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Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion

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ABSTRACT

This paper establishes the global existence and uniqueness of smooth solutions to the two-dimensional compressible magnetohydrodynamic system when the initial data is close to an equilibrium state. In addition, explicit large-time decay rates for various Sobolev norms of the solutions are also obtained. These results are achieved through a new approach of diagonalizing a system of coupled linearized equations. The standard method of diagonalization via the eigenvalues and eigenvectors of the matrix symbol is very difficult to implement here. This new process allows us to obtain an integral representation of the full system through explicit kernels. In addition, in order to overcome various difficulties such as the anisotropicity and criticality, we fully exploit the structure of the integral representation and employ extremely delicate Fourier analysis and associated estimates.

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Contents

1. Introduction

This paper examines the global (in time) existence and uniqueness of solutions to the two-dimensional (2D) compressible magnetohydrodynamic (MHD) system, namely

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) - \Delta \vec{u} - \lambda \nabla(\nabla \cdot \vec{u}) + \nabla P = -\frac{1}{2}\nabla\left(|\vec{b}|^2\right) + \vec{b} \cdot \nabla \vec{b}, \\ \partial_t \vec{b} + \vec{u} \cdot \nabla \vec{b} = \vec{b} \cdot \nabla \vec{u}, \\ \nabla \cdot \vec{b} = 0 \end{cases} \tag{1.1}$$

with the initial data

$$\rho|_{t=0} = \rho_0(x, y), \quad \vec{u}|_{t=0} = \vec{u}_0(x, y), \quad \vec{b}|_{t=0} = \vec{b}_0(x, y).$$

Here $\rho \in \mathbb{R}^+$ denotes the density, $\vec{u} = (u, v) \in \mathbb{R}^2$ represents the velocity field, $P = P(\rho)$ the pressure, $\vec{b} \in \mathbb{R}^2$ the magnetic field and λ is a constant with $|\lambda| < 1$. Moreover, the pressure term $P(\rho)$ is assumed to obey the following polytropic law,

$$P(\rho) = A\rho^\gamma,$$

where A is the entropy constant and $\gamma \geq 1$ is called the adiabatic index. We remark that there is a large literature on the compressible MHD equations (with both velocity dissipation and magnetic diffusion) due to their physical importance and mathematical challenges (see, e.g., [1,4–7]).

This paper aims to achieve three goals. The first is to establish the global well-posedness of smooth solutions of (1.1) when the initial data $(\rho_0, \vec{u}_0, \vec{b}_0)$ is smooth and close to the equilibrium state $(1, \vec{0}, \vec{e}_1)$, where we denote $\vec{0} = (0, 0)$ and $\vec{e}_1 = (1, 0)$. To this end, we may assume that $\gamma = 3$ and $A = \frac{1}{3}$, since the other cases can be essentially reduced to this case, after omitting some high-order terms. The second is to explore the hidden structure in the system of the linearized equations and offer a new way of diagonalizing a complex system of linearized equations. The third is to obtain explicit and sharp large-time decay rates for the solutions in various Sobolev spaces. As we shall see in the subsequent sections, we need to overcome several major difficulties including the anisotropicity and the criticality associated with the 2D compressible MHD equations with only velocity dissipation and no magnetic diffusion.

In the 2D case, $\nabla \cdot \vec{b} = 0$ implies that for a scalar function ϕ,

$$\vec{b} = \nabla^{\perp}\phi \equiv \big(\partial_y \phi, -\partial_x \phi\big).$$

With this substitution, (1.1) becomes

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \quad (t,\, x,\, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) - \Delta \vec{u} - \lambda \nabla(\nabla \cdot \vec{u}) + \rho^2 \nabla \rho = -\nabla \phi \Delta \phi, \\ \partial_t \phi + \vec{u} \cdot \nabla \phi = 0. \end{cases} \tag{1.2}$$

We will mainly work with this form of the equations, which is more convenient for the estimates. However, this form is not essential for our main result.

To precisely state our main result, we introduce the functional settings. The operator notation $\langle \nabla \rangle$ is standard. Let

$$U = (n, u, v, \psi) \quad \text{and} \quad U_0 = (n_0, u_0, v_0, \psi_0).$$

Let M be a big integer ($M \geq 8$ is sufficient, as a careful computation would show). Let $\epsilon > 0$ be a small parameter, $\gamma \in (\frac{1}{2}, 1]$, and $\frac{\gamma}{2} < \bar{\gamma} < 1 + \frac{\gamma}{2}$. We define $X_n, X_{\mathbf{u}}, X_\psi$ with their norms given by

$$\begin{aligned} \|n\|_{X_n} = \sup_{t\geq 0} \Big\{ & \langle t \rangle^{-\epsilon} \|\langle \nabla \rangle^M n\|_{L^2_{xy}} + \langle t \rangle^{\frac{1}{4}} \|\langle \nabla \rangle^3 n\|_{L^2_{xy}} \\ & + \langle t \rangle^{\frac{1}{2}} \|\langle \nabla \rangle^{\frac{3}{2}} n\|_{L^\infty_{xy}} + \langle t \rangle^{\frac{3}{4}} \|\langle \nabla \rangle \partial_x n\|_{L^2_{xy}} \Big\}; \\ \|\vec{u}\|_{X_{\mathbf{u}}} = \sup_{t\geq 0} \Big\{ & \langle t \rangle^{-\epsilon} \|\langle \nabla \rangle^M \vec{u}\|_{L^2_{xy}} + \langle t \rangle^{\frac{1}{2}} \|\vec{u}\|_{L^2_{xy}} + \langle t \rangle \|\langle \nabla \rangle \vec{u}\|_{L^\infty_{xy}} + \langle t \rangle^{\frac{3}{4}} \|v\|_{L^2_x L^\infty_y} \\ & + \langle t \rangle^{\frac{3}{4}} \||\nabla|^{\gamma} \vec{u}\|_{L^2_{xy}} + \langle t \rangle \|\langle \nabla \rangle \partial_x u\|_{L^2_{xy}} + \langle t \rangle \|\langle \nabla \rangle \nabla v\|_{L^2_{xy}} \Big\}; \\ \|\psi\|_{X_\psi} = \sup_{t\geq 0} \Big\{ & \langle t \rangle^{-\epsilon} \|\langle \nabla \rangle^M \nabla \psi\|_{L^2_{xy}} + \langle t \rangle^{\frac{1}{4}} \big\|\langle \nabla \rangle^4 |\nabla|^{\gamma} \psi\big\|_{L^2_{xy}} + \langle t \rangle^{\frac{1}{2}} \big\|\partial_x \psi\big\|_{L^2_{xy}} \\ & + \langle t \rangle^{\frac{3}{4}} \big\|\partial_x \nabla \psi\big\|_{L^2_{xy}} + \langle t \rangle^{\frac{1}{2}} \big\||\nabla|^{\bar{\gamma}} \langle \nabla \rangle \psi\big\|_{L^\infty_{xy}} \Big\}. \end{aligned}$$

Now we define the working space $X = X_n \times X_{\mathbf{u}} \times X_\psi$, whose norm is given by

$$\|U\|_X = \|n\|_{X_n} + \|\vec{u}\|_{X_{\mathbf{u}}} + \|\psi\|_{X_\psi}. \tag{1.3}$$

Note that we have incorporated the large-time behaviors of various Sobolev norms into the definition of X and these rates will become clear in the subsequent sections. Moreover,

$$\|U_0\|_{X_0} = \|\langle\nabla\rangle^M (n_0, \vec{u}_0, \nabla\psi_0)\|_{L^2_{xy}} + \|\langle\nabla\rangle^5 (n_0, \vec{u}_0, \nabla\psi_0)\|_{L^1_{xy}}. \tag{1.4}$$

Our main result can then be stated as follows. We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \le CB$ for some absolute constant $C > 0$.

Theorem 1.1. *Assume $|\lambda| \le c_0$ for some absolute constant $c_0 > 0$, and let $n = \rho - 1, \psi = \phi - y$, $n_0 = \rho_0 - 1, \psi_0 = \phi_0 - y$. Then there exists a small constant $\delta > 0$ such that, if the initial data $(n_0, \vec{u}_0, \phi_0)$ satisfies $\|(n_0, \vec{u}_0, \psi_0)\|_{X_0} \le \delta$, then there exists a unique global solution $(\rho, u, v, \phi) \in X$ to the system (1.1). Moreover,*

$$\|(n, u, v, \phi)\|_X \lesssim \delta.$$

Especially, the following decay estimates hold

$$\|n(t)\|_{L^\infty_{xy}} \lesssim \delta\, t^{-\frac{1}{2}}; \quad \|\vec{u}(t)\|_{L^\infty_{xy}} \lesssim \delta\, t^{-1}; \quad \|\nabla\psi(t)\|_{L^\infty_{xy}} \lesssim \delta\, t^{-\frac{1}{2}}.$$

We remark that the smallness condition on λ is not essential for our main result. The term $\lambda\nabla(\nabla \cdot \vec{u})$ is treated as a nonlinear term in order to simplify the treatment on the linearized equations. We could have included $\lambda\nabla(\nabla \cdot \vec{u})$ as a linear term. That would make the diagonalization process more sophisticated and our main idea murky. Moreover, we remark that the treatment on the "nonlinear" term $\lambda\nabla(\nabla \cdot \vec{u})$ is rather non-trivial and has of independence interest, even though λ is small.

Due to the lack of dissipation or damping in the transport equation for ϕ, the global existence of smooth solutions to (1.2) is an extremely difficult problem and is currently open. Recent strategy has been to seek solutions near an equilibrium state. This has been very successful in the incompressible counterpart of (1.2). The paper of Lin, Xu and Zhang [8] appears to be the very first to establish the global existence of smooth solutions (near an equilibrium) for the 2D incompressible MHD equations without magnetic diffusion. They resort to Lagrangian coordinates and anisotropic Besov spaces. [8] inspired several different approaches ([3,9–11]). Little has been done for the compressible MHD equations. The only work currently available is a very recent preprint of Hu [2], in which the author constructs a global solution for the compressible MHD equation without magnetic diffusion starting from an initial data near a constant background and the Lagrangian deformation gradient near the identity matrix. [2] involves hybrid Besov spaces. The approach of this paper is completely different from that in [2].

Besides the global existence of smooth solutions of (1.2), this paper also aims at thoroughly understanding the structure of the linearized system and how this structure leads to the exact large-time decay rates for various Sobolev norms of the solutions. Keeping all nonlinear terms on the right-hand side, the equations for the perturbation $U = (n, u, v, \psi)$ can be written as

$$\partial_t \overrightarrow{U} = A\overrightarrow{U} + \overrightarrow{N}, \tag{1.5}$$

where

$$\overrightarrow{U} = \begin{pmatrix} n \\ u \\ v \\ \psi \end{pmatrix}, \; A = \begin{pmatrix} 0 & -\partial_x & -\partial_y & 0 \\ -\partial_x & \Delta + \lambda\partial_{xx} & \lambda\partial_{xy} & 0 \\ -\partial_y & \lambda\partial_{xy} & \Delta + \lambda\partial_{yy} & -\Delta \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; \overrightarrow{N} = \begin{pmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix}$$

with N_0, N_1, N_2 and N_3 given in (2.3)–(2.6). The terms involving λ are treated as nonlinear terms. The natural next step would be to diagonalize the system, but the standard method via the eigenvalues and eigenvectors of A appears to be impossible to carry out. We provide a new systematic approach. The diagonalization here is obtained by suitable differentiation of (1.5) and magically all the resulting equations have the same structure,

$$\begin{aligned}
\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)n &= F_0, \\
\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)u &= F_1, \\
\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)v &= F_2, \\
\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)\psi &= F_3,
\end{aligned}$$

where F_0, F_1, F_2 and F_3 are given in (2.11)–(2.14). By factorizing and inverting the differential operator in the equation

$$\Big[(\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big]\Phi = F,$$

we obtain the integral representation

$$\Phi = L(\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0}, \Phi_{4,0}) + \int_0^t K(t-s, \partial_x, \partial_y)F(s)\, ds,$$

where $\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0}, \Phi_{4,0}$ and K are given in Lemma 2.2. This integral representation and its simplified variants form the foundation for proving Theorem 1.1. Due to the standard continuity argument, the proof of Theorem 1.1 is then reduced to verifying the integral representation obeys

$$\|U\|_X \le C_0\|U_0\|_{X_0} + Q(\|U\|_X), \tag{1.6}$$

where C_0 is an absolute positive constant and $Q(r)$ represents a polynomial of r with the lowest order at least quadratic. In this paper, we shall prove that

$$\|U\|_X \le \frac{1}{2}C_0\|U_0\|_{X_0} + C_*|\lambda|\|U\|_X + \frac{1}{2}Q(\|U\|_X), \tag{1.7}$$

for some absolute positive constant C_*. Indeed, by choosing $|\lambda| \le c_0$, for $c_0 = \frac{1}{2C_*}$, we obtain (1.6). To prove (1.7), we first need to understand the exact decay properties of L and K. Because $K = K(t, x, y)$ is anisotropic, the decay rates of K and its various spatial and time derivatives depend on the Fourier frequencies. These decay rates are explicitly given in Section 3. In order to prove (1.7), we further recast the integral representation and estimate each component of $U = (n, u, v, \psi)$ and verify (1.7). This is a complicated and lengthy process.

We need to deal with several difficulties. The first is the tanglement in the linear operator. The linear operator under study is

$$(\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy},$$

which is a four-order (both in time and spatial variables) wave type operator. In general, the number of the unknown functions gives the order of the linearized equations. In light of this, the problem in the incompressible case is much easier than the compressible one, since, under the incompressibility condition, the number of the unknown function members reduces to 2. But in the compressible case, we have four unknown members $U = (n, u, v, \psi)$. This marks the first challenge in this paper and it takes extraordinary efforts to completely understand the properties of the operator and obtain the decay estimates. Another difficulty is the anisotropicity. It is easy to see from the expansion

$$\begin{aligned}&(\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\\ &\quad = \left((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\right)\left((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\right),\end{aligned}$$

that the operator has different behaviors along different directions. Indeed, by a delicate analysis, we observe that kernel of the linear operator obeys the estimate that

$$|\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\ge 1}\frac{1}{A^4}e^{-ct} + \chi_{A\le 1}\frac{1}{A|\xi||\eta|}e^{-\frac{1}{4}A^2 t} + \frac{1}{A^4}\chi_{|\xi|\le A^2}e^{-\frac{\xi^2}{2A^2}t},$$

with $A = \sqrt{\xi^2 + \eta^2}$. It is singular and anisotropic. More precisely, it has a high-order negative regularity in the kernel, and has weaker behavior in y-direction than in x-direction. These properties of the operator are reflected in the definition of the working space, kernel properties and various estimates.

Another difficulty is the criticality, due to the slow dispersion and quasilinearity. Indeed, the spatial L^∞-norm of $\vec{u}$ behaves like t^{-1}, which is not integrable in time.

What's more, the spatial L^∞-norm of n and ψ decay like $t^{-1/2}$, which is even far from being integrable. To overcome this difficulty, we make full use of the structure of the system. In particular, for n and ψ, we obtain the cubic type nonlinearities, thanks to the structure of the velocity diffusion. In contrast to the incompressible case, the compressible MHD problem is much harder. Indeed, by using $\nabla \cdot \vec{u} = 0$ in the incompressible case, one may transfer the derivative ∂_y to ∂_x and then faster decay rates result due to the presence of ∂_x. However, in the compressible case, the fast decay rate associate with ∂_x can not be converted into that for ∂_y due to the lack of incompressibility condition. This criticality prompts us to use extremely fine Fourier analysis and associated estimates.

The rest of this paper is divided into seven sections and an appendix. The second section derives the integral representation for the perturbation $U = (n, u, v, \psi)$ through the kernels K and L. The third section provides pointwise as well as L^p-estimates for $\widehat{K}(t, \xi, \eta)$ (defined in (2.42)) and the Fourier transforms of various derivatives of K. The fourth section establishes the decaying estimates of the linear flow. The fifth section contains the local existence and uniqueness theory and an energy estimate serving as a part of the proof for Theorem 1.1. Section 6 through Section 9 verify (1.7). The appendix proves Lemma 3.4 and several inequalities used in the previous sections.

2. Integral representation

This section derives the integral representation for the perturbation $U = (n, u, v, \psi)$. The derivation consists of four steps, which are presented in four subsections. The first subsection writes the equations for U with the nonlinear terms kept on the right-hand side. The second subsection diagonalizes the equations through suitable differentiation and magically all the equations have the same structure. The third subsection obtains a preliminary integral representation through factoring the differential operator. This representation involves some intermediate variables. The last subsection eliminates the intermediate variables to reach the final integral representation.

2.1. Linearization

First of all, by a simple computation, the second equation in (1.2) can be rewritten as

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} - \frac{\Delta \vec{u} + \lambda \nabla (\nabla \cdot \vec{u})}{\rho} + \rho \nabla \rho = -\frac{\nabla \phi \Delta \phi}{\rho}. \tag{2.1}$$

Setting

$$\rho = n + 1, \quad \vec{u} = (u, v), \quad \phi = \psi + y$$

converts (1.2) into the following equivalent system of equations for (n, u, v, ψ),

$$\begin{cases} \partial_t n + \partial_x u + \partial_y v = N_0, \\ \partial_t u - \Delta u - \lambda(\partial_{xx} u + \partial_{xy} v) + \partial_x n = N_1, \\ \partial_t v - \Delta v - \lambda(\partial_{xy} u + \partial_{yy} v) + \partial_y n + \Delta\psi = N_2, \\ \partial_t \psi + v = N_3, \end{cases} \tag{2.2}$$

where

$$N_0 = -\partial_x(nu) - \partial_y(nv); \tag{2.3}$$

$$N_1 = -(u\partial_x u + v\partial_y u) - \frac{n\Delta u + n\lambda(\partial_{xx}u + \partial_{xy}v)}{\rho} - \frac{\partial_x\psi\Delta\psi}{\rho} - n\partial_x n; \tag{2.4}$$

$$N_2 = -(u\partial_x v + v\partial_y v) - \frac{n\Delta v + n\lambda(\partial_{xy}u + \partial_{yy}v) - n\Delta\psi}{\rho} - \frac{\partial_y\psi\Delta\psi}{\rho} - n\partial_y n; \tag{2.5}$$

$$N_3 = -u\partial_x\psi - v\partial_y\psi. \tag{2.6}$$

2.2. Diagonalization

From (2.2), we can write the system in the form

$$\partial_t \vec{U} = A\vec{U} + \vec{N},$$

where

$$\vec{U} = \begin{pmatrix} n \\ u \\ v \\ \psi \end{pmatrix}, \; A = \begin{pmatrix} 0 & -\partial_x & -\partial_y & 0 \\ -\partial_x & \Delta + \lambda\partial_{xx} & \lambda\partial_{xy} & 0 \\ -\partial_y & \lambda\partial_{xy} & \Delta + \lambda\partial_{yy} & -\Delta \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; \vec{N} = \begin{pmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{pmatrix}.$$

The aim of this subsection is to diagonalize A. However, the standard approach involving the eigenvalues and eigenvectors of the symbol of A appears to demand extremely tedious calculations. Our strategy is to diagonalize through carefully selected differentiations. As mentioned before, we treat the terms involving the parameter λ as nonlinear terms for the brevity of the presentation. The main result of this subsection is

Proposition 2.1. *The functions (n, u, v, ψ) obey the following equations,*

$$\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)n = F_0, \tag{2.7}$$

$$\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)u = F_1, \tag{2.8}$$

$$\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)v = F_2, \tag{2.9}$$

$$\Big((\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big)\psi = F_3, \tag{2.10}$$

where

$$F_0 = (\partial_{tt} - \Delta\partial_t - \Delta)\Pi_0 + \Delta\partial_y\Pi_3; \tag{2.11}$$

$$F_1 = \big(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}\big)\Pi_1 + \partial_{xy}\Pi_2; \tag{2.12}$$

$$F_2 = \big(\partial_{tt} - \Delta\partial_t - \partial_{xx}\big)\Pi_2 + \partial_{xy}\Pi_1; \tag{2.13}$$

$$F_3 = (\partial_{tt} - \Delta\partial_t - \Delta)\Pi_3 + \partial_y\Pi_0, \tag{2.14}$$

and

$$\Pi_0 = \partial_t N_0 - \Delta N_0 - \partial_x N_1 - \partial_y N_2 - \lambda(\partial_x\Delta u + \partial_y\Delta v); \tag{2.15}$$

$$\Pi_1 = \partial_t N_1 - \partial_x N_0 + \lambda\partial_t(\partial_{xx}u + \partial_{xy}v); \tag{2.16}$$

$$\Pi_2 = -\partial_y N_0 + \partial_t N_2 - \Delta N_3 + \lambda\partial_t(\partial_{xy}u + \partial_{yy}v); \tag{2.17}$$

$$\Pi_3 = -N_2 + (\partial_t - \Delta)N_3 - \lambda(\partial_{xy}u + \partial_{yy}v). \tag{2.18}$$

Proof. First, we give some reductions. Taking the time derivative on the first equations of (2.2), then using the second, third and also the first equation of (2.2), we obtain

$$\begin{aligned}
0 &= \partial_{tt}n + \partial_x\partial_t u + \partial_y\partial_t v - \partial_t N_0\\
&= \partial_{tt}n + \partial_x\big[\Delta u + \lambda(\partial_{xx}u + \partial_{xy}v) - \partial_x n + N_1\big]\\
&\quad + \partial_y\big[\Delta v + \lambda(\partial_{xy}u + \partial_{yy}v) - \partial_y n - \Delta\psi + N_2\big] - \partial_t N_0\\
&= \partial_{tt}n + \Delta(\partial_x u + \partial_y v) - \Delta n - \partial_y\Delta\psi\\
&\quad + \partial_x N_1 + \partial_y N_2 - \partial_t N_0 + \lambda(\partial_x\Delta u + \partial_y\Delta v)\\
&= \partial_{tt}n - \Delta\partial_t n - \Delta n - \partial_y\Delta\psi\\
&\quad + (\Delta - \partial_t)N_0 + \partial_x N_1 + \partial_y N_2 + \lambda(\partial_x\Delta u + \partial_y\Delta v).
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
\partial_{tt}n - \Delta\partial_t n - \Delta n = \Delta\partial_y\psi + (\partial_t - \Delta)N_0 - \partial_x N_1 - \partial_y N_2&\\
- \lambda(\partial_x\Delta u + \partial_y\Delta v).&
\end{aligned} \tag{2.19}$$

Now we denote

$$\Pi_0 = (\partial_t - \Delta)N_0 - \partial_x N_1 - \partial_y N_2 - \lambda(\partial_x\Delta u + \partial_y\Delta v),$$

then (2.19) becomes

$$\partial_{tt}n - \Delta\partial_t n - \Delta n = \Delta\partial_y\psi + \Pi_0. \tag{2.20}$$

Taking the time derivative on the second equation of (2.2), and then using the first equation, we obtain

$$\begin{aligned}0 &= \partial_{tt}u - \Delta\partial_t u + \partial_x\partial_t n - \partial_t N_1 - \lambda\partial_t(\partial_{xx}u + \partial_{xy}v)\\ &= \partial_{tt}u - \Delta\partial_t u - \partial_x(\partial_x u + \partial_y v) + \partial_x N_0 - \partial_t N_1 - \lambda\partial_t(\partial_{xx}u + \partial_{xy}v).\end{aligned}$$

Thus, we get

$$\partial_{tt}u - \Delta\partial_t u - \partial_{xx}u = \partial_{xy}v - \partial_x N_0 + \partial_t N_1 + \lambda\partial_t(\partial_{xx}u + \partial_{xy}v). \tag{2.21}$$

Again we denote

$$\Pi_1 = -\partial_x N_0 + \partial_t N_1 + \lambda\partial_t(\partial_{xx}u + \partial_{xy}v),$$

then (2.21) turns to

$$\partial_{tt}u - \Delta\partial_t u - \partial_{xx}u = \partial_{xy}v + \Pi_1. \tag{2.22}$$

Taking the time derivative on the third equation of (2.2), then by the first and the fourth equations we find

$$\begin{aligned}0 &= \partial_{tt}v - \Delta\partial_t v + \partial_y\partial_t n + \Delta\partial_t\psi - \partial_t N_2 - \lambda\partial_t(\partial_{xy}u + \partial_{yy}v)\\ &= \partial_{tt}v - \Delta\partial_t v - \partial_y(\partial_x u + \partial_y v) + \partial_y N_0\\ &\quad - \Delta v + \Delta N_3 - \partial_t N_2 - \lambda\partial_t(\partial_{xx}u + \partial_{xy}v).\end{aligned}$$

Therefore, we obtain that

$$\partial_{tt}v - \Delta\partial_t v - \Delta v - \partial_{yy}v = \partial_{xy}u - \partial_y N_0 + \partial_t N_2 - \Delta N_3 + \lambda\partial_t(\partial_{xy}u + \partial_{yy}v). \tag{2.23}$$

Again we denote

$$\Pi_2 = -\partial_y N_0 + \partial_t N_2 - \Delta N_3 + \lambda\partial_t(\partial_{xy}u + \partial_{yy}v),$$

then (2.23) turns to

$$\partial_{tt}v - \Delta\partial_t v - \Delta v - \partial_{yy}v = \partial_{xy}u + \Pi_2. \tag{2.24}$$

At last, taking the time derivative on the fourth equation of (2.2), and using the third and also the fourth equations, we find

$$\begin{aligned}0 &= \partial_{tt}\psi + \partial_t v - \partial_t N_3\\ &= \partial_{tt}\psi + \Delta v + \lambda(\partial_{xy}u + \partial_{yy}v) - \partial_y n - \Delta\psi + N_2 - \partial_t N_3\\ &= \partial_{tt}\psi - \Delta\partial_t\psi - \Delta\psi - \partial_y n + N_2 + \Delta N_3 - \partial_t N_3 + \lambda(\partial_{xy}u + \partial_{yy}v).\end{aligned}$$

Thus we have

$$\partial_{tt}\psi - \Delta\partial_t\psi - \Delta\psi = \partial_y n - N_2 + (\partial_t - \Delta)N_3 - \lambda(\partial_{xy}u + \partial_{yy}v). \tag{2.25}$$

Again we denote

$$\Pi_3 = -N_2 + (\partial_t - \Delta)N_3 - \lambda(\partial_{xy}u + \partial_{yy}v),$$

then (2.23) turns to

$$\partial_{tt}\psi - \Delta\partial_t\psi - \Delta\psi = \partial_y n + \Pi_3. \tag{2.26}$$

In conclusion, we get (2.20), (2.22), (2.24), (2.26) at this step. To diagonalize the systems, some further analysis on these equations are needed.

Applying the operator $\partial_{tt} - \Delta\partial_t - \Delta$ on (2.20), and using (2.26) we obtain that

$$\begin{aligned}(\partial_{tt} - \Delta\partial_t - \Delta)^2 n &= (\partial_{tt} - \Delta\partial_t - \Delta)\Delta\partial_y\psi + (\partial_{tt} - \Delta\partial_t - \Delta)\Pi_0 \\ &= \Delta\partial_{yy}n + F_0,\end{aligned}$$

which is (2.7). Applying the operator $\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}$ on (2.22), and then using (2.24), we obtain that

$$\begin{aligned}&(\partial_{tt} - \Delta\partial_t - \partial_{xx})(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})u \\ &\quad= (\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})\partial_{xy}v + \big(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}\big)\Pi_1 \\ &\quad= \partial_{xxyy}u + F_1.\end{aligned}$$

Using the formula,

$$\begin{aligned}&(\partial_{tt} - \Delta\partial_t - \partial_{xx})(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}) \\ &\quad= (\partial_{tt} - \Delta\partial_t - \Delta + \partial_{yy})(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}) \\ &\quad= (\partial_{tt} - \Delta\partial_t - \Delta)^2 - \partial_{yyyy},\end{aligned}$$

we obtain

$$(\partial_{tt} - \Delta\partial_t - \Delta)^2 u - \partial_{yyyy}u = \partial_{xxyy}u + F_1.$$

So we have (2.8). Similarly, applying the operator $\partial_{tt} - \Delta\partial_t - \partial_{xx}$ on (2.24), and then using (2.22), we obtain that

$$\begin{aligned}&(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})(\partial_{tt} - \Delta\partial_t - \partial_{xx})v \\ &\quad= (\partial_{tt} - \Delta\partial_t - \partial_{xx})\partial_{xy}u + \big(\partial_{tt} - \Delta\partial_t - \partial_{xx}\big)\Pi_2 \\ &\quad= \partial_{xxyy}v + F_2,\end{aligned}$$

which gives (2.9). Finally, applying the operator $\partial_{tt} - \Delta\partial_t - \Delta$ on (2.26), and then using (2.20), we obtain that

$$\begin{aligned}(\partial_{tt} - \Delta\partial_t - \Delta)^2\psi &= (\partial_{tt} - \Delta\partial_t - \Delta)\partial_y n + (\partial_{tt} - \Delta\partial_t - \Delta)\Pi_3 \\ &= \Delta\partial_{yy}\psi + F_3,\end{aligned}$$

which is (2.10). This finishes the diagonalization process. □

2.3. Preliminary integral representation

Since the linear parts in (2.7), (2.8), (2.9) and (2.10) all have the same structure, it suffices to consider the inhomogeneous equation

$$\Big[(\partial_{tt} - \Delta\partial_t - \Delta)^2 - \Delta\partial_{yy}\Big]\Phi = F, \tag{2.27}$$

and our aim here is to derive an equivalent integral representation of (2.27) by inverting the differentiation operator. The main result is presented in Lemma 2.2. As we shall see, the integral representation here involves some intermediate variables. Rather than using the eigenvalues and eigenfunctions, we present a basic method to resolve (2.27).

We now start the derivation process. First, we set

$$\begin{aligned}&\Big((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Phi = \Psi_1;\\ &\Big((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Phi = \Psi_2.\end{aligned} \tag{2.28}$$

Then

$$2\sqrt{\Delta\partial_{yy}}\Phi = \Psi_2 - \Psi_1. \tag{2.29}$$

Moreover, one may find from (2.27) that

$$\begin{cases}\Big(\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Psi_1 = F;\\ \Big(\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Psi_2 = F.\end{cases} \tag{2.30}$$

According to the first equation in (2.30), we set

$$\Big(\partial_t - \frac{1}{2}\Delta - \sqrt{\frac{1}{4}\Delta^2 + \Delta - \sqrt{\Delta\partial_{yy}}}\Big)\Psi_1 = \Phi_1; \tag{2.31}$$

$$\Big(\partial_t - \frac{1}{2}\Delta + \sqrt{\frac{1}{4}\Delta^2 + \Delta - \sqrt{\Delta\partial_{yy}}}\Big)\Psi_1 = \Phi_2. \tag{2.32}$$

Then

$$2\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}}\Psi_1=\Phi_2-\Phi_1. \tag{2.33}$$

Moreover, from the first equation in (2.30), we have

$$\begin{cases}\Big(\partial_t-\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}}\Big)\Phi_1=F;\\ \Big(\partial_t-\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}}\Big)\Phi_2=F.\end{cases} \tag{2.34}$$

Now according to the second equation in (2.30), we set

$$\Big(\partial_t-\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}\Big)\Psi_2=\Phi_3; \tag{2.35}$$

$$\Big(\partial_t-\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}\Big)\Psi_2=\Phi_4. \tag{2.36}$$

Then

$$2\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}\Psi_2=\Phi_4-\Phi_3. \tag{2.37}$$

Moreover, it holds that

$$\begin{cases}\Big(\partial_t-\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}\Big)\Phi_3=F;\\ \Big(\partial_t-\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}\Big)\Phi_4=F.\end{cases} \tag{2.38}$$

Now by (2.34), (2.38) and the common Duhamel's Principle, we have

$$\Phi_1=e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Phi_{1,0}+\int_0^t e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})(t-s)}F(s)\,ds;$$

$$\Phi_2=e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Phi_{2,0}+\int_0^t e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})(t-s)}F(s)\,ds;$$

$$\Phi_3=e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Phi_{3,0}+\int_0^t e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})(t-s)}F(s)\,ds;$$

$$\Phi_4=e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Phi_{4,0}+\int_0^t e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})(t-s)}F(s)\,ds.$$

Furthermore, by (2.29), (2.33) and (2.37), we find

$$\Phi = \frac{1}{2\sqrt{\Delta\partial_{yy}}}\left[\frac{\Phi_4-\Phi_3}{2\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}} - \frac{\Phi_2-\Phi_1}{2\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}}}\right].$$

Therefore, we thus have obtained the following lemma.

Lemma 2.2. *Let Φ be the solution of (2.27) with the initial datum of $\Phi_j : \Phi_j(0) = \Phi_{j,0}$, $j = 1, 2, 3, 4$. Then it obeys the following formula,*

$$\Phi = L(\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0}, \Phi_{4,0}) + \int_0^t K(t-s, \partial_x, \partial_y)F(s)\, ds, \tag{2.39}$$

where the linear part

$$\begin{aligned} &L(\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0}, \Phi_{4,0}) \qquad (2.40)\\ &= \frac{1}{2\sqrt{\Delta\partial_{yy}}}\left[\frac{1}{2\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Phi_{4,0}\right.\\ &\quad - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Phi_{3,0}\Big)\\ &\quad - \frac{1}{2\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Phi_{2,0}\\ &\quad \left. - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Phi_{1,0}\Big)\right], \qquad (2.41)\end{aligned}$$

and the operator

$$\begin{aligned} &K(t, \partial_x, \partial_y)\\ &= \frac{1}{2\sqrt{\Delta\partial_{yy}}}\left[\frac{1}{2\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\right.\\ &\quad - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Big)\\ &\quad - \frac{1}{2\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\\ &\quad \left. - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big)\right]. \end{aligned} \tag{2.42}$$

2.4. Final integral representation

The previous integral representation involves some intermediate variables $\Phi_{1,0}$, $\Phi_{2,0}$, $\Phi_{3,0}$ and $\Phi_{4,0}$. This subsection eliminates these variables and replaces them by the initial data $\Phi(0)$. In addition, we also eliminate the intermediate variables Π_0, Π_1, Π_2 and Π_3 (defined in (2.15)–(2.18)) from $F = (F_0, F_1, F_2, F_3)$ and rewrite F directly in terms of N_0, N_1, N_2 and N_3.

Lemma 2.3. *The operator L defined in (2.41) can be written as*

$$L(\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0}, \Phi_{4,0}) = K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t\Phi(0)\big] \tag{2.43}$$

$$+ (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_t\Phi(0)\big] \tag{2.44}$$

$$+ \big(\partial_t - \Delta\big)K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\Phi(0)\big] \tag{2.45}$$

$$- \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)\big[\Phi(0)\big] + K_1(t)[\Phi(0)], \tag{2.46}$$

where K_1 is given by

$$K_1 = \frac{1}{4}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}$$
$$+ e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big) \tag{2.47}$$

or, with $A = \sqrt{\xi^2 + \eta^2}$,

$$\widehat{K_1}(t, \xi, \eta) = \frac{1}{4}\Big(e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t} + e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t}$$
$$+ e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t} + e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t}\Big). \tag{2.48}$$

Proof. By (2.36) and (2.28), we have

$$\begin{aligned}\Phi_{4,0} &= \Big(\partial_t - \frac{1}{2}\Delta + \sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}\Big)\Psi_2(0)\\ &= \partial_t\Big((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Phi(0)\\ &\quad - \Big(\frac{1}{2}\Delta - \sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}\Big)\\ &\quad \times \Big((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Phi(0);\end{aligned} \tag{2.49}$$

similarly, by (2.35), (2.32), (2.31) and (2.28),

$$\Phi_{3,0} = \partial_t\Big((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Phi(0) - \Big(\frac{1}{2}\Delta + \sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}\Big)\Big((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Phi(0); \tag{2.50}$$

$$\Phi_{2,0} = \partial_t\Big((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Phi(0) - \Big(\frac{1}{2}\Delta - \sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}\Big)\Big((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Phi(0); \tag{2.51}$$

$$\Phi_{1,0} = \partial_t\Big((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Phi(0) - \Big(\frac{1}{2}\Delta + \sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}\Big)\Big((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Phi(0). \tag{2.52}$$

Inserting (2.49)–(2.52) into (2.41), we have

$$\begin{aligned}
&L(\Phi_{1,0}, \Phi_{2,0}, \Phi_{3,0}, \Phi_{4,0}) \\
&= \frac{1}{2\sqrt{\Delta\partial_{yy}}}\Big[\frac{1}{2\sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} \\
&\quad - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Big) \\
&\quad \cdot \Big(\partial_t(\partial_{tt} - \Delta\partial_t - \Delta) + \partial_t\sqrt{\Delta\partial_{yy}}\Big)\Phi(0) \\
&\quad - \frac{1}{2\sqrt{\frac{1}{4}\Delta^2 + \Delta - \sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t} - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big) \\
&\quad \cdot \Big(\partial_t(\partial_{tt} - \Delta\partial_t - \Delta) - \partial_t\sqrt{\Delta\partial_{yy}}\Big)\Phi(0)\Big] && (2.53) \\
&\quad - \frac{\Delta}{4\sqrt{\Delta\partial_{yy}}}\Big[\frac{1}{2\sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} \\
&\quad - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Big) \\
&\quad \cdot \Big((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Phi(0) \\
&\quad - \frac{1}{2\sqrt{\frac{1}{4}\Delta^2 + \Delta - \sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t} - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big) \\
&\quad \cdot \Big((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Phi(0)\Big] && (2.54) \\
&\quad - \frac{1}{4\sqrt{\Delta\partial_{yy}}}\Big[\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Big) \\
&\quad \cdot \Big((\partial_{tt} - \Delta\partial_t - \Delta) + \sqrt{\Delta\partial_{yy}}\Big)\Phi(0)
\end{aligned}$$

$$- \Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big)$$
$$\cdot \Big((\partial_{tt} - \Delta\partial_t - \Delta) - \sqrt{\Delta\partial_{yy}}\Big)\Phi(0)\Big]. \tag{2.55}$$

Further, by using the definition of K in (3.3) and the formula (3.21), we have

$$\begin{aligned}
(2.53) &= K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t\Phi(0)\big] \\
&\quad + \Big[\frac{1}{4\sqrt{\frac{1}{4}\Delta^2 + \Delta + \sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} \\
&\quad - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Big) \\
&\quad + \frac{1}{4\sqrt{\frac{1}{4}\Delta^2 + \Delta - \sqrt{\Delta\partial_{yy}}}}\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t} \\
&\quad - e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big)\Big]\partial_t\Phi(0) \\
&= K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t\Phi(0)\big] + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_t\Phi(0)\big];
\end{aligned}$$

similarly, we find

$$(2.54) = -\frac{1}{2}\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\Phi(0)\big] - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)\big[\Phi(0)\big];$$

and by the definition of $\partial_t K$ in (3.19) and K_1 in (2.48), we get

$$\begin{aligned}
(2.55) &= -\frac{1}{4\sqrt{\Delta\partial_{yy}}}\Big[\Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t}\Big) \\
&\quad - \Big(e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big)\Big](\partial_{tt} - \Delta\partial_t - \Delta)|\Phi(0) \\
&\quad + \frac{1}{4}\Big[e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta+\sqrt{\Delta\partial_{yy}}})t} \\
&\quad + e^{(\frac{1}{2}\Delta+\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t} + e^{(\frac{1}{2}\Delta-\sqrt{\frac{1}{4}\Delta^2+\Delta-\sqrt{\Delta\partial_{yy}}})t}\Big]\Phi(0) \\
&= -\frac{1}{2}\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\Phi(0)\big] + \partial_t K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\Phi(0)\big] \\
&\quad + K_1(t)[\Phi(0)].
\end{aligned}$$

Therefore, collecting the estimates above, we get (2.46). □

We now derive the final form of F by eliminating the intermediate variables Π_0, Π_1, Π_2 and Π_3 (defined in (2.15)–(2.18)). First, we consider F_0. By (2.15)–(2.18), we have

$$\begin{aligned}
F_0 &= (\partial_{tt} - \Delta\partial_t - \Delta)\Pi_0 + \Delta\partial_y\Pi_3 \\
&= (\partial_{tt} - \Delta\partial_t - \Delta)(\partial_t N_0 - \Delta N_0 - \partial_x N_1 - \partial_y N_2) + \Delta\partial_y(-N_2 + (\partial_t - \Delta)N_3) \\
&\quad + \lambda\Big(\partial_{xxx}\Delta u + \partial_{xxy}\Delta v + \partial_x\partial_t\Delta^2 u + \partial_y\partial_t\Delta^2 v - \partial_x\partial_{tt}\Delta u - \partial_y\partial_{tt}\Delta v\Big) \\
&= \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)N_0 - \Delta(\partial_{tt} - \Delta\partial_t - \Delta)N_0 - \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)N_1 \\
&\quad - \partial_y(\partial_{tt} - \Delta\partial_t)N_2 + \Delta\partial_y(\partial_t - \Delta)N_3 \\
&\quad - \lambda(\partial_{tt} - \Delta\partial_t - \partial_{xx})(\partial_x\Delta u + \partial_y\Delta v). \qquad (2.56)
\end{aligned}$$

Second, we consider F_1.

$$\begin{aligned}
F_1 &= \big(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}\big)\Pi_1 + \partial_{xy}\Pi_2 \\
&= (\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})(\partial_t N_1 - \partial_x N_0) + \partial_{xy}(-\partial_y N_0 + \partial_t N_2 - \Delta N_3) \\
&\quad + \lambda(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t\partial_x(\partial_x u + \partial_y v) \\
&= -\partial_x(\partial_{tt} - \Delta\partial_t - \Delta)N_0 + \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})N_1 + \partial_{xy}\partial_t N_2 - \partial_{xy}\Delta N_3 \\
&\quad + \lambda(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t\partial_x(\partial_x u + \partial_y v). \qquad (2.57)
\end{aligned}$$

Third, we consider F_2.

$$\begin{aligned}
F_2 &= \big(\partial_{tt} - \Delta\partial_t - \partial_{xx}\big)\Pi_2 + \partial_{xy}\Pi_1 \\
&= (\partial_{tt} - \Delta\partial_t - \partial_{xx})(-\partial_y N_0 + \partial_t N_2 - \Delta N_3) + \partial_{xy}(\partial_t N_1 - \partial_x N_0) \\
&\quad + \lambda(\partial_{tt} - \Delta\partial_t)\partial_t\partial_y(\partial_x u + \partial_y v) \\
&= -\partial_y(\partial_{tt} - \Delta\partial_t)N_0 + \partial_t\partial_{xy}N_1 + \partial_t(\partial_{tt} - \Delta\partial_t - \partial_{xx})N_2 - \Delta(\partial_{tt} - \Delta\partial_t - \partial_{xx})N_3 \\
&\quad + \lambda(\partial_{tt} - \Delta\partial_t)\partial_t(\partial_{xy}u + \partial_{yy}v). \qquad (2.58)
\end{aligned}$$

At last, we consider F_3.

$$\begin{aligned}
F_3 &= (\partial_{tt} - \Delta\partial_t - \Delta)\Pi_3 + \partial_y\Pi_0 \\
&= (\partial_{tt} - \Delta\partial_t - \Delta)(-N_2 + (\partial_t - \Delta)N_3) + \partial_y(\partial_t N_0 - \Delta N_0 - \partial_x N_1 - \partial_y N_2) \\
&\quad - \lambda(\partial_{tt} - \Delta\partial_t)\partial_y(\partial_x u + \partial_y v) \\
&= \partial_y\partial_t N_0 - \partial_y\Delta N_0 - \partial_{xy}N_1 - (\partial_{tt} - \Delta\partial_t - \partial_{xx})N_2 + (\partial_t - \Delta)(\partial_{tt} - \Delta\partial_t - \Delta)N_3 \\
&\quad - \lambda(\partial_{tt} - \Delta\partial_t)\partial_y(\partial_x u + \partial_y v). \qquad (2.59)
\end{aligned}$$

3. Fourier analysis on the linear flow

This section provides pointwise as well as L^p-estimates for $\widehat{K}(t,\xi,\eta)$ (defined in (2.42)) and the Fourier transforms of various derivatives of K. The pointwise estimates are stated in Lemma 3.3 while the L^p-estimates are stated in Lemma 3.7 through Lemma 3.22.

The last lemma of this section provides the estimates for $\widehat{K}_1$ defined in (2.48). Before presenting these estimates, we provide several notations and basic tool inequalities.

The definition of the Fourier transform is standard, namely

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^d} e^{-ix\cdot\zeta} f(x)\,dx, \qquad \text{for any} \qquad \zeta \in \mathbb{R}^d.$$

In the 2D case, $\mathcal{F}_\xi f$ and $\mathcal{F}_\eta f$ are used to denote the corresponding Fourier transforms with respect to the x and y variables, respectively. Furthermore, for each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned}
\widehat{P_{\leq N} f}(\zeta) &:= \chi_{\leq N}(\zeta)\hat{f}(\zeta), \\
\widehat{P_{> N} f}(\zeta) &:= \chi_{> N}(\zeta)\hat{f}(\zeta), \\
\widehat{P_N f}(\zeta) &:= (\chi_{\leq N} - \chi_{\leq N/2})(\zeta)\hat{f}(\zeta)
\end{aligned}$$

and similarly $P_{<N}$ and $P_{\geq N}$. Here, we use χ to denote a smooth bump function such that

$$\begin{cases} \chi(x) = 1, & |x| \leq 1, \\ \chi(x) = 0, & |x| \geq 1 + 10^{-4}, \end{cases}$$

and denote $\chi_R = \chi(\cdot/R)$. We also define, for $0 < N_1 < N_2$

$$P_{N_1 < \cdot \leq N_2} := P_{\leq N_2} - P_{\leq N_1}.$$

We will use the following Bernstein's inequality.

