\|\|u(x,y)\|_{H_y^1}\|\partial_x\theta(x,y)\|_{H_y^1}\Big\|_{L_x^2} \\ &\le C\|u(x,y)\|_{L_x^\infty H_y^1}\Big\|\|\partial_x\theta(x,y)\|_{L_y^2} + \|\partial_y\partial_x\theta(x,y)\|_{L_y^2}\Big\|_{L_x^2} \\ &\le C\|u(x,y)\|_{H_x^1 H_y^1}\left(\Big\|\|\partial_x\theta(x,y)\|_{L_y^2}\Big\|_{L_x^2} + \Big\|\|\partial_y\partial_x\theta(x,y)\|_{L_y^2}\Big\|_{L_x^2}\right) \\ &= C\|u\|_{H^1}(\|\partial_x\theta\|_{L^2} + \|\partial_y\partial_x\theta\|_{L^2}). \end{aligned}$$

Therefore, it directly yields

$$I_1 \le \frac{1}{4}\||\partial_y|^{2+s}\theta\|_{L^2}^2 + C\|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}}\|\partial_y\nabla\theta\|_{L^2}^2 + C\|u\|_{H^1}^2(\|\partial_x\theta\|_{L^2} + \|\partial_y\nabla\theta\|_{L^2})^2.$$

Finally, by Young inequality, we arrive at

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|}(\partial_y v \partial_y \theta)|\partial_y|^{1+2s}\theta \, dxdy \\ &= \int_{\mathbb{R}^2} (\partial_y v \partial_y \theta)\frac{\partial_y}{|\partial_y|}|\partial_y|^{1+2s}\theta \, dxdy \\ &\le C\|\partial_y v\|_{L^2}\|\partial_y\theta\|_{L^2}^{\frac{1}{2}}\|\partial_x\partial_y\theta\|_{L^2}^{\frac{1}{2}}\left\|\frac{\partial_y}{|\partial_y|}|\partial_y|^{1+2s}\theta\right\|_{L^2}^{\frac{1-2s}{2-2s}}\left\|\frac{\partial_y}{|\partial_y|}|\partial_y|^{2+s}\theta\right\|_{L^2}^{\frac{1}{2-2s}} \qquad \left(0 < s < \frac{1}{2}\right) \\ &\le C\|\vec{u}\|_{H^1}\|\theta\|_{H^1}^{\frac{1}{2}}\|\partial_y\nabla\theta\|_{L^2}^{\frac{2-3s}{2-2s}}\||\partial_y|^{2+s}\theta\|_{L^2}^{\frac{1}{2-2s}} \qquad \left(0 < s < \frac{1}{2}\right) \\ &\le \frac{1}{8}\||\partial_y|^{2+s}\theta\|_{L^2}^2 + C\|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}}\|\theta\|_{H^1}^{\frac{2-2s}{3-4s}}\|\partial_y\nabla\theta\|_{L^2}^{\frac{4-6s}{3-4s}}. \end{aligned}$$

Combining the estimates for I and J together, we thus conclude

$$\frac{d}{dt}\||\partial_y|^{1+s}\theta(t)\|_{L^2}^2 + \||\partial_y|^{2+s}\theta\|_{L^2}^2 \le C\|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}}\|\partial_y\nabla\theta\|_{L^2}^2$$
$$+ C\|u\|_{H^1}^2(\|\partial_x\theta\|_{L^2} + \|\partial_y\nabla\theta\|_{L^2})^2 + C\|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}}\|\theta\|_{H^1}^{\frac{2-2s}{3-4s}}\|\partial_y\nabla\theta\|_{L^2}^{\frac{4-6s}{3-4s}},$$

which, along with Gronwall's inequality, yields

$$\||\partial_y|^{1+s}\theta(t)\|_{L^2}^2 + \int_0^t \||\partial_y|^{2+s}\theta(\tau)\|_{L^2}^2\, d\tau < \infty, \quad \text{for any } 0 < s < \frac{1}{2}. \tag{4.12}$$

The above key bound allows us to show that

$$\int_0^T \|\partial_y\theta(\tau)\|_{L^\infty}^2\, d\tau < \infty. \tag{4.13}$$

Indeed, we can deduce that

$$\begin{aligned}
\|\partial_y\theta\|_{L^\infty} &\le \|\widehat{\partial_y\theta}\|_{L^1} \\
&\le \int_{\mathbb{R}^2} \left|\xi_y\widehat{\theta}(\xi_x,\xi_y)\right| d\xi_x d\xi_y \\
&\le \left(\int_{\mathbb{R}^2} (1+|\xi_x|^2+|\xi_y|^{2+2s})|\xi_y|^2|\widehat{\theta}(\xi_x,\xi_y)|^2\, d\xi_x d\xi_y\right)^{\frac{1}{2}} \\
&\quad\times \left(\int_{\mathbb{R}^2} (1+|\xi_x|^2+|\xi_y|^{2+2s})^{-1}\, d\xi_x d\xi_y\right)^{\frac{1}{2}} \qquad (s>0) \\
&\le C(\|\partial_y\theta\|_{L^2} + \|\partial_y\partial_x\theta\|_{L^2} + \||\partial_y|^{2+s}\theta\|_{L^2}),
\end{aligned}$$

where we have used

$$\int_{\mathbb{R}^2} (1+|\xi_x|^2+|\xi_y|^{2+2s})^{-1}\, d\xi_x d\xi_y = \int_{\mathbb{R}^2} (1+|\xi_x|^2)^{\frac{-(1+2s)}{2+2s}}(1+\eta^{2+2s})^{-1}\, d\xi_x d\eta < \infty,$$

by making the change of variable $\xi_y = (1+|\xi_x|^2)^{\frac{1}{2+2s}}\eta$. Thus, we get the desired bound (4.13).

