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The 2D Magnetohydrodynamic Equations with Partial or Fractional Dissipation*

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Abstract

This paper surveys recent developments on the global regularity and related problems on the 2D incompressible magnetohydrodynamic (MHD) equations with partial or fractional dissipation. The MHD equations with partial or fractional dissipation are physically relevant and mathematically important. The global regularity and related problems have attracted considerable interests in recent years and there have been substantial developments. In addition to reviewing the existing results, this paper also explains the difficulties associated with several open problems and supply some new results.

1 Introduction

In the last few years there have been substantial developments on the global regularity problem concerning the magnetohydrodynamic (MHD) equations, especially when there is only partial or fractional dissipation. This paper reviews some of these recent results and explains the difficulties associated with several open problems in this direction. Attention will be focused on the two dimensional (2D) whole space case.

The MHD equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. They consist of a coupled system of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Since their initial derivation by the Nobel Laureate H. Alfvén in 1924, the MHD equations have played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [4, 18]).

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The standard incompressible MHD equations can be written as

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.1}$$

where u denotes the velocity field, b the magnetic field, p the pressure, $\nu \geq 0$ the kinematic viscosity and $\eta \geq 0$ the magnetic diffusivity. Besides their wide physical applicability, the MHD equations are also of great interest in mathematics. As a coupled system, the MHD equations contain much richer structures than the Navier-Stokes equations. They are not merely a combination of two parallel Navier-Stokes type equations but an interactive and integrated system. Their distinctive features make analytic studies a great challenge but offer new opportunities.

Our attention will be focused on the initial-value problem of the MHD equations with a given initial data (u_0, b_0) satisfying

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \qquad \nabla \cdot u_0 = 0, \quad \nabla \cdot b_0 = 0.$$

One of the fundamental problems concerning the MHD equations is whether physically relevant regular solutions remain smooth for all time or they develop finite time singularities. This problem can be extremely difficult, even in the 2D case. In recent years the MHD equations have attracted considerable interests and one focus has been on the 2D MHD equations with partial or fractional dissipation. Important progress has been made.

We first explain why the study of the MHD equations with partial or fractional dissipation is relevant and important. When there is no kinematic dissipation or magnetic diffusion, the MHD equations become inviscid and the global regularity problem appears to be out of reach at this moment. In contrast, when both the dissipation and the magnetic diffusion are present, the MHD equations are fully dissipative and the global regularity problem in the 2D case can be solved in a similar way as the one for the 2D Navier-Stokes equations. Mathematically it is natural to explore the intermediate equations that bridge the two extremes: the inviscid MHD and the fully dissipative MHD equations. The MHD equations with partial or fractional dissipation exactly fill this gap. Physically, some of the partially dissipative MHD equations are important models in geophysical or astrophysical applications.

We elaborate on this point in more precise terms and summarize some of the main results on the 2D MHD equations with partial or fractional dissipation. One standard approach on the global regularity problem on the incompressible MHD equations is to divide the process into two steps. The first step is to show the local well-posedness. This step is in general based on the contraction mapping principle and its variants such as successive approximations. In most circumstances we need to restrict to small time interval in order to verify the conditions of the contraction mapping principle. For many incompressible hydrodynamic models such as the Navier-Stokes and the MHD equations, the local well-posedness can be accomplished in this fashion when the initial data are sufficiently regular. The second step is to extend the local solution of the first step into a global (in time) one by establishing suitable global *a priori* bounds on the solutions. That is, one needs to prove that the solution remains bounded at any later time t, even though the bound in general grows in time. Once we have the global bounds, the standard Picard type extension theorem allows us to extend the local solution into a global one. Therefore, the global regularity problem on the MHD equations boils down to the global *a priori* bounds.

In the extreme case when the MHD equations are inviscid, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.2}$$

global bounds on (u, b) in any Sobolev space are not available. It is not clear if smooth solutions of (1.2) can blow up in a finite time, even though local well-posedness in sufficiently regular Sobolev or Besov type spaces are well-known. We will present in Section 2 this local well-posedness result and a regularity criterion. In addition, this section also provides two alternative formulations of the inviscid MHD equations: the Lagrangian-Eulerian formulation and the formulation in terms of a purely Lagrangian variable. These different formulations have their own advantages and may help understand the global regularity issue.