Lemma 3.1 *(Bernstein's inequality). For $1 \leq p \leq q \leq \infty$ and $M > 0$,*

$$\begin{aligned}
\big\| |\nabla|^{\pm s} P_M f \big\|_{L^p(\mathbb{R}^d)} &\sim M^{\pm s} \|P_M f\|_{L^p(\mathbb{R}^d)}, \\
\|P_{\leq M} f\|_{L^q(\mathbb{R}^d)} &\lesssim M^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq M} f\|_{L^p(\mathbb{R}^d)}, \\
\|P_M f\|_{L^q(\mathbb{R}^d)} &\lesssim M^{\frac{d}{p} - \frac{d}{q}} \|P_M f\|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

We define differential operator $P(D)$ as

$$\widehat{P(D) f}(\xi, \eta) = P(\xi, \eta)\widehat{f}(\xi, \eta), \qquad \text{for any } (\xi, \eta) \in \mathbb{R}^2.$$

Then we have the following generalized Young's inequality.

Lemma 3.2 *(Generalized Young's inequality). Let $1 \leq r_1, r_2 \leq 2 \leq p \leq \infty$, $q_1, q_2 \geq p'$ be the numbers satisfying $\frac{1}{p} = \frac{1}{r_1} - \frac{1}{q_1} = \frac{1}{r_2} - \frac{1}{q_2} = 1 - \frac{1}{p'}$, then for any two-variable function $f(x, y) \in L_x^{r_1} L_y^{r_2}(\mathbb{R} \times \mathbb{R})$,*

$$\big\|P(D)f\big\|_{L^p_{xy}} \lesssim \|P(\xi,\eta)\|_{L^{q_1}_\xi L^{q_2}_\eta}\|f\|_{L^{r_1}_x L^{r_2}_y}. \tag{3.1}$$

In particular, let $1 \le r \le 2 \le p \le \infty$, $q \ge p'$ *be the numbers satisfying* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, *then for any* $f \in L^r(\mathbb{R}^2)$,

$$\big\|P(D)f\big\|_{L^p_{xy}} \lesssim \|P(\xi,\eta)\|_{L^q_{\xi\eta}}\|f\|_{L^r_{xy}}. \tag{3.2}$$

Proof. Since $p \ge 2$ and $r' \ge 2$, by Young's and Hölder's inequalities, we have

$$\big\|P(D)f\big\|_{L^p_{xy}} \lesssim \big\|P(\xi,\eta)\hat{f}\big\|_{L^{p'}_{\xi\eta}} \lesssim \big\|P(\xi,\eta)\big\|_{L^{q_1}_\xi L^{q_2}_\eta}\big\|\hat{f}\big\|_{L^{r_1'}_\xi L^{r_2'}_\eta}.$$

Since $r_1' \ge 2, r_2' \ge 2$ and $r_1' \ge r_2$, by Fubini's, and Young's inequalities again, we further have

$$\begin{aligned}\big\|P(D)f\big\|_{L^p_{xy}} &\lesssim \big\|P(\xi,\eta)\big\|_{L^{q_1}_\xi L^{q_2}_\eta}\Big\|\big\|\mathcal{F}_\xi f(\xi,y)\big\|_{L^{r_2}_y}\Big\|_{L^{r_1'}_\xi}\\ &\lesssim \big\|P(\xi,\eta)\big\|_{L^{q_1}_\xi L^{q_2}_\eta}\Big\|\big\|\mathcal{F}_\xi f(\xi,y)\big\|_{L^{r_1'}_\xi}\Big\|_{L^{r_2}_y}\\ &\lesssim \big\|P(\xi,\eta)\big\|_{L^{q_1}_\xi L^{q_2}_\eta}\big\|f(x,y)\big\|_{L^{r_1}_x L^{r_2}_y}.\end{aligned}$$

This proves (3.1). In particular, letting $q_1 = q_2 = q, r_1 = r_2 = r$ yields (3.2). □

The rest of this section is divided into two subsections with the first devoted to the pointwise estimates and the second to the L^p-estimates. To simplify the notation, we again write $A = \sqrt{\xi^2+\eta^2}$. Then, by (2.42),

$$\begin{aligned}\widehat{K}(t,\xi,\eta) = \frac{1}{2A|\eta|}\Bigg[&\frac{1}{2\sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|}}\Big(e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t}\\ &- e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t}\Big)\\ &- \frac{1}{2\sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|}}\Big(e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t}\\ &- e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t}\Big)\Bigg].\end{aligned} \tag{3.3}$$

3.1. Pointwise estimates

The main results in this subsection are stated as follows.

Proposition 3.3. *Let* $\widehat{K}$ *be defined in* (3.3). *Then there exists a constant* $c > 0$ *such that the following estimates hold:*

1. $$|\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^4}e^{-ct} + \chi_{A\leq 1}\min\Big\{\frac{1}{A|\xi||\eta|},\frac{1}{A^4}\Big\}e^{-\frac{1}{4}A^2 t} + \frac{1}{A^4}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.4}$$

2. $$|\partial_t\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^4}e^{-ct} + \chi_{A\leq 1}\min\Big\{\frac{1}{A^3},\frac{1}{A|\eta|}\Big\}e^{-\frac{1}{4}A^2 t} + \frac{\xi^2}{A^6}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.5}$$

3. $$|\partial_{tt}\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^4}e^{-ct} + \chi_{A\leq 1}\min\Big\{\frac{1}{A^2},\frac{1}{|\eta|}\Big\}e^{-\frac{1}{4}A^2 t} + \frac{\xi^4}{A^8}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.6}$$

4. $$|\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^2}e^{-ct} + \chi_{A\leq 1}\min\Big\{\frac{1}{|\xi|},\frac{1}{A^2}\Big\}e^{-\frac{1}{4}A^2 t} + \frac{1}{A^2}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.7}$$

5. $$|\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^2}e^{-ct} + \chi_{A\leq 1}e^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^4}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.8}$$

6. $$|\partial_{tt}\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^2}e^{-ct} + \chi_{A\leq 1}Ae^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}\frac{\xi^4}{A^6}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.9}$$

7. $$|\big(\partial_{tt}+A^2\partial_t+\xi^2\big)\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^2}e^{-ct} + \chi_{A\leq 1}\frac{1}{A}e^{-\frac{1}{4}A^2 t} + \frac{\xi^2}{A^4}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.10}$$

8. $$|\big(\partial_{tt}+A^2\partial_t+\xi^2\big)\partial_t\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}\frac{1}{A^2}e^{-ct} + \chi_{A\leq 1}e^{-\frac{1}{4}A^2 t} + \frac{\xi^4}{A^6}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.11}$$

We make several remarks. As shown in the proposition, we pointwise estimates are split into three parts contained: $\chi_{A\geq 1}, \chi_{A\leq 1}, \chi_{|\xi|\lesssim A^2}$. They reflect the different behaviors of the operator in different regions. The parts $\chi_{A\geq 1}\cdot$ tells the behavior of the operator in high-frequence; the parts $\chi_{A\geq 1}\cdot$ tells the behavior of the operator in low-frequence, also it tells the strength of the singularity of the operator; the parts $\chi_{|\xi|\lesssim A^2}\cdot$ will tell us how the ξ-direction affects the decaying of the linear flow.

In contrast to the incompressible MHD equations studied in [10], the estimates for K sensitively depend on the spatial dimension and as we shall see in the later sections, the 2D compressible MHD equations is critical in the sense that the spatial L^∞ norms behave like t^{-1} and is barely time integrable. One may expect from the parabolic structure that $\partial_t \sim \partial_{xx}$ in the large time behavior, but this proposition implies that $\partial_t \sim \partial_x$. This weaker regularity effect of ∂_t is due to the dispersive effect of $e^{i\sqrt{A^2-\frac{1}{4}A^4+\pm A|\eta|}t}$, as indicated in the proof of this proposition. We also point out that the operator $\big(\partial_{tt} - \Delta\partial_t - \Delta\big)K$ behaves like that of $\sqrt{-\Delta}\partial_y K$, which is better than ΔK. That is,

$$\big(\partial_{tt} + A^2\partial_t + A^2\big) \sim A|\eta|.$$

We emphasize the anisotropicity here because of its non-obvious and its improtant role in our analysis. This anisotropicity will be frequently used in the subsequent sections.

Proof of Proposition 3.3. First, we estimate $\widehat{K}$. According to the singularity of $\frac{1}{A|\eta|}$ from the expression (3.3), we split into the following two cases.

$$\textbf{Case 1:}\quad A|\eta| \le \frac{1}{2}A^2;\quad \textbf{Case 2:}\quad A|\eta| \ge \frac{1}{2}A^2.$$

Case 1: $A|\eta| \le \frac{1}{2}A^2$. It is divided by the following two subcases,

$$\textbf{Subcase 11:}\quad \frac{1}{4}A^4 - A^2 + A|\eta| \ge 0;\quad \textbf{Subcase 12:}\quad \frac{1}{4}A^4 - A^2 + A|\eta| < 0.$$

Subcase 11: $\frac{1}{4}A^4 - A^2 + A|\eta| \ge 0$, then $A \ge 1$, and

$$-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|} \le -\frac{1}{2}.$$

Now we need the following lemma, which will be proved in Appendix A.1.

Lemma 3.4. *Let $a > 0, b \in \mathbb{R}, c > 0$. Then there exists a constant $C > 0$, such that*

$$\left|\frac{1}{c}\left\{\frac{1}{\sqrt{b+c}}\Big[e^{(-a+\sqrt{b+c})t} - e^{(-a-\sqrt{b+c})t}\Big] - \frac{1}{\sqrt{b-c}}\Big[e^{(-a+\sqrt{b-c})t} - e^{(-a-\sqrt{b-c})t}\Big]\right\}\right|$$
$$\le \begin{cases} C\min\left\{t^3, \frac{t}{c}\right\}e^{-at} & \text{when}\quad b+c<0,\\ C\min\left\{t^3, \frac{1}{c\sqrt{b+c}}, \frac{\langle t\rangle}{b+c}\right\}e^{(-a+\sqrt{b+c})t} & \text{when}\quad b+c\ge 0.\end{cases}$$

Thus, by Lemma 3.4, if $A \sim 1$ we have

$$|\widehat{K}(t,\xi,\eta)| \lesssim t^3 e^{-\frac{1}{2}t} \le e^{-ct}, \qquad \text{for some small constant } c > 0;$$

if $A \gg 1$, then $\frac{1}{4}A^4 - A^2 + A|\eta| \sim A^4$, and

$$|\widehat{K}(t,\xi,\eta)| \lesssim \frac{\langle t\rangle}{\frac{1}{4}A^4 - A^2 + A|\eta|}e^{-\frac{1}{2}t} \lesssim \frac{1}{A^4}e^{-ct}.$$

Therefore, no matter in which case,

$$|\widehat{K}(t,\xi,\eta)| \le \chi_{A\ge 1}\frac{1}{A^4}e^{-ct}. \tag{3.12}$$

Subcase 12: $\frac{1}{4}A^4 - A^2 + A|\eta| < 0$. If $A \gtrsim 1$, then by Lemma 3.4, we have

$$|\widehat{K}(t,\xi,\eta)| \lesssim t^3 e^{-\frac{1}{2}A^2 t} \lesssim \frac{1}{A^4}e^{-\frac{1}{4}A^2 t}.$$

So we only need to consider $A \ll 1$. Then, we have

$$\begin{aligned}
&\frac{1}{4}A^4 - A^2 - A|\eta| < 0, \quad \text{and} \\
&\big|\frac{1}{4}A^4 - A^2 + A|\eta|\big| \sim \big|\frac{1}{4}A^4 - A^2 - A|\eta|\big| \sim A^2.
\end{aligned} \tag{3.13}$$

So an immediately estimate from (3.3) is that

$$|\widehat{K}(t,\xi,\eta)| \lesssim \frac{1}{A^2|\eta|}e^{-\frac{1}{4}A^2 t} \lesssim \frac{1}{A|\xi||\eta|}e^{-\frac{1}{4}A^2 t}.$$

Moreover, using the following elementary inequality (see Appendix A.2 for its proof),

$$\Big|\frac{\sin x}{x} - \frac{\sin y}{y}\Big| \lesssim \big||x|-|y|\big| \min\Big\{|x|+|y|, \frac{1}{|x|+|y|}\Big\}, \quad \text{for any } x,y\in\mathbb{R}, \tag{3.14}$$

and (3.13), we have

$$\begin{aligned}
&|\widehat{K}(t,\xi,\eta)| \\
&\quad = \left|\frac{1}{2A|\eta|}e^{-\frac{1}{2}A^2 t}\left(\frac{\sin\big(\sqrt{|\frac{1}{4}A^4 - A^2 + A|\eta||}\,t\big)}{\sqrt{|\frac{1}{4}A^4 - A^2 + A|\eta||}} - \frac{\sin\big(\sqrt{|\frac{1}{4}A^4 - A^2 - A|\eta||}\,t\big)}{\sqrt{|\frac{1}{4}A^4 - A^2 - A|\eta||}}\right)\right| \\
&\quad \lesssim \frac{t}{A|\eta|}e^{-\frac{1}{2}A^2 t}\left(t\sqrt{\big|\frac{1}{4}A^4 - A^2 + A|\eta|\big|} - t\sqrt{\big|\frac{1}{4}A^4 - A^2 - A|\eta|\big|}\right) \\
&\qquad \times \Big(\chi_{At\lesssim 1}At + \chi_{At\gtrsim 1}\frac{1}{At}\Big) \\
&\quad \lesssim \Big(t^3\chi_{At\lesssim 1} + t\frac{1}{A^2}\chi_{At\gtrsim 1}\Big)e^{-\frac{1}{2}A^2 t} \\
&\quad \lesssim \Big(\frac{1}{A^3} + \frac{1}{A^2}t\Big)e^{-\frac{1}{2}A^2 t} \lesssim \frac{1}{A^4}e^{-\frac{1}{4}A^2 t}.
\end{aligned}$$

Therefore, we proved that in this subcase and $A \ll 1$,

$$|\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\le 1}\min\Big\{\frac{1}{A|\xi||\eta|}, \frac{1}{A^4}\Big\}e^{-\frac{1}{4}A^2 t}.$$

Case 2: $A|\eta| \geq \frac{1}{2}A^2$. Again, we consider the following two subcases respectively,

$$\textbf{Subcase 21:}\quad \frac{1}{4}A^4 - A^2 + A|\eta| < \frac{1}{36}A^4; \quad \textbf{Subcase 22:}\quad \frac{1}{4}A^4 - A^2 + A|\eta| \geq \frac{1}{36}A^4.$$

Subcase 21: $\frac{1}{4}A^4 - A^2 + A|\eta| < \frac{1}{36}A^4$. In this subcase, we have

$$(\frac{1}{4} - \frac{1}{36})A^4 \leq A^2 - A|\eta| = A\frac{\xi^2}{A+|\eta|} \leq \xi^2.$$

That is, $A^4 \lesssim \xi^2$, and thus $A \lesssim 1$. Then by mean value theorem,

$$\begin{aligned}
|\widehat{K}(t,\xi,\eta)| &= \left|\frac{1}{2A|\eta|}e^{-\frac{1}{2}A^2t}\left[e^{\sqrt{\frac{1}{4}A^4-A^2+A|\eta|}t}\,\frac{1-e^{-2\sqrt{\frac{1}{4}A^4-A^2+A|\eta|}t}}{2\sqrt{\frac{1}{4}A^4-A^2+A|\eta|}}\right.\right.\\
&\qquad\left.\left. - e^{\sqrt{\frac{1}{4}A^4-A^2-A|\eta|}t}\,\frac{1-e^{-2\sqrt{\frac{1}{4}A^4-A^2-A|\eta|}t}}{2\sqrt{\frac{1}{4}A^4-A^2-A|\eta|}}\right]\right| \qquad (3.15)\\
&\lesssim \frac{t}{A^2}e^{-\frac{1}{2}A^2t}\left|e^{\sqrt{\frac{1}{4}A^4-A^2+A|\eta|}t}\right| + \frac{t}{A^2}e^{-\frac{1}{2}A^2t}\left|e^{\sqrt{\frac{1}{4}A^4-A^2-A|\eta|}t}\right|\\
&\lesssim \frac{t}{A^2}e^{-\frac{1}{3}A^2t} \lesssim \frac{1}{A^4}e^{-\frac{1}{4}A^2t}, \qquad (3.16)
\end{aligned}$$

where we have used $\frac{1}{4}A^4 - A^2 - A|\eta| \leq \frac{1}{4}A^4 - A^2 + A|\eta| < \frac{1}{36}A^4$. Moreover, if $A^4 \sim \xi^2$, then by (3.16),

$$|\widehat{K}(t,\xi,\eta)| \lesssim \frac{1}{A|\xi||\eta|}e^{-\frac{1}{4}A^2t}.$$

If $A^4 \ll \xi^2$, then

$$0 < -\big(\frac{1}{4}A^4 - A^2 + A|\eta|\big) \sim \xi^2, \quad \text{and } 0 < -\big(\frac{1}{4}A^4 - A^2 - A|\eta|\big) \sim A^2.$$

Hence, by (3.15),

$$\begin{aligned}
|\widehat{K}(t,\xi,\eta)| &\lesssim \frac{1}{A|\xi||\eta|}e^{-\frac{1}{2}A^2t}\left|e^{\sqrt{\frac{1}{4}A^4-A^2+A|\eta|}t}\right| + \frac{1}{A^2|\eta|}e^{-\frac{1}{2}A^2t}\left|e^{\sqrt{\frac{1}{4}A^4-A^2-A|\eta|}t}\right|\\
&\lesssim \frac{1}{A|\xi||\eta|}e^{-\frac{1}{2}A^2t}.
\end{aligned}$$

Therefore, no matter in which case, we also obtain that

$$|\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\lesssim 1}\min\Big\{\frac{1}{A|\xi||\eta|}, \frac{1}{A^4}\Big\}e^{-\frac{1}{4}A^2t}.$$

Subcase 22: $\frac{1}{4}A^4 - A^2 + A|\eta| \geq \frac{1}{36}A^4$, then $\xi^2 \lesssim A^4$. Further, since

$$-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|} = -\frac{A}{A+|\eta|}\frac{\xi^2}{\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|}},$$

we have

$$-\frac{2\xi^2}{A^2} \leq -\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|} \leq -\frac{\xi^2}{2A^2}. \tag{3.17}$$

Thus, by (3.17) and Lemma 3.4, we have

$$|\widehat{K}(t,\xi,\eta)| \lesssim \chi_{|\xi|\lesssim A^2}\frac{1}{A|\eta|\sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|}}e^{-\frac{\xi^2}{2A^2}t} \lesssim \frac{1}{A^4}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}.$$

To collect the estimates above, we obtain (3.4).

Now we turn to $\partial_t K(t,\partial_x,\partial_y)$. A direct computation gives us that

$$\begin{aligned}
&\partial_t \widehat{K}(t-s,\xi,\eta) \\
&= \frac{1}{2A|\eta|}\Bigg\{\frac{1}{2\sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|}}\Big[\Big(-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|}\Big) \\
&\quad\times e^{(-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|})t} \\
&\quad - \Big(-\frac{1}{2}A^2 - \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|}\Big)e^{(-\frac{1}{2}A^2 - \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|})t}\Big] \\
&\quad - \frac{1}{2\sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|}}\Big[\Big(-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|}\Big) \\
&\quad\times e^{(-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|})t} \\
&\quad - \Big(-\frac{1}{2}A^2 - \sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|}\Big)e^{(-\frac{1}{2}A^2 - \sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|})t}\Big]\Bigg\} \qquad (3.18) \\
&= -\frac{A^2}{2}\widehat{K}(t,\xi,\eta) + \frac{1}{4A|\eta|}\Big[e^{(-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|})t} \\
&\quad + e^{(-\frac{1}{2}A^2 - \sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|})t} \\
&\quad - e^{(-\frac{1}{2}A^2 + \sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|})t} - e^{(-\frac{1}{2}A^2 - \sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|})t}\Big]. \qquad (3.19)
\end{aligned}$$

Therefore, by (3.19) and the similar estimates as those for (3.4), we obtain (3.5). By a similar way, we also get the following the estimates on $\partial_{tt}K(t,\partial_x,\partial_y)$ in (3.6).

We now give the estimates on the special forms. First, we consider $\big(\partial_{tt} - \Delta\partial_t - \Delta\big)K(t, \partial_x, \partial_y)$. To this end, we derive an identity. Let

$$\phi_{\mu_1,\mu_2} = e^{(\frac{1}{2}\Delta+\mu_1\sqrt{\frac{1}{4}\Delta^2+\Delta+\mu_2\sqrt{\Delta\partial_{yy}}})t},$$

where $\mu_1, \mu_2 = \pm 1$, then we have

$$\begin{aligned}
\partial_{tt}\phi_{\mu_1,\mu_2} &= \left(\frac{1}{2}\Delta + \mu_1\sqrt{\frac{1}{4}\Delta^2 + \Delta + \mu_2\sqrt{\Delta\partial_{yy}}}\right)^2 \phi_{\mu_1,\mu_2} \\
&= \Big(\frac{1}{2}\Delta^2 + \mu_1\Delta\sqrt{\frac{1}{4}\Delta^2 + \Delta + \mu_2\sqrt{\Delta\partial_{yy}}} + \Delta + \mu_2\sqrt{\Delta\partial_{yy}}\Big)\phi_{\mu_1,\mu_2} \\
&= \Delta\partial_t\phi_{\mu_1,\mu_2} + \Delta\phi_{\mu_1,\mu_2} + \mu_2\sqrt{\Delta\partial_{yy}}\phi_{\mu_1,\mu_2},
\end{aligned}$$

that is,

$$\partial_{tt}\phi_{\mu_1,\mu_2} - \Delta\partial_t\phi_{\mu_1,\mu_2} - \Delta\phi_{\mu_1,\mu_2} = \mu_2\sqrt{\Delta\partial_{yy}}\phi_{\mu_1,\mu_2}. \tag{3.20}$$

Now using (3.20), we have

$$\begin{aligned}
&\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) \\
&\quad = \frac{1}{4\sqrt{\frac{1}{4}A^4 - A^2 + A|\eta|}}\Big(e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t} \\
&\qquad - e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t}\Big) \\
&\qquad + \frac{1}{4\sqrt{\frac{1}{4}A^4 - A^2 - A|\eta|}}\Big(e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t} \\
&\qquad - e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t}\Big).
\end{aligned} \tag{3.21}$$

Compared with $\partial_{tt}\widehat{K}(t,\xi,\eta), A^2\widehat{K}(t,\xi,\eta)$, the form $\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta)$ erases the bad factor of $\frac{1}{|\eta|}$, and thus gives (3.7). Further,

$$\begin{aligned}
&\partial_t\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) \\
&\quad = -\frac{1}{2}A^2\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) + \frac{1}{4}\Big(e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t} \\
&\qquad + e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t} \\
&\qquad + e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t} + e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t}\Big).
\end{aligned} \tag{3.22}$$

From (3.22), and by the similar argument as the proof of (3.4), we have (3.8).

Next, we consider another special form $\big(\partial_{tt} - \Delta\partial_t - \partial_{xx}\big)K(t, \partial_x, \partial_y)$. It obeys the same estimate as the operator $\big(\partial_{tt} - \Delta\partial_t - \Delta\big)K(t, \partial_x, \partial_y)$ when $|\eta|$ is small. Indeed, we have

$$\big(\partial_{tt} - \Delta\partial_t - \partial_{xx}\big)K(t, \partial_x, \partial_y) = \big(\partial_{tt} - \Delta\partial_t - \Delta\big)K(t, \partial_x, \partial_y) + \partial_{yy}K(t, \partial_x, \partial_y).$$

But on the other hand, from the form itself, we see that it is better when $|\xi|$ is small. Thus similar as above, we have (3.10) and (3.11). □

Recalling $\widehat{K_1}(t, \xi, \eta)$ defined in (2.48), we have

$$\begin{aligned} &\partial_t\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t, \xi, \eta) \\ &\quad = -\frac{1}{2}A^2\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t, \xi, \eta) + \widehat{K_1}(t, \xi, \eta). \end{aligned} \tag{3.23}$$

This equality is useful in the following sections.

3.2. Estimates in L^p-space

This subsection provides estimates for $\widehat{K}$ and the Fourier transforms of various derivatives of K in L^p-spaces. The pointwise estimates in the previous subsections will be used here. We first state and prove an elementary lemma, which will be used repeatedly in the estimates.

Lemma 3.5. *Let c be a positive constant, and let $N \geq 0$ be the dyadic number ($N = 2^j$ for some $j \in \mathbb{Z}$), then*

$$\iint\limits_{A\leq 1} A^\beta e^{-cA^2 t}\, d\xi d\eta \lesssim \langle t\rangle^{-1-\frac{\beta}{2}}, \qquad \textit{for any } \beta > -2; \tag{3.24}$$

and

$$\iint\limits_{A\sim N} \frac{\xi^\beta}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t}\, d\xi d\eta \lesssim N^{\beta+2-\alpha}\langle t\rangle^{-\frac{1+\beta}{2}}, \qquad \textit{for any } \alpha \in \mathbb{R}, \beta > -1. \tag{3.25}$$

In particular, for any $\alpha \in \mathbb{R}, \beta' \in \mathbb{R}, \beta > -1$ with $\beta' \geq \beta, 2\beta' - \beta + 2 - \alpha > 0$, we have

$$\iint\limits_{A\leq 1} \chi_{|\xi|\lesssim A^2} \frac{\xi^{\beta'}}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t}\, d\xi d\eta \lesssim \langle t\rangle^{-\frac{1+\beta}{2}}. \tag{3.26}$$

Proof. First, we prove (3.24). If $t \geq 1$, we set $\xi_t = \xi\sqrt{t}, \eta_t = \eta\sqrt{t}$. Then by changing variable, we have

$$\iint\limits_{A\le 1} A^\beta e^{-cA^2 t}\, d\xi d\eta = t^{-1-\frac{\beta}{2}} \iint\limits_{\xi_t^2+\eta_t^2\le t} (\xi_t^2+\eta_t^2)^{\frac{\beta}{2}} e^{-c(\xi_t^2+\eta_t^2)}\, d\xi_t d\eta_t$$

$$\le t^{-1-\frac{\beta}{2}} \iint\limits_{\mathbb{R}^2} (\xi^2+\eta^2)^{\frac{\beta}{2}} e^{-c(\xi^2+\eta^2)}\, d\xi d\eta \lesssim t^{-1-\frac{\beta}{2}}.$$

If $0 \le t \le 1$, then

$$\iint\limits_{A\le 1} A^\beta e^{-\frac{1}{4}A^2 t}\, d\xi d\eta \le \iint\limits_{A\le 1} A^\beta\, d\xi d\eta \lesssim 1.$$

Now we prove (3.25). Similarly, by changing variable, we obtain that for some positive constant $\tilde{c}$,

$$\iint\limits_{A\sim N} \frac{\xi^\beta}{A^\alpha} e^{-c\frac{\xi^2 t}{A^2}}\, d\xi d\eta \sim \iint\limits_{A\sim N} \frac{\xi^\beta}{N^\alpha} e^{-\tilde{c}\frac{\xi^2 t}{N^2}}\, d\xi d\eta$$

$$\le t^{-\frac{1+\beta}{2}} N^{\beta+1-\alpha} \int\limits_{|\eta|\lesssim N} \int\limits_{\xi\in\mathbb{R}} \Big(\frac{\xi\sqrt{t}}{N}\Big)^\beta e^{-\tilde{c}\frac{\xi^2 t}{N^2}}\, d(\frac{\xi\sqrt{t}}{N}) d\eta$$

$$\lesssim t^{-\frac{1+\beta}{2}} N^{\beta+2-\alpha}.$$

If $0 \le t \le 1$, then

$$\iint\limits_{A\sim N} \frac{\xi^\beta}{A^\alpha} e^{-\frac{\xi^2}{2A^2} t}\, d\xi d\eta \le \iint\limits_{A\sim N} \frac{\xi^\beta}{A^\alpha}\, d\xi d\eta \lesssim N^{\beta+2-\alpha}.$$

Further, since $\chi_{|\xi|\lesssim A^2} \frac{\xi^{\beta'}}{A^\alpha} \lesssim \frac{\xi^\beta}{A^{\alpha-2(\beta'-\beta)}}$, by using (3.25) and the dyadic decomposition, we also have (3.26). □

Remark 3.6. The L^q estimates can also be easily obtained from the conclusions in Lemma 3.5. For example, from (3.24), let $\tilde{c} = cq$, we obtain

$$\big\|A^\beta e^{-cA^2 t}\big\|_{L^q_{\xi\eta}} = \big\|A^{\beta q} e^{-\tilde{c}A^2 t}\big\|^{\frac{1}{q}}_{L^1_{\xi\eta}} \lesssim \langle t\rangle^{-(1+\frac{\beta q}{2})\frac{1}{q}}.$$

3.2.1. Estimates on $\widehat{K(t)}$

Lemma 3.7. *Let $1 \le q \le \infty$, and $N \gtrsim 1$. Then*

$$\big\|A^4 \widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2q}};$$

$$\big\|\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{\frac{2}{q}-4} \langle t\rangle^{-\frac{1}{2q}}. \tag{3.27}$$

Moreover, for any $\beta > \frac{1}{2}$,

$$\left\|A^{3+\beta}\widehat{K}(t,\xi,\eta)\right\|_{L^2_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{4}}. \tag{3.28}$$

Proof. By (3.4), we have

$$\left|A^4\widehat{K}(t,\xi,\eta)\right| \lesssim \chi_{A\ge 1}e^{-ct} + \chi_{A\le 1}e^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.29}$$

Note that from (3.29),

$$\left\|A^4\widehat{K}(t,\xi,\eta)\right\|_{L^\infty_{\xi\eta}} \lesssim 1. \tag{3.30}$$

Further, when $A \ge 1$,

$$\left|A^4\widehat{K}(t,\xi,\eta)\right| \lesssim \chi_{A\ge 1}e^{-ct} + \chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}.$$

Thus, by (3.25),

$$\begin{aligned}\left\|A^4\widehat{K}(t,\xi,\eta)\right\|_{L^1_{\xi\eta}(A\sim N)} &\lesssim \left\|\chi_{A\ge 1}e^{-ct}\right\|_{L^1_{\xi\eta}(A\sim N)} + \left\|e^{-\frac{\xi^2}{2A^2}t}\right\|_{L^1_{\xi\eta}(A\sim N)}\\ &\lesssim N^2e^{-ct} + N^2\langle t\rangle^{-\frac{1}{2}} \lesssim N^2\langle t\rangle^{-\frac{1}{2}}.\end{aligned}$$

That is,

$$\left\|\widehat{K}(t,\xi,\eta)\right\|_{L^1_{\xi\eta}(A\sim N)} \lesssim N^{-2}\langle t\rangle^{-\frac{1}{2}}. \tag{3.31}$$

Now, when $A \le 1$,

$$\left|A^4\widehat{K}(t,\xi,\eta)\right| \lesssim \chi_{A\le 1}e^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.32}$$

Similarly, by (3.24) and (3.26),

$$\iint\limits_{A\le 1} e^{-\frac{1}{4}(\xi^2+\eta^2)t}\,d\xi d\eta \lesssim \langle t\rangle^{-1}; \quad \iint\limits_{A\le 1} e^{-\frac{\xi^2}{2A^2}t}\,d\xi d\eta \lesssim \langle t\rangle^{-\frac{1}{2}},$$

thus,

$$\left\|A^4\widehat{K}(t,\xi,\eta)\right\|_{L^1_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2}}. \tag{3.33}$$

Now the conclusion (3.27) follows from interpolation between (3.30) and (3.33), (3.31) respectively. For (3.28), by (3.32), we have

$$\left|A^{3+\beta}\widehat{K}(t,\xi,\eta)\right| \lesssim \chi_{A\le 1}A^{\beta-1}e^{-\frac{1}{4}A^2 t} + A^{\beta-1}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t},$$

which is square integrable. Thus direct integration and using (3.24), (3.26) give (3.28). □

We give the estimates on the operator $\partial_{xy}\nabla K(t)$, which read as

Lemma 3.8. *Let $1 \le q \le \infty$ and $N \gtrsim 1$. Then*

$$\big\|\xi\eta A\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{q}}; \tag{3.34}$$

$$\big\|\xi\eta \widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim \langle N\rangle^{-2+\frac{2}{q}}\langle t\rangle^{-\frac{1}{2}-\frac{1}{2q}}. \tag{3.35}$$

Moreover, for any $\beta \in [\frac{1}{2}, \frac{3}{2}]$,

$$\begin{aligned} &\big\|A^{\beta+1}\xi\eta \widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2}};\\ &\big\|A^{\beta}\xi\eta \widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{\beta}{2}}; \end{aligned} \tag{3.36}$$

Proof. By (3.4), we have

$$\big|\xi\eta\widehat{K}(t,\xi,\eta)\big| \lesssim \chi_{A\ge 1}\frac{1}{A^2}e^{-ct} + \chi_{A\le 1}\frac{1}{A}e^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}\frac{\xi}{A^3}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.37}$$

Thus, we further have, for any α,

$$\begin{aligned} \big|A^\alpha\xi\eta\widehat{K}(t,\xi,\eta)\big| \lesssim{}& \chi_{A\ge 1}\frac{1}{A^{2-\alpha}}e^{-ct} + \chi_{A\le 1}A^{\alpha-1}e^{-\frac{1}{4}A^2 t}\\ &+ \chi_{|\xi|\lesssim A^2}\frac{\xi}{A^{3-\alpha}}e^{-\frac{\xi^2}{2A^2}t}. \end{aligned} \tag{3.38}$$

Similar as the proof of Lemma 3.29, by (3.38), (3.24) and (3.26), we have

$$\big\|\xi\eta A\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)} \lesssim 1; \qquad \big\|\xi\eta A\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-1},$$

and thus (3.34) follows from interpolation. For $N \gtrsim 1$, by (3.37) and (3.25),

$$\big\|\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\sim N)} \lesssim N^{-2}\langle t\rangle^{-\frac{1}{2}}; \qquad \big\|\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\sim N)} \lesssim \langle t\rangle^{-1},$$

and thus (3.35) follows from interpolation again. Using (3.38) and Lemma 3.5, (3.36) is also easily followed. □

Lemma 3.9. *Let $N \gtrsim 1$ and $1 \le q \le \infty$, then*

$$\big\|A\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2}-\frac{1}{q}}; \tag{3.39}$$

$$\big\|\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{\frac{2}{q}-1}\langle t\rangle^{-1-\frac{1}{2q}}. \tag{3.40}$$

Moreover,

$$\big\|A^2\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-1}; \tag{3.41}$$

$$\big\|A\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)} \lesssim \langle t\rangle^{-1}. \tag{3.42}$$

Proof. By (3.37), we have

$$\big|\xi^2\eta A\widehat{K}(t,\xi,\eta)\big| \lesssim \chi_{A\ge 1}e^{-ct} + \chi_{A\le 1}Ae^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.43}$$

Then by (3.24) and (3.26), we have

$$\big\|\xi^2\eta A\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2}};$$

$$\big\|\xi^2\eta A\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{3}{2}},$$

and by (3.25),

$$\big\|\xi^2\eta A\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\sim N)} \lesssim \langle t\rangle^{-1};$$

$$\big\|\xi^2\eta A\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\sim N)} \lesssim N^2\langle t\rangle^{-\frac{3}{2}},$$

thus the conclusions (3.39) and (3.40) follow from interpolation. Further, (3.41) easily follows from (3.43). □

Since

$$A^2 - A|\eta| \sim \xi^2,$$

we have

$$\big|A^2(A^2 - A|\eta|)\widehat{K}(t,\xi,\eta)\big| \lesssim \chi_{A\ge 1}e^{-ct} + \chi_{A\le 1}e^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.44}$$

Thus similar as above, we have

Lemma 3.10. *Let $1\le q\le\infty$. Then*

$$\big\|A^2(A^2 - A|\eta|)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim \langle N\rangle^{\frac{2}{q}}\langle t\rangle^{-1-\frac{1}{2q}}. \tag{3.45}$$

Moreover, for any $\beta\in[0, 2-\frac{1}{q}]$,

$$\big\|A^\beta A^2(A^2 - A|\eta|)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{q}-\frac{\beta}{2}}. \tag{3.46}$$

3.2.2. Estimates on $\partial_t \widehat{K(t)}$ and $\partial_{tt}\widehat{K(t)}$

Lemma 3.11. *Let $1 \le q \le \infty$ and $N \gtrsim 1$. Then if $\beta \in (\frac{1}{2}, \frac{5}{2}]$,*

$$\big\|\eta\partial_t \widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{-2(1-\frac{1}{q})-1}\langle t\rangle^{-1-\frac{1}{2q}}; \tag{3.47}$$

$$\big\|A\eta\partial_t \widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\lesssim 1)} \lesssim \langle t\rangle^{-\frac{1}{q}}; \tag{3.48}$$

$$\big\|A^\beta\eta\partial_t \widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{\beta}{2}}. \tag{3.49}$$

Proof. By (3.5), we have

$$\big|\eta\widehat{\partial_t K}(t,\xi,\eta)\big| \lesssim \chi_{A\gtrsim 1}\frac{1}{A^3}e^{-ct} + \chi_{A\lesssim 1}\frac{1}{A}e^{-\frac{1}{4}A^2 t} + \frac{\xi^2}{A^5}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.50}$$

Then by (3.50),

$$\big|\chi_{A\sim N}\eta\widehat{\partial_t K}(t,\xi,\eta)\big| \lesssim \frac{1}{A^3}e^{-ct} + \frac{\xi^2}{A^5}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.51}$$

Thus, by (3.51) and (3.25),

$$\begin{aligned}
\big|\chi_{A\sim N}\eta\widehat{\partial_t K}(t,\xi,\eta)\big| &\lesssim \frac{1}{N^3}e^{-ct} + \frac{1}{N^3 t}\frac{\xi^2 t}{2A^2}e^{-\frac{\xi^2}{2A^2}t} \lesssim N^{-3}\langle t\rangle^{-1};\\
\big\|\chi_{A\sim N}\eta\widehat{\partial_t K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\sim N)} &\lesssim \iint\limits_{A\sim N}\frac{1}{A^3}e^{-ct}\,d\xi d\eta + \iint\limits_{A\sim N}\frac{\xi^2}{A^5}e^{-\frac{\xi^2}{2A^2}t}\,d\xi d\eta\\
&\lesssim N^{-1}\langle t\rangle^{-\frac{3}{2}}.
\end{aligned}$$

This proves (3.47) by interpolation. When $A \le 1$, by (3.50) again,

$$\big|\chi_{A\le 1}A\eta\widehat{\partial_t K}(t,\xi,\eta)\big| \lesssim e^{-\frac{1}{4}A^2 t} + \frac{|\xi|}{A^2}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t} \lesssim 1.$$

That is,

$$\big\|A\eta\widehat{\partial_t K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)} \lesssim 1.$$

Further, by (3.24) and (3.26),

$$\begin{aligned}
\big\|A\eta\widehat{\partial_t K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)} &\lesssim \iint\limits_{A\le 1} e^{-\frac{1}{4}A^2 t}\,d\xi d\eta + \iint\limits_{A\le 1}\frac{|\xi|}{A^2}e^{-\frac{\xi^2}{2A^2}t}\,d\xi d\eta\\
&\lesssim \langle t\rangle^{-1}.
\end{aligned}$$

Then (3.48) follows from interpolation. Similarly, since

$$\begin{aligned}\left|A^\beta \eta \widehat{\partial_t K}(t,\xi,\eta)\right| &\lesssim \chi_{A\gtrsim 1}\frac{1}{A^{3-\beta}}e^{-ct} + \chi_{A\lesssim 1}A^{\beta-1}e^{-\frac{1}{4}A^2 t} \\ &\quad + \frac{\xi^2}{A^{5-\beta}}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}. \end{aligned} \tag{3.52}$$

Note that it is integrable when $\beta > \frac{1}{2}$, then integration and Lemma 3.5 give (3.49). □

Moreover, from (3.52), we have for any $\beta \in [0,2]$,

$$\left\|A^\beta A\eta \widehat{\partial_t K}(t,\xi,\eta)\right\|_{L^\infty_{\xi\eta}} \lesssim \langle t\rangle^{-\frac{\beta}{2}}. \tag{3.53}$$

We also need the following estimates.

Lemma 3.12. *Let $\beta \in [0,3]$, then*

$$\left\|A^\beta \eta \partial_{tt}\widehat{K}(t-s,\xi,\eta)\right\|_{L^\infty_{\xi\eta}} \lesssim \langle t\rangle^{-\frac{\beta}{2}};$$

$$\left\|A\eta \partial_{tt}\widehat{K}(t-s,\xi,\eta)\right\|_{L^2_{\xi\eta}} \lesssim \langle t\rangle^{-1}.$$

Proof. It follows from (3.6) and Lemma 3.5 directly. □

Now we consider some related estimates about $(\partial_{tt} + A^2\partial_t)\widehat{K}$.

Lemma 3.13. *Let $1 \le q \le \infty$, $\beta \in [0,1]$ and $N \gtrsim 1$. Then*

$$\left\|\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{-1+\frac{2}{q}}\langle t\rangle^{-1-\frac{1}{2q}}; \tag{3.54}$$

$$\left\|\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{q}}; \tag{3.55}$$

$$\left\|A^\beta\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right\|_{L^2_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{\beta+1}{2}}; \tag{3.56}$$

$$\left\|A\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right\|_{L^\infty_{\xi\eta}(A\ge 1)} \lesssim \langle t\rangle^{-1}; \tag{3.57}$$

$$\left\|A^{1+\beta}\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right\|_{L^\infty_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{\beta+1}{2}}. \tag{3.58}$$

Proof. By (3.5) and (3.6) we have

$$\left|\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right| \lesssim \chi_{A\ge 1}\frac{1}{A}e^{-ct} + \chi_{A\le 1}e^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^3}e^{-\frac{\xi^2}{2A^2}t}. \tag{3.59}$$

Thus, for $N \gtrsim 1$, by (3.25),

$$\left\|\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right\|_{L^\infty_{\xi\eta}(A\sim N)} \lesssim N^{-1}\langle t\rangle^{-1};$$

$$\left\|\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\right\|_{L^1_{\xi\eta}(A\sim N)} \lesssim N\langle t\rangle^{-\frac{3}{2}},$$

and by (3.24) and (3.26),

$$\big\|\eta(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)}\lesssim 1;$$
$$\big\|\eta(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)}\lesssim \langle t\rangle^{-1}.$$

Then the conclusions (3.54) and (3.55) follow from interpolation. Moreover,

$$\big|A^\beta\eta(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big|\lesssim A^{\beta-1}\chi_{A\ge 1}e^{-ct}+\chi_{A\le 1}A^\beta e^{-\frac14 A^2 t}+\chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^{3-\beta}}e^{-\frac{\xi^2}{2A^2}t}.$$

Then (3.56), (3.57) and (3.58) follow from integration directly. □

Furthermore, since

$$\big|A^2(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big|\lesssim \chi_{A\ge 1}e^{-ct}+\chi_{A\le 1}e^{-\frac14 A^2 t}+\chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^2}e^{-\frac{\xi^2}{2A^2}t}. \quad (3.60)$$

Thus one may find that it has the same bound with $A^2(A^2-A|\eta|)\widehat{K}(t,\xi,\eta)$. Thus the same as Lemma 3.10, we have

Lemma 3.14. *Let $1\le q\le\infty$ and $N\gtrsim 1$. Then*

$$\big\|A^2(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)}\lesssim \langle N\rangle^{\frac{2}{q}}\langle t\rangle^{-1-\frac{1}{2q}}. \quad (3.61)$$

Moreover, for $\beta\in[0,2-\frac1q]$,

$$\big\|A^\beta A^2(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)}\lesssim \langle t\rangle^{-\frac1q-\frac\beta2}. \quad (3.62)$$

Proof. By (3.60), (3.24) and (3.26), we have

$$\big\|A^\beta A^2(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)}\lesssim \langle t\rangle^{-\frac\beta2},\quad \text{for any } \beta\in[0,2];$$
$$\big\|A^\beta A^2(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)}\lesssim \langle t\rangle^{-1-\frac\beta2},\quad \text{for any } \beta\in[0,1],$$

then (3.62) immediately follows by interpolation. For $N\gtrsim 1$, by (3.25),

$$\big\|A^2(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\sim N)}\lesssim \langle t\rangle^{-1};$$
$$\big\|A^2(\partial_{tt}+A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\sim N)}\lesssim N^2\langle t\rangle^{-\frac32}.$$

Then the conclusion (3.61) follows from interpolation again. □

3.2.3. Estimates on $(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)$

To this end, we also need some special basic estimates.