Taking ∂_x on the temperature equation and multiplying the resulting equation by $|\partial_x\theta|^{q-2}\partial_x\theta$, we have

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\|\partial_x\theta\|_{L^q}^q + (q-1)\int_{\mathbb{R}^2}(\partial_y\partial_x\theta)^2|\partial_x\theta|^{q-2}\, dxdy &= \int_{\mathbb{R}^2}\partial_x(u\partial_x\theta + v\partial_y\theta)|\partial_x\theta|^{q-2}\partial_x\theta\, dxdy \\
&= \int_{\mathbb{R}^2}\partial_x u\partial_x\theta|\partial_x\theta|^{q-2}\partial_x\theta\, dxdy + \int_{\mathbb{R}^2}\partial_x v\partial_y\theta|\partial_x\theta|^{q-2}\partial_x\theta\, dxdy,
\end{aligned} \tag{4.14}$$

where in the last line we have used the following fact due to the incompressibility of $\vec{u}$

$$\int_{\mathbb{R}^2}(u\partial_x\partial_x\theta + v\partial_y\partial_x\theta)|\partial_x\theta|^{q-2}\partial_x\theta\, dxdy = 0.$$

Integration by parts and Young inequality allow us to show

$$\begin{aligned}
\int_{\mathbb{R}^2}\partial_x u\partial_x\theta|\partial_x\theta|^{q-2}\partial_x\theta\, dxdy &= -\int_{\mathbb{R}^2}\partial_y v\partial_x\theta|\partial_x\theta|^{q-2}\partial_x\theta\, dxdy \\
&\le q\int_{\mathbb{R}^2}|v|\,|\partial_x\theta|^{q-1}|\partial_y\partial_x\theta|\, dxdy \\
&\le \frac{q-1}{2}\int_{\mathbb{R}^2}(\partial_y\partial_x\theta)^2|\partial_x\theta|^{q-2}\, dxdy + Cq\int_{\mathbb{R}^2}|v|^2\,|\partial_x\theta|^q\, dxdy \\
&\le \frac{q-1}{2}\int_{\mathbb{R}^2}(\partial_y\partial_x\theta)^2|\partial_x\theta|^{q-2}\, dxdy + Cq\|v\|_{L^\infty}^2\|\partial_x\theta\|_{L^q}^q.
\end{aligned} \tag{4.15}$$

Invoking Sobolev interpolation and Young inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_x v \partial_y \theta |\partial_x \theta|^{q-2} \partial_x \theta \, dxdy &\le C\|\partial_y \theta\|_{L^\infty} \|\partial_x v\|_{L^q} \|\partial_x \theta\|_{L^q}^{q-1} \\ &\le C\sqrt{q}\|\partial_y \theta\|_{L^\infty} \|\partial_x v\|_{L^2}^{\frac{2}{q}} \|\partial_x \nabla v\|_{L^2}^{\frac{q-2}{q}} \|\partial_x \theta\|_{L^q}^{q-1}, \end{aligned} \tag{4.16}$$

where the following interpolation has been used

$$\|f\|_{L^q} \le C\sqrt{q}\|f\|_{L^2}^{\frac{2}{q}} \|\nabla f\|_{L^2}^{\frac{q-2}{q}}, \quad 2 \le q < \infty,$$

for some absolute constant C independent of q.

Substituting (4.15) and (4.16) into (4.14), we immediately obtain

$$\frac{d}{dt}\|\partial_x \theta\|_{L^q} \le Cq\|v\|_{L^\infty}^2 \|\partial_x \theta\|_{L^q} + C\sqrt{q}\|\partial_y \theta\|_{L^\infty} \|\partial_x v\|_{L^2}^{\frac{2}{q}} \|\partial_x \nabla v\|_{L^2}^{\frac{q-2}{q}}. \tag{4.17}$$

By (4.8), it follows that

$$\int_0^T \|v(\tau)\|_{L^\infty}^2 \, d\tau < \infty. \tag{4.18}$$

Noticing the bounds (4.8), (4.12) and (4.18), then making use of Gronwall inequality, one can deduce from the inequality (4.17) that

$$\|\partial_x \theta(t)\|_{L^q} \le C(T, q, \vec{u}_0, \theta_0) < \infty, \quad 2 \le q < \infty.$$

Therefore, this concludes the proof of Theorem 1.6. □

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