Lemma 3.15. *Let c be a positive constant, $\beta \geq 0$ and let $N \geq 0$ be the dyadic number ($N = 2^j$ for some $j \in \mathbb{Z}$), then*

$$\left\|A^\beta e^{-cA^2t}\right\|_{L^\infty_\xi L^1_\eta(A\leq 1)} + \left\|A^\beta e^{-cA^2t}\right\|_{L^1_\xi L^\infty_\eta(A\leq 1)} \lesssim \langle t\rangle^{-\frac{1+\beta}{2}}, \tag{3.63}$$

and

$$\left\|\frac{\xi^\beta}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t}\right\|_{L^1_\xi L^\infty_\eta(A\sim N)} \lesssim N^{\beta+1-\alpha}\langle t\rangle^{-\frac{1+\beta}{2}}; \tag{3.64}$$

$$\left\|\frac{\xi^\beta}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t}\right\|_{L^\infty_\xi L^1_\eta(A\sim N)} \lesssim N^{\beta+1-\alpha}\langle t\rangle^{-\frac{\beta}{2}}. \tag{3.65}$$

Moreover, for any $\alpha \in \mathbb{R}, \beta' \in \mathbb{R}, \beta \geq 0$ with $\beta' \geq \beta, 2\beta' - \beta + 1 - \alpha > 0$,

$$\left\|\chi_{|\xi|\lesssim A^2}\frac{\xi^{\beta'}}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t}\right\|_{L^1_\xi L^\infty_\eta(A\leq 1)} \lesssim \langle t\rangle^{-\frac{1+\beta}{2}};$$

$$\left\|\chi_{|\xi|\lesssim A^2}\frac{\xi^{\beta'}}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t}\right\|_{L^\infty_\xi L^1_\eta(A\leq 1)} \lesssim \langle t\rangle^{-\frac{\beta}{2}}. \tag{3.66}$$

Proof. First, we prove (3.63). Since

$$A^\beta e^{-cA^2t} \lesssim \langle t\rangle^{-\frac{\beta}{2}} e^{-\frac{c}{2}A^2t} \leq \langle t\rangle^{-\frac{\beta}{2}} e^{-\frac{c}{2}\xi^2 t},$$

we have

$$\left\|A^\beta e^{-cA^2t}\right\|_{L^1_\xi L^\infty_\eta(A\leq 1)} \lesssim \langle t\rangle^{-\frac{\beta}{2}}\left\|e^{-\frac{c}{2}\xi^2t}\right\|_{L^1_\xi} \lesssim \langle t\rangle^{-\frac{1+\beta}{2}}.$$

By the symmetry, we also have $\left\|A^\beta e^{-cA^2t}\right\|_{L^\infty_\xi L^1_\eta(A\leq 1)} \lesssim \langle t\rangle^{-\frac{1+\beta}{2}}$. This gives (3.63).

Now we turn to prove (3.64) and (3.65). For $A \sim N$, there exists a constant $\tilde{c} > 0$, such that

$$\frac{\xi^\beta}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t} \sim \frac{\xi^\beta}{N^\alpha} e^{-\tilde{c}\frac{\xi^2}{N^2}t} \lesssim N^{\beta-\alpha}\langle t\rangle^{-\frac{\beta}{2}}.$$

Therefore,

$$\left\|\frac{\xi^\beta}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t}\right\|_{L^\infty_\xi L^1_\eta(A\sim N)} \lesssim N^{\beta+1-\alpha}\langle t\rangle^{-\frac{\beta}{2}}.$$

Moreover,

$$\left\| \frac{\xi^\beta}{A^\alpha} e^{-c\frac{\xi^2}{A^2}t} \right\|_{L^1_\xi L^\infty_\eta(A\sim N)} \lesssim \left\| \frac{\xi^\beta}{N^\alpha} e^{-\tilde{c}\frac{\xi^2}{N^2}t} \right\|_{L^1_\xi(|\xi|\lesssim N)} \lesssim N^{\beta+1-\alpha} \langle t\rangle^{-\frac{1+\beta}{2}}.$$

Furthermore, using (3.64), (3.65) and the dyadic decomposition, we also have (3.66). □

Lemma 3.16. *Let $1 \le q, r \le \infty$ and $N \gtrsim 1$. Then*

$$\left\| A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta) \right\|_{L^q_\xi L^r_\eta(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2q}};$$
$$\left\| A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta) \right\|_{L^q_\xi L^r_\eta(A\sim N)} \lesssim N^{\frac{1}{q}+\frac{1}{r}} \langle t\rangle^{-\frac{1}{2q}}.$$

Proof. By (3.7), we have

$$|A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\ge 1} e^{-ct} + \chi_{A\le 1} e^{-\frac{1}{4}A^2 t} + \chi_{|\xi|\lesssim A^2} e^{-\frac{\xi^2}{2A^2}t}.$$

Then by (3.63) and (3.66), we have

$$\left\| A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta) \right\|_{L^\infty_\xi L^r_\eta(A\le 1)} \lesssim 1;$$
$$\left\| A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta) \right\|_{L^1_\xi L^r_\eta(A\le 1)} \lesssim t^{-\frac{1}{2}},$$

and by (3.64) and (3.65),

$$\left\| A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta) \right\|_{L^\infty_\xi L^r_\eta(A\sim N)} \lesssim N^{\frac{1}{r}};$$
$$\left\| A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta) \right\|_{L^1_\xi L^r_\eta(A\sim N)} \lesssim N^{1+\frac{1}{r}} t^{-\frac{1}{2}}.$$

Then the conclusion follows from interpolation. □

Lemma 3.17. *Let $1 \le q \le 2$, and $N \gtrsim 1$. Then*

$$\left\| \xi\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) \right\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{\frac{2}{q}-1} \langle t\rangle^{-\frac{1}{2}-\frac{1}{2q}}; \tag{3.67}$$
$$\left\| \xi\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) \right\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{q}}. \tag{3.68}$$

Moreover,

$$\left\| A\xi\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) \right\|_{L^2_\xi L^\infty_\eta(A\sim N)} \tag{3.69}$$
$$\left\| A\xi\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) \right\|_{L^2_\xi L^\infty_\eta(A\le 1)}. \tag{3.70}$$

Further, for any $\beta \geq \frac{1}{2}$,

$$\big\|A^\beta\xi\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\leq 1)} \lesssim \langle t\rangle^{-\frac{1}{2}-\frac{1}{2q}}; \tag{3.71}$$

$$\big\|A^\beta\xi\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^1_\xi L^2_\eta(A\sim N)} \lesssim N^{\frac{1}{2}+\beta}\langle t\rangle^{-1}; \tag{3.72}$$

$$\big\|A^\beta\xi\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^1_\xi L^2_\eta(A\leq 1)} \lesssim \langle t\rangle^{-1}. \tag{3.73}$$

Proof. By (3.7), we find that the function $\xi\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)$ has the same estimates as $\xi\eta A\widehat{K}(t,\xi,\eta)$ in (3.38) ($\alpha = 1$). Thus, we get the estimates (3.67) and (3.68) as Lemma 3.8. Moreover, by (3.7) again, we have

$$\begin{aligned}&|A^\beta\xi\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)|\\&\lesssim \chi_{A\geq 1}A^{-1+\beta}e^{-ct}+\chi_{A\leq 1}A^\beta e^{-\frac{1}{4}A^2t}+\frac{\xi}{A^{2-\beta}}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}.\end{aligned}$$

Then (3.69)–(3.73) follow by integration directly and using Lemma 3.15. □

Lemma 3.18. *Let* $1 \leq q \leq \infty$, $N \gtrsim 1$. *Then for any* $\beta \in [\frac{1}{2}, 1]$,

$$\big\|\xi^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{\frac{2}{q}}\langle t\rangle^{-1-\frac{1}{2q}}; \tag{3.74}$$

$$\big\|\xi^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\leq 1)} \lesssim \langle t\rangle^{-\frac{1}{2}-\frac{1}{q}}; \tag{3.75}$$

$$\big\|A^\beta\xi^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\leq 1)} \lesssim \langle t\rangle^{-\frac{1+\beta}{2}}. \tag{3.76}$$

Moreover,

$$\big\|\xi^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\leq 1)} \lesssim \langle t\rangle^{-\frac{3}{4}}; \tag{3.77}$$

$$\big\|\xi^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\sim N)} \lesssim N^{\frac{1}{2}}\langle t\rangle^{-1}. \tag{3.78}$$

Proof. By (3.7),

$$\begin{aligned}&|\xi^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)|\\&\lesssim \chi_{A\geq 1}e^{-ct}+\chi_{A\leq 1}Ae^{-\frac{1}{4}A^2t}+\frac{\xi^2}{A^2}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t};\end{aligned}\tag{3.79}$$

$$\begin{aligned}&|A^\beta\xi^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)|\\&\lesssim \chi_{A\geq 1}A^\beta e^{-ct}+\chi_{A\leq 1}A^{1+\beta}e^{-\frac{1}{4}A^2t}+\frac{\xi^2}{A^{2-\beta}}\chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}.\end{aligned}\tag{3.80}$$

Then similar as the proof in Lemma 3.9, we obtain (3.74) and (3.75). (3.77) and (3.78) follow from (3.79) and Lemma 3.15 directly. Also, from (3.80) and Lemma 3.15, (3.76) are followed. □

Now we give the estimates on $(\partial_{tt} - \Delta\partial_t - \partial_{xx})$.

Lemma 3.19. *Let $1 \le q \le \infty$ and $N \gtrsim 1$. Then*

$$\big\|(\partial_{tt} + A^2\partial_t + \xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{-2+\frac{2}{q}}\langle t\rangle^{-1-\frac{1}{2q}}; \tag{3.81}$$

$$\big\|A(\partial_{tt} + A^2\partial_t + \xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{q}}; \tag{3.82}$$

$$\big\|A^2(\partial_{tt} + A^2\partial_t + \xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)} \lesssim \langle t\rangle^{-1}; \tag{3.83}$$

and for any $\beta_1 \in [\frac{1}{2}, 2]$, $\beta_2 \in [0,2]$,

$$\big\|A^{\beta_1}(\partial_{tt} + A^2\partial_t + \xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{\beta_1}{2}}; \tag{3.84}$$

$$\big\|A^{1+\beta_2}(\partial_{tt} + A^2\partial_t + \xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{\beta_2}{2}}. \tag{3.85}$$

Proof. From (3.10) and (3.59), the function $A(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)$ obeys the similar estimates on $\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)$. So the estimates can be obtained by the same way in the proof of Lemma 3.13. □

Much similar as $A^2(\partial_{tt}+A^2\partial_t)\widehat{K}$, we have the estimate on the operator $\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)$ as following.

Lemma 3.20. *Let $1 \le q \le \infty$ and $N \gtrsim 1$. Then*

$$\big\|\partial_t(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{q}}; \tag{3.86}$$

$$\big\|A\partial_t(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2}-\frac{1}{q}}; \tag{3.87}$$

$$\big\|\partial_t(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\sim N)} \lesssim N^{-2(1-\frac{1}{q})}\langle t\rangle^{-1-\frac{1}{2q}}. \tag{3.88}$$

Moreover, for $\beta \in [0, \frac{3}{2}]$,

$$\big\|A^\beta\partial_t(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-\frac{1}{2}-\frac{\beta}{2}}; \tag{3.89}$$

$$\big\|A^2\partial_t(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^q_{\xi\eta}(A\le 1)} \lesssim \langle t\rangle^{-1-\frac{1}{2q}}. \tag{3.90}$$

Proof. By (3.8) and (3.50), the function $\partial_t(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)$ obeys the similar estimates on $A\eta\partial_t\widehat{K}(t,\xi,\eta)$. So (3.86) and (3.88) are followed similarly as (3.47) and (3.48). Again, by (3.8),

$$|A\partial_t\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta)| \lesssim \chi_{A\ge 1}\frac{1}{A}e^{-ct} + \chi_{A\le 1}Ae^{-\frac{1}{4}A^2t} + \chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^3}e^{-\frac{\xi^2}{2A^2}t}.$$

Therefore, by (3.24) and (3.26),

$$\big\|A\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)}\lesssim t^{-\frac{1}{2}};$$
$$\big\|A\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)}\lesssim t^{-\frac{3}{2}}.$$

Then the conclusion (3.87) follows from interpolation again. By (3.8), when $A\le 1$, we have

$$|A^\beta\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)|\lesssim \chi_{A\le 1}A^\beta e^{-\frac{1}{4}A^2t}+\chi_{|\xi|\lesssim A^2}\frac{\xi^2}{A^{4-\beta}}e^{-\frac{\xi^2}{2A^2}t}.$$

Then the conclusions (3.89) and (3.90) follow from Lemma 3.5. □

In particular, one may find from (3.8) that for any $\beta\in[0,2]$,

$$\big\|\langle A\rangle^{2-\beta}A^\beta\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\lesssim\langle t\rangle^{-\frac{\beta}{2}}. \tag{3.91}$$

Furthermore,

Lemma 3.21. *Let* $1\le q\le\infty$. *Then*

$$\big\|A\xi\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)}\lesssim\langle t\rangle^{-\frac{3}{2}}; \tag{3.92}$$
$$\big\|A^2\xi\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)}\lesssim\langle t\rangle^{-\frac{3}{2}}. \tag{3.93}$$

Proof. By (3.8), we have when $A\ge 1$,

$$|A\xi\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)|\lesssim\chi_{A\ge 1}e^{-ct}+\chi_{|\xi|\lesssim A^2}\frac{\xi^3}{A^3}e^{-\frac{\xi^2}{2A^2}t}\lesssim\langle t\rangle^{-\frac{3}{2}},$$

while when $A\le 1$,

$$|A^2\xi\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}(t,\xi,\eta)|\lesssim\chi_{A\le 1}A^3e^{-\frac{1}{4}A^2t}+\chi_{|\xi|\lesssim A^2}\frac{\xi^3}{A^2}e^{-\frac{\xi^2}{2A^2}t}\lesssim\langle t\rangle^{-\frac{3}{2}}.$$

This proves the conclusions. □

We also need the following estimates.

Lemma 3.22. *Let* $\beta\in[0,2]$, *then*

$$\big\|\langle A\rangle^{2-\beta}A^\beta\partial_{tt}\big(\partial_{tt}+A^2\partial_t+A^2+\eta^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\lesssim\langle t\rangle^{-\frac{1}{2}-\frac{\beta}{2}};$$
$$\big\|\partial_{tt}\big(\partial_{tt}+A^2\partial_t+A^2+\eta^2\big)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}}\lesssim\langle t\rangle^{-1}.$$

Proof. Since

$$\partial_{tt}\big(\partial_{tt} + A^2\partial_t + A^2 + \eta^2\big)\widehat{K}(t,\xi,\eta) = \partial_{tt}\big(\partial_{tt} + A^2\partial_t + A^2\big)\widehat{K}(t,\xi,\eta) + \eta^2\partial_{tt}\widehat{K}(t,\xi,\eta),$$

the conclusions follow directly from (3.9), (3.6) and Lemma 3.5. □

3.2.4. Estimates on $\widehat{K}_1(t,\xi,\eta)$

We also need the estimates on the operator $K_1(t)$. Note that by the definition of $\widehat{K}_1(t,\xi,\eta)$, we have

$$|\widehat{K_1}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}e^{-ct} + \chi_{A\leq 1}e^{-\frac{1}{4}A^2t} + \chi_{|\xi|\lesssim A^2}e^{-\frac{\xi^2}{2A^2}t}.$$

Moreover,

$$|\xi\widehat{K_1}(t,\xi,\eta)| \lesssim \chi_{A\geq 1}Ae^{-ct} + \chi_{A\leq 1}Ae^{-\frac{1}{4}A^2t} + \chi_{|\xi|\lesssim A^2}\xi e^{-\frac{\xi^2}{2A^2}t}.$$

Thus by integration and Lemma 3.5 we have

Lemma 3.23. *Let $N \gtrsim 1$. Then*

$$\big\|\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\leq 1)} + N^{-\frac{1}{2}}\big\|\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\sim N)} \lesssim \langle t\rangle^{-\frac{1}{4}};$$

$$\big\|A^{-1}\xi\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\leq 1)} \lesssim \langle t\rangle^{-\frac{1}{2}};$$

$$\big\|\xi\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\leq 1)} + N^{-2}\big\|\xi\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)} \lesssim \langle t\rangle^{-\frac{3}{4}}.$$

4. Linear estimates on the operator $K(t)$

In this section, we establish the decaying estimates of the linear flow, by using the conclusions obtained in the previous section.

Proposition 4.1. *For any smooth function f, $\beta > \frac{1}{2}$,*

$$\big\||\nabla|^{3+\beta}K(t)f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^{\beta+}f\big\|_{L^1_{xy}}.$$

Proof. By Lemma 3.7 and Lemma 3.2,

$$\begin{aligned}\big\||\nabla|^{3+\beta}K(t)f\big\|_{L^2_{xy}} &\lesssim \big\|P_{\leq 1}|\nabla|^{3+\beta}K(t)f\big\|_{L^2_{xy}} + \sum_{N\geq 1}\big\|P_N|\nabla|^{3+\beta}K(t)f\big\|_{L^2_{xy}}\\ &\lesssim \big\|A^{3+\beta}\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\leq 1)}\|P_{\leq 1}f\|_{L^1_{xy}}\\ &\quad + \sum_{N\geq 1}N^{3+\beta}\big\|\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)}\big\|P_Nf\big\|_{L^1_{xy}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \langle t\rangle^{-\frac{1}{4}}\|P_{\le 1}f\|_{L^1_{xy}} + \langle t\rangle^{-\frac{1}{4}}\sum_{N\ge 1} N^{\beta}\big\|P_N f\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^{\beta+} f\big\|_{L^1_{xy}}. \qquad \square
\end{aligned}
$$

Proposition 4.2. *For any smooth function f, $\beta' \ge \beta$ and $\beta \in [\frac{1}{2}, \frac{3}{2}]$,*

$$
\begin{aligned}
&\text{(i)}, \quad \big\||\nabla|^{\beta'}\partial_{xy}K(t)f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{\beta'-1+} f\big\|_{L^1_{xy}};\\
&\text{(ii)}, \quad \big\||\nabla|^{\beta'+1}\partial_{xy}K(t)f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big\|\langle\nabla\rangle^{\beta'-1+} f\big\|_{L^2_{xy}};\\
&\text{(iii)}, \quad \big\|\nabla\partial_{xy}K(t)f\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{1+} f\big\|_{L^1_{xy}};
\end{aligned}
$$

Proof. By Lemma 3.8 and Lemma 3.2,

$$
\begin{aligned}
\big\||\nabla|^{\beta'}\partial_{xy}K(t)f\big\|_{L^2_{xy}} &\lesssim \big\|P_{\le 1}|\nabla|^{\beta'}\partial_{xy}K(t)f\big\|_{L^2_{xy}} + \sum_{N\ge 1}\big\|P_N|\nabla|^{\beta'}\partial_{xy}K(t)f\big\|_{L^2_{xy}}\\
&\lesssim \big\|A^{\beta}\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^1_{xy}}\\
&\quad + \sum_{N\ge 1} N^{\beta'}\big\|\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta}{2}}\|P_{\le 1}f\|_{L^1_{xy}} + \langle t\rangle^{-\frac{3}{4}}\sum_{N\ge 1} N^{\beta'-1}\big\|P_N f\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{\beta'-1+} f\big\|_{L^1_{xy}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\big\||\nabla|^{\beta'+1}\partial_{xy}K(t)f\big\|_{L^2_{xy}} &\lesssim \big\|A^{\beta+1}\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^2_{xy}}\\
&\quad + \sum_{N\ge 1} N^{\beta'+1}\big\|\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{1}{2}}\|P_{\le 1}f\|_{L^2_{xy}} + \langle t\rangle^{-\frac{1}{2}}\sum_{N\ge 1} N^{\beta'-1}\big\|P_N f\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{\beta'-1+} f\big\|_{L^1_{xy}};
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|\nabla\partial_{xy}K(t)f\big\|_{L^\infty_{xy}} &\lesssim \big\|A\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^1_{xy}}\\
&\quad + \sum_{N\ge 1} N\big\|\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-1}\|P_{\le 1}f\|_{L^1_{xy}} + \langle t\rangle^{-1}\sum_{N\ge 1} N\big\|P_N f\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{1+} f\big\|_{L^1_{xy}}. \qquad \square
\end{aligned}
$$

Proposition 4.3. *For any smooth function f,*

$$\text{(i)},\quad \big\|\nabla\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} \lesssim \min\Big\{\langle t\rangle^{-\frac34}\|\langle\nabla\rangle^{\frac12+}f\|_{L^{\frac43}}, \langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^1_{xy}}\Big\};$$

$$\text{(ii)},\quad \big\|\Delta\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle f\big\|_{L^2_{xy}}.$$

Proof. The estimates are followed from Lemma 3.9, Lemma 3.2 and Sobolev's inequality. First,

$$\begin{aligned}
\big\|\nabla\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} &\lesssim \big\|P_{\le 1}\nabla\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} + \big\|P_{\ge 1}\nabla\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}}\\
&\lesssim \big\|A\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^1_{xy}}\\
&\quad + \big\|A\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)}\big\|P_{\ge 1}f\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^1_{xy}}.
\end{aligned}$$

On the other hand, by (3.39) and (3.40) ($q = 4$) instead,

$$\begin{aligned}
\big\|\nabla\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} &\lesssim \big\|P_{\le 1}\nabla\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} + \sum_{N\ge 1}\big\|P_N\nabla\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}}\\
&\lesssim \big\|A\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^4_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^{\frac43}_{xy}}\\
&\quad + \sum_{N\ge 1}N^{\frac12}\big\|\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^4_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^{\frac43}_{xy}}\\
&\lesssim \langle t\rangle^{-\frac34}\|\langle\nabla\rangle^{\frac12+}f\|_{L^{\frac43}}.
\end{aligned}$$

Furthermore, by (3.41) and (3.42),

$$\begin{aligned}
\big\|\Delta\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} &\lesssim \big\|P_{\le 1}\Delta\partial_{xx}\partial_y K(t)f\big\|_{L^2_{xy}} + \big\|P_{\ge 1}\nabla\cdot\partial_{xx}\partial_y K(t)\nabla f\big\|_{L^2_{xy}}\\
&\lesssim \big\|A^2\xi^2\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^2_{xy}}\\
&\quad + \big\|A\xi\eta\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)}\big\|P_{\ge 1}\nabla f\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle f\big\|_{L^2_{xy}}. \qquad \square
\end{aligned}$$

Proposition 4.4. *For any smooth function f, $p \in [2,\infty]$, $\beta \in [0, 2-\frac1q]$,*

$$\big\||\nabla|^\beta\Delta\big(\Delta+\sqrt{\Delta\partial_{yy}}\big)K(t)f\big\|_{L^p_{xy}} \lesssim \langle t\rangle^{-\frac{1}{p'}-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{\frac{2}{p'}+\beta+}f\big\|_{L^1_{xy}}.$$

Proof. From Lemma 3.10 and Lemma 3.2,

$$
\begin{aligned}
&\big\||\nabla|^\beta \Delta\big(\Delta+\sqrt{\Delta\partial_{yy}}\big)K(t)f\big\|_{L^p_{xy}}\\
&\lesssim \big\|P_{\le 1}|\nabla|^\beta \Delta\big(\Delta+\sqrt{\Delta\partial_{yy}}\big)K(t)f\big\|_{L^p_{xy}}+\sum_{N\ge 1}N^\beta\big\|P_N\Delta\big(\Delta+\sqrt{\Delta\partial_{yy}}\big)K(t)f\big\|_{L^p_{xy}}\\
&\lesssim \big\|A^\beta A^2(A^2-A|\eta|)\widehat{K}(t,\xi,\eta)\big\|_{L^{p'}_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^1_{xy}}\\
&\quad+\sum_{N\ge 1}N^\beta\big\|A^2(A^2-A|\eta|)\widehat{K}(t,\xi,\eta)\big\|_{L^{p'}_{\xi\eta}(A\sim N)}\big\|P_Nf\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{1}{p'}-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{\frac{2}{p'}+\beta+}f\big\|_{L^1_{xy}}. \qquad \square
\end{aligned}
$$

Proposition 4.5. *For any smooth function f, $\beta\in(\frac12,\frac52]$,*

$$
\begin{aligned}
&\text{(i)},\quad \big\||\nabla|^\beta\partial_y\partial_tK(t)f\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{\beta-2+}f\big\|_{L^1_{xy}};\\
&\text{(ii)},\quad \big\|\nabla\partial_y\partial_tK(t)f\big\|_{L^\infty_{xy}}\lesssim\langle t\rangle^{-1}\big\|\langle\nabla\rangle^{0+}f\big\|_{L^1_{xy}}.
\end{aligned}
$$

Moreover, for any $\beta'\in[0,2]$,

$$
\text{(iii)},\qquad \big\||\nabla|^{\beta'}\nabla\partial_y\partial_tK(t)f\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{\beta}{2}}\big\|f\big\|_{L^2_{xy}}. \tag{4.1}
$$

Proof. From Lemma 3.11 and Lemma 3.2,

$$
\begin{aligned}
\big\||\nabla|^\beta\partial_y\partial_tK(t)f\big\|_{L^2_{xy}}&\lesssim\big\|P_{\le 1}|\nabla|^\beta\partial_y\partial_tK(t)f\big\|_{L^2_{xy}}+\sum_{N\ge1}N^\beta\big\|P_N\partial_y\partial_tK(t)f\big\|_{L^2_{xy}}\\
&\lesssim\big\|A^\beta\eta\partial_t\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\lesssim1)}\|P_{\le1}f\|_{L^1_{xy}}\\
&\quad+\sum_{N\ge1}N^\beta\big\|\eta\partial_t\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)}\big\|P_Nf\big\|_{L^1_{xy}}\\
&\lesssim\langle t\rangle^{-\frac{\beta}{2}}\|P_{\le1}f\|_{L^1_{xy}}+\langle t\rangle^{-1-\frac{1}{2q}}\sum_{N\ge1}N^{\beta-2}\big\|P_Nf\big\|_{L^1_{xy}}\\
&\lesssim\langle t\rangle^{-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{\beta-2+}f\big\|_{L^1_{xy}}.
\end{aligned}
$$

By the similar way, using (3.47) and (3.48) ($q=1$) instead, we get (ii). For (4.1), we use the special estimate (3.53), and get

$$
\begin{aligned}
\big\||\nabla|^{\beta'}\nabla\partial_y\partial_tK(t)f\big\|_{L^2_{xy}}&\lesssim\big\|A^\beta A\eta\widehat{\partial_tK}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\big\|f\big\|_{L^2_{xy}}\\
&\lesssim\langle t\rangle^{-\frac{\beta}{2}}\big\|f\big\|_{L^2_{xy}}. \qquad\square
\end{aligned}
$$

Proposition 4.6. *For any smooth function f, $\beta \in [0,1]$,*

$$\text{(i)}, \quad \big\||\nabla|^\beta \partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{\beta+1}{2}}\big\|\langle\nabla\rangle^{\beta-1+}f\big\|_{L^1_{xy}};$$

$$\text{(ii)}, \quad \big\||\nabla|^{\beta+1} \partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{\beta+1}{2}}\big\|\langle\nabla\rangle^{\beta}f\big\|_{L^2_{xy}};$$

$$\text{(iii)}, \quad \big\|\nabla\partial_y(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{1+}f\big\|_{L^2_{xy}}.$$

Proof. From Lemma 3.13 and Lemma 3.2,

$$\begin{aligned}
&\big\||\nabla|^\beta \partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^2_{xy}}\\
&\lesssim \big\|P_{\le 1}|\nabla|^\beta \partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^2_{xy}} + \sum_{N\ge 1} N^\beta\big\|P_N\partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^2_{xy}}\\
&\lesssim \big\|A^\beta\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\lesssim 1)}\|P_{\le 1}f\|_{L^1_{xy}} + \sum_{N\ge 1} N^\beta\big\|\eta(\partial_{tt}\\
&\quad + A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta+1}{2}}\|P_{\le 1}f\|_{L^1_{xy}} + \langle t\rangle^{-1-\frac{1}{2q}}\sum_{N\ge 1} N^{\beta-1}\big\|P_N f\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta+1}{2}}\big\|\langle\nabla\rangle^{\beta-1+}f\big\|_{L^1_{xy}}.
\end{aligned}$$

Moreover, by (3.54) ($q=\infty$) and (3.58) instead,

$$\begin{aligned}
&\big\||\nabla|^{\beta+1} \partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^2_{xy}}\\
&\lesssim \big\|P_{\le 1}|\nabla|^{\beta+1} \partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^2_{xy}} + \big\|P_{\ge 1}\nabla\partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)|\nabla|^\beta f\big\|_{L^2_{xy}}\\
&\lesssim \big\|A^{\beta+1}\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\lesssim 1)}\|P_{\le 1}f\|_{L^2_{xy}}\\
&\quad + \big\|A\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)}\big\|\langle\nabla\rangle^\beta f\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta+1}{2}}\|f\|_{L^2_{xy}} + \langle t\rangle^{-1}\big\|\langle\nabla\rangle^\beta f\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta+1}{2}}\big\|\langle\nabla\rangle^\beta f\big\|_{L^2_{xy}}.
\end{aligned}$$

Similarly, by (3.54) ($q=2$), (3.58), and Sobolev's inequality,

$$\begin{aligned}
&\big\|\nabla\partial_y(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^\infty_{xy}}\\
&\lesssim \big\|P_{\le 1}\nabla\partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^\infty_{xy}} + \big\|P_{\ge 1}\nabla\partial_y\big(\partial_{tt} - \Delta\partial_t\big)K(t)f\big\|_{L^\infty_{xy}}\\
&\lesssim \big\|A\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\lesssim 1)}\|P_{\le 1}f\|_{L^2_{xy}}\\
&\quad + \big\|A\eta(\partial_{tt} + A^2\partial_t)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)}\big\|f\big\|_{L^\infty_{xy}}\\
&\lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{1+}f\big\|_{L^2_{xy}}. \qquad \square
\end{aligned}$$

Proposition 4.7. *For any smooth function f, $p \in [2, \infty]$, $\beta \in [0, 2 - \frac{1}{q}]$,*

$$\big\| |\nabla|^{\beta} \Delta \big(\partial_{tt} - \Delta \partial_t\big) K(t) f \big\|_{L^p_{xy}} \lesssim \langle t \rangle^{-\frac{1}{p'} - \frac{\beta}{2}} \big\| \langle \nabla \rangle^{\frac{2}{p'} + \beta +} f \big\|_{L^1_{xy}}.$$

Proof. From Lemma 3.10 and Lemma 3.14, the operator $\Delta\big(\partial_{tt} - \Delta\partial_t\big)K(t)$ obeys the same estimates as $\Delta\big(\Delta + \sqrt{\Delta \partial_{yy}}\big)K(t)$. Hence the lemma follows from the same way as in the proof of Proposition 4.4. □

Proposition 4.8. *For any smooth function f,*

$$\big\|\Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^2_{xy}} \lesssim \langle t \rangle^{-\frac{1}{4}} \min\Big\{ \|\langle \nabla \rangle^{\frac{1}{2}+} f\|_{L^1_x L^2_y}, \|\langle \nabla \rangle^{1+} f\|_{L^1_{xy}} \Big\}.$$

Proof. From Lemma 3.16 and Lemma 3.2,

$$\begin{aligned}
&\big\|\Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^2_{xy}} \\
&\lesssim \big\|P_{\le 1}\Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^2_{xy}} + \sum_{N \ge 1} \big\|P_N \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^2_{xy}} \\
&\lesssim \big\|A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta (A \le 1)} \|P_{\le 1} f\|_{L^1_x L^2_y} \\
&\quad + \sum_{N \ge 1} \big\|A^2(\partial_{tt} + A^2\partial_t + A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta (A \sim N)} \big\|P_N f\big\|_{L^1_x L^2_y} \\
&\lesssim \langle t \rangle^{-\frac{1}{4}} \big\|\langle \nabla \rangle^{\frac{1}{2}+} f\big\|_{L^1_x L^2_y}.
\end{aligned}$$

Further, by the Sobolev inequality, $\big\|\langle \nabla \rangle^{\frac{1}{2}+} f\big\|_{L^1_x L^2_y} \lesssim \|\langle \nabla \rangle^{1+} f\|_{L^1_{xy}}$, we prove the lemma. □

Proposition 4.9. *For any smooth function f, $2 \le p \le \infty$, and $\beta \ge \frac{1}{2}$,*

$$\begin{aligned}
&\text{(i)}, \quad \big\|\partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^p_{xy}} \lesssim \langle t \rangle^{-\frac{1}{p'}} \|\langle \nabla \rangle^{1-\frac{2}{p}+} f\|_{L^1_{xy}}; \\
&\text{(ii)}, \quad \big\|\nabla\partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^2_{xy}} \lesssim \langle t \rangle^{-\frac{3}{4}} \|\langle \nabla \rangle^{\frac{1}{2}+} f\|_{L^1_x L^2_y}; \\
&\text{(iii)}, \quad \big\||\nabla|^{\beta}\partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^2_{xy}} \lesssim \langle t \rangle^{-\frac{3}{4}} \|\langle \nabla \rangle^{1+} f\|_{L^1_{xy}}; \\
&\text{(iv)}, \quad \big\||\nabla|^{\beta}\partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t)f\big\|_{L^\infty_{xy}} \\
&\qquad\qquad \lesssim \langle t \rangle^{-1} \min\Big\{ \|\langle \nabla \rangle^{1+\beta+} f\|_{L^1_{xy}} + \|\langle \nabla \rangle^{\frac{1}{2}+\beta+} f\|_{L^1_x L^2_y} \Big\}.
\end{aligned}$$

Proof. From Lemma 3.17 and Lemma 3.8, we find that $\xi(\partial_{tt} + A^2\partial_t + A^2)$ and $A\xi\eta$ obey the same L^p estimates. Hence, the parts (i) and (iii) in the present proposition can be obtained by the same way as Proposition 4.2 (i) and (iii). For part (ii), by (3.69), (3.70) and Lemma 3.2,

$$
\begin{aligned}
&\big\|\nabla\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
\lesssim\ &\big\|P_{\le 1}\nabla\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
&+\sum_{N\ge 1}\big\|P_N\nabla\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
\lesssim\ &\big\|A\xi(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\le 1)}\|P_{\le 1}f\|_{L^1_xL^2_y}\\
&+\sum_{N\ge 1}\big\|A\xi(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\sim N)}\big\|P_Nf\big\|_{L^1_xL^2_y}\\
\lesssim\ &\langle t\rangle^{-\frac34}\|\langle\nabla\rangle^{\frac12+}f\|_{L^1_xL^2_y}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\big\||\nabla|^\beta\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}}\\
\lesssim\ &\big\|P_{\le 1}|\nabla|^\beta\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}}\\
&+\sum_{N\ge 1}\big\|P_N|\nabla|^\beta\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}}\\
\lesssim\ &\big\|A^\beta\xi(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^1_\xi L^2_\eta(A\le 1)}\|P_{\le 1}f\|_{L^1_xL^2_y}\\
&+\sum_{N\ge 1}\big\|A^\beta\xi(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^1_\xi L^2_\eta(A\sim N)}\big\|P_Nf\big\|_{L^1_xL^2_y}\\
\lesssim\ &\langle t\rangle^{-1}\|\langle\nabla\rangle^{\frac12+\beta+}f\|_{L^1_xL^2_y}.
\end{aligned}
$$

At last, we use the Sobolev inequality, $\|g\|_{L^1_xL^2_y}\lesssim\|\langle\nabla\rangle^{\frac12+}g\|_{L^1_{xy}}$, to complete the proof of the lemma. □

Proposition 4.10. *For any smooth function f, $\beta\in[\frac12,1]$,*

(i), $\big\|\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\lesssim\min\Big\{\langle t\rangle^{-\frac34}\|\langle\nabla\rangle^{\frac12+}f\|_{L^1_xL^2_y},\langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^1_{xy}}\Big\}$;

(ii), $\big\||\nabla|^\beta\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{\beta+1}{2}}\|\langle\nabla\rangle^{\beta+}f\|_{L^2_{xy}}$;

(iii), $\big\|\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}}\lesssim\langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^2_{xy}}$.

Proof. From Lemma 3.18 and Lemma 3.9, we note that $\xi^2(\partial_{tt}+A^2\partial_t+A^2)$ and $A\xi^2\eta$ obey the same L^p estimates. Thus, the same way in the proof of Proposition 4.3 (i) gives part (i) in this proposition. Moreover, for part (ii) from (3.74) ($q=\infty$), (3.76) and Lemma 3.2,

$$\begin{aligned}
&\big\||\nabla|^{\beta}\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
\lesssim&\big\|P_{\leq 1}|\nabla|^{\beta}\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
&+\sum_{N\geq 1}\big\|P_N|\nabla|^{\beta}\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
\lesssim&\big\|A^{\beta}\xi^2(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^{\infty}_{\xi\eta}(A\leq 1)}\|P_{\leq 1}f\|_{L^2_{xy}}\\
&+\sum_{N\geq 1}N^{\beta}\big\|\xi^2(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^{\infty}_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^2_{xy}}\\
\lesssim&\langle t\rangle^{-\frac{1+\beta}{2}}\|\langle\nabla\rangle^{\beta+}f\|_{L^2_{xy}}.
\end{aligned}$$

Similarly as (i), and using (3.74) and (3.74) ($q=2$) instead, we get (iii). □

Proposition 4.11. *For any smooth function f, $\beta_1\in[\frac{1}{2},2],\beta_2\in[0,2]$,*

$$\begin{aligned}
&\text{(i)},\quad \big\||\nabla|^{\beta_1}(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{\beta_1}{2}}\|\langle\nabla\rangle^{\beta_1-1+}f\|_{L^1_{xy}};\\
&\text{(ii)},\quad \big\||\nabla|^{\beta_2}\nabla(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{\beta_2}{2}}\|\langle\nabla\rangle^{\beta_2-1}f\|_{L^2_{xy}};\\
&\text{(iii)},\quad \big\|\nabla(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^\infty_{xy}}\lesssim\langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^1_{xy}};\\
&\text{(iv)},\quad \big\|\Delta(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^\infty_{xy}}\lesssim\langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^2_{xy}}.
\end{aligned}$$

Proof. From Lemma 3.19 and Lemma 3.2,

$$\begin{aligned}
&\big\||\nabla|^{\beta_1}(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}\\
\lesssim&\big\|P_{\leq 1}|\nabla|^{\beta_1}(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}\\
&+\sum_{N\geq 1}\big\|P_N|\nabla|^{\beta_1}(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}\\
\lesssim&\big\|A^{\beta_1}(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\leq 1)}\|P_{\leq 1}f\|_{L^1_{xy}}\\
&+\sum_{N\geq 1}N^{\beta_1}\big\|(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^1_{xy}}\\
\lesssim&\langle t\rangle^{-\frac{\beta_1}{2}}\|\langle\nabla\rangle^{\beta_1-1+}f\|_{L^1_{xy}}.
\end{aligned}$$

By (3.83) and (3.85) instead, we have

$$\begin{aligned}
&\big\||\nabla|^{\beta_2}\nabla(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}\\
\lesssim&\big\|P_{\leq 1}|\nabla|^{\beta_2}(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}\\
&+\big\|P_{\geq 1}|\nabla|^{\beta_2}(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^2_{xy}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \big\|A^{\beta_2+1}(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^2_{xy}}\\
&\quad+\big\|A^2(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}(A\ge 1)}\big\|P_{\ge 1}\langle\nabla\rangle^{\beta_2-1}f\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-\frac{\beta_2}{2}}\|\langle\nabla\rangle^{\beta_2-1}f\|_{L^2_{xy}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\big\|\nabla(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^\infty_{xy}}\\
&\lesssim \big\|A(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^1_{xy}}\\
&\quad+\sum_{N\ge 1}N\big\|(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\sim N)}\big\|P_Nf\big\|_{L^1_{xy}}\\
&\lesssim \langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^2_{xy}};
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\Delta(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)f\big\|_{L^\infty_{xy}}\\
&\lesssim \big\|A^2(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^2_{xy}}\\
&\quad+\sum_{N\ge 1}N^2\big\|(\partial_{tt}+A^2\partial_t+\xi^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)}\big\|P_Nf\big\|_{L^2_{xy}}\\
&\lesssim \langle t\rangle^{-1}\|\langle\nabla\rangle^{1+}f\|_{L^2_{xy}}; \qquad \square
\end{aligned}
$$

Proposition 4.12. *For any smooth function f, $\beta\in[0,2]$,*

$$
\begin{aligned}
&\text{(i)},\quad \big\||\nabla|^\beta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
&\qquad\lesssim \min\Big\{\langle t\rangle^{-\min\{\frac54,\frac{\beta+1}{2}\}}\|\langle\nabla\rangle^{\beta-1+}f\|_{L^1_{xy}},\langle t\rangle^{-\frac{\beta}{2}}\|\langle\nabla\rangle^{\beta-2}f\|_{L^2_{xy}}\Big\};\\
&\text{(ii)},\quad \big\|\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}}\lesssim \langle t\rangle^{-1}\|\langle\nabla\rangle^{0+}f\|_{L^1_{xy}};\\
&\text{(iii)},\quad \big\|\nabla\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}}\lesssim \langle t\rangle^{-1}\|\langle\nabla\rangle^{0+}f\|_{L^2_{xy}};\\
&\text{(iv)},\quad \big\|\Delta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}}\lesssim \min\Big\{\langle t\rangle^{-\frac54}\|\langle\nabla\rangle^{1+}f\|_{L^2_{xy}},\langle t\rangle^{-1}\|f\|_{L^\infty_{xy}}\Big\}
\end{aligned}
$$

Proof. From Lemma 3.20 and Lemma 3.2,

$$
\begin{aligned}
&\big\||\nabla|^\beta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
&\lesssim \big\|P_{\le 1}|\nabla|^\beta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
&\quad+\sum_{N\ge 1}\big\|P_N|\nabla|^\beta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \big\|A^\beta \partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^1_{xy}} \\
&\quad + \sum_{N\ge 1} N^\beta \big\|\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^1_{xy}} \\
&\lesssim \langle t\rangle^{-\min\{\frac54,\frac{\beta+1}{2}\}}\|P_{\le 1}f\|_{L^1_{xy}} + \langle t\rangle^{-\frac54}\sum_{N\ge 1} N^{\beta-1}\big\|P_N f\big\|_{L^1_{xy}} \\
&\lesssim \langle t\rangle^{-\min\{\frac54,\frac{\beta+1}{2}\}}\|\langle\nabla\rangle^{\beta-1+}f\|_{L^1_{xy}}.
\end{aligned}
$$

Similarly, using (3.91),

$$
\begin{aligned}
&\big\||\nabla|^\beta \partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}} \\
&\lesssim \big\|\langle A\rangle^{2-\beta}A^\beta\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\|\langle\nabla\rangle^{2-\beta}f\|_{L^2_{xy}} \\
&\lesssim \langle t\rangle^{-\frac{\beta}{2}}\|\langle\nabla\rangle^{\beta-2}f\|_{L^2_{xy}}.
\end{aligned}
$$

Moreover, by (3.86) and (3.88) ($q=1$),

$$
\begin{aligned}
&\big\|\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}} \\
&\lesssim \big\|\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\le 1)}\|P_{\le 1}f\|_{L^1_{xy}} \\
&\quad + \sum_{N\ge 1}\big\|\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^1_{\xi\eta}(A\sim N)}\big\|P_N f\big\|_{L^1_{xy}} \\
&\lesssim \langle t\rangle^{-1}\|\langle\nabla\rangle^{0+}f\|_{L^1_{xy}}.
\end{aligned}
$$

Using (3.87) and (3.88) ($q=2$) instead, we have (iii). While using (3.88) and (3.90) ($q=2$), we have $\big\|\Delta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-\frac54}\|\langle\nabla\rangle^{1+}f\|_{L^2_{xy}}$. At last, by the special estimate (3.91), we have

$$
\begin{aligned}
\big\|\Delta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^\infty_{xy}} &\lesssim \big\|A^2\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\|f\|_{L^\infty_{xy}} \\
&\lesssim \langle t\rangle^{-1}\|f\|_{L^\infty_{xy}}.
\end{aligned}
$$

This finishes the proof of the lemma. □

Proposition 4.13. *For any smooth function f,*

$$
\big\|\Delta\partial_x\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac32}\|\langle\nabla\rangle f\|_{L^2_{xy}}.
$$

Proof. From Lemma 3.21 and Lemma 3.2,

$$\begin{aligned}
&\big\|\Delta\partial_x\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)f\big\|_{L^2_{xy}}\\
&\quad=\big\|\langle\nabla\rangle^{-1}\Delta\partial_x\partial_t(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\langle\nabla\rangle f\big\|_{L^2_{xy}}\\
&\quad\lesssim\big\|\langle A\rangle^{-1}A^2\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\|\langle\nabla\rangle f\|_{L^2_{xy}}\\
&\quad\lesssim\langle t\rangle^{-\frac{3}{2}}\|\langle\nabla\rangle f\|_{L^2_{xy}}. \qquad\square
\end{aligned}$$

Proposition 4.14. *For any smooth function f, $\beta\in[0,2]$,*

$$\text{(i)},\quad \big\|\langle\nabla\rangle^{2-\beta}|\nabla|^\beta\partial_{tt}(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t)f\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{\beta+1}{2}}\|f\|_{L^2_{xy}};$$

$$\text{(ii)},\quad \big\|\partial_{tt}(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t)f\big\|_{L^\infty_{xy}}\lesssim\langle t\rangle^{-1}\|f\|_{L^2_{xy}}.$$

Proof. From Lemma 3.22 and Lemma 3.2,

$$\begin{aligned}
&\big\|\langle\nabla\rangle^{2-\beta}|\nabla|^\beta\partial_{tt}(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t)f\big\|_{L^2_{xy}}\\
&\quad\lesssim\big\|\langle A\rangle^{2-\beta}A^\beta\partial_{tt}(\partial_{tt}+A^2\partial_t+A^2+\eta^2)\widehat{K}(t,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\|f\|_{L^2_{xy}}\\
&\quad\lesssim\langle t\rangle^{-\frac{\beta+1}{2}}\|f\|_{L^2_{xy}};
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\partial_{tt}(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t)f\big\|_{L^\infty_{xy}}\\
&\quad\lesssim\big\|\partial_{tt}(\partial_{tt}+A^2\partial_t+A^2+\eta^2)\widehat{K}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}}\|f\|_{L^2_{xy}}\\
&\quad\lesssim\langle t\rangle^{-1}\|f\|_{L^2_{xy}}.
\end{aligned}$$

This proves (i) and (ii). $\square$

Proposition 4.15. *For any smooth function f,*

$$\text{(i)},\quad \big\|K_1(t)f\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^{\frac{1}{2}+}|\nabla|^{\frac{1}{2}-}f\big\|_{L^1_{xy}};$$

$$\text{(ii)},\quad \big\|\partial_x K_1(t)f\big\|_{L^2_{xy}}\lesssim\min\Big\{\langle t\rangle^{-\frac{1}{2}}\|\langle\nabla\rangle^{1+}\nabla f\|_{L^1_{xy}},\langle t\rangle^{-\frac{3}{4}}\|\langle\nabla\rangle^{2+}f\|_{L^1_{xy}}\Big\}.$$

Proof. From Lemma 3.23 and Lemma 3.2,

$$\begin{aligned}
\big\|K_1(t)f\big\|_{L^2_{xy}}&\lesssim\big\|P_{\le1}K_1(t)f\big\|_{L^2_{xy}}+\sum_{N\ge1}\big\|P_NK_1(t)f\big\|_{L^2_{xy}}\\
&\lesssim\big\|\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\le1)}\|P_{\le1}f\|_{L^1_xL^2_y}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{N\geq 1} \big\|\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_\xi L^\infty_\eta(A\sim N)} \|P_N f\|_{L^1_x L^2_y} \\
&\lesssim \langle t\rangle^{-\frac{1}{4}} \|\langle\nabla\rangle^{\frac{1}{2}+}|\nabla|^{\frac{1}{2}-} f\|_{L^1_{xy}},
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|\partial_x K_1(t) f\big\|_{L^2_{xy}} &\lesssim \big\|P_{\leq 1}\frac{\nabla}{-\Delta}\partial_x K_1(t)\cdot\nabla f\big\|_{L^2_{xy}} + \sum_{N\geq 1}\big\|P_N \partial_x K_1(t) f\big\|_{L^2_{xy}} \\
&\lesssim \big\|A^{-1}\xi\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\leq 1)} \|P_{\leq 1}\nabla f\|_{L^1_{xy}} \\
&\quad + \sum_{N\geq 1}\big\|\xi\widehat{K_1}(t,\xi,\eta)\big\|_{L^2_{\xi\eta}(A\sim N)} \|P_N f\|_{L^1_{xy}} \\
&\lesssim \langle t\rangle^{-\frac{1}{2}} \|\langle\nabla\rangle^{1+}\nabla f\|_{L^1_{xy}}.
\end{aligned}
$$

Similarly, using the third estimates in Lemma 3.23, we also have

$$
\big\|\partial_x K_1(t) f\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} \|\langle\nabla\rangle^{2+} f\|_{L^1_{xy}}. \qquad \square
$$

5. Local theory and energy estimates

This section presents the local (in time) existence and uniqueness result for (2.2), which can be deduced from the work of Kawashima [6,7]. More importantly we obtain an energy inequality needed in the proof of Theorem 1.1.

Proposition 5.1. *Let $1+\lambda > 0$. Assume that $(n_0, \vec{u}_0, \nabla\psi_0) \in H^\sigma(\mathbb{R}^2)$ with $\sigma \geq 3$. Assume*

$$
\overline{\rho} \geq \rho_0 = 1 + n_0 \geq \underline{\rho}
$$

for two constants $\overline{\rho} > \underline{\rho} > 0$. Then there exists $T_0 = T_0(\|(n_0, \vec{u}_0, \nabla\psi_0)\|_{H^\sigma}) > 0$, and a unique smooth local solution $(n, \vec{u}, \nabla\psi) \in C^0_t([0,T_0]; H^\sigma(\mathbb{R}^2))$ to (2.2) with

$$
\overline{\rho} \geq \rho = 1 + n \geq \underline{\rho}. \tag{5.1}
$$

Furthermore, for ϵ and M given in the definition of X in (1.3),

$$
\langle t\rangle^{-\epsilon} \big\|\langle\nabla\rangle^M (n, \vec{u}, \nabla\psi)(t)\big\|_2 \leq \big\|\langle\nabla\rangle^M (n_0, \vec{u}_0, \nabla\psi_0)\big\|_2 + Q(\|U\|_X), \tag{5.2}
$$

where $Q(r)$ represents a polynomial of r with the lowest order at least quadratic.

Proof of Proposition 5.1. The existence of the local solution $(n, \vec{u}, \nabla\psi)$ with the property (5.1) follows from [6,7]. The main effort of this proof is devoted to the bound in (5.2).

For any $\sigma \geq 0$, we have, from $\partial_t n + \nabla\cdot\vec{u} + \nabla\cdot(n\vec{u}) = 0$,

$$\frac{1}{2}\partial_t \int \left|\langle\nabla\rangle^\sigma n\right|^2 dx = -\int \big(\langle\nabla\rangle^\sigma \nabla\cdot\vec{u}\big)\langle\nabla\rangle^\sigma n\, dx - \int \big(\langle\nabla\rangle^\sigma \nabla\cdot(n\vec{u})\big)\langle\nabla\rangle^\sigma n\, dx.$$

Using the commutator notation

$$[\langle\nabla\rangle^\sigma\nabla\cdot,\,\vec{u}]n \equiv \langle\nabla\rangle^\sigma\nabla\cdot(n\vec{u}) - \vec{u}\cdot\nabla\langle\nabla\rangle^\sigma n$$

and integrating by parts, we have

$$\int \big(\langle\nabla\rangle^\sigma \nabla\cdot(n\vec{u})\big)\langle\nabla\rangle^\sigma n\, dx = \int \big([\langle\nabla\rangle^\sigma\nabla\cdot,\,\vec{u}]n\big)\langle\nabla\rangle^\sigma n\, dx - \frac{1}{2}\int (\nabla\cdot\vec{u})\big(\langle\nabla\rangle^\sigma n\big)^2 dx.$$

It then follows from the commutator estimate

$$\|[\langle\nabla\rangle^\sigma\nabla\cdot,\,\vec{u}]n\|_2 \lesssim \|n\|_\infty \left\|\langle\nabla\rangle^\sigma\nabla\vec{u}\right\|_2 + \left\|\nabla\vec{u}\right\|_\infty \|\langle\nabla\rangle^\sigma n\|_2$$

and Hölder's inequality that

$$\begin{aligned}
\frac{1}{2}\partial_t\left(\|\langle\nabla\rangle^\sigma n\|_2^2\right) &\leq -\int \big(\langle\nabla\rangle^\sigma\nabla\cdot\vec{u}\big)\langle\nabla\rangle^\sigma n\, dx + C\|n\|_\infty \left\|\langle\nabla\rangle^\sigma n\right\|_2 \left\|\langle\nabla\rangle^\sigma\nabla\vec{u}\right\|_2 \\
&\quad + \left\|\nabla\vec{u}\right\|_\infty \|\langle\nabla\rangle^\sigma n\|_2^2 \\
&\leq -\int \big(\langle\nabla\rangle^\sigma\nabla\cdot\vec{u}\big)\langle\nabla\rangle^\sigma n\, dx \\
&\quad + C\Big(\|n\|_\infty^2 + \left\|\nabla\vec{u}\right\|_\infty\Big)\left\|\langle\nabla\rangle^\sigma n\right\|_2^2 + \frac{\bar{\lambda}}{5\underline{\rho}}\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2^2,
\end{aligned} \tag{5.3}$$

where C is a large constant which may depend on $\underline{\rho}, \overline{\rho}$ and may vary from line to line, and $\bar{\lambda} = 1 + \lambda$. Applying $\langle\nabla\rangle^\sigma$ to (2.1) and then dotting with $\langle\nabla\rangle^\sigma\vec{u}$, we have, after integration by parts,

$$\frac{1}{2}\partial_t \int \left|\langle\nabla\rangle^\sigma \vec{u}\right|^2 dx = J_1 + J_2 + J_3 + J_4 + J_5,$$

where

$$\begin{aligned}
J_1 &= -\int \langle\nabla\rangle^\sigma\big(\vec{u}\cdot\nabla\vec{u}\big)\cdot\langle\nabla\rangle^\sigma\vec{u}\, dx, \\
J_2 &= \int \langle\nabla\rangle^\sigma n\, \langle\nabla\rangle^\sigma\big(\nabla\cdot\vec{u}\big)\, dx - \int \langle\nabla\rangle^\sigma\big(n\nabla n\big)\cdot\langle\nabla\rangle^\sigma\vec{u}\, dx, \\
J_3 &= \int \langle\nabla\rangle^\sigma(\rho^{-1}\Delta\vec{u})\cdot\langle\nabla\rangle^\sigma\vec{u}\, dx, \\
J_4 &= \lambda\int \langle\nabla\rangle^\sigma(\rho^{-1}\nabla(\nabla\cdot\vec{u}))\cdot\langle\nabla\rangle^\sigma\vec{u}\, dx,
\end{aligned}$$

$$J_5 = -\int \langle\nabla\rangle^\sigma(\rho^{-1}\nabla\phi\Delta\phi)\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx.$$

We bound the terms on the right-hand side. Writing

$$J_1 = -\int [\langle\nabla\rangle^\sigma, \vec{u}\cdot\nabla]\vec{u}\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx + \frac{1}{2}\int(\nabla\cdot\vec{u})|\langle\nabla\rangle^\sigma\vec{u}|^2\,dx,$$

we have, by a commutator estimate,

$$|J_1| \lesssim \|\nabla\vec{u}\|_\infty \big\|\langle\nabla\rangle^\sigma\vec{u}\big\|_2^2.$$

After integration by parts,

$$\begin{aligned}\left|\int\langle\nabla\rangle^\sigma\big(n\nabla n\big)\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx\right| &\lesssim \|\langle\nabla\rangle^\sigma(n^2)\|_2\,\|\langle\nabla\rangle^\sigma\big(\nabla\cdot\vec{u}\big)\|_2\\ &\lesssim \|n\|_\infty\,\|\langle\nabla\rangle^\sigma n\|_2\,\|\langle\nabla\rangle^\sigma\big(\nabla\cdot\vec{u}\big)\|_2.\end{aligned}$$

That is,

$$\begin{aligned}J_2 &\le \int\langle\nabla\rangle^\sigma n\,\langle\nabla\rangle^\sigma\big(\nabla\cdot\vec{u}\big)\,dx + \|n\|_\infty\,\|\langle\nabla\rangle^\sigma n\|_2\,\big\|\langle\nabla\rangle^\sigma\nabla\vec{u}\big\|_2\\ &\le \int\langle\nabla\rangle^\sigma n\,\langle\nabla\rangle^\sigma\big(\nabla\cdot\vec{u}\big)\,dx + \|n\|_\infty^2\,\|\langle\nabla\rangle^\sigma n\|_2^2 + \frac{\bar{\lambda}}{5\underline{\rho}}\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2^2.\end{aligned}$$

To estimate J_3, we write

$$\begin{aligned}J_3 &= -\int\langle\nabla\rangle^\sigma(\rho^{-1}\nabla\vec{u})\cdot\langle\nabla\rangle^\sigma\nabla\vec{u}\,dx + \int\langle\nabla\rangle^\sigma(\nabla(\rho^{-1})\cdot\nabla\vec{u})\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx\\ &= -\int[\langle\nabla\rangle^\sigma,\rho^{-1}]\nabla\vec{u}\cdot\langle\nabla\rangle^\sigma\nabla\vec{u}\,dx - \int\rho^{-1}\,|\langle\nabla\rangle^\sigma\nabla\vec{u}|^2\,dx\\ &\quad + \int\langle\nabla\rangle^\sigma(\nabla(\rho^{-1})\cdot\nabla\vec{u})\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx.\end{aligned}$$

Therefore, by a commutator estimate,

$$\begin{aligned}J_3 &\le -\int\rho^{-1}\,|\langle\nabla\rangle^\sigma\nabla\vec{u}|^2\,dx\\ &\quad + C\|\nabla\vec{u}\|_\infty\,\|\langle\nabla\rangle^\sigma(\rho^{-1})\|_2\,\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2 + C\|\nabla(\rho^{-1})\|_\infty\,\|\langle\nabla\rangle^\sigma\vec{u}\|_2\,\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2.\end{aligned}$$

Invoking (5.1) and using Sobolev's inequality,

$$\begin{aligned}&\|\langle\nabla\rangle^\sigma(\rho^{-1})\|_2 \le C(\underline{\rho},\overline{\rho})\|\langle\nabla\rangle^\sigma n\|_2,\\ &\|\nabla(\rho^{-1})\|_\infty \le C(\underline{\rho})\,\|\nabla n\|_\infty,\end{aligned}\tag{5.4}$$

we obtain, by Young's inequality,

$$\begin{aligned}
J_3 &\le -\int \rho^{-1}\, |\langle\nabla\rangle^\sigma \nabla \vec{u}|^2\, dx + C\|\nabla \vec{u}\|_\infty\, \|\langle\nabla\rangle^\sigma n\|_2 \|\langle\nabla\rangle^\sigma \nabla \vec{u}\|_2 \\
&\quad + C\|\nabla n\|_\infty\, \|\langle\nabla\rangle^\sigma \vec{u}\|_2\, \|\langle\nabla\rangle^\sigma \nabla \vec{u}\|_2 \\
&\le -\int \rho^{-1}\, |\langle\nabla\rangle^\sigma \nabla \vec{u}|^2\, dx + C\|\nabla \vec{u}\|_\infty^2\, \|\langle\nabla\rangle^\sigma n\|_2^2 \\
&\quad + C\|\nabla n\|_\infty^2 \|\langle\nabla\rangle^\sigma \vec{u}\|_2^2 + \frac{\bar{\lambda}}{10\underline{\rho}} \|\langle\nabla\rangle^\sigma \nabla \vec{u}\|_2^2.
\end{aligned}$$

J_4 is similarly estimated as J_3. Writing J_4 as

$$\begin{aligned}
J_4 &= -\lambda \int \rho^{-1}\, |\langle\nabla\rangle^\sigma \nabla\cdot \vec{u}|^2\, dx - \lambda \int [\langle\nabla\rangle^\sigma, \rho^{-1}] \nabla\cdot\vec{u} \cdot \langle\nabla\rangle^\sigma \nabla \vec{u}\, dx \\
&\quad - \lambda \int \langle\nabla\rangle^\sigma (\nabla(\rho^{-1})\, \nabla\cdot\vec{u}) \cdot \langle\nabla\rangle^\sigma \vec{u}\, dx,
\end{aligned}$$

we obtain, after similar estimates as for J_3,

$$\begin{aligned}
J_4 &\le -\lambda \int \rho^{-1}\, |\langle\nabla\rangle^\sigma \nabla\cdot \vec{u}|^2\, dx + C\|\nabla \vec{u}\|_\infty^2\, \|\langle\nabla\rangle^\sigma n\|_2^2 \\
&\quad + C\|\nabla n\|_\infty^2 \|\langle\nabla\rangle^\sigma \vec{u}\|_2^2 + \frac{\bar{\lambda}}{10\underline{\rho}} \|\langle\nabla\rangle^\sigma \nabla \vec{u}\|_2^2.
\end{aligned}$$

Since $|\langle\nabla\rangle^\sigma \nabla\cdot\vec{u}|^2 \le |\langle\nabla\rangle^\sigma \nabla\vec{u}|^2$, for any $\lambda + 1 > 0$, we have

$$\begin{aligned}
J_3 + J_4 &\le -\int \rho^{-1}\, |\langle\nabla\rangle^\sigma \nabla \vec{u}|^2\, dx - \lambda \int \rho^{-1}\, |\langle\nabla\rangle^\sigma \nabla\cdot \vec{u}|^2\, dx \\
&\quad + C\|\nabla \vec{u}\|_\infty^2\, \|\langle\nabla\rangle^\sigma n\|_2^2 + C\|\nabla n\|_\infty^2 \|\langle\nabla\rangle^\sigma \vec{u}\|_2^2 + \frac{\bar{\lambda}}{5\underline{\rho}} \|\langle\nabla\rangle^\sigma \nabla \vec{u}\|_2^2 \\
&\le -\frac{\bar{\lambda}}{\underline{\rho}} \int |\langle\nabla\rangle^\sigma \nabla \vec{u}|^2\, dx \\
&\quad + C\|\nabla \vec{u}\|_\infty^2\, \|\langle\nabla\rangle^\sigma n\|_2^2 + C\|\nabla n\|_\infty^2 \|\langle\nabla\rangle^\sigma \vec{u}\|_2^2 + \frac{\bar{\lambda}}{5\underline{\rho}} \|\langle\nabla\rangle^\sigma \nabla \vec{u}\|_2^2.
\end{aligned}$$

To bound J_5, after integration by parts and using $\phi = \psi + y$, we write

$$\begin{aligned}
J_5 &= \int \langle\nabla\rangle^\sigma \nabla\psi \cdot \langle\nabla\rangle^\sigma \nabla v\, dx - \int \rho^{-1} \langle\nabla\rangle^\sigma \big(\nabla\psi \Delta\psi\big) \cdot \langle\nabla\rangle^\sigma \vec{u}\, dx \\
&\quad - \int [\langle\nabla\rangle^\sigma, \rho^{-1}](\nabla\psi\Delta\psi) \cdot \langle\nabla\rangle^\sigma \vec{u}\, dx - \int \langle\nabla\rangle^\sigma (\frac{n}{\rho}\Delta\psi)\, \langle\nabla\rangle^\sigma v\, dx.
\end{aligned}$$

Since $\nabla\psi\Delta\psi = \nabla\cdot(\nabla\psi\nabla\psi) - \frac{1}{2}\nabla(|\nabla\psi|^2)$, we have

$$\Big|\int \rho^{-1}\langle\nabla\rangle^\sigma\big(\nabla\psi\Delta\psi\big)\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx\Big| \le C(\underline{\rho})\|\nabla\psi\|_\infty\big\|\langle\nabla\rangle^\sigma\nabla\vec{u}\big\|_2\big\|\langle\nabla\rangle^\sigma\nabla\psi\big\|_2$$

$$\le C\|\nabla\psi\|_\infty^2\,\|\langle\nabla\rangle^\sigma\nabla\psi\|_2^2 + \frac{\bar{\lambda}}{15\underline{\rho}}\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2^2.$$

By a commutator estimate and (5.4),

$$\begin{aligned}
&\Big|\int[\langle\nabla\rangle^\sigma,\rho^{-1}](\nabla\psi\Delta\psi)\cdot\langle\nabla\rangle^\sigma\vec{u}\,dx\Big| \\
&\le C\|\nabla(\rho^{-1})\|_\infty\,\|\nabla\psi\|_\infty\,\|\langle\nabla\rangle^\sigma\nabla\psi\|_2\,\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2 \\
&\quad + C\|\langle\nabla\rangle\nabla\psi\|_\infty^2\,\|\langle\nabla\rangle^\sigma(\rho^{-1})\|_2\,\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2 \\
&\le C\big(\|\nabla n\|_\infty^2 + \|\nabla\psi\|_\infty^2\big)\,\|\langle\nabla\rangle^\sigma\nabla\psi\|_2^2 \\
&\quad + C\|\langle\nabla\rangle\nabla\psi\|_\infty^2\,\|\langle\nabla\rangle^\sigma n\|_2^2 + \frac{\bar{\lambda}}{15\underline{\rho}}\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2^2.
\end{aligned}$$

Since $\langle\nabla\rangle^\sigma v = \langle\nabla\rangle^\sigma P_{\le 1}v + \langle\nabla\rangle^\sigma P_{\ge 1}v$,

$$\begin{aligned}
&\Big|\int\langle\nabla\rangle^\sigma(\frac{n}{\rho}\Delta\psi)\,\langle\nabla\rangle^\sigma v\,dx\Big| \\
&\le \Big|\int\langle\nabla\rangle^\sigma(\frac{n}{\rho}\Delta\psi)\,\langle\nabla\rangle^\sigma P_{\le 1}v\,dx\Big| + \Big|\int\langle\nabla\rangle^\sigma(\frac{n}{\rho}\Delta\psi)\,\langle\nabla\rangle^\sigma P_{\ge 1}v\,dx\Big| \\
&\le C\|v\|_\infty\|n\|_2\|\Delta\psi\|_2 + C\|\langle\nabla\rangle n\|_\infty\big\|\langle\nabla\rangle^\sigma\nabla\psi\big\|_2\big\|\langle\nabla\rangle^\sigma\nabla v\big\|_2 \\
&\quad + C\big\|\langle\nabla\rangle^\sigma n\big\|_2\,\big\|\langle\nabla\rangle\nabla\psi\big\|_\infty\big\|\langle\nabla\rangle^\sigma P_{\ge 1}v\big\|_2 \\
&\le C\Big(\|v\|_\infty + \|\langle\nabla\rangle n\|_\infty^2 + \big\|\langle\nabla\rangle\nabla\psi\big\|_\infty^2\Big)\,\big(\|\langle\nabla\rangle^\sigma n\|_2^2 + \|\langle\nabla\rangle^\sigma\nabla\psi\|_2^2\big) \\
&\quad + \frac{\bar{\lambda}}{15\underline{\rho}}\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_5 &\le C\Big(\|v\|_\infty + \|\langle\nabla\rangle n\|_\infty^2 + \big\|\langle\nabla\rangle\nabla\psi\big\|_\infty^2\Big)\,\big(\|\langle\nabla\rangle^\sigma n\|_2^2 + \|\langle\nabla\rangle^\sigma\nabla\psi\|_2^2\big) \\
&\quad + \frac{\bar{\lambda}}{5\underline{\rho}}\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2^2.
\end{aligned}$$

From $\partial_t\psi + v + \vec{u}\cdot\nabla\psi = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\partial_t \int \left|\langle\nabla\rangle^\sigma \nabla\psi\right|^2 dx \\
&= -\int \langle\nabla\rangle^\sigma \nabla v \cdot \langle\nabla\rangle^\sigma \nabla\psi\, dx - \int \langle\nabla\rangle^\sigma \nabla\big(\vec{u}\cdot\nabla\psi\big)\cdot\langle\nabla\rangle^\sigma\nabla\psi\, dx \\
&= -\int \langle\nabla\rangle^\sigma \nabla v \cdot \langle\nabla\rangle^\sigma \nabla\psi\, dx - \int \langle\nabla\rangle^\sigma \big(\nabla\vec{u}\cdot\nabla\psi\big)\cdot\langle\nabla\rangle^\sigma\nabla\psi\, dx \\
&\quad + \frac{1}{2}\int (\nabla\cdot\vec{u})\left|\langle\nabla\rangle^\sigma\nabla\psi\right|^2 dx - \int \big[\langle\nabla\rangle^\sigma, \vec{u}\cdot\nabla\big]\nabla\psi\cdot\langle\nabla\rangle^\sigma\nabla\psi\, dx.
\end{aligned}$$

Similar as above, we obtain

$$\begin{aligned}
\frac{1}{2}\partial_t \int \left|\langle\nabla\rangle^\sigma \nabla\psi\right|^2 dx \le{}& -\int \langle\nabla\rangle^\sigma \nabla v\cdot\langle\nabla\rangle^\sigma\nabla\psi\, dx \\
&+ \big(\|\nabla\vec{u}\|_\infty + \|\nabla\psi\|_\infty^2\big)\|\langle\nabla\rangle^\sigma\nabla\psi\|_2^2 + \frac{\bar{\lambda}}{5\underline{\rho}}\|\langle\nabla\rangle^\sigma\nabla\vec{u}\|_2^2.
\end{aligned}$$

Adding the estimates above, and taking $\sigma = M$ as in the definition of X in (1.3), we have

$$\begin{aligned}
\big\|\langle\nabla\rangle^M(n,\vec{u},\nabla\psi)(t)\big\|_2 \lesssim{}& \big\|\langle\nabla\rangle^M(n_0,\vec{u}_0,\nabla\psi_0)\big\|_2 \\
&+ \int_0^t \Big(\|\langle\nabla\rangle\vec{u}\|_\infty + \big\|\langle\nabla\rangle(n,\nabla\psi,\vec{u})\big\|_\infty^2\Big)\big\|\langle\nabla\rangle^M(n,\nabla\psi,\vec{u})\big\|_2\, ds \\
\lesssim{}& \big\|\langle\nabla\rangle^M(n_0,\vec{u}_0,\nabla\psi_0)\big\|_2^2 + \int_0^t \langle s\rangle^{-1+\epsilon}\, ds Q(\|U\|_X) \\
\lesssim{}& \big\|\langle\nabla\rangle^M(n_0,\vec{u}_0,\nabla\psi_0)\big\|_2 + \langle t\rangle^\epsilon Q(\|U\|_X),
\end{aligned}$$

which gives (5.2). This completes the proof of Proposition 5.1. □

6. The estimates on n

In this section, we shall prove that there exists some small constant $\epsilon_0 > 0$, such that

$$\|\langle\nabla\rangle^3 n(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\big(\|U_0\|_{X_0} + \epsilon_0\|U\|_X + Q(\|U\|_X)\big); \tag{6.1}$$

$$\|\langle\nabla\rangle^{\frac{3}{2}} n(t)\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0} + \epsilon_0\|U\|_X + Q(\|U\|_X)\big); \tag{6.2}$$

$$\|\langle\nabla\rangle\partial_x n(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\big(\|U_0\|_{X_0} + \epsilon_0\|U\|_X + Q(\|U\|_X)\big). \tag{6.3}$$

Moreover, it follows from Nash's inequality, that

$$\|f\|_{L^\infty_{xy}} \lesssim \|\partial_x f\|^{\frac{1}{2}}_{L^2_{xy}} \|\langle\nabla\rangle^{1+} f\|^{\frac{1}{2}}_{L^2_{xy}}.$$

Using this inequality,

$$\|\langle\nabla\rangle^{\frac{3}{2}} n(t)\|_{L^\infty_{xy}} \lesssim \|\langle\nabla\rangle^3 n(t)\|^{\frac{1}{2}}_{L^2_{xy}} \|\langle\nabla\rangle \partial_x n(t)\|^{\frac{1}{2}}_{L^2_{xy}}.$$

Thus, we only need to show (6.1) and (6.3). The rest of this section is divided into three subsections. The first subsection recasts the integral representation given in Lemma 2.3, which we call "reexpression" of n. This new representation further reveals the structure of the model and is suitable for the desired estimates. The second subsection estimates the linear parts while the third section bounds the nonlinear part.

6.1. The reexpression of n

This subsection recasts the representation given in Lemma 2.3. More precisely, we prove the following proposition.

Proposition 6.1. *The unknown function n obeys the formula,*

$$n(t,x,y) = (L_n + B_n)(t; n_0, \vec{u}_0, \vec{b}_0) + \mathcal{N}_n(t; n, \vec{u}, \psi), \tag{6.4}$$

where $(L_n + B_n)$ is given by

$$\begin{aligned} &L_n(t; n_0, \vec{u}_0, \vec{b}_0) + B_n(t; n_0, \vec{u}_0, \vec{b}_0) \\ &\quad = -K(t)\big[\partial_y \Delta^2 \psi_0\big] + \partial_t K(t)\big[\Delta \partial_y \psi_0\big] - (\partial_{tt} - \Delta\partial_t - \Delta) K(t)\big[\partial_x u_0\big] \\ &\qquad - (\partial_{tt} - \Delta\partial_t) K(t)\big[\partial_y v_0\big] - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}} K(t) n_0 + K_1(t) n_0 \end{aligned} \tag{6.5}$$

and $\mathcal{N}_n(t; n, \vec{u}, \psi)$ is defined as

$$\begin{aligned} \mathcal{N}_n(t; n, \vec{u}, \psi) = &\int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) N_0(s)\, ds \\ &- \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) N_0(s)\, ds \\ &- \int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) N_1(s)\, ds \end{aligned}$$

$$+\int_0^t \partial_y(\partial_{tt}-\Delta\partial_t)K(t-s)N_2(s)\,ds$$
$$+\int_0^t \Delta\partial_y(\partial_t-\Delta)K(t-s)N_3(s)\,ds$$
$$-\lambda\int_0^t (\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)(\partial_x\Delta u+\partial_y\Delta v)\,ds. \tag{6.6}$$

Now we begin to prove this proposition. According to (2.39), (2.43)–(2.46) and (2.56),

$$n(t,x,y)=L_n(t;n_0,\vec{u}_0,\vec{b}_0)+\int_0^t K(t-s)F_0(s)\,ds,$$

where

$$L_n(t;n_0,\vec{u}_0,\vec{b}_0)$$
$$=K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t n(0)\big] \tag{6.7}$$
$$+(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\big[\partial_t n(0)\big] \tag{6.8}$$
$$-\Delta K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)n(0)\big]+\partial_t K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)n(0)\big] \tag{6.9}$$
$$-\frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[n_0]+K_1(t)[n_0] \tag{6.10}$$

and

$$F_0(s)=\big(\partial_s-\Delta\big)(\partial_{ss}-\Delta\partial_s-\Delta)N_0-\partial_x(\partial_{ss}-\Delta\partial_s-\Delta)N_1$$
$$-\partial_y(\partial_{ss}-\Delta\partial_s)N_2+\Delta\partial_y(\partial_s-\Delta)N_3$$
$$-\lambda(\partial_{ss}-\Delta\partial_s-\partial_{xx})(\partial_x\Delta u+\partial_y\Delta v).$$

To prove Proposition 6.1, we essentially replace $\partial_t n(0)$ and $\partial_{tt}n(0)$ in L_n by the terms in the equation of $\partial_t n$. The terms in $B_n(t;n_0,\vec{u}_0,\vec{b}_0)$ are the boundary terms that come from integration by parts in the time integral $\int_0^t K(t-s)F_0(s)\,ds$. The details for deriving the expressions of L_n and B_n are given in the following two sub-subsections.

6.1.1. $L_n(t;n_0,\vec{u}_0,\vec{b}_0)$

Since $\partial_t n(0)=-\partial_x u(0)-\partial_y v(0)+N_0(0)$, by (2.21) and (2.23) we have

$$(6.7)=-K(t)(\partial_{tt}-\Delta\partial_t-\Delta)(\partial_x u(0)+\partial_y v(0))+K(t)(\partial_{tt}-\Delta\partial_t-\Delta)N_0(0)$$
$$=-K(t)\big(-\partial_{xyy}u(0)+\partial_{xxy}v(0)+\partial_t\partial_x N_1(0)$$

$$
\begin{aligned}
&- \partial_{xx} N_0(0) + \lambda \partial_t (\partial_{xxx} u(0) + \partial_{xxy} v(0)) \\
&- K(t)\big(\partial_{yyy} v + \partial_{xyy} u - \partial_{yy} N_0 + \partial_t \partial_y N_2 - \Delta \partial_y N_3 + \lambda \partial_t (\partial_{xyy} u + \partial_{yyy} v)\big) \\
&+ K(t)(\partial_{tt} - \Delta \partial_t - \Delta) N_0(0) \\
= {}& K(t)\big[\Delta N_0(0) - \partial_t \partial_x N_1(0) - \partial_t \partial_y N_2(0) + \partial_y \Delta N_3(0)\big] \\
&+ K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) N_0(0)\big] \\
&- K(t)\big[\partial_y \Delta v(0) + \lambda \partial_t \partial_x \Delta u(0) + \lambda \partial_t \partial_y \Delta v(0)\big].
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(6.8) &= (\partial_{tt} - \Delta \partial_t - \Delta) K(t)\big[\partial_t n(0)\big] \\
&= (\partial_{tt} - \Delta \partial_t - \Delta) K(t)\big[- \partial_x u(0) - \partial_y v(0) + N_0(0)\big] \\
&= -\partial_x (\partial_{tt} - \Delta \partial_t - \Delta) K(t) u(0) - \partial_y (\partial_{tt} - \Delta \partial_t - \Delta) K(t) v(0) \\
&\quad + (\partial_{tt} - \Delta \partial_t - \Delta) K(t)\big[N_0(0)\big].
\end{aligned}
$$

Further, since by (2.19),

$$
\begin{aligned}
&\Delta K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) n(0)\big] \\
&\quad = \Delta K(t)\big[\Delta \partial_y \phi(0) + \partial_t N_0(0) - \Delta N_0(0) - \partial_x N_1(0) \\
&\qquad - \partial_y N_2(0) - \lambda(\partial_x \Delta u(0) + \partial_y \Delta v(0))\big]; \\
&\partial_t K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) n(0)\big] \\
&\quad = \partial_t K(t)\big(\Delta \partial_y \phi(0) + \partial_t N_0(0) - \Delta N_0(0) - \partial_x N_1(0) \\
&\qquad - \partial_y N_2(0) - \lambda(\partial_x \Delta u(0) + \partial_y \Delta v(0))\big),
\end{aligned}
$$

we have

$$
\begin{aligned}
(6.9) &= \partial_t K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) n(0)\big] - \Delta K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) n(0)\big] \\
&= \big(\partial_t - \Delta\big) K(t)\big[\partial_t N_0(0) - \Delta N_0(0) - \partial_x N_1(0) - \partial_y N_2(0)\big] \\
&\quad + \big(\partial_t - \Delta\big) K(t)\big[\Delta \partial_y \phi(0) - \lambda(\partial_x \Delta u(0) + \partial_y \Delta v(0))\big].
\end{aligned}
$$

Collecting the equalities above, we have

$$
\begin{aligned}
&L_n(t; n_0, \vec{u}_0, \vec{b}_0) \\
&= K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) N_0(0)\big] - (\partial_t - \Delta) \Delta K(t) N_0(0) \qquad (6.11) \\
&\quad - K(t)\big[\partial_t \Delta N_0(0) + \partial_t \partial_x N_1(0) + \partial_t \partial_y N_2(0)\big] \\
&\quad + K(t)\big[\Delta N_0(0) + \partial_x \Delta N_1(0) + \partial_y \Delta N_2(0) + \partial_y \Delta N_3(0)\big] \\
&\quad + \partial_t K(t)\big[\partial_t N_0(0) - \partial_x N_1(0) - \partial_y N_2(0)\big] + (\partial_{tt} - \Delta \partial_t - \Delta) K(t)\big[N_0(0)\big]
\end{aligned}
$$

$$
\begin{aligned}
&- K(t)\big[\partial_y \Delta v(0) + \partial_y \Delta^2 \phi(0)\big] + \partial_t K(t) \Delta \partial_y \phi(0) \\
&- (\partial_{tt} - \Delta \partial_t - \Delta) K(t)\big[\partial_x u(0) + \partial_y v(0)\big] \\
&- \lambda K(t)\big[\partial_x \partial_t \Delta u(0) + \partial_y \partial_t \Delta v(0) - \partial_x \Delta^2 u(0) - \partial_y \Delta^2 v(0)\big] \\
&- \lambda \partial_t K(t)\big[\partial_x \Delta u(0) + \partial_y \Delta v(0)\big] \\
&- \frac{1}{2} \Delta \sqrt{\Delta \partial_{yy}} K(t)[n_0] + K_1(t)[n_0].
\end{aligned}
\tag{6.12}
$$

6.1.2. $B_n(t; n_0, \vec{u}_0, \vec{b}_0)$

Now we consider the boundary term $B_n(t; n_0, \vec{u}_0, \vec{b}_0)$. Since $K(0) = \partial_t K(0) = \partial_{tt} K(0) = 0$, we have

$$
\begin{aligned}
&\int_0^t K(t-s) \partial_s (\partial_{ss} - \Delta \partial_s - \Delta) N_0(s)\, ds \\
&\quad = K(t-s)\Big[(\partial_{ss} - \Delta \partial_s - \Delta) N_0(s)\Big|_0^t\Big] - \int_0^t \partial_s K(t-s)(\partial_{ss} - \Delta \partial_s - \Delta) N_0(s)\, ds \\
&\quad = -K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) N_0(0)\big] + \int_0^t \partial_t K(t-s)(\partial_{ss} - \Delta \partial_s - \Delta) N_0(s)\, ds \\
&\quad = -K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) N_0(0)\big] - \partial_t K(t)\big[(\partial_t - \Delta) N_0(0)\big] \\
&\qquad - \int_0^t \partial_t K(t-s) \Delta N_0(s)\, ds + \int_0^t \partial_{tt} K(t-s)(\partial_s - \Delta) N_0(s)\, ds \\
&\quad = -K(t)\big[(\partial_{tt} - \Delta \partial_t - \Delta) N_0(0)\big] - \partial_t K(t)\big[(\partial_t - \Delta) N_0(0)\big] - \partial_{tt} K(t)\big[N_0(0)\big] \\
&\qquad + \int_0^t \partial_t (\partial_{tt} - \Delta \partial_t - \Delta) K(t-s) N_0(s)\, ds.
\end{aligned}
$$

By a similar treatment, we have

$$
\begin{aligned}
&- \int_0^t K(t-s) \Delta (\partial_{ss} - \Delta \partial_s - \Delta) N_0(s)\, ds \\
&\quad = K(t)\big[(\partial_t - \Delta) \Delta N_0(0)\big] + \partial_t K(t)\big[\Delta N_0(0)\big] \\
&\qquad - \int_0^t \Delta (\partial_{tt} - \Delta \partial_t - \Delta) K(t-s) N_0(s)\, ds;
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^t K(t-s)\partial_x(\partial_{ss}-\Delta\partial_s-\Delta)N_1(s)\,ds\\
&\quad = K(t)\big[(\partial_t-\Delta)\partial_x N_1(0)\big]+\partial_t K(t)\big[\partial_x N_1(0)\big]\\
&\qquad -\int_0^t \partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)N_1(s)\,ds;
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^t K(t-s)\partial_y(\partial_{ss}-\Delta\partial_s)N_2(s)\,ds\\
&\quad = K(t)\big[(\partial_t-\Delta)\partial_y N_2(0)\big]+\partial_t K(t)\big[\partial_y N_2(0)\big]\\
&\qquad -\int_0^t \partial_y(\partial_{tt}-\Delta\partial_t)K(t-s)N_2(s);
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t K(t-s)\Delta\partial_y(\partial_s-\Delta)N_3(s)\,ds\\
&\quad = -K(t)\big[\Delta\partial_y N_3(0)\big]+\int_0^t \Delta\partial_y\big(\partial_t-\Delta\big)K(t-s)N_3(s)\,ds;
\end{aligned}
$$

$$
\begin{aligned}
&-\lambda\int_0^t K(t-s)(\partial_{ss}-\Delta\partial_s-\partial_{xx})(\partial_x\Delta u+\partial_y\Delta v)\,ds\\
&\quad = \lambda K(t)\big[(\partial_t-\Delta)(\partial_x\Delta u(0)+\partial_y\Delta v(0))\big]+\lambda\partial_t K(t)\big[\partial_x\Delta u(0)+\partial_y\Delta v(0)\big]\\
&\qquad -\lambda\int_0^t(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)(\partial_x\Delta u(s)+\partial_y\Delta v(s))\,ds.
\end{aligned}
$$

Collecting the estimates, we obtain that

$$
\begin{aligned}
&B_n(t;n_0,\vec{u}_0,\vec{b}_0)\\
&\quad = -K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)N_0(0)\big]+K(t)\big[(\partial_t-\Delta)\Delta N_0(0)\big]\\
&\qquad +\partial_t K(t)\big[\Delta N_0(0)\big]+K(t)\big[\partial_t\partial_x N_1(0)+\partial_t\partial_y N_2(0)\big]\\
&\qquad -K(t)\big[\partial_x\Delta N_1(0)+\partial_y\Delta N_2(0)+\partial_y\Delta N_3(0)\big]\\
&\qquad -\partial_t K(t)\big[\partial_t N_0(0)-\partial_x N_1(0)-\partial_y N_2(0)-\Delta N_0(0)\big]-\partial_{tt}K(t)\big[N_0(0)\big]\\
&\qquad +\lambda K(t)\big[(\partial_t-\Delta)(\partial_x\Delta u(0)+\partial_y\Delta v(0))\big]\\
&\qquad +\lambda\partial_t K(t)\big[\partial_x\Delta u(0)+\partial_y\Delta v(0)\big].
\end{aligned}
\tag{6.13}
$$

The recasting above allowed us to cancel some of the troubling terms from (6.12) and (6.13) to get (6.5). This property is also valid for $\vec{u}$ and ψ, which is important in analysis. The rest of this section is split into two subsections to estimate the linear and nonlinear parts, respectively.

6.2. Estimates on the linear parts $L_n + B_n$

In this subsection, we prove that

Lemma 6.2.

$$\big\|\langle\nabla\rangle^3(L_n+B_n)(t;n_0,\vec{u}_0,\vec{b}_0)\big\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{4}}\|U_0\|_{X_0}; \tag{6.14}$$

$$\big\|\langle\nabla\rangle\partial_x(L_n+B_n)(t;n_0,\vec{u}_0,\vec{b}_0)\big\|_{L^2} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U_0\|_{X_0}. \tag{6.15}$$

Now we estimate the terms in (6.5). First of all, we consider $K(t)\big[\partial_y\Delta^2\psi_0\big]$. By Proposition 4.1, we have

$$\begin{aligned}\big\|\langle\nabla\rangle^3K(t)\big[\partial_y\Delta^2\psi_0\big]\big\|_{L^2_{xy}} &= \big\|\Delta^2K(t)\big[\partial_y\langle\nabla\rangle^3\psi_0\big]\big\|_{L^2_{xy}}\\ &\lesssim \langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^{4+}\nabla\psi_0\big\|_{L^1_{xy}}.\end{aligned} \tag{6.16}$$

Moreover, almost the same as (6.16), and using Proposition 4.2 (i) ($\beta'=3$) instead, we have

$$\begin{aligned}\big\|\langle\nabla\rangle\partial_xK(t)\big[\partial_y\Delta^2\psi_0\big]\big\|_{L^2_{xy}} &= \big\|\Delta\nabla\partial_{xy}K(t)\cdot\big[\langle\nabla\rangle\nabla\psi_0\big]\big\|_{L^2_{xy}}\\ &\lesssim \langle t\rangle^{-\frac{3}{4}}\big\|\langle\nabla\rangle^{3+}\nabla\psi_0\big\|_{L^1_{xy}}.\end{aligned} \tag{6.17}$$

Now we consider $\partial_tK(t)[\Delta\partial_y\psi_0]$. From Proposition 4.5 (i) ($\beta=1$), we have

$$\begin{aligned}\big\|\langle\nabla\rangle^3\partial_tK(t)\big[\Delta\partial_y\psi_0\big]\big\|_{L^2_{xy}} &= \big\|\nabla\partial_y\partial_tK(t)\cdot\big[\langle\nabla\rangle^3\nabla\psi_0\big]\big\|_{L^2_{xy}}\\ &\lesssim \langle t\rangle^{-\frac{1}{2}}\big\|\langle\nabla\rangle^{2+}\nabla\psi_0\big\|_{L^1_{xy}}.\end{aligned} \tag{6.18}$$

Similarly, by Proposition 4.5 (i) again ($\beta=2$),

$$\begin{aligned}\big\|\langle\nabla\rangle\partial_x\partial_tK(t)\big[\Delta\partial_y\psi_0\big]\big\|_{L^2_{xy}} &= \big\|\Delta\partial_y\partial_tK(t)\big[\langle\nabla\rangle\partial_x\psi_0\big]\big\|_{L^2_{xy}}\\ &\lesssim \langle t\rangle^{-\frac{3}{4}}\big\|\langle\nabla\rangle^{1+}\nabla\psi_0\big\|_{L^1_{xy}}.\end{aligned} \tag{6.19}$$

Now we consider the term $(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\big[\partial_xu_0\big]$. From Proposition 4.9 (i) ($p=2$) and Proposition 4.10 (i), we have

$$\big\|\langle\nabla\rangle^3(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\big[\partial_xu_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big\|\langle\nabla\rangle^{3+}u_0\big\|_{L^1_{xy}}; \tag{6.20}$$

$$\big\|\langle\nabla\rangle\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\big[\partial_x u_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{2+}u_0\big\|_{L^1_{xy}}. \tag{6.21}$$

Now we consider the term $(\partial_{tt}-\Delta\partial_t)K(t)\big[\partial_y v_0\big]$. From Proposition 4.6 (i) ($\beta = 0, 1$),

$$\big\|\langle\nabla\rangle^3(\partial_{tt}-\Delta\partial_t)K(t)\big[\partial_y v_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big\|\langle\nabla\rangle^{3+}v_0\big\|_{L^1_{xy}}; \tag{6.22}$$

$$\big\|\langle\nabla\rangle\partial_x(\partial_{tt}-\Delta\partial_t)K(t)\big[\partial_y v_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{2+}v_0\big\|_{L^1_{xy}}. \tag{6.23}$$

At last, we consider the terms (6.10). For the first term $-\frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[n_0]$, from Proposition 4.1, we have

$$\big\|\langle\nabla\rangle^3\Delta\sqrt{\Delta\partial_{yy}}K(t)[n_0]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^{4+}n_0\big\|_{L^1_{xy}}. \tag{6.24}$$

Similarly, by Proposition 4.2 (i) ($\beta' = 3$),

$$\big\|\langle\nabla\rangle\partial_x\Delta\sqrt{\Delta\partial_{yy}}K(t)[n_0]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\big\|\langle\nabla\rangle^{3+}n_0\big\|_{L^1_{xy}}. \tag{6.25}$$

For the second term $K_1(t)[n_0]$, by Proposition 4.15,

$$\big\|\langle\nabla\rangle^3K_1(t)[n_0]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^{\frac{7}{2}+}n_0\big\|_{L^1_{xy}}; \tag{6.26}$$

$$\big\|\langle\nabla\rangle\partial_xK_1(t)[n_0]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\big\|\langle\nabla\rangle^{\frac{5}{2}+}n_0\big\|_{L^1_{xy}}. \tag{6.27}$$

By the estimates (6.16), (6.18), (6.20), (6.22), (6.24), and (6.26), we have

$$\begin{aligned}
&\big\|\langle\nabla\rangle^3(L_n+B_n)(t;n_0,\vec{u}_0,\vec{b}_0)\big\|_{L^2}\\
&\lesssim \langle t\rangle^{-\frac{1}{4}}\Big(\big\|\langle\nabla\rangle^{4+}\nabla\psi_0\big\|_{L^1_{xy}}+\big\|\langle\nabla\rangle^{3+}\vec{u}_0\big\|_{L^1_{xy}}+\big\|\langle\nabla\rangle^{4+}n_0\big\|_{L^1_{xy}}\Big)\\
&\lesssim \langle t\rangle^{-\frac{1}{4}}\|U_0\|_{X_0}.
\end{aligned}$$

Similarly, by the estimates (6.17), (6.19), (6.21), (6.23), (6.25), and (6.27), we have

$$\big\|\langle\nabla\rangle\partial_x(L_n+B_n)(t;n_0,\vec{u}_0,\vec{b}_0)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U_0\|_{X_0}.$$

6.3. Estimates on the nonlinear parts $\mathcal{N}_n$

By Proposition 6.1 and Lemma 6.2, we are left to prove that

$$\big\|\langle\nabla\rangle^3\mathcal{N}_n(t;n,\vec{u},\psi)\big\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{4}}\Big(\epsilon_0\|U\|_X+Q(\|U\|_X)\Big); \tag{6.28}$$

$$\big\|\langle\nabla\rangle\partial_x\mathcal{N}_n(t;n,\vec{u},\psi)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\Big(\epsilon_0\|U\|_X+Q(\|U\|_X)\Big). \tag{6.29}$$

First of all, we split each of the nonlinearities $N_j, j = 0, 1, 2, 3$ (which were defined in Section 2.1) into the low frequency part and high frequency part, which reads as

$$N_j(n,u,v,\psi) = N_j^l(n,u,v,\psi) + N_j^h(n,u,v,\psi), \tag{6.30}$$

where

$$N_j^l(n,u,v,\psi)(t) = N_j\left(n_{\le \langle t\rangle^{0.01}}, u_{\le \langle t\rangle^{0.01}}, v_{\le \langle t\rangle^{0.01}}, \psi_{\le \langle t\rangle^{0.01}}\right)(t).$$

Here we use the notation $f_{\le N} = P_{\le N} f$. That is, each term in N_j^h contains at least one high frequency part and the terms in N_j^l involve only low frequencies. Then we have

Lemma 6.3.

$$\left\|\langle\nabla\rangle^5 N_j^h(n,u,v,\psi)(t)\right\|_{L^1_{xy}} \lesssim \langle t\rangle^{-1.03}\|U\|_X^2 \quad j = 0, 3; \tag{6.31}$$

$$\left\|\langle\nabla\rangle^4 N_j^h(n,u,v,\psi)(t)\right\|_{L^1_{xy}} \lesssim \langle t\rangle^{-1.03}\frac{\|U\|_X^2}{1-\|U\|_X} \quad j = 1, 2. \tag{6.32}$$

Proof. We only give the corresponding estimate on N_0, since the others can be proved by the same standard way. By the definition of N_0 in (2.3), and choosing M large enough and ϵ small enough, we have, for any $2 \le p \le +\infty$,

$$\begin{aligned}
&\left\|\langle\nabla\rangle^5 N_0^h(n,u,v,\psi)(t)\right\|_{L^1_{xy}} \\
&\lesssim \left\|\langle\nabla\rangle^6 P_{\ge\langle t\rangle^{0.01}} n(t)\right\|_{L^2_{xy}}\left(\|u(t)\|_{L^2_{xy}} + \|v(t)\|_{L^2_{xy}}\right) \\
&\quad + \|n(t)\|_{L^2_{xy}}\left(\left\|\langle\nabla\rangle^6 P_{\ge\langle t\rangle^{0.01}} u(t)\right\|_{L^2_{xy}} + \left\|\langle\nabla\rangle^6 P_{\ge\langle t\rangle^{0.01}} v(t)\right\|_{L^2_{xy}}\right) \\
&\lesssim \langle t\rangle^{-0.01(M-6)}\Big[\left\|\langle\nabla\rangle^M n(t)\right\|_{L^2_{xy}}\left(\|u(t)\|_{L^2_{xy}} + \|v(t)\|_{L^2_{xy}}\right) \\
&\quad + \|n(t)\|_{L^2_{xy}}\left(\left\|\langle\nabla\rangle^M u(t)\right\|_{L^2_{xy}} + \left\|\langle\nabla\rangle^M v(t)\right\|_{L^2_{xy}}\right)\Big] \\
&\lesssim \langle t\rangle^{-0.8}\Big[\langle t\rangle^{\epsilon}\|U\|_X \,\langle t\rangle^{-0.5}\|U\|_X + \langle t\rangle^{-0.25}\|U\|_X \,\langle t\rangle^{\epsilon}\|U\|_X\Big] \\
&\lesssim \langle t\rangle^{-1.03}\|U\|_X^2.
\end{aligned}$$

This proves the estimate on N_0. □

This lemma provides explicit decay rates for the L^1-norm of the high frequency parts in the nonlinearities. Since the decay rates in the estimates are smaller than -1 (-1.03 indeed), the corresponding nonlinearities are time integrable in the light of kernel estimates in Section 3. See Section 6.3.1 as an example.

6.3.1. $\int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds$

In this subsubsection, we will prove that

$$\Big\| \langle\nabla\rangle^3 \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds \Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2; \tag{6.33}$$

$$\Big\| \langle\nabla\rangle\partial_x \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds \Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} \|U\|_X^2. \tag{6.34}$$

First, we consider the piece of N_0^h. By Proposition 4.12 (i) ($\beta = 0$) and (6.31), we have

$$\begin{aligned}
&\Big\| \langle\nabla\rangle^3 \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0^h(s)\,ds \Big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \Big\| \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\langle\nabla\rangle^3 N_0^h(s) \Big\|_{L^2_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{2}} \big\| \langle\nabla\rangle^3 N_0^h(s) \big\|_{L^1_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{2}} \langle s\rangle^{-1.03}\,ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{1}{2}} \|U\|_X^2.
\end{aligned}$$

By the same way, from Proposition 4.12 (i) ($\beta = 1$) again, we have

$$\begin{aligned}
&\Big\| \langle\nabla\rangle\partial_x \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0^h(s)\,ds \Big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \Big\| \nabla\partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\langle\nabla\rangle N_0^h(s) \Big\|_{L^2_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \big\| \langle\nabla\rangle^{1+} N_0^h(s) \big\|_{L^1_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{-1.03}\,ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-1} \|U\|_X^2.
\end{aligned}$$

Remark 6.4. In the following context, we focus our attention on the low-frequency parts most of the time, since the high-frequency parts can be treated standardly as N_0^h above, by using the energy estimates. To simplify the notation, we only write n, u, v, ψ for $n_{\leq \langle s\rangle^{0.01}}, u_{\leq \langle s\rangle^{0.01}}, v_{\leq \langle s\rangle^{0.01}}, \psi_{\leq \langle s\rangle^{0.01}}$ in the low-frequency parts, if there is no confusion.

Now we consider N_0^l piece. By Lemma 3.20,

$$
\begin{aligned}
&\Big\| \langle \nabla \rangle^3 \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) N_0^l(s)\, ds \Big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \Big\| \nabla \partial_t(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) \cdot \langle \nabla \rangle^3 P_{\lesssim \langle s\rangle^{0.01}} (n\vec{u})(s) \Big\|_{L^2_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \big\| \langle \nabla \rangle^3 P_{\lesssim \langle s\rangle^{0.01}} (n\vec{u})(s) \big\|_{L^1_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.04} \big\| n \big\|_{L^2_{xy}} \big\| \vec{u} \big\|_{L^2_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.04-0.25-0.5} ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2.
\end{aligned}
$$

Also, almost the same, and using

$$
\|\partial_x(n\vec{u})\|_{L^1_{xy}} \leq \Big(\|\partial_x n\|_{L^2_{xy}} \|\vec{u}\|_{L^2_{xy}} + \|n\|_{L^2_{xy}} \|\nabla \cdot \vec{u}\|_{L^2_{xy}} \Big) \lesssim \langle s\rangle^{-1.25} \|U\|_X^2, \tag{6.35}
$$

we have

$$
\begin{aligned}
&\Big\| \langle \nabla \rangle \partial_x \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) N_0^l(s)\, ds \Big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \Big\| \nabla \partial_t(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) \cdot P_{\lesssim \langle s\rangle^{0.01}} \langle \nabla \rangle \partial_x (n\vec{u})(s) \Big\|_{L^2_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.03} \|\partial_x(n\vec{u})\|_{L^1_{xy}} \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.03-1.25} ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{3}{4}} \|U\|_X^2.
\end{aligned}
$$

Therefore, we give the estimate (6.33) and (6.34).

6.3.2. $\int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds$

This is one of most tricky terms. Indeed, regardless of the loss of regularity, the best estimate one may get is

$$\big\|N_0(s)\big\|_{L_x^1 L_y^2} \lesssim \langle s\rangle^{-1}\|U\|_X^2,$$

which is critical for integrable (and it is not integrable). We need some new argument to prove the estimates followed,

$$\Big\|\langle\nabla\rangle^3 \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^2; \tag{6.36}$$

$$\Big\|\langle\nabla\rangle\partial_x \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2. \tag{6.37}$$

According to the frequency, and the definition (2.3): $N_0 = \partial_x(nu) + \partial_y(nv)$, we have

$$\begin{aligned}
N_0 &= N_0^h + N_0^l\\
&= N_0^h + \partial_x\big(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}}\big) + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}}\big)\\
&= N_0^h + \partial_x\big(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}}\big) + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, v_{\le\langle s\rangle^{-10}}\big)\,ds\\
&\quad + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, v_{\langle s\rangle^{-0.05}\le\cdot\le\langle s\rangle^{0.01}}\big) + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, v_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\big).
\end{aligned}$$

Similar as before, we denote $f_{\ge N} = P_{\ge N}f$ and $f_{N\le\cdot\le M} = P_{N\le\cdot\le M}f$ here. Then by the fourth equation in (2.2): $v = -\partial_t\psi + N_3$, and the product rule, we further have

$$\begin{aligned}
N_0 &= N_0^h + \partial_x\big(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}}\big) + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, v_{\le\langle s\rangle^{-10}}\big)\\
&\quad + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, v_{\langle s\rangle^{-0.05}\le\cdot\le\langle s\rangle^{0.01}}\big)\\
&\quad + \partial_y\Big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}N_3\Big) + \partial_y\Big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\partial_s\psi\Big)\\
&= N_0^h + \partial_x\big(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}}\big) + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, v_{\le\langle s\rangle^{-10}}\big)\\
&\quad + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, v_{\langle s\rangle^{-0.05}\le\cdot\le\langle s\rangle^{0.01}}\big)\\
&\quad + \partial_y\big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}N_3\big) + \partial_y\partial_s\Big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\Big)\\
&\quad - \partial_y\Big(\partial_s n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\Big) + \textit{other parts},
\end{aligned}\tag{6.38}$$

where the *other parts* include the terms which ∂_s hits $P_{\le\langle s\rangle^{0.01}}$ or $P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}$. By (6.38), we split the term into several parts as follows,

$$\int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds$$

$$= \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0^h(s)\,ds \tag{6.39}$$

$$+ \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\partial_x(n_{\leq\langle s\rangle^{0.01}}\, u_{\leq\langle s\rangle^{0.01}})(s)\,ds \tag{6.40}$$

$$+ \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\partial_y(n_{\leq\langle s\rangle^{0.01}}\, v_{\leq\langle s\rangle^{-10}})(s)\,ds \tag{6.41}$$

$$+ \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\partial_y(n_{\leq\langle s\rangle^{0.01}}\, v_{\geq\langle s\rangle^{-0.05}})(s)\,ds \tag{6.42}$$

$$+ \int_0^t \Delta\partial_y(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\big(n_{\leq\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}N_3\big)(s)\,ds \tag{6.43}$$

$$+ \int_0^t \Delta\partial_y(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\partial_s\big(n_{\leq\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)(s)\,ds \tag{6.44}$$

$$- \int_0^t \Delta\partial_y(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\big(\partial_s n_{\leq\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)(s)\,ds \tag{6.45}$$

$$+\textit{other easy terms}, \tag{6.46}$$

where the *other easy terms* is the corresponding terms from *other parts* in (6.38).

The high frequency piece (6.39) can be treated standardly as in Section 6.3.1, and thus we obtain (the details of the proof are omitted)

$$\Big\|\langle\nabla\rangle^3(6.39)\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^2;$$

$$\Big\|\langle\nabla\rangle\partial_x(6.39)\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2.$$

Now we consider the part (6.40). By Proposition 4.9 (ii) we have

$$\Big\|\langle\nabla\rangle^3(6.40)\Big\|_{L^2_{xy}} \lesssim \int_0^t \big\|P_{\lesssim\langle s\rangle^{0.01}}\nabla\partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\cdot\nabla\langle\nabla\rangle^3(nu)(s)\big\|_{L^2_{xy}}\,ds$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \|P_{\lesssim \langle s\rangle^{0.01}} \langle\nabla\rangle^{4+} nu\|_{L^1_{xy}}\, ds$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.05} \langle s\rangle^{-0.75}\, ds\, \|U\|_X^2$$

$$\lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2.$$

By the same way, and using (6.35) we have

$$\Big\|\langle\nabla\rangle\partial_x (6.40)\Big\|_{L^2_{xy}} \lesssim \int_0^t \langle s\rangle^{0.03} \langle t-s\rangle^{-\frac{3}{4}} \|\partial_x(nu)\|_{L^1_{xy}}\, ds$$

$$\lesssim \int_0^t \langle s\rangle^{0.03-1.25} \langle t-s\rangle^{-\frac{3}{4}}\, ds\, \|U\|_X^2 \lesssim \langle t\rangle^{-\frac{3}{4}} \|U\|_X^2.$$

Next, we consider (6.41) and (6.42). First, by Bernstein's inequality, we have

$$\big\|n_{\leq \langle s\rangle^{0.01}}\, v_{\leq \langle s\rangle^{-10}}\big\|_{L^1_x L^2_y} \lesssim \big\|n\big\|_{L^2_{xy}} \big\|v_{\leq \langle s\rangle^{-10}}\big\|_{L^2_x L^\infty_y} \lesssim \langle s\rangle^{-4.9} \big\|n\big\|_{L^2_{xy}} \big\|v\big\|_{L^2_{xy}}$$

$$\lesssim \langle s\rangle^{-5} \|U\|_X^2.$$

Then by Proposition 4.8,

$$\Big\|\langle\nabla\rangle^3 (6.41)\Big\|_{L^2_{xy}}$$

$$\lesssim \int_0^t \big\|\Delta(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) \partial_y (n_{\leq \langle s\rangle^{0.01}}\, v_{\leq \langle s\rangle^{-10}})(s)\big\|_{L^2_{xy}}\, ds$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{4}} \big\|\langle\nabla\rangle^5 \big(n_{\leq \langle s\rangle^{0.01}}\, v_{\leq \langle s\rangle^{-10}}\big)\big\|_{L^1_x L^2_y}\, ds$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{4}} \langle s\rangle^{0.05} \langle s\rangle^{-5}\, ds \|U\|_X^2$$

$$\lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2.$$

On the other hand, since

$$\big\|n\big\|_{L^2_{xy}} \big\|v_{\geq \langle s\rangle^{-0.05}}\big\|_{L^2_{xy}} \lesssim \langle s\rangle^{0.05} \big\|n\big\|_{L^2_{xy}} \|\nabla v\|_{L^2_{xy}} \lesssim \langle s\rangle^{-1.2} \|U\|_X^2,$$

by the same treatment we also have

$$\left\|\langle\nabla\rangle^3(6.42)\right\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^2.$$

Therefore, we get

$$\left\|\langle\nabla\rangle^3\big((6.41)+(6.42)\big)\right\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^2. \tag{6.47}$$

By using Proposition 4.9 (ii) instead, we also get

$$\left\|\langle\nabla\rangle\partial_x\big((6.41)+(6.42)\big)\right\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2. \tag{6.48}$$

Now we turn to consider the part (6.43), by Proposition 4.8,

$$\begin{aligned}
\|\langle\nabla\rangle^3(6.43)\|_{L^2_{xy}} &\lesssim \int_0^t \big\|\Delta(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s) \\
&\quad \times\partial_y\big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}N_3\big)(s)\big\|_{L^2_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{4}}\langle s\rangle^{0.05}\big\|n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}N_3\big\|_{L^1_{xy}}\,ds \\
&\lesssim \int_0^t \langle s\rangle^{-0.05}\langle t-s\rangle^{-\frac{1}{4}}\big\|n\big\|_{L^2_{xy}}\big\|\nabla\psi\big\|_{L^2_{xy}}\big\|\vec{u}\big\|_{L^\infty_{xy}}\,ds \\
&\lesssim \int_0^t \langle s\rangle^{0.05-1.5}\langle t-s\rangle^{-\frac{1}{4}}\,ds\|U\|_X^3 \\
&\lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^3.
\end{aligned} \tag{6.49}$$

Also, using Proposition 4.9 (ii) instead, we obtain

$$\left\|\langle\nabla\rangle\partial_x(6.43)\right\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2. \tag{6.50}$$

To prove (6.44), we integrate by parts to get

$$(6.44) = -\Delta\partial_y(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\big(P_{\lesssim 1}n_0\, P_{\sim 1}\psi_0\big) \tag{6.51}$$

$$\begin{aligned}
&+\int_0^t \Delta\partial_y(\partial_{tt}-\Delta\partial_t-\Delta) \\
&\quad\times\partial_t K(t-s)\big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)(s)\,ds.
\end{aligned} \tag{6.52}$$

By Proposition 4.8 and Proposition 4.9 (ii), the boundary term (6.51) can be controlled as following,

$$\|\langle\nabla\rangle^3(6.51)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\|\langle\nabla\rangle^3 P_{\lesssim 1}n_0\, P_{\sim 1}\psi_0\|_{L^1_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\|n_0\|_{L^2_{xy}}\|\nabla\psi_0\|_{L^2_{xy}};$$
$$\|\partial_x(6.51)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|\langle\nabla\rangle^3 P_{\lesssim 1}n_0\, P_{\sim 1}\psi_0\|_{L^1_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|n_0\|_{L^2_{xy}}\|\nabla\psi_0\|_{L^2_{xy}}$$

To prove (6.52), we need the following estimate, for any $\epsilon > 0$,

$$\begin{aligned}\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\|_{L^\infty_{xy}} &\lesssim \big\||\nabla|^{1-\epsilon}P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big\|_{L^2_{xy}}\\ &\lesssim \langle s\rangle^{-0.02}\big\||\nabla|^{\frac{1}{2}+\epsilon}\psi\big\|_{L^2_{xy}} \lesssim \langle s\rangle^{-0.27}\|U\|_X. \end{aligned}\tag{6.53}$$

Then using (3.88), (3.91) and (6.53), we have

$$\begin{aligned}\|\langle\nabla\rangle^3(6.52)\|_{L^2_{xy}} &\lesssim \int_0^t \big\|A^2\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t-s,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\\ &\quad\cdot\big\|\langle\nabla\rangle^3\big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)\big\|_{L^2_{xy}}\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03}\big\|n\big\|_{L^2_{xy}}\big\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big\|_{L^\infty_{xy}}\,ds\\ &\lesssim \int_0^t \langle s\rangle^{-0.27+0.03-0.25}\langle t-s\rangle^{-1}\,ds\|U\|_X^2\\ &\lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\langle\nabla\rangle\partial_x(6.52)\|_{L^2_{xy}} &\lesssim \int_0^t \big\|A^2\partial_t(\partial_{tt}+A^2\partial_t+A^2)\widehat{K}(t-s,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\\ &\quad\cdot\big\|\langle\nabla\rangle\partial_x\big(n_{\le\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)\big\|_{L^2_{xy}}\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03}\Big(\big\|\partial_x n\big\|_{L^2_{xy}}\big\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big\|_{L^\infty_{xy}}\\ &\quad+\big\|n\big\|_{L^\infty_{xy}}\big\|\partial_x\psi\big\|_{L^2_{xy}}\Big)\,ds\\ &\lesssim \int_0^t \langle s\rangle^{0.04-1}\langle t-s\rangle^{-1}\,ds\|U\|_X^2\\ &\lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2.\end{aligned}$$

For the term (6.45), since $\partial_t n = -\partial_x u - \partial_y v + N_0$, we have

$$
\begin{aligned}
(6.45) = &-\int_0^t \Delta\partial_y(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s) \\
&\times\big(P_{\le\langle s\rangle^{0.01}}\partial_x u\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)(s)\, ds \qquad (6.54)\\
&-\int_0^t \Delta\partial_y(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s) \\
&\times\big(P_{\le\langle s\rangle^{0.01}}\partial_y v\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)(s)\, ds \qquad (6.55)\\
&+\int_0^t \Delta\partial_y(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s) \\
&\times\big(P_{\le\langle s\rangle^{0.01}}N_0\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)(s)\, ds. \qquad (6.56)
\end{aligned}
$$

These three terms can be treated by the similar way. Now we consider the term (6.54). By Proposition 4.8, Sobolev's and Beinstein's inequalities, we have

$$
\begin{aligned}
\|\langle\nabla\rangle^3(6.54)\|_{L^2_{xy}} &\lesssim \int_0^t \big\|\Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s) \\
&\quad\times \partial_y\big(P_{\le\langle s\rangle^{0.01}}\partial_x u\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)(s)\big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac14}\langle s\rangle^{0.01(3+\frac52)}\big\|P_{\le\langle s\rangle^{0.01}}\partial_x u\, P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big\|_{L^1_xL^2_y}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac14}\langle s\rangle^{0.06}\big\|\partial_x u\big\|_{L^2_{xy}}\,\big\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big\|_{L^2_xL^\infty_y}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac14}\langle s\rangle^{0.06}\big\|\partial_x u\big\|_{L^2_{xy}}\,\big\||\nabla|^{\frac12-}\psi\big\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac14}\langle s\rangle^{0.07-1-0.25}\, ds\|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac14}\|U\|_X^2.
\end{aligned}
$$

Almost the same, we have the estimate on the term (6.55),

$$
\|\langle\nabla\rangle^3(6.55)\|_{L^2_{xy}} \lesssim \int_0^t \langle t-s\rangle^{-\frac14}\langle s\rangle^{0.06}\big\|\partial_y v\big\|_{L^2_{xy}}\,\big\||\nabla|^{\frac12-}\psi\big\|_{L^2_{xy}}\, ds
$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{4}} \langle s\rangle^{0.07-1-0.25}\, ds \|U\|_X^2$$
$$\lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2.$$

Also, for the term (6.56), we have

$$\begin{aligned}
\|\langle\nabla\rangle^3 (6.56)\|_{L^2_{xy}} &\lesssim \int_0^t \langle s\rangle^{0.06} \langle t-s\rangle^{-\frac{1}{4}} \|n\|_{L^2_{xy}} \|\vec{u}\|_{L^2_x L^\infty_y} \left\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\right\|_{L^\infty_{xy}} ds\\
&\lesssim \int_0^t \langle s\rangle^{0.07} \langle t-s\rangle^{-\frac{1}{4}} \|n\|_{L^2_{xy}} \|\vec{u}\|_{L^2_x L^\infty_y} \|\nabla\psi\|_{L^2_{xy}} ds\\
&\lesssim \int_0^t \langle s\rangle^{0.08-0.25-0.75-0.25} \langle t-s\rangle^{-\frac{1}{4}}\, ds \|U\|_X^3\\
&\lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^3.
\end{aligned}$$

Combining these three estimates, we obtain

$$\|\langle\nabla\rangle^3 (6.45)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}} \left(\|U\|_X^2 + \|U\|_X^3\right). \tag{6.57}$$

Replacing the operator $\Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)$ by $\Delta\partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)$, and using Proposition 4.9 (ii) instead, we also get

$$\|\langle\nabla\rangle\partial_x (6.45)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} \left(\|U\|_X^2 + \|U\|_X^3\right). \tag{6.58}$$

At last, we estimate (6.46). Since for any $\beta \in \mathbb{R}$, $1 \le q \le \infty$,

$$\|\partial_s P_{\le\langle s\rangle^\beta} f\|_{L^q} \lesssim \langle s\rangle^{-1} \|P_{\sim\langle s\rangle^\beta} f\|_{L^q}, \tag{6.59}$$

(see Appendix A.3 for its proof) it is easy to prove that

$$\|\langle\nabla\rangle^3 (6.46)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2; \quad \|\partial_x (6.46)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} \|U\|_X^2. \tag{6.60}$$

Collecting the estimates above, we establish (6.36) and (6.37).

6.3.3. $\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_1(s)\, ds$.

We shall prove that

$$\left\|\langle\nabla\rangle^3 \int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_1(s)\, ds\right\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}} Q\left(\|U\|_X\right); \tag{6.61}$$

$$\Big\| \int_0^t \partial_x^2(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_1(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} Q\big(\|U\|_X\big). \tag{6.62}$$

As before, the estimate for the high frequency part is standard and can be obtained by Lemma 3.17 and (6.32). Thus we only consider the low frequency piece N_1^l. We first split N_1 it into two parts,

$$N_1 = N_{11} + N_{12},$$

where

$$\begin{aligned} N_{11} &= -(u\partial_x u + v\partial_y u) - \frac{-\nabla n\cdot\nabla u + n\lambda(\partial_{xx}u + \partial_{xy}v)}{\rho} + \frac{n\,\partial_x\psi\Delta\psi}{\rho}; \\ N_{12} &= -\partial_x\psi\Delta\psi - n\partial_x n - \nabla\cdot(n\nabla u), \end{aligned}$$

and write $N_1^l = N_{11}^l + N_{12}^l$ respectively. Then

$$\begin{aligned} &\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_1^l(s)\,ds \\ &= \int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_{11}^l(s)\,ds \qquad (6.63) \\ &\quad + \int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_{12}^l(s)\,ds. \qquad (6.64) \end{aligned}$$

First, we have

Lemma 6.5.

$$\|N_{11}^l\|_{L^1_{xy}} \lesssim \langle t\rangle^{-1} Q(\|U\|_X).$$

Proof. By Hölder's inequality and Bernstein' inequality, we have

$$\begin{aligned} \|N_{11}^l\|_{L^1_{xy}} &\lesssim \|\nabla u\|_{L^2_{xy}}\|\vec{u}\|_{L^2_{xy}} + \frac{\|\langle\nabla\rangle n\|_{L^2_{xy}}\Big(\|\nabla u\|_{L^2_{xy}} + \|\lambda P_{\leq\langle s\rangle^{0.01}}(\partial_{xx}u + \partial_{xy}v)\|_{L^2_{xy}}\Big)}{1 - \|n\|_{L^\infty_{xy}}} \\ &\quad + \frac{\|n\|_{L^2_{xy}}\|\partial_x\psi\|_{L^2_{xy}}\|P_{\leq\langle s\rangle^{0.01}}\Delta\psi\|_{L^2_{xy}}}{1 - \|n\|_{L^\infty_{xy}}} \\ &\lesssim \|\nabla u\|_{L^2_{xy}}\|\vec{u}\|_{L^2_{xy}} \\ &\quad + \frac{\|\langle\nabla\rangle n\|_{L^2_{xy}}\big(\|\nabla u\|_{L^2_{xy}} + \langle s\rangle^{0.01}\|\nabla\cdot\vec{u}\|_{L^2_{xy}} + \langle s\rangle^{0.01}\|\partial_x\psi\|_{L^2_{xy}}\|\nabla\psi\|_{L^2_{xy}}\big)}{1 - \|n\|_{L^\infty_{xy}}} \end{aligned}$$

$$\lesssim \langle t\rangle^{-0.25-0.75}\frac{\|U\|_X^2+\|U\|_X^3}{1-\|U\|_X}$$
$$\lesssim \langle t\rangle^{-1}Q(\|U\|_X). \qquad \square$$

For (6.63), by Proposition 4.9 (i) and Lemma 6.5,

$$\begin{aligned}\left\|\langle\nabla\rangle^3(6.63)\right\|_{L^2_{xt}} &\lesssim \int_0^t \left\|\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\langle\nabla\rangle^3 N_{11}^l(s)\right\|_{L^2_{xt}} ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac12}\langle s\rangle^{0.04}\|N_{11}^l(s)\|_{L^1_{xy}}\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac12}\langle s\rangle^{0.04-1}Q(\|U\|_X)\\ &\lesssim \langle t\rangle^{-\frac14}Q(\|U\|_X).\end{aligned} \tag{6.65}$$

Replacing $\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)$ by $\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)$, and using Proposition 4.10 (i) instead, we get

$$\left\|\langle\nabla\rangle\partial_x(6.63)\right\|_{L^2_{xt}} \lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.04-1}Q(\|U\|_X) \lesssim \langle t\rangle^{-\frac34}Q(\|U\|_X). \tag{6.66}$$

Now we consider the part (6.64). Since

$$N_{12} = -\nabla\cdot(\partial_x\psi\nabla\psi+n\nabla u)+\frac12\partial_x\big(|\nabla\psi|^2\big)-\frac12\partial_x(n^2),$$

we have,

$$(6.64) = -\int_0^t \partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\nabla\cdot(\partial_x\psi\nabla\psi+n\nabla u)(s)\,ds \tag{6.67}$$
$$+\frac12\int_0^t \partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\partial_x\big(|\nabla\psi|^2-n^2\big)(s)\,ds, \tag{6.68}$$

where ψ and n lie in the low frequency. For (6.67), by Proposition 4.9 (ii), we have

$$\begin{aligned}\left\|\langle\nabla\rangle^3(6.67)\right\|_{L^2_{xt}} \lesssim \int_0^t &\big\|\nabla\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\\ &\cdot P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle^3(\partial_x\psi\nabla\psi+n\nabla u)(s)\big\|_{L^2_{xy}}\,ds\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.05} \big(\|\partial_x \psi \nabla \psi\|_{L^1_{xy}} + \|n\, \nabla u_{\le \langle s\rangle^{0.01}}\|_{L^1_{xy}} \big)\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.06} \big(\|\partial_x \psi\|_{L^2_{xy}} \|\nabla \psi\|_{L^2_{xy}} + \|n\|_{L^2_{xy}} \|u\|_{L^2_{xy}} \big)\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.06-0.5-0.25}\, ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2.
\end{aligned} \tag{6.69}
$$

Moreover, by Proposition 4.10 (ii) ($\beta = 1$),

$$
\begin{aligned}
\Big\| \langle \nabla \rangle \partial_x (6.67) \Big\|_{L^2_{xt}} &\lesssim \int_0^t \big\| \nabla \partial_{xx} (\partial_{tt} - \Delta \partial_t - \Delta) K(t-s) \\
&\qquad \cdot P_{\lesssim \langle s\rangle^{0.01}} \langle \nabla \rangle (\partial_x \psi \nabla \psi + n \nabla u)(s) \big\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.03} \big(\|\partial_x \psi \nabla \psi\|_{L^2_{xy}} + \|n\, \nabla u_{\le \langle s\rangle^{0.01}}\|_{L^2_{xy}} \big)\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.04} \big(\|\partial_x \psi\|_{L^2_{xy}} \|\nabla \psi\|_{L^\infty_{xy}} + \|n\|_{L^2_{xy}} \|u\|_{L^\infty_{xy}} \big)\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.04-0.5-0.5}\, ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{3}{4}} \|U\|_X^2.
\end{aligned} \tag{6.70}
$$

For (6.68), by Proposition 4.10 (i), we have

$$
\begin{aligned}
\Big\| \langle \nabla \rangle^3 (6.68) \Big\|_{L^2_{xt}} &\lesssim \int_0^t \big\| \partial_{xx} (\partial_{tt} - \Delta \partial_t - \Delta) K(t-s) \\
&\qquad \cdot P_{\lesssim \langle s\rangle^{0.01}} \langle \nabla \rangle^3 \big(|\nabla \psi|^2 - n^2 \big)(s) \big\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.05} \big\| |\nabla \psi|^2 - n^2 \big\|_{L^1_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.05} \big(\|\nabla \psi\|_{L^2_{xy}} \|\nabla \psi\|_{L^2_{xy}} + \|n\|_{L^2_{xy}} \|n\|_{L^2_{xy}} \big)\, ds
\end{aligned}
$$

$$\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.05}\langle s\rangle^{-\frac{1}{2}}\,ds\|U\|_X^2$$
$$\lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^2. \tag{6.71}$$

Moreover, Proposition 4.10 (ii) ($\beta = 1$) again,

$$\begin{aligned}\Big\|\langle\nabla\rangle\partial_x(6.68)\Big\|_{L^2_{xt}} &\lesssim \int_0^t \big\|\nabla\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\\ &\quad\cdot P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle\big(|\nabla\psi|^2-n^2\big)(s)\big\|_{L^2_{xy}}\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03}\Big(\big\|\nabla\psi_{\le\langle s\rangle^{0.01}}\cdot\nabla\partial_x\psi_{\le\langle s\rangle^{0.01}}\big\|_{L^2_{xy}}+\big\|\partial_x nn\big\|_{L^2_{xy}}\Big)\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03}\Big(\|\partial_x\psi\|_{L^2_{xy}}\|\nabla\psi\|_{L^\infty_{xy}}+\|\partial_x n\|_{L^2_{xy}}\|n\|_{L^\infty_{xy}}\Big)\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03-0.5-0.5}\,ds\|U\|_X^2\\ &\lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2.\end{aligned} \tag{6.72}$$

Collecting the estimates in (6.69)–(6.72), we have

$$\Big\|\langle\nabla\rangle^3(6.64)\Big\|_{L^p_{xt}} \lesssim \langle t\rangle^{-\frac{1}{4}}Q\big(\|U\|_X\big); \tag{6.73}$$
$$\Big\|\langle\nabla\rangle\partial_x(6.64)\Big\|_{L^2_{xt}} \lesssim \langle t\rangle^{-\frac{3}{4}}Q\big(\|U\|_X\big). \tag{6.74}$$

Therefore, we finish the estimates in this subsubsection and thus give (6.61) and (6.62).

6.3.4. $\int_0^t \partial_y(\partial_{tt}-\Delta\partial_t)K(t-s)N_2(s)\,ds$.

This term is at the same level as $\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)N_1(s)\,ds$. Indeed, the two symbols have the relation

$$\eta(\partial_{tt}+A^2\partial_t)\sim\frac{\xi}{A}\cdot\xi(\partial_{tt}+A^2\partial_t+A^2).$$

Therefore, by the same way as Section 6.3.3 (details are omitted here), we have

$$\Big\|\langle\nabla\rangle^3\int_0^t \partial_y(\partial_{tt}-\Delta\partial_t)K(t-s)N_2(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}Q\big(\|U\|_X\big); \tag{6.75}$$

$$\Big\|\langle\nabla\rangle\partial_x\int_0^t \partial_y(\partial_{tt}-\Delta\partial_t)K(t-s)N_2(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}Q\big(\|U\|_X\big). \tag{6.76}$$

6.3.5. $\int_0^t \Delta\partial_y(\partial_t-\Delta)K(t-s)N_3(s)\,ds$.

This term is at the same level as $\int_0^t(\partial_t-\Delta)(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)N_0(s)\,ds$. Indeed, the operator $\Delta(\partial_t-\Delta)K(t)$ has the same decay estimates as $(\partial_t-\Delta)(\partial_{tt}-\Delta\partial_t-\Delta)K(t)$. Moreover, we shall show that the nonlinearity $\partial_y N_3$ has the same estimates as N_0. Indeed, as the part $\partial_x(nu)$ in N_0, the part $\partial_y(u\partial_x\psi)$ in N_3 is subcritical in integration (decay rate larger than -1), so it is easy to treat. For the other term

$$\partial_y(v\partial_y\psi) = \partial_y v\,\partial_y\psi + v\partial_{yy}\psi,$$

the first piece is easy since it is subcritical in integration again. So one may find that the trouble is from the piece $v\partial_{yy}\psi$. However, by the third equation in (2.2), one may find that

$$\partial_{yy}\psi + \partial_y n = -\partial_{xx}\psi - \partial_t v + \Delta v + \lambda(\partial_{xy}u + \partial_{yy}v) + N_2.$$

Since the right-hand side in the identity is decaying faster than the each piece of the left-hand side, we roughly obtain that

$$\partial_{yy}\psi \sim -\partial_y n.$$

Thus, the estimates on $v\partial_{yy}\psi$ can be reduced to the ones on $v\partial_y n$, which the most trouble piece in N_0 and we have dealt with it as before. So by the same way, we establish that

$$\Big\|\langle\nabla\rangle^3\int_0^t \Delta\partial_y(\partial_t-\Delta)K(t-s)N_3(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}Q\big(\|U\|_X\big); \tag{6.77}$$

$$\Big\|\langle\nabla\rangle\partial_x\int_0^t \Delta\partial_y(\partial_t-\Delta)K(t-s)N_3(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}Q\big(\|U\|_X\big). \tag{6.78}$$

6.3.6. $\lambda\int_0^t(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)(\partial_x\Delta u(s)+\partial_y\Delta v(s))\,ds$.

This indeed is a linear term, to use the continuity argument, we need some small bound. To do this, we set $T_0=\epsilon_0^{-6}$, and assume that $t\ge 2T_0$, otherwise it is concluded in the local theory. Then from Proposition 4.11 (ii) ($\beta_2=2$), we get

$$\begin{aligned}
&\Big\|\langle\nabla\rangle^3\lambda\int_0^t(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)(\partial_x\Delta u(s)+\partial_y\Delta v(s))\,ds\Big\|_{L^2_{xy}}\\
&\le|\lambda|\int_0^t\Big\|\Delta\nabla(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)\cdot\langle\nabla\rangle^3\vec{u}(s)\Big\|_{L^2_{xy}}\,ds\\
&\lesssim\int_0^t\langle t-s\rangle^{-1}\big\|\langle\nabla\rangle^4\vec{u}(s)\big\|_{L^2_{xy}}\,ds\\
&\lesssim\int_0^t\langle t-s\rangle^{-1}\langle s\rangle^{-\frac{1}{3}}\,ds\|U\|_X\\
&\lesssim\langle t\rangle^{-\frac{1}{2}}\|U\|_X\lesssim T_0^{-\frac{1}{4}}\langle t\rangle^{-\frac{1}{4}}\|U\|_X\\
&=\epsilon_0\langle t\rangle^{-\frac{1}{4}}\|U\|_X,
\end{aligned}\tag{6.79}$$

where we have used (by choosing M large enough)

$$\big\|\langle\nabla\rangle^3\vec{u}(s)\big\|_{L^2_{xy}}\lesssim\big\|\vec{u}(s)\big\|_{L^2_{xy}}^{1-\frac{4}{M}}\big\|\langle\nabla\rangle^M\vec{u}(s)\big\|_{L^2_{xy}}^{\frac{4}{M}}\lesssim\langle s\rangle^{-\frac{1}{3}}\|U\|_X.$$

Similarly,

$$\begin{aligned}
&\Big\|\langle\nabla\rangle\partial_x\int_0^t(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)(\partial_x\Delta u(s)+\partial_y\Delta v(s))\,ds\Big\|_{L^2_{xt}}\\
&\lesssim\int_0^t\langle t-s\rangle^{-1}\big\|\langle\nabla\rangle^2\partial_x\vec{u}(s)\big\|_{L^2_{xy}}\,ds\lesssim\epsilon_0\langle t\rangle^{-\frac{3}{4}}\|U\|_X.
\end{aligned}\tag{6.80}$$

Now collecting the estimates obtained in Section 6.3.1–Section 6.3.6, we obtain (6.28). Combination with (6.14), gives us (6.1).

7. The estimates on u

In this section, we shall prove that

$$\|u(t)\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0}+|\lambda|\,\|U\|_X+Q(\|U\|_X)\big);\tag{7.1}$$

$$\|\langle\nabla\rangle u(t)\|_{L^\infty_{xy}}\lesssim\langle t\rangle^{-1}\big(\|U_0\|_{X_0}+|\lambda|\,\|U\|_X+Q(\|U\|_X)\big);\tag{7.2}$$

$$\||\nabla|^\gamma u(t)\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{3}{4}}\big(\|U_0\|_{X_0}+|\lambda|\,\|U\|_X+Q(\|U\|_X)\big);\tag{7.3}$$

$$\|\langle\nabla\rangle\partial_x u(t)\|_{L^2_{xy}}\lesssim\langle t\rangle^{-1}\big(\|U_0\|_{X_0}+|\lambda|\,\|U\|_X+Q(\|U\|_X)\big).\tag{7.4}$$

7.1. The reexpression of u

Similar as the expression n in Section 6.1, we give the reexpression of u and prove that

Proposition 7.1. *The unknown function u obeys the formula,*

$$u(t,x,y) = (L_u + B_u)(t; n_0, \vec{u}_0, \vec{b}_0) + \mathcal{N}_u(t; n, \vec{u}, \psi), \tag{7.5}$$

where $(L_u + B_u)$ is given by

$$\begin{aligned} &L_u(t; n_0, \vec{u}_0, \vec{b}_0) + B_u(t; n_0, \vec{u}_0, \vec{b}_0) \\ &= -K(t)\big[\partial_{xy}\Delta\psi_0\big] - \partial_t K(t)[\partial_{yy}u_0 - \partial_{xy}v_0] \\ &\quad - (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_x n_0\big] + \frac{1}{2}\Delta(\partial_{tt} - \Delta\partial_t)K(t)\big[u_0\big] \\ &\quad - \frac{1}{2}\Delta\big(\Delta + \sqrt{\Delta\partial_{yy}}\big)K(t)\big[u_0\big] + \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[u_0\big], \end{aligned} \tag{7.6}$$

and $\mathcal{N}_u(t; n, \vec{u}, \psi)$ is given by

$$\begin{aligned} \mathcal{N}_u(t; n, \vec{u}, \psi) = &-\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds \\ &+ \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_1(s)\,ds \\ &+ \int_0^t \partial_{xy}\partial_t K(t-s)N_2(s)\,ds - \int_0^t \partial_{xy}\Delta K(t-s)N_3(s)\,ds \\ &+ \lambda\int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\,ds. \end{aligned} \tag{7.7}$$

The proof of this proposition will occupy the rest of this subsection. First, according to (2.39), using (2.57) and integration by parts (see details in Section 7.1.2), we have

$$\begin{aligned} u(t,x,y) &= L_u(t; n_0, \vec{u}_0, \vec{b}_0) + \int_0^t K(t-s)F_1(s)\,ds \\ &= L_u(t; n_0, \vec{u}_0, \vec{b}_0) + \int_0^t K(t-s)\Big[-\partial_x(\partial_{ss} - \Delta\partial_s - \Delta)N_0 \end{aligned}$$

$$
\begin{aligned}
&+ \partial_s(\partial_{ss} - \Delta\partial_s - \Delta - \partial_{yy})N_1 \\
&+ \partial_{xy}\partial_s N_2 - \partial_{xy}\Delta N_3 + \lambda(\partial_{ss} - \Delta\partial_s - \Delta)\partial_s\partial_x(\partial_x u + \partial_y v)\Big]\, ds \\
= {} & L_u(t; n_0, \vec{u}_0, \vec{b}_0) + B_u(t; n_0, \vec{u}_0, \vec{b}_0) + \mathcal{N}_u(t; n, \vec{u}, \psi).
\end{aligned}
$$

Here

$$
\begin{aligned}
& L_u(t; n_0, \vec{u}_0, \vec{b}_0) \\
= {} & K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t u(0)\big] && (7.8)\\
& +(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_t u(0)\big] && (7.9)\\
& -\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)u(0)\big] + \partial_t K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)u(0)\big] && (7.10)\\
& -\frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)u_0 + K_1(t)u_0, && (7.11)
\end{aligned}
$$

and $B_u(t; n_0, \vec{u}_0, \vec{b}_0)$ is the boundary term given below. Now we give the explicit expressions of L_u and B_u respectively.

7.1.1. $L_u(t; n_0, \vec{u}_0, \vec{b}_0)$

By the equations (2.2) and (2.19) at $t = 0$, we have

$$
\begin{aligned}
& (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t u(0) \\
& = \big[(\partial_{tt} - \Delta\partial_t - \Delta)\big(\Delta u(0) + \lambda(\partial_{xx}u(0) + \partial_{xy}v(0)) - \partial_x n(0) + N_1(0)\big)\big] \\
& = (\partial_{tt} - \Delta\partial_t - \Delta)\Delta u(0) + (\partial_{tt} - \Delta\partial_t - \Delta)N_1(0) \\
& \quad - \partial_x\big[\partial_y\Delta\psi(0) + \partial_t N_0(0) - \Delta N_0(0) - \partial_x N_1(0) - \partial_y N_2(0) \\
& \quad - \lambda(\partial_x\Delta u(0) + \partial_y\Delta v(0))\big] \\
& \quad + \lambda(\partial_{tt} - \Delta\partial_t - \Delta)\big[\partial_{xx}u(0) + \partial_{xy}v(0)\big] \\
& = (\partial_{tt} - \Delta\partial_t - \Delta)\Delta u(0) + (\partial_{tt} - \Delta\partial_t - \Delta)N_1(0) - \partial_{xy}\Delta\psi(0) \\
& \quad + \big[-\partial_t\partial_x N_0 + \Delta\partial_x N_0(0) + \partial_{xx}N_1(0) + \partial_{xy}N_2(0)\big] \\
& \quad + \lambda(\partial_{tt} - \Delta\partial_t)\big[\partial_{xx}u(0) + \partial_{xy}v(0)\big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(7.8) = {} & K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\Delta u(0)\big] + K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)N_1(0)\big] - K(t)\big[\partial_{xy}\Delta\psi(0)\big] \\
& + K(t)\big[-\partial_t\partial_x N_0(0) + \partial_x\Delta N_0(0) + \partial_{xx}N_1(0) + \partial_{xy}N_2(0)\big] \\
& + \lambda K(t)\big[(\partial_{tt} - \Delta\partial_t)(\partial_{xx}u(0) + \partial_{xy}v(0))\big].
\end{aligned}
$$

Similarly, by the equations (2.2) and (2.21) at $t = 0$,

$$\begin{aligned}
(7.9) &= (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\Delta u(0) + \lambda(\partial_{xx}u(0) + \partial_{xy}v(0)) - \partial_x n(0) + N_1(0)\big] \\
&= (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\Delta u(0) - \partial_x n(0)\big] + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[N_1(0)\big] \\
&\quad + \lambda(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_{xx}u(0) + \partial_{xy}v(0)\big]; \\
(7.10) &= -\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)u(0)\big] + \partial_t K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)u(0)\big] \\
&= -\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)u(0)\big] - \partial_{yy}\partial_t K(t)[u(0)] \\
&\quad + \partial_t K(t)\big[\partial_{xy}v(0) + \partial_t N_1(0) - \partial_x N_0(0) + \lambda\partial_t(\partial_{xx}u(0) + \partial_{xy}v(0))\big].
\end{aligned}$$

Then collecting the estimates above, we have

$$\begin{aligned}
&L_u(t; n_0, \vec{u}_0, \vec{b}_0) \\
&= K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)N_1(0)\big] \\
&\quad + K(t)\big[-\partial_t\partial_x N_0(0) + \partial_x\Delta N_0(0) + \partial_{xx}N_1(0) + \partial_{xy}N_2(0)\big] \\
&\quad + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[N_1(0)\big] + \partial_t K(t)\big[\partial_t N_1(0) - \partial_x N_0(0)\big] \\
&\quad - K(t)\big[\partial_{xy}\Delta\psi(0)\big] - \partial_t K(t)[\partial_{yy}u(0) - \partial_{xy}v(0)] \\
&\quad + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\Delta u(0) - \partial_x n(0)\big] + \lambda K(t)\big[(\partial_{tt} - \Delta\partial_t)(\partial_{xx}u(0) + \partial_{xy}v(0))\big] \\
&\quad + \lambda(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_{xx}u(0) + \partial_{xy}v(0)\big] + \lambda\partial_t K(t)\big[\partial_t(\partial_{xx}u(0) + \partial_{xy}v(0))\big] \\
&\quad - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[u_0] + K_1(t)[u_0].
\end{aligned}$$

7.1.2. $B_u(t; n_0, \vec{u}_0, \vec{b}_0)$

Now we consider the boundary term $B_u(t; n_0, \vec{u}_0, \vec{b}_0)$. By integration by parts, and arguing similarly as in Section 6.1.2 we have

$$\begin{aligned}
&-\int_0^t K(t-s)\big[\partial_x(\partial_{ss} - \Delta\partial_s - \Delta)N_0(s)\big]\,ds \\
&\quad = K(t)\big[(\partial_t - \Delta)\partial_x N_0(0)\big] + \partial_t K(t)\big[\partial_x N_0(0)\big] \\
&\qquad - \int_0^t \partial_x(\partial_{ss} - \Delta\partial_s - \Delta)K(t-s)N_0(s)\,ds;
\end{aligned}$$

$$\begin{aligned}
&\int_0^t K(t-s)\big[\partial_s(\partial_{ss} - \Delta\partial_s - \Delta - \partial_{yy})N_1(s)\big]\,ds \\
&\quad = -K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})N_1(0)\big] - \partial_t K(t)\big[(\partial_t - \Delta)N_1(0)\big]
\end{aligned}$$

$$- \partial_{tt} K(t)\big[N_1(0)\big] + \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_1(s)\,ds;$$

$$\int_0^t K(t-s)\big[\partial_{xy}\partial_s N_2(s)\big]\,ds$$
$$= -K(t)\big[\partial_{xy}N_2(0)\big] + \int_0^t \partial_{xy}\partial_t K(t-s)N_2(s)\,ds;$$

$$\lambda\int_0^t K(t-s)\big[(\partial_{ss} - \Delta\partial_s - \Delta)\partial_s(\partial_{xx}u(s) + \partial_{xy}v(s))\big]\,ds$$
$$\begin{aligned}
&= -\lambda K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)(\partial_{xx}u(0) + \partial_{xy}v(0))\big]\\
&\quad - \lambda\partial_t K(t)\big[(\partial_t - \Delta)(\partial_{xx}u(0) + \partial_{xy}v(0))\big] - \lambda\partial_{tt}K(t)\big[\partial_{xx}u(0) + \partial_{xy}v(0)\big]\\
&\quad + \lambda\int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\,ds.
\end{aligned}$$

Therefore, we obtain the boundary term $B_u(t; n_0, \vec{u}_0, \vec{b}_0)$ as

$$\begin{aligned}
&B_u(t; n_0, \vec{u}_0, \vec{b}_0)\\
&= -K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})N_1(0)\big] + K(t)\big[\partial_t\partial_x N_0(0) - \partial_x\Delta N_0(0) - \partial_{xy}N_2(0)\big]\\
&\quad + \partial_t K(t)\big[\partial_x N_0(0) - \partial_t N_1(0) + \Delta N_1(0)\big] - \partial_{tt}K(t)\big[N_1(0)\big]\\
&\quad - \lambda K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)(\partial_{xx}u(0) + \partial_{xy}v(0))\big]\\
&\quad - \lambda\partial_t K(t)\big[(\partial_t - \Delta)(\partial_{xx}u(0) + \partial_{xy}v(0))\big] - \lambda\partial_{tt}K(t)\big[\partial_{xx}u(0) + \partial_{xy}v(0)\big].
\end{aligned}$$

Together with the result obtained in Section 7.1.1, we have

$$\begin{aligned}
&L_u(t; n_0, \vec{u}_0, \vec{b}_0) + B_u(t; n_0, \vec{u}_0, \vec{b}_0)\\
&= -K(t)\big[\partial_{xy}\Delta\psi(0)\big] - \partial_t K(t)[\partial_{yy}u(0) - \partial_{xy}v(0)]\\
&\quad + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\Delta u(0) - \partial_x n(0)\big]\\
&\quad - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[u_0] + K_1(t)[u_0].
\end{aligned} \tag{7.12}$$

The terms can be further simplified. Indeed, by (3.23),

$$\begin{aligned}
&(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\Delta u(0)\big] - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)u_0 + K_1(t)u_0\\
&= \Big[\frac{1}{2}\Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t) - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)\Big]\big[u(0)\big]\\
&\quad + \Big[\frac{1}{2}\Delta(\partial_{tt} - \Delta\partial_t - \Delta)K(t) + K_1(t)\Big]\big[u(0)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Delta(\partial_{tt} - \Delta\partial_t)K(t)\big[u(0)\big] - \frac{1}{2}\Delta\big(\Delta + \sqrt{\Delta\partial_{yy}}\big)K(t)\big[u(0)\big] \\
&\quad + \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[u(0)\big]. \qquad (7.13)
\end{aligned}
$$

Using (7.13) and (7.12), we have (7.6). Now we split into the following two subsection to consider the linear parts and nonlinear parts separately.

7.2. Estimates on the linear parts $L_u + B_u$

In this subsection, we prove that

Lemma 7.2.

$$
\begin{aligned}
\big\|L_u(t; n_0, \vec{u}_0, \vec{b}_0) + B_u(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2_{xy}} &\lesssim \langle t\rangle^{-\frac{1}{2}}\|U_0\|_{X_0}; \\
\big\|\langle\nabla\rangle L_u(t; n_0, \vec{u}_0, \vec{b}_0) + B_u(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^\infty_{xy}} &\lesssim \langle t\rangle^{-1}\|U_0\|_{X_0}; \\
\big\||\nabla|^\gamma L_u(t; n_0, \vec{u}_0, \vec{b}_0) + B_u(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2_{xy}} &\lesssim \langle t\rangle^{-\frac{3}{4}}\|U_0\|_{X_0}; \\
\big\|\langle\nabla\rangle\partial_x L_u(t; n_0, \vec{u}_0, \vec{b}_0) + B_u(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2_{xy}} &\lesssim \langle t\rangle^{-1}\|U_0\|_{X_0}.
\end{aligned}
$$

To prove this lemma, we will show that each term in (7.6) obeys the estimates claimed in the lemma. Since it is quite direct by the same argument in Section 6.2, we only give the sketch of proof. For example, the first term $K(t)\big[\partial_{xy}\Delta\psi_0\big]$ can be proved by using Proposition 4.2 and Proposition 4.3 as follows,

$$
\begin{aligned}
\big\|K(t)\big[\partial_{xy}\Delta\psi_0\big]\big\|_{L^2_{xy}} &= \big\|\nabla\partial_{xy}K(t)\cdot\big[\nabla\psi_0\big]\big\|_{L^2_{xy}} \\
&\lesssim \langle t\rangle^{-\frac{1}{2}}\big\|\langle\nabla\rangle^{0+}\nabla\psi_0\big\|_{L^1_{xy}}; \qquad (7.14) \\
\big\|\langle\nabla\rangle K(t)\big[\partial_{xy}\Delta\psi_0\big]\big\|_{L^\infty_{xy}} &= \big\|\nabla\partial_{xy}K(t)\cdot\big[\langle\nabla\rangle\nabla\psi_0\big]\big\|_{L^\infty_{xy}} \\
&\lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{2+}\nabla\psi_0\big\|_{L^1_{xy}}; \qquad (7.15) \\
\big\||\nabla|^\gamma K(t)\big[\partial_{xy}\Delta\psi_0\big]\big\|_{L^2_{xy}} &= \big\||\nabla|^\gamma\nabla\partial_{xy}K(t)\cdot\big[\nabla\psi_0\big]\big\|_{L^2_{xy}} \\
&\lesssim \langle t\rangle^{-\frac{3}{4}}\big\|\langle\nabla\rangle^{\gamma+}\nabla\psi_0\big\|_{L^1_{xy}}; \qquad (7.16) \\
\big\|\langle\nabla\rangle\partial_x K(t)\big[\partial_{xy}\Delta\psi_0\big]\big\|_{L^2_{xy}} &= \big\|\nabla\partial_{xx}\partial_y K(t)\cdot\big[\langle\nabla\rangle\nabla\psi_0\big]\big\|_{L^2_{xy}} \\
&\lesssim \langle t\rangle^{-1}\big\|\langle\nabla\rangle^{2+}\nabla\psi_0\big\|_{L^1_{xy}}. \qquad (7.17)
\end{aligned}
$$

Similarly, the estimates on the term $\partial_t K(t)[\partial_{yy}u_0 - \partial_{xy}v_0]$ follow from Proposition 4.5. The estimates on the term $(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_x n_0\big]$ can be obtained by using Propositions 4.9 and 4.10. The estimates on the term $\Delta(\partial_{tt} - \Delta\partial_t)K(t)\big[u_0\big]$ follow from Proposition 4.7. The estimates on the term $\Delta\big(\Delta + \sqrt{\Delta\partial_{yy}}\big)K(t)\big[u_0\big]$ can be proved by Proposition 4.4. At last, the estimates on the term $\partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[u_0\big]$ can be shown by Proposition 4.12.

7.3. The estimates on nonlinear parts $\mathcal{N}_u$

In this subsection, we establish that

$$\big\|\mathcal{N}_u(t; n, \vec{u}, \psi)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)\big); \tag{7.18}$$

$$\big\|\langle\nabla\rangle\mathcal{N}_u(t; n, \vec{u}, \psi)\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)); \tag{7.19}$$

$$\big\||\nabla|^\gamma\mathcal{N}_u(t; n, \vec{u}, \psi)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)); \tag{7.20}$$

$$\big\|\langle\nabla\rangle\partial_x\mathcal{N}_u(t; n, \vec{u}, \psi)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)). \tag{7.21}$$

By the definition of $\mathcal{N}_u(t; n, \vec{u}, \psi)$, we estimate it terms by terms in (7.7).

7.3.1. $\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds$

In this subsubsection, and we prove that

$$\Big\|\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}Q(\|U\|_X);$$

$$\Big\|\langle\nabla\rangle\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds\Big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}Q(\|U\|_X);$$

$$\Big\||\nabla|^\gamma\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}Q(\|U\|_X);$$

$$\Big\|\langle\nabla\rangle\partial_x\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}Q(\|U\|_X).$$

The treatment is similar as what in Section 6.3.2. First, we rewrite

$$\begin{aligned}
&\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds\\
&= \int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0^h(s)\,ds\\
&\quad + \int_0^t \partial_{xx}(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)(n_{\le\langle s\rangle^{0.01}}\,u_{\le\langle s\rangle^{0.01}})(s)\,ds
\end{aligned}$$

$$+\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)P_{\leq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\,v_{\leq\langle s\rangle^{0.01}})(s)\,ds$$

$$+\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)P_{\geq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\,v_{\leq\langle s\rangle^{0.01}})(s)\,ds.$$

Furthermore, by (6.38), we also split it into several parts as follows,

$$\int_0^t \partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)N_0(s)\,ds$$

$$=\int_0^t \partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)N_0^h(s)\,ds \tag{7.22}$$

$$+\int_0^t \partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)(n_{\leq\langle s\rangle^{0.01}}\,u_{\leq\langle s\rangle^{0.01}})(s)\,ds \tag{7.23}$$

$$+\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)P_{\leq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\,v_{\leq\langle s\rangle^{0.01}})(s)\,ds \tag{7.24}$$

$$+\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)P_{\geq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\,v_{\leq\langle s\rangle^{-10}})(s)\,ds \tag{7.25}$$

$$+\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)P_{\geq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\,v_{\geq\langle s\rangle^{-0.05}})(s)\,ds \tag{7.26}$$

$$+\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)$$

$$\times P_{\geq\langle s\rangle^{-0.04}}\Big(n_{\leq\langle s\rangle^{0.01}}\,P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}N_3\Big)(s)\,ds. \tag{7.27}$$

$$-\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)$$

$$\times P_{\geq\langle s\rangle^{-0.04}}\Big(\partial_s n_{\leq\langle s\rangle^{0.01}}\,P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\Big)(s)\,ds \tag{7.28}$$

$$+\int_0^t \partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)$$

$$\times \partial_s\Big(P_{\geq\langle s\rangle^{-0.04}}\big(n_{\leq\langle s\rangle^{0.01}}\,P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)(s)\Big)\,ds \tag{7.29}$$

$$+ \textit{ other easy terms.} \tag{7.30}$$

Again, the *other easy terms* is the corresponding terms from *other parts* in (6.38).

As before, the high frequency piece is standard, and thus the estimates on (7.22) are omitted here. Now we consider the term (7.23), by Proposition 4.10 (i) and interpolation estimates,

$$\begin{aligned}
\big\|(7.23)\big\|_{L^2_{xy}} &\lesssim \int_0^t \Big\|\partial_{xx}(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}})(s)\Big\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \big\|(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}})(s)\big\|_{L^1_x L^2_y}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \|n(s)\|_{L^2_{xy}} \big\|u(s)\big\|_{L^2_x L^\infty_y}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.01} \|n(s)\|_{L^2_{xy}} \big\|u(s)\big\|^{0+}_{L^2_{xy}} \big\||\nabla|^{\frac{1}{2}+} u(s)\big\|^{1-}_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.02-0.25-0.75}\, ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{1}{2}} \|U\|_X^2.
\end{aligned}$$

Similarly, by Proposition 4.10 (iii) instead,

$$\begin{aligned}
\big\|\langle\nabla\rangle(7.23)\big\|_{L^\infty_{xy}} &\lesssim \int_0^t \Big\|\partial_{xx}(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}})(s)\Big\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.03} \|(n_{\le\langle s\rangle^{0.01}}\, u_{\le\langle s\rangle^{0.01}})(s)\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.03} \|n(s)\|_{L^2_{xy}} \big\|u(s)\big\|_{L^\infty_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.03-0.25-1}\, ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-1} \|U\|_X^2,
\end{aligned}$$

and by Proposition 4.10 (ii), for any $\beta \in [\frac{1}{2}, 1]$,

$$\begin{aligned}
&\big\|\langle\nabla\rangle|\nabla|^{\beta}(7.23)\big\|_{L^2_{xy}}\\
&\lesssim \int_0^t \Big\||\nabla|^{\beta}\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ \langle\nabla\rangle(n_{\le\langle s\rangle^{0.01}}\,u_{\le\langle s\rangle^{0.01}})(s)\Big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1+\beta}{2}}\langle s\rangle^{0.02}\big\|(n_{\le\langle s\rangle^{0.01}}\,u_{\le\langle s\rangle^{0.01}})(s)\big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1+\beta}{2}}\langle s\rangle^{0.02-0.25-1}\,ds\|U\|_X^2\\
&\lesssim \langle t\rangle^{-\frac{1+\beta}{2}}\|U\|_X^2,
\end{aligned}$$

where $\beta=\gamma$ and $\beta=1$ (which turns to ∂_x(7.23)) are the estimates we want.

Now we consider (7.24). By Beinstein's inequality, and Proposition 4.9 (ii), we have

$$\begin{aligned}
&\big\||\nabla|^{\beta}(7.24)\big\|_{L^2_{xy}}\\
&\lesssim \int_0^t \big\||\nabla|^{\beta}\partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ P_{\le\langle s\rangle^{-0.04}}(n_{\le\langle s\rangle^{0.01}}\,v_{\le\langle s\rangle^{0.01}})(s)\big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle s\rangle^{-0.04\beta}\big\|\nabla\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ P_{\le\langle s\rangle^{-0.04}}(n_{\le\langle s\rangle^{0.01}}\,v_{\le\langle s\rangle^{0.01}})(s)\big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}}\langle s\rangle^{-0.04\beta}\big\|(n_{\le\langle s\rangle^{0.01}}\,v_{\le\langle s\rangle^{0.01}})(s)\big\|_{L^1_xL^2_y}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}}\langle s\rangle^{-0.04\beta}\|n(s)\|_{L^2_{xy}}\big\|v(s)\big\|_{L^2_xL^\infty_y}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}}\langle s\rangle^{-0.03\beta-1}\,ds\|U\|_X^2,
\end{aligned}$$

if $\beta=0$, we obtain

$$\big\|(7.24)\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{1}{2}}\|U\|_X^2,$$

if $\beta=\gamma$, we obtain

$$\big\||\nabla|^{\gamma}(7.24)\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{3}{4}}\|U\|_X^2.$$

By using Proposition 4.9 (iii) instead, we have

$$\begin{aligned}
&\big\|\langle\nabla\rangle(7.24)\big\|_{L^\infty_{xy}}\\
&\lesssim \int_0^t \big\|\nabla\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ P_{\leq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\ v_{\leq\langle s\rangle^{0.01}})(s)\big\|_{L^\infty_{xy}}\,ds\\
&\lesssim \int_0^t \langle s\rangle^{-0.02}\big\||\nabla|^{\frac{1}{2}}\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\\
&\quad\times P_{\leq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\ v_{\leq\langle s\rangle^{0.01}})(s)\big\|_{L^\infty_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-0.02}\big\|(n_{\leq\langle s\rangle^{0.01}}\ v_{\leq\langle s\rangle^{0.01}})(s)\big\|_{L^1_xL^2_y}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-0.02}\|n(s)\|_{L^2_{xy}}\big\|v(s)\big\|_{L^2_xL^\infty_y}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-0.01-1}\,ds\|U\|_X^2\\
&\lesssim \langle t\rangle^{-1}\|U\|_X^2.
\end{aligned}$$

By using Proposition 4.10 (ii) ($\beta=1$) instead,

$$\begin{aligned}
&\big\|\langle\nabla\rangle\partial_x(7.24)\big\|_{L^2_{xy}}\\
&\lesssim \int_0^t \big\|\nabla\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ P_{\leq\langle s\rangle^{-0.04}}(n_{\leq\langle s\rangle^{0.01}}\ v_{\leq\langle s\rangle^{0.01}})(s)\big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\|n_{\leq\langle s\rangle^{0.01}}\ v_{\leq\langle s\rangle^{0.01}}\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\|n(s)\|_{L^2_{xy}}\|v(s)\|_{L^\infty_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-0.25-1}\,ds\|U\|_X^2\\
&\lesssim \langle t\rangle^{-1}\|U\|_X^2.
\end{aligned}$$

We consider the terms (7.25)–(7.28) together. First, we show that the nonlinearities in these terms satisfy the following same estimates. Let

$$\Pi(s) \triangleq \left|(n_{\leq\langle s\rangle^{0.01}}\ v_{\leq\langle s\rangle^{-10}})(s)\right| + \left|(n_{\leq\langle s\rangle^{0.01}}\ v_{\geq\langle s\rangle^{-0.05}})(s)\right| \\ + \left|\big(n_{\leq\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}} N_3\big)(s)\right| + \left|\big(\partial_s n_{\leq\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)(s)\right|.$$

Then,

$$\big\|\Pi(s)\big\|_{L^1_x L^2_y} \lesssim \langle s\rangle^{-1.01} Q(\|U\|_X). \tag{7.31}$$

Indeed, by Sobolev' and Beinstein's inequalities, we have

$$\begin{aligned}\|(n_{\leq\langle s\rangle^{0.01}}\ v_{\leq\langle s\rangle^{-10}})\|_{L^1_x L^2_y} &\lesssim \|n\|_{L^2_{xy}}\|v_{\leq\langle s\rangle^{-10}}\|_{L^2_x L^\infty_y} \\ &\lesssim \langle s\rangle^{-4}\|n\|_{L^2_{xy}}\|v\|_{L^2_{xy}} \lesssim \langle s\rangle^{-4}\|U\|_X^2; \\ \|(n_{\leq\langle s\rangle^{0.01}}\ v_{\geq\langle s\rangle^{-0.05}})(s)\|_{L^1_x L^2_y} &\lesssim \|n\|_{L^2_{xy}}\|v_{\geq\langle s\rangle^{-0.05}}\|_{L^2_x L^\infty_y} \\ &\lesssim \langle s\rangle^{0.05}\|n\|_{L^2_{xy}}\|\nabla v\|_{L^2_{xy}} \lesssim \langle s\rangle^{-1.2}\|U\|_X^2;\end{aligned}$$

also, recall $N_3 = \vec{u}\cdot\nabla\psi$,

$$\begin{aligned}\big\|\big(n_{\leq\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}} N_3\big)(s)\big\|_{L^1_x L^2_y} &\lesssim \|n\|_{L^2_{xy}}\|\vec{u}\|_{L^2_x L^\infty_y}\|\nabla\psi\|_{L^\infty_{xy}} \\ &\lesssim \langle s\rangle^{-0.25-0.75-0.25+}\|U\|_X^3 \lesssim \langle s\rangle^{-1.2}\|U\|_X^3.\end{aligned}$$

Since $\partial_t n = -\partial_x u - \partial_y v + N_0$, we have

$$\begin{aligned}&\big\|\partial_s n_{\leq\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big\|_{L^1_x L^2_y} \\ &\quad\lesssim \big\|P_{\leq\langle s\rangle^{0.01}}\partial_s n\big\|_{L^2_{xy}}\big\|P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big\|_{L^2_x L^\infty_y} \\ &\quad\lesssim \langle s\rangle^{0.01}\big(\|\nabla\cdot\vec{u}\|_{L^2_{xy}} + \|P_{\leq\langle s\rangle^{0.01}} N_0\|_{L^2_{xy}}\big)\big\||\nabla|^{\frac12+}\psi\big\|_{L^2_{xy}} \\ &\quad\lesssim \langle s\rangle^{0.02}\big(\|\nabla\cdot\vec{u}\|_{L^2_{xy}} + \|n\|_{L^2_{xy}}\|\vec{u}\|_{L^\infty_{xy}}\big)\big\||\nabla|^{\frac12+}\psi\big\|_{L^2_{xy}} \lesssim \langle s\rangle^{-1.2}(\|U\|_X^2 + \|U\|_X^3).\end{aligned}$$

Thus we have (7.31). Now from Proposition 4.9 (ii), we have

$$\begin{aligned}&\big\|\langle\nabla\rangle\big((7.25)+\cdots+(7.28)\big)\big\|_{L^2_{xy}} \\ &\lesssim \int_0^t \big\|\nabla\partial_x(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle\Pi(s)\big\|_{L^2_{xy}}\,ds \\ &\lesssim \int_0^t \langle t-s\rangle^{-\frac34}\langle s\rangle^{0.02}\|\Pi(s)\|_{L^1_x L^2_y}\,ds\end{aligned}$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.02-1.2}\, ds Q(\|U\|_X)$$
$$\lesssim \langle t\rangle^{-\frac{3}{4}} Q(\|U\|_X).$$

This estimate concludes the estimates what we want on $\big\|(7.25)+\cdots+(7.28)\big\|_{L^2_{xy}}$ and $\big\||\nabla|^\gamma\big((7.25)+\cdots+(7.28)\big)\big\|_{L^2_{xy}}$. Similarly, by Proposition 4.9 (iii), we have

$$\begin{aligned}
&\big\|\langle\nabla\rangle\big((7.25)+\cdots+(7.28)\big)\big\|_{L^\infty_{xy}}\\
&\lesssim \int_0^t \big\|\partial_{xy}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle\Pi(s)\|_{L^\infty_{xy}}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03}\|\Pi(s)\|_{L^1_xL^2_y}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03-1.2}\, dsQ(\|U\|_X)\\
&\lesssim \langle t\rangle^{-1}Q(\|U\|_X).
\end{aligned}$$

Now by Proposition 4.10 (ii) ($\beta=1$) instead, and by Beinstein's inequality we have

$$\begin{aligned}
&\big\|\langle\nabla\rangle\partial_x\big((7.25)+\cdots+(7.28)\big)\big\|_{L^2_{xy}}\\
&\lesssim \int_0^t \big\|\nabla\partial_{xx}(\partial_{tt}-\Delta\partial_t-\Delta)K(t-s)\ P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle\Pi(s)\|_{L^2_{xy}}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.03}\|P_{\lesssim\langle s\rangle^{0.01}}\Pi(s)\|_{L^2_{xy}}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.04}\|\Pi(s)\|_{L^1_xL^2_y}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.04-1.2}\, dsQ(\|U\|_X)\\
&\lesssim \langle t\rangle^{-1}Q(\|U\|_X).
\end{aligned}$$

Similar as (6.60), it is easy to treat the term (7.30), so we omit here. At last, we consider the term (7.29). Integration by parts, we find

$$(7.29) = \partial_{xy}(\partial_{tt} - \Delta\partial_t - \Delta)K(t)P_{\sim 1}\big(P_{\lesssim 1}n_0\, P_{\sim 1}\psi_0\big) \tag{7.32}$$

$$- \int_0^t \partial_y(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)P_{\geq \langle s\rangle^{-0.04}}$$
$$\times\, \partial_x\big(n_{\leq \langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)(s)\, ds. \tag{7.33}$$

By Proposition 4.9 (ii) and (iii) and Proposition 4.10 (i), the boundary term (7.32) can be controlled as following. For $2 \leq p \leq \infty$,

$$\|\langle\nabla\rangle(7.32)\|_{L^p_{xy}} \lesssim \langle t\rangle^{-1+\frac{1}{2p}}\|\langle\nabla\rangle\big(P_{\lesssim 1}n_0\, P_{\sim 1}\psi_0\big)\|_{L^1_{xy}} \lesssim \langle t\rangle^{-1+\frac{1}{2p}}\|n_0\|_{L^2_{xy}}\|\nabla\psi_0\|_{L^2_{xy}};$$
$$\|\langle\nabla\rangle\partial_x(7.32)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\|\langle\nabla\rangle P_{\lesssim 1}n_0\, P_{\sim 1}\psi_0\|_{L^1_{xy}} \lesssim \langle t\rangle^{-1}\|n_0\|_{L^2_{xy}}\|\nabla\psi_0\|_{L^2_{xy}}.$$

The first estimate gives the desirable estimates on $\|(7.32)\|_{L^2_{xy}}$, $\||\nabla|^\gamma(7.32)\|_{L^2_{xy}}$ and $\|(7.32)\|_{L^\infty_{xy}}$.

Now we consider (7.33). By Proposition 4.12 (i) ($\beta = 2$) and (6.53), we have

$$\begin{aligned}
\big\|(7.33)\big\|_{L^2_{xy}} &\lesssim \int_0^t \|\Delta\partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)\big(n_{\leq\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)(s)\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.01}\|n_{\leq\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.01}\|n\|_{L^2_{xy}}\|P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\|_{L^\infty_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{0.01-0.25-0.27}\|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{1}{2}}\|U\|_X^2.
\end{aligned}$$

Moreover, since $\gamma > \frac{1}{2}$, by Proposition 4.13 and (6.53), we have

$$\begin{aligned}
\big\||\nabla|^\gamma(7.33)\big\|_{L^2_{xy}} &\lesssim \int_0^t \||\nabla|^{\gamma+1}\partial_t\big(\partial_{tt} - \Delta\partial_t - \Delta\big)K(t-s) \\
&\quad \times \partial_x\big(n_{\leq\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)(s)\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1+\gamma}{2}}\|\partial_x\big(n_{\leq\langle s\rangle^{0.01}}\, P_{\langle s\rangle^{-10}\leq\cdot\leq\langle s\rangle^{-0.05}}\psi\big)\|_{L^2_{xy}}\, ds
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1+\gamma}{2}} \big(\|\partial_x n\|_{L^2_{xy}} \|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\|_{L^\infty_{xy}} \\
&\quad + \|n\|_{L^\infty_{xy}} \|\partial_x \psi\|_{L^2_{xy}} \big)\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1+\gamma}{2}} \langle s\rangle^{-1}\, ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{3}{4}} \|U\|_X^2.
\end{aligned}
$$

For $\big\|\langle\nabla\rangle(7.33)\big\|_{L^\infty_{xy}}$, we need some special treatment (which will be used frequently below). By Beinstein's inequality and Proposition 4.12 (iv), we have

$$
\begin{aligned}
\big\|\langle\nabla\rangle(7.33)\big\|_{L^\infty_{xy}} &\lesssim \int_0^t \big\|P_{\ge\langle s\rangle^{-0.04}} \partial_y \partial_t \big(\partial_{tt} - \Delta\partial_t - \Delta\big) \\
&\quad \times K(t-s)\ \partial_x \langle\nabla\rangle \big(n_{\le\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)(s)\big\|_{L^\infty_{xy}} ds \\
&\lesssim \int_0^t \big\|\Delta\partial_t \big(\partial_{tt} - \Delta\partial_t - \Delta\big) K(t-s)\ P_{\ge\langle s\rangle^{-0.04}} \frac{\partial_y\langle\nabla\rangle}{-\Delta} \\
&\quad \times \partial_x \big(n_{\le\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)(s)\big\|_{L^\infty_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \Big\|P_{\ge\langle s\rangle^{-0.04}} \frac{\partial_y\langle\nabla\rangle}{-\Delta} \partial_x \big(n_{\langle s\rangle^{-0.04}\lesssim\cdot\le\langle s\rangle^{0.01}} \\
&\quad \times P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)\Big\|_{L^\infty_{xy}} ds.
\end{aligned}
$$

Now we claim that

$$
\begin{aligned}
&\Big\|P_{\ge\langle s\rangle^{-0.04}} \frac{\partial_y\langle\nabla\rangle}{-\Delta} \partial_x \big(n_{\langle s\rangle^{-0.04}\lesssim\cdot\le\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)\Big\|_{L^\infty_{xy}} \\
&\quad \lesssim \langle t\rangle^{-1.01} \|U\|_X^2.
\end{aligned}
\tag{7.34}
$$

Indeed, by Littlewood–Paley's decomposition and (6.53),

$$
\begin{aligned}
&\Big\|P_{\ge\langle s\rangle^{-0.04}} \frac{\partial_y\langle\nabla\rangle}{-\Delta} \partial_x \big(n_{\langle s\rangle^{-0.04}\lesssim\cdot\le\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)\Big\|_{L^\infty_{xy}} \\
\lesssim &\sum_{\langle s\rangle^{-0.04}\le N\le 1} N^{-1} \big\|\partial_x \big(n_N\ P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)\big\|_{L^\infty_{xy}} \\
&+ \big\|P_{\ge 1} \langle\nabla\rangle^{0+} \partial_x \big(n_{1\le\cdot\le\langle s\rangle^{0.01}}\ P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\big)\big\|_{L^\infty_{xy}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{\langle s\rangle^{-0.04}\le N\le 1} N^{-1}\big(\|\partial_x n_N\|_{L^\infty_{xy}}\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\|_{L^\infty_{xy}} \\
&\quad + \|n_N\|_{L^\infty_{xy}}\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\partial_x\psi\|_{L^\infty_{xy}}\big) \\
&\quad + \big(\|\langle\nabla\rangle^{0+}\partial_x n_{\le\langle s\rangle^{0.01}}\|_{L^\infty_{xy}}\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\|_{L^\infty_{xy}} \\
&\quad + \|\langle\nabla\rangle^{0+} n_{\le\langle s\rangle^{0.01}}\|_{L^\infty_{xy}}\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\partial_x\psi\|_{L^\infty_{xy}}\big) \\
&\lesssim \sum_{\langle s\rangle^{-0.04}\le N\le 1} N^{0-}\|\partial_x n\|_{L^2_{xy}}\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\|_{L^\infty_{xy}} \\
&\quad + N^{-1}\|n\|_{L^\infty_{xy}}\|\partial_x\psi\|_{L^2_{xy}}^{\frac12+}\|\nabla\partial_x\psi\|_{L^2_{xy}}^{\frac12-}\big) \\
&\quad + \langle s\rangle^{0+}\big(\|\langle\nabla\rangle\partial_x n\|_{L^2_{xy}}\|P_{\langle s\rangle^{-10}\le\cdot\le\langle s\rangle^{-0.05}}\psi\|_{L^\infty_{xy}} + \langle s\rangle^{-0.02}\|n\|_{L^\infty_{xy}}\|\partial_x\psi\|_{L^2_{xy}}\big) \\
&\lesssim \langle t\rangle^{-1.01}\|U\|_X^2.
\end{aligned}
$$

This gives (7.34). Using (7.34), we further obtain that

$$
\begin{aligned}
\big\|\langle\nabla\rangle(7.33)\big\|_{L^\infty_{xy}} &\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-1.01}\,ds\|U\|_X^2 \\
&\lesssim \langle t\rangle^{-1}\|U\|_X^2.
\end{aligned}
$$

For $\big\|\langle\nabla\rangle\partial_x(7.33)\big\|_{L^2_{xy}}$, similarly, by Proposition 4.13 instead and the same treatment to $\big\|\langle\nabla\rangle(7.33)\big\|_{L^\infty_{xy}}$, we also have

$$
\big\|\langle\nabla\rangle\partial_x(7.33)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\|U\|_X^2.
$$

Collecting the estimates obtained above yields the claimed estimates in this subsubsection.

7.3.2. $\int_0^t \partial_t(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t-s)N_1(s)\,ds$

This is the most troubled term in this section. Indeed, we take one of the terms in N_1, $\partial_x\psi\Delta\psi$ for example. We have the estimate

$$
\|\partial_x\psi\Delta\psi\|_{L^1_{xy}} \le \|\partial_x\psi\|_{L^2_{xy}}\|\Delta\psi\|_{L^2_{xy}} \lesssim \langle s\rangle^{-\frac34}\|U\|_X^2.
$$

But this is far from integrable, even weaker than the term in 6.3.2. Therefore, some special techniques which heavily depends on the structure and the Fourier analysis, shall be employed to deal with it. The main estimates we state in this subsubsection are the following

$$
\Big\|\int_0^t \partial_t(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t-s)N_1(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac12}Q(\|U\|_X);
$$

$$\big\|\langle\nabla\rangle \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_1(s)\,ds\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}Q(\|U\|_X);$$

$$\big\||\nabla|^\gamma \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_1(s)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}Q(\|U\|_X);$$

$$\big\|\langle\nabla\rangle\partial_x \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_1(s)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}Q(\|U\|_X).$$

According to the frequency, we also split N_1 into N_1^l and N_2^h pieces. The estimates on the high frequency piece N_1^h is standard and we omit the details. Now we continue to estimate the low frequency piece N_1^l. However, one may find that direct estimate fails. To overcome the difficulties, firstly, by using the first equation in (2.2), we rewrite N_1 as

$$\begin{aligned}
N_1 &= -(u\partial_x u + v\partial_y u) - n\big[\Delta u + \lambda(\partial_{xx}u + \partial_{xy}v) - \partial_x n\big] - 2n\partial_x n - \partial_x\psi\Delta\psi \\
&\quad + \frac{n^2\Delta u + n^2\lambda(\partial_{xx}u + \partial_{xy}v)}{\rho} + \frac{n\partial_x\psi\Delta\psi}{\rho} \\
&= -n\big(\partial_t u - N_1\big) - \nabla\cdot(\partial_x\psi\nabla\psi) + \partial_x\big(\frac{1}{2}|\nabla\psi|^2 - n^2\big) \\
&\quad - (u\partial_x u + v\partial_y u) + \frac{n^2\Delta u + n^2\lambda(\partial_{xx}u + \partial_{xy}v)}{\rho} + \frac{n\partial_x\psi\Delta\psi}{\rho} \\
&= -\partial_t(nu) + \partial_t n\, u - \nabla\cdot(\partial_x\psi\nabla\psi) + \partial_x\big(\frac{1}{2}|\nabla\psi|^2 - n^2\big) \\
&\quad + nN_1 - (u\partial_x u + v\partial_y u) + \frac{n^2\Delta u + n^2\lambda(\partial_{xx}u + \partial_{xy}v)}{\rho} + \frac{n\partial_x\psi\Delta\psi}{\rho}.
\end{aligned}$$

Let

$$\begin{aligned}
&N_{13} = \partial_t n\, u + nN_1 - (u\partial_x u + v\partial_y u) + \frac{n^2\Delta u + n^2\lambda(\partial_{xx}u + \partial_{xy}v)}{\rho} + \frac{n\partial_x\psi\Delta\psi}{\rho}; \\
&N_{14} = -\partial_t(nu); \quad N_{15} = -\nabla\cdot(\partial_x\psi\nabla\psi); \quad N_{16} = \partial_x\big(\frac{1}{2}|\nabla\psi|^2 - n^2\big),
\end{aligned} \tag{7.35}$$

and write $N_1^l = N_{13}^l + \cdots + N_{16}^l$. Similar as Lemma 6.5, we have

$$\|N_{13}^l\|_{L^1_{xy}} \lesssim \langle s\rangle^{-1.2}Q(\|U\|_X). \tag{7.36}$$

Remark 7.3. Note that

$$\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy} = \big(\partial_{tt} - \Delta\partial_t - \Delta\big) - \partial_{yy},$$

and $\partial_{tt} + A^2\partial_t + A^2$ has almost the same estimates as $A|\eta|$, so one may regard $\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}$ as $\partial_{tt} - \Delta\partial_t - \Delta$ with no differences. Thus the linear estimates concerned

$\partial_{tt} - \Delta\partial_t - \Delta$ also hold for $\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}$. Accordingly, we will not mention the difference below.

Hence by Proposition 4.12 (i) ($\beta = 0$) and (ii), and (7.36), for $p = 2, \infty$ we have

$$\begin{aligned}
&\Big\|\langle\nabla\rangle \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_{13}^l(s)\,ds\Big\|_{L^p_{xy}} \\
&\lesssim \int_0^t \big\|\partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)\langle\nabla\rangle N_{13}^l(s)\big\|_{L^p_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1+\frac{1}{p}} \|\langle\nabla\rangle N_{13}^l\|_{L^1_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1+\frac{1}{p}} \langle s\rangle^{-1.2+0.01}\,ds Q(\|U\|_X) \\
&\lesssim \langle t\rangle^{-1+\frac{1}{p}} Q(\|U\|_X).
\end{aligned}$$

By using Proposition 4.12 (i) ($\beta = \gamma, 1$) instead, we also get the following estimates,

$$\Big\||\nabla|^\gamma \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_{13}^l(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} Q(\|U\|_X);$$

$$\Big\|\langle\nabla\rangle\partial_x \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_{13}^l(s)\,ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1} Q(\|U\|_X).$$

For the piece N_{14}, we integrate by parts, to get

$$\begin{aligned}
&-\int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)\partial_s(nu)(s)\,ds \\
&= \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t)\big(n_0u_0\big) \qquad (7.37) \\
&\quad - \int_0^t \partial_{tt}(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)(nu)(s)\,ds. \qquad (7.38)
\end{aligned}$$

For (7.37), as in Proposition 4.4, we have

$$\big\|(7.37)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}} Q(\|U_0\|_{X_0}); \qquad \big\|\langle\nabla\rangle(7.37)\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1} Q(\|U_0\|_{X_0});$$

$$\big\||\nabla|^\gamma(7.37)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} Q(\|U_0\|_{X_0}); \qquad \big\|\langle\nabla\rangle\partial_x(7.37)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1} Q(\|U_0\|_{X_0}).$$

For (7.38), we find

$$\|(nu)(s)\|_{L^2_{xy}} \lesssim \|n(s)\|_{L^2_{xy}}\|u(s)\|_{L^\infty_{xy}} \lesssim \langle s\rangle^{-1.25}Q(\|U\|_X). \tag{7.39}$$

Now by (7.39) and Proposition 4.14 (i), for $\beta = 0, \gamma, 1$, we have

$$\begin{aligned}
&\big\|\langle\nabla\rangle|\nabla|^\beta(7.38)\big\|_{L^2_{xy}}\\
&\lesssim \int_0^t \|\langle\nabla\rangle|\nabla|^\beta \partial_{tt}(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)(nu)(s)\|_{L^2_{xy}}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1}{2}-\frac{\beta}{2}}\|(nu)(s)\|_{L^2_{xy}}\, ds\\
&\lesssim \langle t\rangle^{-\frac{1}{2}-\frac{\beta}{2}}Q(\|U\|_X).
\end{aligned}$$

This gives the desired estimates of $\big\|(7.38)\big\|_{L^2_{xy}}$, $\big\||\nabla|^\gamma(7.38)\big\|_{L^2_{xy}}$, $\big\|\langle\nabla\rangle\partial_x(7.38)\big\|_{L^2_{xy}}$. Furthermore, from Proposition 4.14 (ii),

$$\begin{aligned}
\big\|\langle\nabla\rangle(7.38)\big\|_{L^\infty_{xy}} &\lesssim \int_0^t \big\|\partial_{tt}(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)\langle\nabla\rangle(nu)(s)\|_{L^2_{xy}}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\big\|P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle(nu)(s)\big\|_{L^2_{xy}}\, ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-1.25+0.02}\, dsQ(\|U\|_X)\\
&\lesssim \langle t\rangle^{-1}Q(\|U\|_X).
\end{aligned}$$

Thus we finish the estimates on the piece N_{14}.

Now we continue to consider the piece N_{15}. First we write

$$\begin{aligned}
\partial_x\psi\nabla\psi = {}& \partial_x\psi_{\geq\langle s\rangle^{-0.05}}\nabla\psi + \partial_x\psi_{\leq\langle s\rangle^{-0.05}}\nabla\psi_{\leq\langle s\rangle^{-0.04}}\\
&+ P_{\gtrsim\langle s\rangle^{-0.04}}\big(\partial_x\psi_{\leq\langle s\rangle^{-0.05}}\nabla\psi_{\geq\langle s\rangle^{-0.04}}\big),
\end{aligned}$$

and thus

$$\int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)N_{15}(s)\,ds$$

$$= \int_0^t \partial_t\nabla(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)$$

$$\cdot \Big(\partial_x\psi_{\geq\langle s\rangle^{-0.05}}\,\nabla\psi + \partial_x\psi_{\leq\langle s\rangle^{-0.05}}\,\nabla\psi_{\leq\langle s\rangle^{-0.04}}\Big)\,ds \tag{7.40}$$

$$+ \int_0^t \partial_t\nabla(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)$$

$$\cdot P_{\gtrsim\langle s\rangle^{-0.04}}\big(\partial_x\psi_{\leq\langle s\rangle^{-0.05}}\,\nabla\psi_{\geq\langle s\rangle^{-0.04}}\big)(s)\,ds. \tag{7.41}$$

Since the first two parts have the estimates

$$\begin{aligned}
&\big\|\partial_x\psi_{\geq\langle s\rangle^{-0.05}}\,\nabla\psi\big\|_{L^2_{xy}}\\
&\lesssim \big\|\partial_x\psi_{\geq\langle s\rangle^{-0.05}}\big\|_{L^2_{xy}}\big\|\nabla\psi\big\|_{L^\infty_{xy}} \lesssim \langle s\rangle^{0.05}\big\|\partial_x\nabla\psi\big\|_{L^2_{xy}}\big\|\nabla\psi\big\|_{L^\infty_{xy}}\\
&\lesssim \langle s\rangle^{0.05-0.75-0.5}\|U\|_X^2 \lesssim \langle s\rangle^{-1.2}\|U\|_X^2;\\
&\big\|\partial_x\psi_{\leq\langle s\rangle^{-0.05}}\,\nabla\psi_{\leq\langle s\rangle^{-0.04}}\big\|_{L^2_{xy}}\\
&\lesssim \big\|\partial_x\psi\big\|_{L^2_{xy}}\big\|\nabla\psi_{\leq\langle s\rangle^{-0.04}}\big\|_{L^\infty_{xy}} \lesssim \langle s\rangle^{-0.03+}\big\|\partial_x\psi\big\|_{L^2_{xy}}\big\||\nabla|^{\frac14+}\psi\big\|_{L^\infty_{xy}}\\
&\lesssim \langle s\rangle^{-0.03-0.5-0.5+}\|U\|_X^2 \lesssim \langle s\rangle^{-1.02}\|U\|_X^2,
\end{aligned}$$

we can treat them together. Therefore, by Proposition 4.12 (i) ($\beta = 1$) and (iii), for $p = 2, \infty$, we have

$$\begin{aligned}
&\big\|\langle\nabla\rangle(7.40)\big\|_{L^p_{xy}}\\
&\lesssim \int_0^t \Big\|\partial_t\nabla(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy})K(t-s)\\
&\quad\cdot\langle\nabla\rangle\Big(\partial_x\psi_{\geq\langle s\rangle^{-0.05}}\,\nabla\psi + \partial_x\psi_{\leq\langle s\rangle^{-0.05}}\,\nabla\psi_{\leq\langle s\rangle^{-0.04}}\Big)(s)\Big\|_{L^p_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1+\frac1p}\Big\|\langle\nabla\rangle^{1+}\big(\partial_x\psi_{\geq\langle s\rangle^{-0.05}}\,\nabla\psi + \partial_x\psi_{\leq\langle s\rangle^{-0.05}}\,\nabla\psi_{\leq\langle s\rangle^{-0.04}}\big)\Big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-1+\frac1p}\langle s\rangle^{0.01-1.02+}\|U\|_X^2\,ds\\
&\lesssim \langle t\rangle^{-1+\frac1p}\|U\|_X^2.
\end{aligned}$$

By the same way, and using Proposition 4.12 (i) ($\beta = 1+\gamma, 2$) instead, we also get

$$\big\||\nabla|^\gamma(7.40)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2; \quad \big\|\langle\nabla\rangle\partial_x(7.40)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\|U\|_X^2.$$

For (7.41), we use the argument to treat $\big\|\langle\nabla\rangle(7.33)\big\|_{L^\infty_{xy}}$ before. Indeed, by Proposition 4.12 (i) ($\beta = 2$) and (iv),

$$\begin{aligned}\big\|\langle\nabla\rangle(7.41)\big\|_{L^p_{xy}} &\lesssim \int_0^t \big\|\Delta\partial_t(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t-s)\\
&\qquad P_{\gtrsim\langle s\rangle^{-0.04}}\frac{\langle\nabla\rangle\nabla}{-\Delta}\cdot\big(\partial_x\psi_{\le\langle s\rangle^{-0.05}}\,\nabla\psi_{\ge\langle s\rangle^{-0.04}}\big)(s)\big\|_{L^p_{xy}}\\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\big\|P_{\gtrsim\langle s\rangle^{-0.04}}\frac{\langle\nabla\rangle\nabla}{-\Delta}\cdot\big(\partial_x\psi_{\le\langle s\rangle^{-0.05}}\,\nabla\psi_{\ge\langle s\rangle^{-0.04}}\big)(s)\big\|_{L^p_{xy}}.\end{aligned}$$

Similar as (7.34), we have

$$\begin{aligned}&\big\|P_{\gtrsim\langle s\rangle^{-0.04}}\frac{\langle\nabla\rangle\nabla}{-\Delta}\cdot\big(\partial_x\psi_{\le\langle s\rangle^{-0.05}}\,\nabla\psi_{\ge\langle s\rangle^{-0.04}}\big)(s)\big\|_{L^p_{xy}}\,ds\\
&\quad\lesssim \langle s\rangle^{0.05}\big\|\big(\partial_x\psi_{\le\langle s\rangle^{-0.05}}\,\nabla\psi_{\ge\langle s\rangle^{-0.04}}\big)(s)\big\|_{L^p_{xy}}\,ds,\end{aligned}$$

and thus it is less than $\langle s\rangle^{-0.95}\|U\|_X^2$ when $p=2$; less than $\langle s\rangle^{-1.05}\|U\|_X^2$ when $p=\infty$. Hence, we obtain that

$$\begin{aligned}&\big\|(7.41)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\|U\|_X^2; \quad \big\||\nabla|^\gamma(7.41)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2;\\
&\big\|\langle\nabla\rangle(7.41)\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}\|U\|_X^2.\end{aligned}$$

For $\big\|\langle\nabla\rangle\partial_x(7.41)\big\|_{L^2_{xy}}$, we use Proposition 4.13 instead, to get

$$\big\|\langle\nabla\rangle\partial_x(7.41)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\|U\|_X^2.$$

Thus we finish the estimates on the piece N_{15}.

Now we consider the estimates on the piece N_{16}. To do this, we swap the places of the operators ∂_t and ∂_x via integration by parts, and get

$$\begin{aligned}&\int_0^t \partial_t(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t-s)\partial_x\big(\frac{1}{2}|\nabla\psi|^2-n^2\big)(s)\,ds\\
&= -\partial_x(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t)\big(\frac{1}{2}|\nabla\psi_0|^2-n_0^2\big) \qquad (7.42)\end{aligned}$$

$$+\int_0^t \partial_x(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t-s)\partial_s\big(\frac{1}{2}|\nabla\psi|^2-n^2\big)(s)\,ds. \tag{7.43}$$

For (7.42), as in Proposition 4.4, we have

$$\big\|(7.42)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}Q(\|U_0\|_{X_0}); \qquad \big\|(7.42)\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}Q(\|U_0\|_{X_0});$$
$$\big\||\nabla|^\gamma(7.42)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}Q(\|U_0\|_{X_0}); \qquad \big\|\partial_x(7.42)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}Q(\|U_0\|_{X_0}).$$

For (7.43), by the first and fourth equations in (2.2), we have

$$\begin{aligned}\partial_t\big(\frac{1}{2}|\nabla\psi|^2-n^2\big) &= \nabla\psi\cdot(-\nabla v+\nabla N_3)-2n(-\nabla\cdot\vec{u}+N_0)\\ &= -\nabla\psi\cdot\nabla v+2n\nabla\cdot\vec{u}+\nabla\psi\cdot\nabla N_3-2nN_0.\end{aligned}$$

Thus similar as Lemma 6.5, and using interpolation, we have

$$\|\langle\nabla\rangle^3\partial_s\big(\frac{1}{2}|\nabla\psi|^2-n^2\big)\|_{L^1_{xy}} \lesssim \langle s\rangle^{-1.1}Q(\|U\|_X). \tag{7.44}$$

Then by Proposition 4.9 (i), we have

$$\begin{aligned}\big\|\langle\nabla\rangle(7.43)\big\|_{L^p_{xy}} &\lesssim \int_0^t \Big\|\partial_x(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t-s)\ \langle\nabla\rangle\partial_s\big(\frac{1}{2}|\nabla\psi|^2-n^2\big)(s)\Big\|_{L^p_{xy}}\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-1+\frac{1}{p}}\big\|\langle\nabla\rangle^3\partial_s\big(\frac{1}{2}|\nabla\psi|^2-n^2\big)\big\|_{L^1_{xy}}\,ds\\ &\lesssim \int_0^t \langle t-s\rangle^{-1+\frac{1}{p}}\langle s\rangle^{-1.1}\,ds\|U\|_X^2\\ &\lesssim \langle t\rangle^{-1+\frac{1}{p}}\|U\|_X^2.\end{aligned}$$

By using Proposition 4.9 (iii) ($\beta=\gamma$) instead, we obtain

$$\big\||\nabla|^\gamma(7.43)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2.$$

While by using Proposition 4.10 (i) instead, we obtain

$$\big\|\langle\nabla\rangle\partial_x(7.43)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\|U\|_X^2.$$

Thus we finish the estimates on the piece N_{16}. Now collecting the estimates obtained above, we finish the proof of the claimed estimates in this subsubsection.

7.3.3. $\int_0^t \partial_{xy}\partial_t K(t-s)N_2(s)\,ds$

This term is in the same level as $\int_0^t \partial_t(\partial_{tt}-\Delta\partial_t-\Delta-\partial_{yy})K(t-s)N_1(s)\,ds$, so we just give the sketch of proof. Indeed, similar as above, we rewrite N_2 as

$$\begin{aligned}
N_2 &= -(u\partial_x v + v\partial_y v) - n\big[\Delta v + \lambda(\partial_{xy}u + \partial_{yy}v) - \Delta\psi - \partial_y n\big] - 2n\partial_y n - \partial_y\psi\Delta\psi \\
&\quad + \frac{n^2\Delta v + n^2\lambda(\partial_{xy}u + \partial_{yy}v) - n^2\Delta\psi}{\rho} + \frac{n\partial_y\psi\Delta\psi}{\rho} \\
&= -n\big(\partial_t v - N_2\big) - \nabla\cdot(\partial_y\psi\nabla\psi) + \partial_y\big(\frac{1}{2}|\nabla\psi|^2 - n^2\big) \\
&\quad - (u\partial_x v + v\partial_y v) + \frac{n^2\Delta v + n^2\lambda(\partial_{xy}u + \partial_{yy}v) - n^2\Delta\psi}{\rho} + \frac{n\partial_y\psi\Delta\psi}{\rho} \\
&= -\partial_t(nv) + \partial_t n\, v - \nabla\cdot(\partial_y\psi\nabla\psi) + \partial_y\big(\frac{1}{2}|\nabla\psi|^2 - n^2\big) \\
&\quad + nN_2 - (u\partial_x v + v\partial_y v) + \frac{n^2\Delta v + n^2\lambda(\partial_{xy}u + \partial_{yy}v) - n^2\Delta\psi}{\rho} + \frac{n\partial_y\psi\Delta\psi}{\rho}.
\end{aligned}$$

Let

$$\begin{aligned}
&N_{23} = \partial_t n\, v + nN_2 - (u\partial_x v + v\partial_y v) + \frac{n^2\Delta v + n^2\lambda(\partial_{xy}u + \partial_{yy}v) - n^2\Delta\psi}{\rho} + \frac{n\partial_y\psi\Delta\psi}{\rho}; \\
&N_{24} = -\partial_t(nv); \quad N_{25} = -\nabla\cdot(\partial_y\psi\nabla\psi) + \partial_y\big(\frac{1}{2}|\nabla\psi|^2 - n^2\big). \qquad (7.45)
\end{aligned}$$

By the same arguments to treat the piece N_{13} and N_{14} respectively as above, we obtain the disable estimates on N_{23} and N_{24}. Moreover, the piece N_{25} is similar as N_{16}. Indeed, we change the places between ∂_x and ∇, then

$$\begin{aligned}
&\int_0^t \partial_{xy}\partial_t K(t-s)N_{25}(s)\,ds \\
&= \int_0^t \partial_y\nabla\cdot\partial_t K(t-s)\partial_x(\partial_y\psi\nabla\psi)(s)\,ds + \int_0^t \partial_{yy}\partial_t K(t-s)\partial_x\big(\frac{1}{2}|\nabla\psi|^2 - n^2\big)(s)\,ds.
\end{aligned}$$

Since the operator $\partial_{xy}\nabla K(t)$ obeys the same decaying estimates as $\partial_x(\partial_t - \Delta\partial_t - \Delta - \partial_{yy})\nabla K(t)$, we can obtain the disable estimates by the way to treat N_{16}. Therefore, we get that

$$\begin{aligned}
\Big\|\int_0^t \partial_{xy}\partial_t K(t-s)N_2(s)\,ds\Big\|_{L^2_{xy}} &\lesssim \langle t\rangle^{-\frac{1}{2}}Q(\|U\|_X); \\
\Big\|\langle\nabla\rangle\int_0^t \partial_{xy}\partial_t K(t-s)N_2(s)\,ds\Big\|_{L^\infty_{xy}} &\lesssim \langle t\rangle^{-1}Q(\|U\|_X);
\end{aligned}$$

$$\big\||\nabla|^\gamma \int_0^t \partial_{xy}\partial_t K(t-s)N_2(s)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} Q(\|U\|_X);$$

$$\big\|\langle\nabla\rangle\partial_x \int_0^t \partial_{xy}\partial_t K(t-s)N_2(s)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1} Q(\|U\|_X).$$

7.3.4. $\int_0^t \partial_{xy}\Delta K(t-s)N_3(s)\,ds$

The same reason as in Section 6.3.5, we find

$$\partial_{xy}\nabla \sim \partial_x(\partial_t - \Delta\partial_t - \Delta), \quad \text{ and } \nabla N_3 \sim N_0.$$

Therefore, the term $\int_0^t \partial_{xy}\Delta K(t-s)N_3(s)\,ds$ can be treated by the similar way as $\int_0^t \partial_x(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_0(s)\,ds$, and thus we get the estimates (the details are omitted here)

$$\big\|\int_0^t \partial_{xy}\Delta K(t-s)N_3(s)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}} Q(\|U\|_X);$$

$$\big\|\langle\nabla\rangle \int_0^t \partial_{xy}\Delta K(t-s)N_3(s)\,ds\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1} Q(\|U\|_X);$$

$$\big\||\nabla|^\gamma \int_0^t \partial_{xy}\Delta K(t-s)N_3(s)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} Q(\|U\|_X);$$

$$\big\|\langle\nabla\rangle\partial_x \int_0^t \partial_{xy}\Delta K(t-s)N_3(s)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1} Q(\|U\|_X).$$

7.3.5. $\lambda\int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)(\partial_{xx}u(s) + \partial_{xy}v(s))\,ds$

We will prove in this subsubsection that

$$\big\|\lambda \int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}} |\lambda|\, \|U\|_X;$$

$$\big\|\langle\nabla\rangle\lambda \int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\,ds\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1} |\lambda|\, \|U\|_X;$$

$$\big\||\nabla|^\gamma\lambda \int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\,ds\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}} |\lambda|\, \|U\|_X;$$

$$\Big\|\langle\nabla\rangle\partial_x\lambda\int_0^t(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t K(t-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds\Big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-1}|\lambda|\,\|U\|_X.$$

To do this, we first write

$$\begin{aligned}&\lambda\int_0^t(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t K(t-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds\\ &=\lambda\int_0^{t/2}(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t K(t-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds \qquad (7.46)\\ &+\lambda\int_{t/2}^t(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t K(t-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds. \qquad (7.47)\end{aligned}$$

By Proposition 4.12 (i) ($\beta=2$), we have

$$\begin{aligned}\big\|(7.46)\big\|_{L^2_{xy}}&\lesssim|\lambda|\int_0^{t/2}\big\|\Delta(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t K(t-s)\vec{u}(s)\big\|_{L^2_{xy}}\,ds\\ &\lesssim|\lambda|\int_0^{t/2}\langle t-s\rangle^{-1}\big\|\vec{u}(s)\big\|_{L^2_{xy}}\,ds\lesssim|\lambda|\int_0^{t/2}\langle t-s\rangle^{-1}\langle s\rangle^{-\frac12}\,ds\|U\|_X\\ &\lesssim|\lambda|\langle t\rangle^{-\frac12}\|U\|_X.\end{aligned}$$

By using Proposition 4.12 (i) ($\beta=2$), (iv) and (iv) respectively, we also get

$$\begin{aligned}\big\||\nabla|^\gamma(7.46)\big\|_{L^2_{xy}}&\lesssim|\lambda|\langle t\rangle^{-\frac34}\|U\|_X;\\ \big\|\langle\nabla\rangle(7.46)\big\|_{L^\infty_{xy}}&\lesssim|\lambda|\langle t\rangle^{-1}\|U\|_X;\\ \big\|\langle\nabla\rangle\partial_x(7.46)\big\|_{L^2_{xy}}&\lesssim|\lambda|\langle t\rangle^{-1}\|U\|_X.\end{aligned}$$

While by Proposition 4.12 (i) ($\beta=1$),

$$\begin{aligned}\big\|(7.47)\big\|_{L^2_{xy}}&\lesssim|\lambda|\int_{t/2}^t\big\|\nabla(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t K(t-s)\ \nabla\cdot\vec{u}(s)\big\|_{L^2_{xy}}\,ds\\ &\lesssim|\lambda|\int_{t/2}^t\langle t-s\rangle^{-\frac12}\langle s\rangle^{-1}\,ds\|U\|_X\\ &\lesssim|\lambda|\langle t\rangle^{-\frac12}\|U\|_X.\end{aligned}$$

Thus, we give that

$$|\lambda| \Big\| \int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\, ds \Big\|_{L^2_{xy}} \lesssim |\lambda| \langle t\rangle^{-\frac{1}{2}} \|U\|_X.$$

Similar argument, we obtain that

$$|\lambda| \Big\| |\nabla|^\gamma \int_0^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\, ds \Big\|_{L^2_{xy}} \lesssim |\lambda| \langle t\rangle^{-\frac{3}{4}} \|U\|_X.$$

However, to estimate $\big\|\langle\nabla\rangle(7.47)\big\|_{L^\infty_{xy}}$ and $\big\|\partial_x(7.47)\big\|_{L^2_{xy}}$, they are much difficult. The reason is that we do not have

$$\int_{t/2}^t \langle t-s\rangle^{-1}\langle s\rangle^{-1}\, ds \|U\|_X \le C\langle t\rangle^{-1}\|U\|_X, \tag{7.48}$$

due to the unboundedness of the integral on the left-hand side as $t \to \infty$. It is also worthing to note that $\int_{t/2}^t \langle t-s\rangle^{-1-\delta}\langle s\rangle^{-1}\, ds\|U\|_X$ is bounded by the right-hand side for any positive δ. However, we have such a δ-loss in (7.48). To overcome the difficulties, we split the operator $K(t)$ into two parts,

$$K(t) = K_a(t) + K_b(t),$$

where K_a, K_b are defined by

$$\widehat{K_a}(t,\xi,\eta) = \chi_{|\xi|\le A^2}\widehat{K}(t,\xi,\eta); \quad \widehat{K_b}(t,\xi,\eta) = \chi_{|\xi|\ge A^2}\widehat{K}(t,\xi,\eta).$$

Therefore,

$$(7.47) = \lambda \int_{t/2}^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K_a(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\, ds \tag{7.49}$$

$$+\lambda \int_{t/2}^t (\partial_{tt} - \Delta\partial_t - \Delta)\partial_t K_b(t-s)\big(\partial_{xx}u(s) + \partial_{xy}v(s)\big)\, ds. \tag{7.50}$$

In the following, we only consider $\big\|\langle\nabla\rangle\partial_x(7.47)\big\|_{L^2_{xy}}$. The estimate $\big\|\langle\nabla\rangle(7.47)\big\|_{L^\infty_{xy}}$ can be treated by the same way, or one may get it by interpolation directly.

By (3.8), we have

$$
\begin{aligned}
&|A\xi\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K_a}(t,\xi,\eta)| \\
&\lesssim \chi_{|\xi|\le A^2}\big(\chi_{A\ge 1}e^{-ct}+A|\xi|\chi_{A\le 1}e^{-\frac{1}{4}A^2t}+\frac{\xi^3}{A^3}e^{-\frac{\xi^2}{2A^2}t}\big) \\
&\lesssim \chi_{A\ge 1}\frac{1}{A}e^{-ct}+A^3\chi_{A\le 1}e^{-\frac{1}{4}A^2t}+\frac{\xi^3}{A^3}e^{-\frac{\xi^2}{2A^2}t} \\
&\lesssim \langle t\rangle^{-\frac{3}{2}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\big\|\langle\nabla\rangle\partial_x(7.49)\big\|_{L^2_{xy}} \\
&\lesssim |\lambda|\int_{t/2}^{t}\big\|A\xi\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}_a(t-s,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\big\|\langle\nabla\rangle\nabla\cdot\vec{u}(s)\big\|_{L^2_{xy}}\,ds \\
&\lesssim |\lambda|\int_{t/2}^{t}\langle t-s\rangle^{-\frac{3}{2}}\langle s\rangle^{-1}\,ds\|U\|_X \\
&\lesssim |\lambda|\langle t\rangle^{-1}\|U\|_X. \qquad (7.51)
\end{aligned}
$$

Now we consider the term (7.50). To this end, we give some analysis on $K_b(t)$ first. By (3.23), we have

$$
\begin{aligned}
&\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}_b(t,\xi,\eta) \\
&=-\frac{1}{2}A^2\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}_b(t,\xi,\eta)+\chi_{|\xi|\ge A^2}\widehat{K}_1(t,\xi,\eta), \qquad (7.52)
\end{aligned}
$$

where K_1 is defined in (2.48) at the beginning of Section 2.4. Therefore, we have

$$
\begin{aligned}
(7.50)=\frac{1}{2}\lambda\int_{t/2}^{t}&\Delta(\partial_{tt}-\Delta\partial_t-\Delta)\partial_tK_b(t-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds \qquad (7.53)\\
&+\lambda\int_{t/2}^{t}K_{1,b}(t-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds, \qquad (7.54)
\end{aligned}
$$

where $\widehat{K_{1,b}}(t,\xi,\eta)=\chi_{|\xi|\ge A^2}\widehat{K_1}(t,\xi,\eta)$. By (3.7), we have

$$
|\xi A^3\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K}_b(t,\xi,\eta)|\lesssim A^3\chi_{A\le 1}e^{-\frac{1}{4}A^2t}\lesssim\langle t\rangle^{-\frac{3}{2}}.
$$

It obeys the same estimate as $A\xi\partial_t\big(\partial_{tt}+A^2\partial_t+A^2\big)\widehat{K_a}(t,\xi,\eta)$. Thus, we have the same estimates on (7.53) as (7.51). By the definition of $K_{1,b}$, we have

$$\widehat{K_{1,b}}(t,\xi,\eta)=\frac{1}{4}\chi_{|\xi|\geq A^2}e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t}+\frac{1}{4}\chi_{|\xi|\geq A^2}e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2+A|\eta|})t}$$
$$+\frac{1}{4}\chi_{|\xi|\geq A^2}e^{(-\frac{1}{2}A^2+\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t}+\frac{1}{4}\chi_{|\xi|\geq A^2}e^{(-\frac{1}{2}A^2-\sqrt{\frac{1}{4}A^4-A^2-A|\eta|})t}.$$

It includes four parts, and each part has the bound of $e^{-\frac{1}{4}A^2t}$. So we may only consider one of them. In particular, let the operator $K_c(t)$ be defined by

$$\widehat{K_c}(t,\xi,\eta)=\frac{1}{4}\chi_{|\xi|\geq A^2}e^{(-\frac{1}{2}A^2+\mu_1\sqrt{\frac{1}{4}A^4-A^2+\mu_2A|\eta|})t},$$

for $\mu_1,\mu_2=\pm1$. Then we have

$$K_c(t-s)=K_c(t)K_c(-s).$$

Since K_c is bounded from L^2 to L^2, we have

$$\begin{aligned}
&\Big\|\langle\nabla\rangle\partial_x\lambda\int_{t/2}^{t}K_c(t-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds\Big\|_{L^2_{xy}}\\
&=|\lambda|\Big\|\langle\nabla\rangle\partial_x\int_{t/2}^{t}K_c(t)K_c(-s)\big(\partial_{xx}u(s)+\partial_{xy}v(s)\big)\,ds\Big\|_{L^2_{xy}}\\
&\lesssim|\lambda|\int_{t/2}^{t}\big\|K_c(t)\partial_{xx}K_c(-s)\big(\langle\nabla\rangle\nabla\cdot\vec{u}(s)\big)\big\|_{L^2_{xy}}\,ds\\
&\lesssim|\lambda|\int_{t/2}^{t}\big\|\partial_{xx}K_c(-s)\langle\nabla\rangle\big(\nabla\cdot u(s)\big)\big\|_{L^2_{xy}}\,ds\\
&\lesssim|\lambda|\int_{t/2}^{t}\big\|A^2\widehat{K_c}(-s,\xi,\eta)\big\|_{L^\infty_{\xi\eta}}\big\|\langle\nabla\rangle\big(\nabla\cdot u(s)\big)\big\|_{L^2_{xy}}\,ds\\
&\lesssim|\lambda|\int_{t/2}^{t}\langle s\rangle^{-2}\,ds\|U\|_X\lesssim|\lambda|\langle t\rangle^{-1}\|U\|_X.
\end{aligned}$$

Therefore, we prove that

$$\big\|\langle\nabla\rangle\partial_x(7.54)\big\|_{L^2_{xy}}\lesssim|\lambda|\langle t\rangle^{-1}\|U\|_X.$$

Now collecting the estimates obtained in Section 7.3.1–Section 7.3.5, we obtain (7.18)–(7.21). Combining with the estimates in Section 7.2 gives (7.1)–(7.4).

8. The estimates on v

In this section, we shall prove that

$$\|v(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)\big); \tag{8.1}$$

$$\|\langle\nabla\rangle v(t)\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}\big(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)\big); \tag{8.2}$$

$$\|v(t)\|_{L^2_x L^\infty_y} \lesssim \langle t\rangle^{-\frac{3}{4}}\big(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)\big); \tag{8.3}$$

$$\||\nabla|^\gamma v(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\big(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)\big); \tag{8.4}$$

$$\|\langle\nabla\rangle\nabla v(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\big(\|U_0\|_{X_0} + |\lambda|\,\|U\|_X + Q(\|U\|_X)\big). \tag{8.5}$$

Note that by Nash's inequality, and Sobolev's inequality, we have

$$\|v(t)\|_{L^2_x L^\infty_y} \lesssim \|v(t)\|^{\frac{1}{2}}_{L^2_{xy}}\|\partial_y v(t)\|^{\frac{1}{2}}_{L^2_{xy}} \lesssim \|v(t)\|^{\frac{1}{2}}_{L^2_{xy}}\|\langle\nabla\rangle\nabla v(t)\|^{\frac{1}{2}}_{L^2_{xy}};$$

$$\big\||\nabla|^\gamma v(t)\big\|_{L^2_{xy}} \lesssim \|v(t)\|^{\frac{1}{2}}_{L^2_{xy}}\|\langle\nabla\rangle\nabla v(t)\|^{\frac{1}{2}}_{L^2_{xy}}.$$

Thus (8.3) and (8.4) can be established by (8.1) and (8.5), and we only need to prove (8.1), (8.2) and (8.5).

8.1. The reexpression of v

Now we give the reexpression of v as

Proposition 8.1. *The unknown function v obeys the formula,*

$$v(t,x,y) = (L_v + B_v)(t; n_0, \vec{u}_0, \vec{b}_0) + \mathcal{N}_v(t; n, \vec{u}, \psi), \tag{8.6}$$

where $(L_v + B_v)$ is given by

$$\begin{aligned}
&L_v(t; n_0, \vec{u}_0, \vec{b}_0) + B_v(t; n_0, \vec{u}_0, \vec{b}_0)\\
&\quad = \partial_t K(t)[\partial_{xy}u_0 + \partial_{yy}v_0] - (\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t)\big[\Delta\psi_0\big]\\
&\qquad + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)[\Delta v(0)]\\
&\qquad - (\partial_{tt} - \Delta\partial_t)K(t)\big[\partial_y n_0\big] - \frac{1}{2}\Delta\big(\Delta + \sqrt{\Delta\partial_{yy}}\big)K(t)\big[v_0\big]\\
&\qquad + \partial_t(\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t)\big[v_0\big],
\end{aligned} \tag{8.7}$$

and $\mathcal{N}_v(t; n, \vec{u}, \psi)$ is defined as

$$\mathcal{N}_v(t;n,\vec{u},\psi) = -\int_0^t \partial_y(\partial_{tt}-\Delta\partial_t)K(t-s)N_0(s)\,ds$$
$$+\int_0^t \partial_t\partial_{xy}K(t-s)N_1(s)\,ds + \int_0^t \partial_t(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)N_2(s)\,ds$$
$$-\int_0^t \Delta(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)N_3(s)\,ds$$
$$+\lambda\int_0^t \partial_t(\partial_{tt}-\Delta\partial_t)K(t-s)\big[\partial_{xy}u(s)+\partial_{yy}v(s)\big]\,ds. \tag{8.8}$$

Now we begin to prove this proposition. Again, using (2.58) and integration by parts (see below for the details), we have

$$\begin{aligned}
v(t,x,y) &= L_v(t;n_0,\vec{u}_0,\vec{b}_0) + \int_0^t K(t-s)F_2(s)\,ds \\
&= L_v(t;n_0,\vec{u}_0,\vec{b}_0) + \int_0^t K(t-s)\Big[-\partial_y(\partial_{ss}-\Delta\partial_s)N_0(s)+\partial_s\partial_{xy}N_1(s) \\
&\quad + \partial_s(\partial_{ss}-\Delta\partial_s-\partial_{xx})N_2(s) - \Delta(\partial_{ss}-\Delta\partial_s-\partial_{xx})N_3(s) \\
&\quad + \lambda(\partial_{ss}-\Delta\partial_s)\partial_s\big(\partial_{xy}u(s)+\partial_{yy}v(s)\big)\Big]\,ds \\
&= L_v(t;n_0,\vec{u}_0,\vec{b}_0) + B_v(t;n_0,\vec{u}_0,\vec{b}_0) + \mathcal{N}_v(t;n,\vec{u},\psi).
\end{aligned}$$

Here according to (2.39), we denote

$$L_v(t;n_0,\vec{u}_0,\vec{b}_0)$$
$$= K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)\partial_t v(0)\big] \tag{8.9}$$
$$+(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\big[\partial_t v(0)\big] \tag{8.10}$$
$$-\Delta K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)v(0)\big] + \partial_t K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)v(0)\big] \tag{8.11}$$
$$-\frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[v_0] + K_1(t)[v_0], \tag{8.12}$$

and $B_v(t;n_0,\vec{u}_0,\vec{b}_0)$ is the boundary term given below.

Next we will simply $L_v + B_v$. To do this, we give the explicit expressions of L_v and B_v respectively.

8.1.1. $L_v(t; n_0, \vec{u}_0, \vec{b}_0)$

By the equations (2.2), (2.19) and (2.25) at $t = 0$, we have

$$\begin{aligned}
&(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t v(0) \\
&\quad = (\partial_{tt} - \Delta\partial_t - \Delta)\big(\Delta v(0) + \lambda(\partial_{xy}u(0) + \partial_{yy}v(0)) - \partial_y n(0) - \Delta\psi(0) + N_2(0)\big) \\
&\quad = (\partial_{tt} - \Delta\partial_t - \Delta)\Delta v(0) + (\partial_{tt} - \Delta\partial_t - \Delta)N_2(0) \\
&\qquad - \partial_y\big[\partial_y\Delta\psi(0) + \partial_t N_0(0) - \Delta N_0(0) - \partial_x N_1(0) - \partial_y N_2(0) \\
&\qquad - \lambda(\partial_x\Delta u(0) + \partial_y\Delta v(0))\big] \\
&\qquad - \Delta\big[\partial_y n(0) - N_2(0) + (\partial_t - \Delta)N_3(0) - \lambda(\partial_{xy}u(0) + \partial_{yy}v(0))\big] \\
&\qquad + \lambda(\partial_{tt} - \Delta\partial_t - \Delta)\big[\partial_{xy}u(0) + \partial_{yy}v(0)\big] \\
&\quad = (\partial_{tt} - \Delta\partial_t - \Delta)\Delta v(0) + (\partial_{tt} - \Delta\partial_t - \Delta)N_2(0) - \partial_{yy}\Delta\psi(0) - \partial_y\Delta n(0) \\
&\qquad + \big[-\partial_t\partial_y N_0 + \Delta\partial_y N_0(0) + \partial_{xy}N_1(0) + (\Delta + \partial_{yy})N_2(0) - (\partial_t - \Delta)\Delta N_3(0)\big] \\
&\qquad + \lambda(\partial_{tt} - \Delta\partial_t + \Delta)\big[\partial_{xy}u(0) + \partial_{yy}v(0)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
(8.9) &= K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\Delta v(0)\big] + K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)N_2(0)\big] \\
&\quad - K(t)\big[\partial_{yy}\Delta\psi(0) + \partial_y\Delta n(0)\big] \\
&\quad + K(t)\big[-\partial_t\partial_y N_0 + \partial_y\Delta N_0(0) + \partial_{xy}N_1(0) \\
&\quad + (\Delta + \partial_{yy})N_2(0) - (\partial_t - \Delta)\Delta N_3(0)\big] \\
&\quad + \lambda K(t)\big[(\partial_{tt} - \Delta\partial_t + \Delta)(\partial_{xy}u(0) + \partial_{yy}v(0))\big].
\end{aligned}$$

Now we consider (8.10) and (8.11). Similarly,

$$\begin{aligned}
(8.10) &= (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\Delta v(0) + \lambda(\partial_{xy}u(0) + \partial_{yy}v(0)) \\
&\quad - \partial_y n(0) - \Delta\psi(0) + N_2(0)\big] \\
&= (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\Delta v(0) - \partial_y n(0) - \Delta\psi(0)\big] \\
&\quad + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[N_2(0)\big] \\
&\quad + \lambda(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_{xy}u(0) + \partial_{yy}v(0)\big]; \\
(8.11) &= -\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)v(0)\big] + \partial_t K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)v(0)\big] \\
&= -\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)v(0)\big] + \partial_{yy}\partial_t K(t)[v(0)] \\
&\quad + \partial_t K(t)\big[\partial_{xy}u(0) - \partial_y N_0(0) + \partial_t N_2(0) - \Delta N_3(0) \\
&\quad + \lambda\partial_t(\partial_{xy}u(0) + \partial_{yy}v(0))\big].
\end{aligned}$$

Then collecting the estimates above, we have

$$\begin{aligned}
&L_v(t;n_0,\vec{u}_0,\vec{b}_0)\\
&= K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)N_2(0)\big]+(\partial_{tt}-\Delta\partial_t-\Delta)K(t)\big[N_2(0)\big]\\
&\quad + K(t)\big[-\partial_t\partial_y N_0+\partial_y\Delta N_0(0)+\partial_{xy}N_1(0)+(\Delta+\partial_{yy})N_2(0)-(\partial_t-\Delta)\Delta N_3(0)\big]\\
&\quad + \partial_t K(t)\big[-\partial_y N_0(0)+\partial_t N_2(0)-\Delta N_3(0)\big]+(\partial_{tt}-\Delta\partial_t-\Delta)K(t)[\Delta v(0)]\\
&\quad + \partial_{xy}\partial_t K(t)[u_0]-(\partial_{tt}-\Delta\partial_t)K(t)\big[\partial_y n_0\big]-(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t)\big[\Delta\psi_0\big]\\
&\quad + \partial_{yy}\partial_t K(t)[v_0]+\lambda K(t)\big[(\partial_{tt}-\Delta\partial_t)(\partial_{xy}u(0)+\partial_{yy}v(0))\big]\\
&\quad + \lambda(\partial_{tt}-\Delta\partial_t)K(t)\big[\partial_{xy}u(0)+\partial_{yy}v(0)\big]+\lambda\partial_t K(t)\big[\partial_t(\partial_{xy}u(0)+\partial_{yy}v(0))\big]\\
&\quad - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[v_0]+K_1(t)[v_0].
\end{aligned}$$

8.1.2. $B_v(t;n_0,\vec{u}_0,\vec{b}_0)$

Now we consider the boundary term $B_v(t;n_0,\vec{u}_0,\vec{b}_0)$. By integration by parts, and arguing similarly as in Section 6.1.2 we have

$$\begin{aligned}
&-\int_0^t K(t-s)\big[\partial_y(\partial_{ss}-\Delta\partial_s)N_0(s)\big]\,ds\\
&\quad = K(t)\big[\partial_y(\partial_t-\Delta)N_0(0)\big]+\partial_t K(t)\big[\partial_y N_0(0)\big]\\
&\qquad - \int_0^t \partial_y(\partial_{tt}-\Delta\partial_t)K(t-s)N_0(s)\,ds;
\end{aligned}$$

$$\begin{aligned}
&\int_0^t K(t-s)\big[\partial_s\partial_{xy}N_1(s)\big]\,ds\\
&\quad = -K(t)\big[\partial_{xy}N_1(0)\big]+\int_0^t \partial_t\partial_{xy}K(t-s)N_1(s);
\end{aligned}$$

$$\begin{aligned}
&\int_0^t K(t-s)\big[\partial_s(\partial_{ss}-\Delta\partial_s-\partial_{xx})N_2(s)\big]\,ds\\
&\quad = -K(t)\big[(\partial_{tt}-\Delta\partial_t-\partial_{xx})N_2(0)\big]-\partial_t K(t)\big[(\partial_t-\Delta)N_2(0)\big]-\partial_{tt}K(t)\big[N_2(0)\big]\\
&\qquad + \int_0^t \partial_t(\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)N_2(s)\,ds;
\end{aligned}$$

$$-\int_0^t K(t-s)\big[(\partial_{ss}-\Delta\partial_s-\partial_{xx})\Delta N_3(s)\big]\,ds$$

$$= K(t)\big[(\partial_t - \Delta)\Delta N_3(0)\big] + \partial_t K(t)\big[\Delta N_3(0)\big] - \int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t-s)N_3(s)\,ds;$$

$$\lambda \int_0^t K(t-s)\big[(\partial_{ss} - \Delta\partial_s)\partial_s\big(\partial_{xy}u(s) + \partial_{yy}v(s)\big)\big]\,ds$$
$$= -\lambda K(t)\big[(\partial_{tt} - \Delta\partial_t)\big(\partial_{xy}u(0) + \partial_{yy}v(0)\big)\big]$$
$$- \lambda\partial_t K(t)\big[(\partial_t - \Delta)\big(\partial_{xy}u(0) + \partial_{yy}v(0)\big)\big] - \lambda\partial_{tt}K(t)\big[\partial_{xy}u(0) + \partial_{yy}v(0)\big]$$
$$+ \lambda \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t)K(t-s)\big[\partial_{xy}u(s) + \partial_{yy}v(s)\big]\,ds.$$

Therefore, we obtain the boundary term $B_v(t; n_0, \vec{u}_0, \vec{b}_0)$ as

$$\begin{aligned}
&B_v(t; n_0, \vec{u}_0, \vec{b}_0)\\
&= -K(t)\big[(\partial_{tt} - \Delta\partial_t - \partial_{xx})N_2(0)\big]\\
&\quad + K(t)\big[\partial_t\partial_y N_0(0) - \partial_y\Delta N_0(0) - \partial_{xy}N_1(0) + \partial_t\Delta N_3(0) - \Delta^2 N_3(0)\big]\\
&\quad + \partial_t K(t)\big[\partial_y N_0(0) - \partial_t N_2(0) + \Delta N_2(0) + \Delta N_3(0)\big] - \partial_{tt}K(t)\big[N_2(0)\big]\\
&\quad - \lambda K(t)\big[(\partial_{tt} - \Delta\partial_t)\big(\partial_{xy}u(0) + \partial_{yy}v(0)\big)\big]\\
&\quad - \lambda\partial_t K(t)\big[(\partial_t - \Delta)\big(\partial_{xy}u(0) + \partial_{yy}v(0)\big)\big] - \lambda\partial_{tt}K(t)\big[\partial_{xy}u(0) + \partial_{yy}v(0)\big].
\end{aligned}$$

Now combining with the result obtained in Section 8.1.1, and by the same argument in (7.6), we have (8.7).

Again, we split into the following two subsection to consider the linear parts and nonlinear parts separately. Most of the terms are in the same level as the corresponding ones in Section 7, and can be treated similarly, so we only give the sketch of the proof.

8.2. Estimates on the linear parts $L_v + B_v$

In this subsection, we shall prove that

Lemma 8.2.

$$\big\|(L_v + B_v)(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\|U_0\|_{X_0}; \tag{8.13}$$

$$\big\|\langle\nabla\rangle(L_v + B_v)(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}\|U_0\|_{X_0}; \tag{8.14}$$

$$\big\|\langle\nabla\rangle\nabla(L_v + B_v)(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\|U_0\|_{X_0}. \tag{8.15}$$

Now we show that the each in (8.7) obeys the estimates in (8.13)–(8.15).

The estimates on the term $\partial_t K(t)[\partial_{xy} u_0 + \partial_{yy} v_0]$ can be obtained by Proposition 4.5. Also, for the term $(\partial_{tt} - \Delta\partial_t)K(t)\big[\partial_y n_0\big]$, we use Proposition 4.7 to obtain the desirable estimates. For the term $(\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t)\big[\Delta\psi_0\big]$, we use Proposition 4.11 to obtain the desirable estimates. The estimates on the terms $\frac{1}{2}\Delta\big(\Delta + \sqrt{\Delta\partial_{yy}}\big)K(t)\big[v_0\big]$ and $\partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[v_0\big]$ are obtained by Propositions 4.4 and 4.12 (i) (ii) respectively.

8.3. Estimates on the nonlinear parts $\mathcal{N}_v$

In this subsection, we shall prove that

Lemma 8.3.

$$\big\|\mathcal{N}_v(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\|U_0\|_{X_0}; \tag{8.16}$$

$$\big\|\langle\nabla\rangle\mathcal{N}_v(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-1}\|U_0\|_{X_0}; \tag{8.17}$$

$$\big\|\langle\nabla\rangle\nabla(\mathcal{N}_v)(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-1}\|U_0\|_{X_0}. \tag{8.18}$$

We estimate $\mathcal{N}_v(t; n, \vec{u}, \psi)$ in (8.8) terms by terms.

8.3.1. $\int_0^t \partial_y(\partial_{tt} - \Delta\partial_t)K(t-s)N_0(s)\,ds$

The high frequency piece N_0^h can be treated standard, so we only treat N_0^l. Recall that $N_0 = \nabla\cdot(n\vec{u})$. First, we have

$$\big\|(n\vec{u})(s)\big\|_{L^2_{xy}} \lesssim \|n(s)\|_{L^2_{xy}}\|\vec{u}(s)\|_{L^\infty_{xy}} \lesssim \langle s\rangle^{-1.25}\|U\|_X^2. \tag{8.19}$$

Then by Proposition 4.6 (ii), for $\beta = 0, 1$, and (8.19) we have

$$\begin{aligned}
&\Big\||\nabla|^\beta \int_0^t \partial_y(\partial_{tt} - \Delta\partial_t)K(t-s)N_0^l(s)\,ds\Big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \Big\||\nabla|^\beta \nabla\partial_y(\partial_{tt} - \Delta\partial_t)K(t-s)\cdot(n\vec{u})(s)\,ds\Big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1+\beta}{2}}\big\|P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle^\beta(n\vec{u})(s)\big\|_{L^2_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{1+\beta}{2}}\langle s\rangle^{-1.25+0.01}\|U\|_X^2\,ds \\
&\lesssim \langle t\rangle^{-\frac{1+\beta}{2}}\|U\|_X^2.
\end{aligned}$$

Similarly, using Proposition 4.6 (iii) instead, we have

$$\begin{aligned}
&\Big\| \int_0^t \partial_y(\partial_{tt} - \Delta\partial_t)K(t-s)N_0^l(s)\,ds\Big\|_{L^\infty_{xy}} \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \big\| P_{\lesssim \langle s\rangle^{0.01}} \langle\nabla\rangle^{1+}(n\vec{u})(s)\big\|_{L^2_{xy}}\,ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-1.25+0.02}\|U\|_X^2\,ds \\
&\lesssim \langle t\rangle^{-1}\|U\|_X^2.
\end{aligned}$$

Thus, we obtain the claimed results.

8.3.2. $\int_0^t \partial_t\partial_{xy}K(t-s)N_1(s)\,ds$

We use the same argument as in Section 7.3.2, and decompose N_1 into four parts as (7.35). Then all the desirable estimates on the terms $N_{13}, N_{14}, N_{15}, N_{16}$ can be obtained by the corresponding ways. More precisely, since

$$\partial_t\partial_x\partial_y \sim \partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}); \quad \nabla\partial_t\partial_x\partial_y \sim \partial_x\partial_t(\partial_{tt} - \Delta\partial_t - \Delta - \partial_{yy}),$$

here $\sim$ means that the two operator obey the similar decaying estimates presented in Section 3, the details are just mimicked there and so are omitted here.

8.3.3. $\int_0^t \partial_t(\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t-s)N_2(s)\,ds$

Also, using the argument in Section 7.3.3, and noting that

$$\partial_t(\partial_{tt} - \Delta\partial_t - \partial_{xx}) \sim \partial_t\partial_x\partial_y; \quad \nabla\partial_t(\partial_{tt} - \Delta\partial_t - \partial_{xx}) \sim \partial_x\partial_t\partial_x\partial_y$$

(indeed, the former behaviors slightly better than the latter), we can obtain the desirable estimates here. The details are omitted again.

8.3.4. $\int_0^t \Delta(\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t-s)N_3(s)\,ds$

By Lemma 3.13 and Lemma 3.19, we note that

$$\nabla(\partial_{tt} - \Delta\partial_t - \partial_{xx}) \sim \partial_y(\partial_{tt} - \Delta\partial_t).$$

Moreover,

$$\|\nabla\psi\cdot\vec{u}\|_{L^2_{xy}} \lesssim \|\nabla\psi\|_{L^2_{xy}}\|\vec{u}\|_{L^\infty_{xy}} \lesssim \langle s\rangle^{-1.25}\|U\|_X^2.$$

Therefore, by the same way as in Section 8.3.1, we obtain the desirable estimates.

8.3.5. $\lambda \int_0^t \partial_t(\partial_{tt} - \Delta\partial_t)K(t-s)[(\partial_{xy}u(s) + \partial_{yy}v(s))]\,ds$

To prove this term, we use the similar precess as in Section 7.3.5. Indeed, we rewrite

$$\partial_t(\partial_{tt} - \Delta\partial_t)K_b(t) = \partial_t(\partial_{tt} - \Delta\partial_t - \Delta)K_b(t) + \partial_t\Delta K_b(t).$$

Moreover, by (3.19), we have

$$\partial_t A^2\widehat{K_b}(t) = -\frac{A^4}{2}\widehat{K_b}(t) + \frac{A}{|\eta|}\widehat{K_{1,b}}(t).$$

Therefore, we have the similar structure as the first part. Then by the same way as in Section 8.3.1, we get the claimed estimates.

Collecting all the estimates in Section 8.3.1–Section 8.3.5, we obtain Lemma 8.3. Together with Lemma 8.2, we establish (8.1)–(8.5).

9. The estimates on ψ

In this section, we shall prove that there exists some small constant $\epsilon > 0$, such that

$$\|\langle\nabla\rangle^4|\nabla|^\gamma\psi(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\big(\|U_0\|_{X_0} + \epsilon_0\|U\|_X + Q(\|U\|_X)\big); \tag{9.1}$$

$$\big\||\nabla|^{\bar\gamma}\langle\nabla\rangle\psi(t)\big\|_{L^\infty_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0} + \epsilon_0\|U\|_X + Q(\|U\|_X)\big); \tag{9.2}$$

$$\|\partial_x\psi(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0} + \epsilon_0\|U\|_X + Q(\|U\|_X)\big); \tag{9.3}$$

$$\|\partial_x\nabla\psi(t)\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\big(\|U_0\|_{X_0} + \epsilon_0\|U\|_X + Q(\|U\|_X)\big). \tag{9.4}$$

Again, it follows from Nash's inequality that for any $\frac{\gamma}{2} < \bar\gamma < 1 + \frac{\gamma}{2}$,

$$\big\||\nabla|^{\bar\gamma}\langle\nabla\rangle\psi(t)\big\|_{L^\infty_{xy}} \lesssim \|\langle\nabla\rangle^4|\nabla|^\gamma\psi(t)\|^{\frac{1}{2}}_{L^2_{xy}}\|\nabla\partial_x\psi(t)\|^{\frac{1}{2}}_{L^2_{xy}}, \tag{9.5}$$

(see its proof in Appendix A.4). Thus, we only need to show (9.1), (9.3) and (9.4).

9.1. The reexpression of ψ

Now we give the reexpression of ψ and obtain

Proposition 9.1. *The unknown function ψ obeys the formula,*

$$\psi(t,x,y) = (L_\psi + B_\psi)(t; n_0, \vec{u}_0, \vec{b}_0) + \mathcal{N}_\psi(t; n, \vec{u}, \psi), \tag{9.6}$$

where $(L_\psi + B_\psi)$ is given by

$$\begin{aligned}
&L_\psi(t; n_0, \vec{u}_0, \vec{b}_0) + B_\psi(t; n_0, \vec{u}_0, \vec{b}_0) \\
&\quad = -K(t)\big[\partial_{xy} u_0 + \partial_y \Delta n_0\big] + \partial_y \partial_t K(t)[n_0] \\
&\qquad - (\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t)\big[v_0\big] - \frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[\psi_0] + K_1(t)[\psi_0],
\end{aligned} \tag{9.7}$$

and

$$\begin{aligned}
\mathcal{N}_\psi(t; n, \vec{u}, \psi) = &\int_0^t \partial_y \partial_t K(t-s) N_0(s)\, ds - \int_0^t \partial_y \Delta K(t-s) N_0(s)\, ds \\
&- \int_0^t \partial_{xy} K(t-s) N_1(s)\, ds - \int_0^t (\partial_{tt} - \Delta\partial_t - \partial_{xx}) K(t-s) N_2(s)\, ds \\
&- \int_0^t (\partial_t - \Delta)(\partial_{tt} - \Delta\partial_t - \Delta) K(t-s) N_3(s)\, ds \\
&-\lambda \int_0^t (\partial_{tt} - \Delta\partial_t) K(t-s)\big(\partial_{xy} u(s) + \partial_{yy} v(s)\big)\, ds.
\end{aligned} \tag{9.8}$$

The proof of this proposition is similar as the one in the previous three sections. Again, using (2.59) and integration by parts (see below for details), we have

$$\begin{aligned}
\psi(t, x, y) &= L_\psi(t; n_0, \vec{u}_0, \vec{b}_0) + \int_0^t K(t-s) F_3(s)\, ds \\
&= L_\psi(t; n_0, \vec{u}_0, \vec{b}_0) + \int_0^t K(t-s)\Big[\partial_y \partial_s N_0(s) - \partial_y \Delta N_0(s) - \partial_{xy} N_1(s) \\
&\quad - (\partial_{ss} - \Delta\partial_s - \partial_{xx}) N_2(s) + (\partial_s - \Delta)(\partial_{ss} - \Delta\partial_s - \Delta) N_3(s) \\
&\quad - \lambda(\partial_{ss} - \Delta\partial_s)(\partial_{xy} u(s) + \partial_{yy} v(s))\Big]\, ds \\
&= L_\psi(t; n_0, \vec{u}_0, \vec{b}_0) + B_\psi(t; n_0, \vec{u}_0, \vec{b}_0) + \mathcal{N}_\psi(t; n, \vec{u}, \psi).
\end{aligned}$$

Here according to (2.39),

$$L_\psi(t; n_0, \vec{u}_0, \vec{b}_0)$$

$$= K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t \psi(0)\big] \tag{9.9}$$

$$+(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[\partial_t \psi(0)\big] \tag{9.10}$$

$$-\Delta K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\psi(0)\big] + \partial_t K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\psi(0)\big] \tag{9.11}$$

$$-\frac{1}{2}\Delta\sqrt{\Delta\partial_{yy}}K(t)[\psi_0] + K_1(t)[\psi_0], \tag{9.12}$$

and $B_\psi(t; n_0, \vec{u}_0, \vec{b}_0)$ is the boundary term given below.

Next we will simply $L_\psi + B_\psi$. To this end, we give the explicit expressions of L_ψ and B_ψ respectively.

9.1.1. $L_\psi(t; n_0, \vec{u}_0, \vec{b}_0)$

By the equations (2.2) and (2.23) at $t = 0$, we have

$$\begin{aligned}(\partial_{tt} - \Delta\partial_t - \Delta)\partial_t\psi(0) &= \big[(\partial_{tt} - \Delta\partial_t - \Delta)(-v(0) + N_3(0))\big]\\ &= -\big(\partial_{yy}v(0) + \partial_{xy}u(0) - \partial_y N_0(0) + \partial_t N_2(0) - \Delta N_3(0)\\ &\quad + \lambda\partial_t(\partial_{xy}u(0) + \partial_{yy}v(0))\big) + (\partial_{tt} - \Delta\partial_t - \Delta)N_3(0).\end{aligned}$$

Thus,

$$\begin{aligned}(9.9) = &-K(t)\big[\partial_{yy}v(0) + \partial_{xy}u(0) - \partial_y N_0(0) + \partial_t N_2(0) - \Delta N_3(0)\\ &+ \lambda\partial_t(\partial_{xy}u(0) + \partial_{yy}v(0))\big] + K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)N_3(0)\big].\end{aligned}$$

Now we consider (9.10) and (9.11), similarly,

$$\begin{aligned}(9.10) &= (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[-v(0) + N_3(0)\big]\\ &= -(\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[v(0)\big] + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[N_3(0)\big];\\ (9.11) &= \big(\partial_t - \Delta\big)K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)\psi(0)\big]\\ &= \big(\partial_t - \Delta\big)K(t)\big[\partial_y n(0) - N_2(0) + (\partial_t - \Delta)N_3(0) - \lambda(\partial_{xy}u(0) + \partial_{yy}v(0))\big].\end{aligned}$$

Then collecting the estimates above, we have

$$\begin{aligned}&L_\psi(t; n_0, \vec{u}_0, \vec{b}_0)\\ &= K(t)\big[(\partial_{tt} - \Delta\partial_t - \Delta)N_3(0)\big] + (\partial_{tt} - \Delta\partial_t - \Delta)K(t)\big[N_3(0)\big]\\ &\quad + K(t)\big[\partial_y N_0(0) - \partial_t N_2(0) + \Delta N_2(0) + \Delta N_3(0) - (\partial_t - \Delta)\Delta N_3(0)\big]\\ &\quad + \partial_t K(t)\big[-N_2(0) + (\partial_t - \Delta)N_3(0)\big]\\ &\quad - K(t)\big[\partial_{xy}u(0) + \partial_y\Delta n(0)\big] + \partial_y\partial_t K(t)[n(0)]\\ &\quad - (\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t)\big[v(0)\big] - \lambda K(t)\big[(\partial_t - \Delta)(\partial_{xy}u(0) + \partial_{yy}v(0))\big]\\ &\quad - \lambda\partial_t K(t)\big[\partial_{xy}u(0) + \partial_{yy}v(0)\big] + (9.12).\end{aligned}$$

9.1.2. $B_\psi(t; n_0, \vec{u}_0, \vec{b}_0)$

Now we consider the boundary term $B_\psi(t; n_0, \vec{u}_0, \vec{b}_0)$. By integration by parts, and arguing similarly as in Section 6.1.2 we have

$$\int_0^t K(t-s)\big[\partial_y\partial_s N_0(s)\big]\,ds$$
$$= -K(t)\big[\partial_y N_0(0)\big] + \int_0^t \partial_y\partial_t K(t-s)N_0(s)\,ds;$$
$$-\int_0^t K(t-s)\big[(\partial_{ss}-\Delta\partial_s-\partial_{xx})N_2(s)\big]\,ds$$
$$= K(t)\big[(\partial_t-\Delta)N_2(0)\big] + \partial_t K(t)\big[N_2(0)\big] - \int_0^t (\partial_{tt}-\Delta\partial_t-\partial_{xx})K(t-s)N_2(s)\,ds;$$
$$\int_0^t K(t-s)\big[(\partial_s-\Delta)(\partial_{ss}-\Delta\partial_s-\Delta)N_3(s)\big]\,ds$$
$$= -K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)N_3(0)\big] - \partial_t K(t)\big[(\partial_t-\Delta)N_3(0)\big] - \partial_{tt}K(t)\big[N_3(0)\big] + K(t)\big[(\partial_t-\Delta)\Delta N_3(0)\big] + \Delta\partial_t K(t)\big[N_3(0)\big] + (\partial_t-\Delta)(\partial_{tt}-\Delta\partial_t-\Delta)\int_0^t K(t-s)N_3(s)\,ds;$$
$$-\lambda\int_0^t K(t-s)\big[(\partial_{tt}-\Delta\partial_t)(\partial_{xy}u(s)+\partial_{yy}v(s))\big]\,ds$$
$$= \lambda K(t)\big[(\partial_t-\Delta)(\partial_{xy}u(0)+\partial_{yy}v(0))\big] + \lambda\partial_t K(t)\big[\partial_{xy}u(0)+\partial_{yy}v(0)\big] - \lambda\int_0^t (\partial_{tt}-\Delta\partial_t)K(t-s)(\partial_{xy}u(s)+\partial_{yy}v(s))\,ds.$$

Therefore, we obtain the boundary term $B_\psi(t; n_0, \vec{u}_0, \vec{b}_0)$ as

$$\begin{aligned} &B_\psi(t; n_0, \vec{u}_0, \vec{b}_0) \\ &= -K(t)\big[(\partial_{tt}-\Delta\partial_t-\Delta)N_3(0)\big] \\ &\quad + K(t)\big[-\partial_y N_0(0) + (\partial_t-\Delta)N_2(0) + (\partial_t-\Delta)\Delta N_3(0)\big] \\ &\quad + \partial_t K(t)\big[N_2(0) - (\partial_t-\Delta)N_3(0)\big] + \Delta\partial_t K(t)\big[N_3(0)\big] - \partial_{tt}K(t)\big[N_3(0)\big] \\ &\quad + \lambda K(t)\big[(\partial_t-\Delta)(\partial_{xy}u(0)+\partial_{yy}v(0))\big] + \lambda\partial_t K(t)\big[\partial_{xy}u(0)+\partial_{yy}v(0)\big]. \end{aligned}$$

Combining with the result obtained in Section 9.1.1, we have (9.7).

Again, we split into two subsections to consider the linear parts and nonlinear parts separately.

9.2. Estimates on the linear parts $L_\psi + B_\psi$

In this subsection, we prove that

Lemma 9.2. *For $\gamma > \frac{1}{2}$,*

$$\big\|\langle\nabla\rangle^4|\nabla|^\gamma(L_\psi + B_\psi)(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{4}}\|U_0\|_{X_0}; \tag{9.13}$$

$$\big\|\partial_x(L_\psi + B_\psi)(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{2}}\|U_0\|_{X_0}. \tag{9.14}$$

$$\big\|\nabla\partial_x(L_\psi + B_\psi)(t; n_0, \vec{u}_0, \vec{b}_0)\big\|_{L^2} \lesssim \langle t\rangle^{-\frac{3}{4}}\|U_0\|_{X_0}. \tag{9.15}$$

Now we estimate the terms in (9.7). For the term $\partial_{xy}K(t)[u_0]$, by Proposition 4.2 (i), we have

$$\big\|\langle\nabla\rangle^4|\nabla|^\gamma\partial_{xy}K(t)[u_0]\big\|_{L^2_{xy}} = \big\||\nabla|^\gamma\partial_{xy}K(t)\langle\nabla\rangle^4[u_0]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^4 u_0\big\|_{L^1_{xy}};$$
$$\big\|\partial_x\partial_{xy}K(t)[u_0]\big\|_{L^2_{xy}} \lesssim \big\|\nabla\partial_{xy}K(t)[u_0]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big\|\langle\nabla\rangle^{0+}u_0\big\|_{L^1_{xy}};$$
$$\big\|\nabla\partial_x\partial_{xy}K(t)[u_0]\big\|_{L^2_{xy}} \lesssim \big\||\nabla|^2\partial_{xy}K(t)[u_0]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\big\|\langle\nabla\rangle^{1+}u_0\big\|_{L^1_{xy}}.$$

Now we consider the term $K(t)\big[\partial_y\Delta n_0\big]$. By Proposition 4.1 (i) and Proposition 4.2 (i), for $\beta = 0$ or 1, we have (since $\gamma > \frac{1}{2}$),

$$\big\|\langle\nabla\rangle^4|\nabla|^\gamma K(t)\big[\partial_y\Delta n_0\big]\big\|_{L^2_{xy}} \lesssim \big\||\nabla|^{3+\gamma}K(t)\big[\langle\nabla\rangle^4 n_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{4}}\big\|\langle\nabla\rangle^5 n_0\big\|_{L^1_{xy}};$$
$$\big\||\nabla|^\beta\partial_x K(t)\big[\partial_y\Delta n_0\big]\big\|_{L^2_{xy}} \lesssim \big\||\nabla|^{2+\beta}\partial_{xy}K(t)\big[n_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{3}{4}}\big\|\langle\nabla\rangle^{2+}n_0\big\|_{L^1_{xy}}.$$

Next, we consider the term $\partial_y\partial_t K(t)[n_0]$. Similarly, by Proposition 4.5 (i), for $\beta = \gamma$, $1, 2$,

$$\big\|\langle\nabla\rangle^4|\nabla|^\beta\partial_t K(t)\big[\partial_y n_0\big]\big\|_{L^2_{xy}} \lesssim \big\||\nabla|^\beta\partial_y\partial_t K(t)\big[\langle\nabla\rangle^4 n_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{4+}n_0\big\|_{L^1_{xy}}.$$

Choosing $\beta = \gamma, 1, 2$, we have the claimed estimates.

Now we consider the term $(\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t)\big[v_0\big]$. From Proposition 4.11 (i),

$$\big\|\langle\nabla\rangle^4|\nabla|^\beta(\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t)\big[v_0\big]\big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{\beta}{2}}\big\|\langle\nabla\rangle^{4+}v_0\big\|_{L^1_{xy}}.$$

Again, letting $\beta = \gamma, 1, 2$, we have the claimed estimates.

At last, we consider the terms in (9.12). The same as the term $K(t)\big[\partial_y\Delta n_0\big]$, for $\beta = 0$ or 1, we have

$$\big\|\langle\nabla\rangle^4|\nabla|^\gamma\Delta\sqrt{\Delta\partial_{yy}}K(t)[\psi_0]\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac14}\big\|\langle\nabla\rangle^4\nabla\psi_0\big\|_{L^1_{xy}};$$
$$\big\||\nabla|^\beta\partial_x\Delta\sqrt{\Delta\partial_{yy}}K(t)[\psi_0]\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac34}\big\|\langle\nabla\rangle^2\nabla\psi_0\big\|_{L^1_{xy}}.$$

Similarly, using Proposition 4.15, we have

$$\big\|\langle\nabla\rangle^4|\nabla|^\gamma K_1(t)[\psi_0]\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac14}\big\|\langle\nabla\rangle^5\nabla\psi_0\big\|_{L^1_{xy}};$$
$$\big\||\nabla|^\beta\partial_x K_1(t)[\psi_0]\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac12-\frac\beta4}\big\|\langle\nabla\rangle^2\nabla\psi_0\big\|_{L^1_{xy}}.$$

Collecting the estimates above, we give the proof of Lemma 9.13.

9.3. Estimates on the nonlinear parts $\mathcal{N}_\psi$

In this subsection, we shall prove that

Lemma 9.3.

$$\big\|\langle\nabla\rangle^4|\nabla|^\gamma\mathcal{N}_\psi(t;n_0,\vec u_0,\vec b_0)\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac12}\big(\|U_0\|_{X_0}+\epsilon_0\|U\|_X+Q(\|U\|_X)\big);\quad(9.16)$$
$$\big\|\partial_x\mathcal{N}_\psi(t;n_0,\vec u_0,\vec b_0)\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-1}\big(\|U_0\|_{X_0}+\epsilon_0\|U\|_X+Q(\|U\|_X)\big);\quad(9.17)$$
$$\big\|\partial_x\nabla(\mathcal{N}_\psi)(t;n_0,\vec u_0,\vec b_0)\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-1}\big(\|U_0\|_{X_0}+\epsilon_0\|U\|_X+Q(\|U\|_X)\big).\quad(9.18)$$

We estimate $\mathcal{N}_\psi(t;n,\vec u,\psi)$ in (9.8) terms by terms.

9.3.1. $\int_0^t\partial_y\partial_t K(t-s)N_0(s)\,ds$

The estimates on this term are much similar as the ones in Section 6.3.1. Since the estimates on N_0^h are standard, we only consider N_0^l. Then by Proposition 4.5 (iii), for $\beta=\gamma,1,2$ we have

$$\begin{aligned}&\Big\|\langle\nabla\rangle^4|\nabla|^\beta\int_0^t\partial_y\partial_t K(t-s)N_0^l(s)\,ds\Big\|_{L^2_{xy}}\\
&\lesssim\int_0^t\Big\||\nabla|^\beta\nabla\partial_y\partial_t K(t-s)\cdot\langle\nabla\rangle^4(n_{\le\langle s\rangle^{0.01}}\vec u_{\le\langle s\rangle^{0.01}})(s)\Big\|_{L^2_{xy}}ds\\
&\lesssim\int_0^t\langle s\rangle^{0.04}\langle t-s\rangle^{-\frac\beta2}\|n(s)\|_{L^2_{xy}}\|\vec u(s)\|_{L^\infty_{xy}}\,ds\\
&\lesssim\int_0^t\langle t-s\rangle^{-\frac\beta2}\langle s\rangle^{-1.21}\|U\|_X^2\end{aligned}$$

$$\lesssim \langle t\rangle^{-\frac{\beta}{2}}\|U\|_X^2.$$

Letting $\beta=\gamma,1,2$, we have the claimed estimates.

9.3.2. $\int_0^t \partial_y \Delta K(t-s)N_0(s)\,ds$

Note that the operator $\partial_y\nabla\Delta K(t)$ has the similar estimates with $(\partial_{tt}-\Delta\partial_t-\Delta)\Delta K(t)$. One just repeats the precess in Section 6.3.2, to get the following estimates,

$$\big\|\langle\nabla\rangle^4|\nabla|^\gamma\int_0^t \partial_y\Delta K(t-s)N_0(s)\,ds\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0}+Q(\|U\|_X)\big);$$

$$\big\|\partial_x\int_0^t \partial_y\Delta K(t-s)N_0(s)\,ds\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-1}\big(\|U_0\|_{X_0}+Q(\|U\|_X)\big);$$

$$\big\|\nabla\partial_x\int_0^t \partial_y\Delta K(t-s)N_0(s)\,ds\big\|_{L^2_{xy}}\lesssim\langle t\rangle^{-1}\big(\|U_0\|_{X_0}+Q(\|U\|_X)\big).$$

9.3.3. $\int_0^t \partial_{xy}K(t-s)N_1(s)\,ds$

Using (7.35), we have

$$N_1=N_{13}+N_{14}+N_{15}+N_{16}.$$

Again, we only consider $N^l_{1j}, j=3,\cdots,6$. N^l_{13} and N^l_{14} behave well, we just give the sketch of the estimation. By (7.36), $\|N^l_{13}\|_{L^1_{xy}}$ decays faster than $\langle s\rangle^{-1}$, it is easy. By using the corresponding way in Section 7.3.2, we can get the desirable estimates. For N^l_{14}, similar as (7.37) and (7.38), and by the corresponding ways there, we also easily get the desirable estimates and thus the details are omitted here. Now we focus our attentions on the terms N^l_{15} and N^l_{16}.

For N^l_{15}, by Proposition 4.2 (ii) ($\beta'=\gamma$) we have

$$\begin{aligned}
&\Big\|\langle\nabla\rangle^4|\nabla|^\gamma\int_0^t \partial_{xy}K(t-s)N^l_{15}(s)\,ds\Big\|_{L^2_{xy}}\\
&\lesssim\int_0^t\Big\||\nabla|^\gamma\nabla\partial_{xy}K(t-s)\cdot P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle^4(\partial_x\psi\nabla\psi)(s)\,ds\Big\|_{L^2_{xy}}\,ds\\
&\lesssim\int_0^t\langle t-s\rangle^{-\frac{\gamma}{2}}\big\|\langle\nabla\rangle^4P_{\lesssim\langle s\rangle^{0.01}}(\partial_x\psi\nabla\psi)(s)\big\|_{L^2_{xy}}\,ds\\
&\lesssim\int_0^t\langle t-s\rangle^{-\frac{\gamma}{2}}\langle s\rangle^{0.04}\|\partial_x\psi\|_{L^2_{xy}}\|\nabla\psi\|_{L^\infty_{xy}}\,ds
\end{aligned}$$

$$\lesssim \int_0^t \langle t-s\rangle^{-\frac{\gamma}{2}} \langle s\rangle^{0.04-1} \|U\|_X^2 \, ds$$

$$\lesssim \langle t\rangle^{-\frac{1}{4}} \|U\|_X^2.$$

Moreover, from Proposition 4.3 (i),

$$\begin{aligned}
\Big\|\partial_x \int_0^t \partial_{xy} K(t-s) N_{15}^l(s)\, ds\Big\|_{L^2_{xy}} &\lesssim \int_0^t \Big\|\partial_{xxy}\nabla K(t-s)\cdot(\partial_x\psi\nabla\psi)(s)\Big\|_{L^2_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \big\|\langle\nabla\rangle^{\frac{1}{2}+}(\partial_x\psi\nabla\psi)(s)\big\|_{L^{\frac{4}{3}}_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \|\langle\nabla\rangle\partial_x\psi\|_{L^2_{xy}} \|\nabla\psi\|^{\frac{1}{2}}_{L^2_{xy}} \|\langle\nabla\rangle^4\nabla\psi\|^{\frac{1}{2}}_{L^\infty_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{-\frac{1}{2}-\frac{1}{8}-\frac{1}{4}} \|U\|_X^2 \, ds \\
&\lesssim \langle t\rangle^{-\frac{1}{2}} \|U\|_X^2.
\end{aligned}$$

Similarly, by Proposition 4.3 (ii) instead,

$$\begin{aligned}
\Big\|\nabla\partial_x \int_0^t \partial_{xy} K(t-s) N_{15}^l(s)\, ds\Big\|_{L^2_{xy}} &\lesssim \int_0^t \Big\|\partial_{xxy}\nabla\nabla K(t-s)\cdot(\partial_x\psi\nabla\psi)(s)\Big\|_{L^2_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \big\|P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle(\partial_x\psi\nabla\psi)(s)\big\|_{L^2_{xy}} ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{0.01-1} \|U\|_X^2 \, ds \\
&\lesssim \langle t\rangle^{-\frac{3}{4}} \|U\|_X^2.
\end{aligned}$$

For N_{16}^l, we need some special handling. Since $\gamma > \frac{1}{2}$, by Proposition 4.2 (i) and (ii) (in low and high frequence cases respectively), and Bernstein's inequality,

$$
\begin{aligned}
&\Big\|\langle\nabla\rangle^4|\nabla|^\gamma\int_0^t \partial_{xy}K(t-s)N_{16}^l(s)\,ds\Big\|_{L^2_{xy}}\\
&\lesssim \int_0^t \Big\|P_{\le 1}|\nabla|^\gamma\partial_{xy}K(t-s)\ \partial_x(\frac{1}{2}|\nabla\psi|^2-n^2)(s)\Big\|_{L^2_{xy}}\,ds\\
&\quad+\int_0^t \Big\|P_{\ge 1}|\nabla|^{\gamma+1}\partial_{xy}K(t-s)\ P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle^3\partial_x(\frac{1}{2}|\nabla\psi|^2-n^2)(s)\Big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{\gamma}{2}}\big\|\partial_x(\frac{1}{2}|\nabla\psi|^2-n^2)(s)\big\|_{L^1_{xy}}\\
&\quad+\int_0^t \langle t-s\rangle^{-\frac{1}{2}}\big\|P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle^{4+\gamma}\partial_x(\frac{1}{2}|\nabla\psi|^2-n^2)(s)\big\|_{L^2_{xy}}\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{\gamma}{2}}\big(\|\nabla\psi\|_{L^2_{xy}}\|\nabla\partial_x\psi\|_{L^2_{xy}}+\|n\|_{L^2_{xy}}\|\partial_x n\|_{L^2_{xy}}\big)\\
&\quad+\int_0^t \langle t-s\rangle^{-\frac{1}{2}}\langle s\rangle^{0.05}\big(\|\nabla\psi\|_{L^\infty_{xy}}\|\nabla\partial_x\psi\|_{L^2_{xy}}+\|n\|_{L^\infty_{xy}}\|\partial_x n\|_{L^2_{xy}}\big)\,ds\\
&\lesssim \int_0^t \langle t-s\rangle^{-\frac{\gamma}{2}}\langle s\rangle^{-1}\|U\|_X^2\,ds+\int_0^t \langle t-s\rangle^{-\frac{1}{2}}\langle s\rangle^{-1.25+0.05}\|U\|_X^2\,ds\\
&\lesssim \langle t\rangle^{-\frac{1}{4}}\|U\|_X^2.
\end{aligned}
$$

Further, we use the argument in Section 7.3.5, and write

$$
\begin{aligned}
&\Big\|\partial_x\int_0^t \partial_{xy}K(t-s)N_{16}(s)\,ds\Big\|_{L^2_{xy}}\\
&\le \Big\|\int_0^{t/2} \partial_{xxy}K(t-s)N_{16}(s)\,ds\Big\|_{L^2_{xy}}+\Big\|\int_{t/2}^t \partial_{xxy}K(t-s)N_{16}(s)\,ds\Big\|_{L^2_{xy}}\\
&= \Big\|\int_0^{t/2} \partial_{xxxy}K(t-s)(\frac{1}{2}|\nabla\psi|^2-n^2)(s)\,ds\Big\|_{L^2_{xy}} \qquad (9.19)\\
&\quad+\Big\|\int_{t/2}^t \partial_{xxy}K(t-s)\partial_x(\frac{1}{2}|\nabla\psi|^2-n^2)(s)\,ds\Big\|_{L^2_{xy}}. \qquad (9.20)
\end{aligned}
$$

For (9.19), using Proposition 4.3 (i),

$$
\begin{aligned}
(9.19) &\lesssim \int_0^{t/2} \left\| \nabla \partial_{xxy} K(t-s)(\frac{1}{2}|\nabla\psi|^2 - n^2)(s) \right\|_{L^2_{xy}} ds \\
&\lesssim \int_0^{t/2} \langle t-s\rangle^{-1} \left\| \langle\nabla\rangle^{1+}(\frac{1}{2}|\nabla\psi|^2 - n^2)(s) \right\|_{L^1_{xy}} ds \\
&\lesssim \int_0^{t/2} \langle t-s\rangle^{-1} \left(\|\langle\nabla\rangle^2 \nabla\psi\|^2_{L^2_{xy}} + \|\langle\nabla\rangle^2 n\|^2_{L^2_{xy}} \right) ds \\
&\lesssim \langle t\rangle^{-1} \int_0^{t/2} \langle s\rangle^{-\frac{1}{2}} \|U\|_X^2 \, ds \\
&\lesssim \langle t\rangle^{-\frac{1}{2}} \|U\|_X^2 .
\end{aligned}
$$

While by Beinstein's inequality, and Proposition 4.2 (i) ($\beta = 1, \frac{3}{2}$ in the low and high frequence cases respectively),

$$
\begin{aligned}
(9.20) &\lesssim \int_{t/2}^{t} \left\| P_{\le 1} \nabla \partial_{xy} K(t-s)\ \partial_x(\frac{1}{2}|\nabla\psi|^2 - n^2)(s) \right\|_{L^2_{xy}} ds \\
&\quad + \int_{t/2}^{t} \left\| P_{\ge 1} |\nabla|^{\frac{3}{2}} \partial_{xy} K(t-s)\ P_{\lesssim \langle s\rangle^{0.01}} \partial_x(\frac{1}{2}|\nabla\psi|^2 - n^2)(s) \right\|_{L^2_{xy}} ds \\
&\lesssim \int_{t/2}^{t} \langle t-s\rangle^{-\frac{1}{2}} \left(\|\nabla\psi\|_{L^2_{xy}} \|\nabla\partial_x\psi\|_{L^2_{xy}} + \|n\|_{L^2_{xy}} \|\partial_x n\|_{L^2_{xy}} \right) ds \\
&\quad + \int_{t/2}^{t} \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.01} \left(\|\nabla\psi\|_{L^2_{xy}} \|\nabla\partial_x\psi\|_{L^2_{xy}} + \|n\|_{L^2_{xy}} \|\partial_x n\|_{L^2_{xy}} \right) ds \\
&\lesssim \int_{t/2}^{t} \langle t-s\rangle^{-\frac{1}{2}} \langle s\rangle^{-1} \, ds \|U\|_X^2 + \int_{t/2}^{t} \langle t-s\rangle^{-\frac{3}{4}} \langle s\rangle^{0.01-1} \, ds \|U\|_X^2 \\
&\lesssim \langle t\rangle^{-\frac{1}{2}} \|U\|_X^2 .
\end{aligned}
$$

Thus combining the two estimates above, we get that

$$\Big\|\partial_x \int_0^t \partial_{xy} K(t-s) N_{16}^l(s)\, ds\Big\|_{L^2_{xy}} \lesssim \langle t\rangle^{-\frac{1}{2}} \|U\|_X^2.$$

Similarly, by Proposition 4.3 (i),

$$\begin{aligned}
&\Big\|\nabla\partial_x \int_0^t \partial_{xy} K(t-s) N_{16}(s)\, ds\Big\|_{L^2_{xy}} \\
&\lesssim \int_0^t \Big\|\nabla\partial_{xxy} K(t-s)\partial_x(\frac{1}{2}|\nabla\psi|^2 - n^2)(s)\Big\|_{L^2_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\big\|P_{\lesssim\langle s\rangle^{0.01}}\langle\nabla\rangle^{1+}\partial_x(\frac{1}{2}|\nabla\psi|^2 - n^2)(s)\big\|_{L^1_{xy}}\, ds \\
&\lesssim \int_0^t \langle t-s\rangle^{-1}\langle s\rangle^{-1+0.01}\|U\|_X^2\, ds \\
&\lesssim \langle t\rangle^{-\frac{3}{4}}\|U\|_X^2.
\end{aligned}$$

Therefore, we complete all the estimates in this subsubsection and give the desirable results.

9.3.4. $\int_0^t (\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t-s)N_2(s)\, ds$

Also, by (7.45) in Section 7.3.3, we have

$$N_2 = N_{23} + N_{24} + N_{25}.$$

The treatments on N_{23}, N_{24}, N_{25} are the same as the treatments on N_{13}, N_{14}, N_{16} in Section 9.3.3, so repeating the precess, we have the claimed result as follows,

$$\begin{aligned}
\big\|\langle\nabla\rangle^4|\nabla|^\gamma \int_0^t (\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t-s)N_2(s)\, ds\big\|_{L^2_{xy}} &\lesssim \langle t\rangle^{-\frac{1}{2}}\big(\|U_0\|_{X_0} + Q(\|U\|_X)\big); \\
\big\|\partial_x \int_0^t (\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t-s)N_2(s)\, ds\big\|_{L^2_{xy}} &\lesssim \langle t\rangle^{-1}\big(\|U_0\|_{X_0} + Q(\|U\|_X)\big); \\
\big\|\nabla\partial_x \int_0^t (\partial_{tt} - \Delta\partial_t - \partial_{xx})K(t-s)N_2(s)\, ds\big\|_{L^2_{xy}} &\lesssim \langle t\rangle^{-1}\big(\|U_0\|_{X_0} + Q(\|U\|_X)\big).
\end{aligned}$$

9.3.5. $\int_0^t(\partial_t - \Delta)(\partial_{tt} - \Delta\partial_t - \Delta)K(t-s)N_3(s)\,ds$

As the same reason in Section 6.3.5, this term is in the same level as $\int_0^t \partial_y(\partial_t - \Delta)K(t-s)N_0(s)\,ds$, which has been shown in Sections 9.3.1 and 9.3.2. Thus we have the desirable estimates by the same way and omit the details.

9.3.6. $\lambda\int_0^t(\partial_{tt} - \Delta\partial_t)K(t-s)(\partial_{xy}u(s) + \partial_{yy}v(s))\,ds$

This term can be treated by the same way as in Section 6.3.6, thus we omit the cumbersome details again.

Therefore, we finish all the estimates in this subsection, and obtain Lemma 9.3. Combining with Lemma 9.2, we establish the claimed results in (9.1)–(9.4).

Combining with the estimates obtained in Section 7–9, we finish the proof of (1.7), and thus prove Theorem 1.1.

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Appendix A

A.1. Proof of Lemma 3.4

We denote

$$I = \frac{1}{c}\left\{\frac{1}{\sqrt{b+c}}\left[e^{(-a+\sqrt{b+c})t} - e^{(-a-\sqrt{b+c})t}\right] - \frac{1}{\sqrt{b-c}}\left[e^{(-a+\sqrt{b-c})t} - e^{(-a-\sqrt{b-c})t}\right]\right\},$$

then

$$\begin{aligned}
I &= e^{-at}\frac{1}{c}\int_{-t}^{t}\left(e^{\sqrt{b+cx}} - e^{\sqrt{b-cx}}\right)dx \\
&= e^{-at}\frac{1}{c}\int_{0}^{t}\left(e^{\sqrt{b+cx}} - e^{\sqrt{b-cx}}\right)dx + e^{-at}\frac{1}{c}\int_{-t}^{0}\left(e^{\sqrt{b+cx}} - e^{\sqrt{b-cx}}\right)dx \\
&= e^{-at}\frac{1}{c}\int_{0}^{t}\left(e^{\sqrt{b+cx}} - e^{\sqrt{b-cx}}\right)dx + e^{-at}\frac{1}{c}\int_{0}^{t}\left(e^{-\sqrt{b+cx}} - e^{-\sqrt{b-cx}}\right)dx \\
&:= I_1 + I_2.
\end{aligned}$$

Note that

$$I_1 = e^{-at}\frac{1}{c}\int_0^t e^{\sqrt{b+c}x}\left[1 - e^{(\sqrt{b-c}-\sqrt{b+c})x}\right]dx,$$

and

$$I_2 = -e^{-at}\frac{1}{c}\int_0^t e^{-\sqrt{b-c}x}\left[1 - e^{(\sqrt{b-c}x-\sqrt{b+c})x}\right]dx.$$

Therefore,

$$I = e^{-at}\frac{1}{c}\int_0^t e^{\sqrt{b+c}x}\left[1 - e^{-(\sqrt{b-c}+\sqrt{b+c})x}\right]\left[1 - e^{(\sqrt{b-c}-\sqrt{b+c})x}\right]dx. \tag{A.1}$$

First, since $c > 0$, we have

$$\left|1 - e^{-(\sqrt{b-c}+\sqrt{b+c})x}\right|, \quad \left|1 - e^{(\sqrt{b-c}-\sqrt{b+c})x}\right| \lesssim 1.$$

Hence we obtain from (A.1) that if $b + c \geq 0$, then

$$\begin{aligned}|I| &\lesssim e^{-at}\frac{1}{c}\int_0^t e^{\sqrt{b+c}x}\,dx = e^{-at+\sqrt{b+c}t}\frac{1}{c}\int_0^t e^{\sqrt{b+c}(x-t)}\,dx \\ &\lesssim \frac{1}{c\sqrt{b+c}}e^{-at+\sqrt{b+c}t};\end{aligned} \tag{A.2}$$

if $b + c < 0$, then

$$|I| \lesssim \frac{t}{c}e^{-at}.$$

Second, using the following two inequalities,

$$\begin{aligned}\left|1 - e^{-(\sqrt{b-c}+\sqrt{b+c})x}\right| &\lesssim \left|(\sqrt{b-c}+\sqrt{b+c})x\right|; \\ \left|1 - e^{(\sqrt{b-c}-\sqrt{b+c})x}\right| &\lesssim \left|(\sqrt{b-c}-\sqrt{b+c})x\right|,\end{aligned}$$

we obtain that if $b + c \geq 0$, then

$$|I| \lesssim e^{-at}\frac{1}{c}\int_0^t e^{\sqrt{b+c}x}cx^2\,dx \lesssim e^{-at+\sqrt{b+c}t}\int_0^t e^{\sqrt{b+c}(x-t)}x^2\,dx \lesssim t^3e^{-at+\sqrt{b+c}t};$$

if $b+c<0$, then

$$|I| \lesssim e^{-at}\frac{1}{c}\int_0^t cx^2\,dx \lesssim t^3 e^{-at}.$$

Third, we particularly consider the case of $|c| \ll |b+c|$, then $|c| \ll |b|$ and

$$\Big|1-e^{-(\sqrt{b-c}+\sqrt{b+c})x}\Big| \lesssim 1; \quad \Big|1-e^{(\sqrt{b-c}-\sqrt{b+c})x}\Big| \lesssim \frac{cx}{\sqrt{|b|}}.$$

If $b+c \geq 0$, then using the inequality above,

$$\begin{aligned}|I| &\lesssim e^{-at}\frac{1}{c}\int_0^t e^{\sqrt{b+c}x}\frac{cx}{\sqrt{|b|}}\,dx = e^{-at+\sqrt{b+c}t}\int_0^t e^{\sqrt{b+c}(x-t)}\frac{x}{\sqrt{|b|}}\,dx \\ &\lesssim \frac{t}{\sqrt{|b|}\sqrt{b+c}}e^{-at+\sqrt{b+c}t} \sim \frac{t}{b+c}e^{-at+\sqrt{b+c}t}.\end{aligned}$$

Combining with (A.2), no matter when $|c| \ll |b+c|$ or $|c| \gtrsim |b+c|$, we have

$$|I| \lesssim \frac{\langle t\rangle}{b+c}e^{-at+\sqrt{b+c}t}.$$

In conclusion, we proved that when $b+c<0$,

$$|I| \lesssim \min\{t^3, \frac{t}{c}\}e^{-at};$$

while when $b+c \geq 0$,

$$|I| \lesssim \min\{t^3, \frac{1}{c\sqrt{b+c}}, \frac{\langle t\rangle}{b+c}\}e^{-at+\sqrt{b+}t}.$$

A.2. Proof of (3.14)

We may assume that $x>0, y>0$, otherwise one may replace it by $|x|, |y|$, also we assume that $x \geq y$ by symmetry. Now we prove it by splitting several cases.

Case 1: $x \ll 1, y \ll 1$, then it follows by Tayor's expansion.

Case 2: $x \gtrsim 1, x \gg y$. Then

$$|\frac{\sin x}{x} - \frac{\sin y}{y}| \lesssim |\frac{\sin x}{x}| + |\frac{\sin y}{y}| \lesssim 1,$$

which is less than the right-hand side of (3.14).

Case 3: $x \gtrsim 1, x \sim y$. Then by the mean value theorem,

$$\left|\frac{\sin x}{x} - \frac{\sin y}{y}\right| \lesssim \left|\frac{\bar{x}\cos\bar{x} - \sin\bar{x}}{\bar{x}^2}\right|(x-y) \lesssim \frac{(x-y)}{x},$$

where $\bar{x}$ is middle point satisfying $\bar{x} \in (y, x)$.

Combining with these three cases, we prove (3.14).

A.3. Proof of (6.59)

For any $\xi \in \mathbb{R}^d$, we have

$$\widehat{P_{\leq \langle s\rangle^\beta}} f(\xi) = \chi_{\leq 1}\Big(\frac{\xi}{\langle s\rangle^\beta}\Big)\hat{f}(\xi).$$

Therefore,

$$\partial_s \widehat{P_{\leq \langle s\rangle^\beta}} f(\xi) = |\xi|\langle s\rangle^{-\beta-1}\chi'_{\leq 1}\Big(\frac{\xi}{\langle s\rangle^\beta}\Big)\hat{f}(\xi).$$

Since supp $\chi'_{\leq 1} \in \{|\xi| \sim 1\}$, we have

$$\partial_s P_{\leq \langle s\rangle^\beta} f = \langle s\rangle^{-\beta-1} P_{\sim \langle s\rangle^\beta}|\nabla| f.$$

Thus by Beinstein's inequality, for any $\beta \in \mathbb{R}$, $1 \leq q \leq \infty$,

$$\|\partial_s P_{\leq \langle s\rangle^\beta} f\|_{L^q} \lesssim \langle s\rangle^{-\beta-1}\|P_{\sim\langle s\rangle^\beta}|\nabla| f\|_{L^q} \sim \langle s\rangle^{-1}\|P_{\sim\langle s\rangle^\beta} f\|_{L^q}.$$

A.4. Proof of (9.5)

By the Littlewood–Paley decomposition,

$$\Big\||\nabla|^{\bar{\gamma}}\langle\nabla\rangle\psi\Big\|_{L^\infty_{xy}} \lesssim \sum_{N\leq 1} N^{\bar{\gamma}}\big\|P_N\psi\big\|_{L^\infty_{xy}} + \sum_{N\geq 1} N^{\bar{\gamma}+1}\big\|P_N\psi\big\|_{L^\infty_{xy}},$$

where N is the dyadic number. By Nash's inequality,

$$\begin{aligned}\|P_N\psi\|_{L^\infty_{xy}} &\lesssim \|\partial_{xy}P_N\psi\|^{\frac14}_{L^2_{xy}}\|\partial_x P_N\psi\|^{\frac14}_{L^2_{xy}}\|\partial_y P_N\psi\|^{\frac14}_{L^2_{xy}}\|P_N\psi\|^{\frac14}_{L^2_{xy}}\\ &\lesssim N^{\frac12}\|\partial_x P_N\psi\|^{\frac12}_{L^2_{xy}}\|P_N\psi\|^{\frac12}_{L^2_{xy}}.\end{aligned}$$

Therefore, by Beinstein's inequality,

$$\Big\||\nabla|^{\bar{\gamma}}\langle\nabla\rangle\psi\Big\|_{L^\infty_{xy}} \lesssim \sum_{N\leq 1} N^{\bar{\gamma}+\frac12}\|\partial_x P_N\psi\|^{\frac12}_{L^2_{xy}}\|P_N\psi\|^{\frac12}_{L^2_{xy}}$$

$$
\begin{aligned}
&+ \sum_{N\geq 1} N^{\bar{\gamma}+1+\frac{1}{2}} \|\partial_x P_N \psi\|_{L^2_{xy}}^{\frac{1}{2}} \|P_N \psi\|_{L^2_{xy}}^{\frac{1}{2}} \\
\lesssim &\sum_{N\leq 1} N^{\bar{\gamma}-\frac{\gamma}{2}} \|P_N \partial_x \nabla \psi\|_{L^2_{xy}}^{\frac{1}{2}} \|P_N |\nabla|^{\gamma} \psi\|_{L^2_{xy}}^{\frac{1}{2}} \\
&+ \sum_{N\geq 1} N^{\bar{\gamma}-1-\frac{\gamma}{2}} \|P_N \partial_x \nabla \psi\|_{L^2_{xy}}^{\frac{1}{2}} \big\|P_N \langle\nabla\rangle^4 |\nabla|^{\gamma} \psi\big\|_{L^2_{xy}}^{\frac{1}{2}}.
\end{aligned}
$$

Hence, when $\frac{\gamma}{2} < \bar{\gamma} < 1 + \frac{\gamma}{2}$, the sums are finite. So we have

$$
\big\||\nabla|^{\bar{\gamma}} \langle\nabla\rangle \psi\big\|_{L^\infty_{xy}} \lesssim \|\partial_x \nabla \psi\|_{L^2_{xy}}^{\frac{1}{2}} \big\|\langle\nabla\rangle^4 |\nabla|^{\gamma} \psi\big\|_{L^2_{xy}}^{\frac{1}{2}}.
$$

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