In another extreme case when the MHD equations are fully dissipative, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \end{cases} \tag{1.3}$$

with $\nu > 0$ and $\eta > 0$, the global regularity problem can be solved similarly as for the 2D Navier-Stokes equations. In fact, any initial data $(u_0, b_0) \in L^2(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$ leads to a unique global solution (u, b) that becomes infinitely smooth, namely $(u, b) \in C^\infty(\mathbb{R}^2 \times (t_0, \infty))$ for any $t_0 > 0$. This simple global result is explained in Section 3.

Mathematically it is natural to examine intermediate equations that fill the gap between (1.2) and (1.3). One type of such equations are the MHD equations with partial dissipation,

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu_1 u_{xx} + \nu_2 u_{yy} + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta_1 b_{xx} + \eta_2 b_{yy} + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.4}$$

where the parameters $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\eta_1 \geq 0$ and $\eta_2 \geq 0$. Clearly, when $\nu_1 = \nu_2 = \eta_1 = \eta_2 = 0$, (1.4) reduces to (1.2) while (1.4) with $\nu_1 = \nu_2 > 0$ and $\eta_1 = \eta_2 > 0$ reduces to (1.3). Various partial dissipation cases arise when some of the four parameters but not all are zero. Another type of intermediate equations that bridge (1.2) and (1.3) are the 2D MHD equations with fractional dissipation

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p - \nu(-\Delta)^\alpha u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = -\eta(-\Delta)^\beta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.5}$$

where the fractional Laplacian operator is defined via the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

When $\alpha = \beta = 1$, (1.5) becomes (1.3) while $\alpha = \beta = 0$, or more precisely $\nu = \eta = 0$, corresponds to the inviscid MHD equations in (1.2). Recent efforts are devoted to seeking the global regularity of (1.5) for smallest possible parameters $\alpha \geq 0$ and $\beta \geq 0$.

One special partial dissipation case is the 2D resistive MHD equations, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.6}$$

where $\eta > 0$ denotes the magnetic diffusivity (resistivity). (1.6) is applicable when the fluid viscosity can be ignored while the role of resistivity is important such as in magnetic reconnection and magnetic turbulence. Magnetic reconnection refers to the breaking and reconnecting of oppositely directed magnetic field lines in a plasma and is at the heart of many spectacular events in our solar system such as solar flares and northern lights. The mathematical study of (1.6) may help understand the Sweet-Parker model arising in magnetic reconnection theory [41]. The global regularity problem is not completely solved at this moment, but recent efforts on this problem have significantly advanced our understanding. Global *a priori* bounds in very regular functional settings have been obtained, but the global bound for $\omega \in L^\infty(0,T; L^\infty)$ is lacking. As a consequence, the uniqueness and the higher regularity can not be achieved. Section 4 reviews several recent results, explains the difficulty involved, details two hopeful attempts and discusses paths that may potentially yield the solution to this intriguing problem. In particular, we will present the work of Q. Jiu, D. Niu, J. Wu, X. Xu and H. Yu [30] as well as some *a priori* estimates obtained in a work in progress with P. Zhang [54]. In addition, we state a theorem of C. Cao, J. Wu and B. Yuan [12] on a slightly more regular system that points to the criticality of this global regularity problem.

Another significant partial dissipation case is the 2D MHD equations with velocity dissipation and no resistivity, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.7}$$

(1.7) models fluids that can be treated as perfect conductors such as strongly collisional plasmas. In addition, the breakdown of ideal MHD is known to be the cause of solar flares, the largest explosions in the solar system [40]. The intriguing problem of whether (1.7) can blow up in a finite time has recently attracted considerable interests. Recent strategy has been to seek global solutions near an equilibrium. Since the pioneering work of F. Lin, L. Xu and P. Zhang [36], this direction has flourished and a rich array of results have been obtained. Section 5 presents these recent results. We start with a local well-posedness result of P. Constantin achieved via the Lagrangian-Eulerian approach. We then state and describe a global result (near an equilibrium) of Lin, Xu and Zhang [36] and their Lagrangian approach. We then present the main result of X. Ren, J. Wu, Z. Xiang and Z. Zhang [43] and outline the proof. A closely related work of J. Wu, Y. Wu and X. Xu [55] via the method of dispersive equations is then supplied. We also briefly summarize the results of X. Hu and F. Lin [27], and of T. Zhang [62].

Section 6 explains the global existence and uniqueness result of C. Cao and J. Wu [11] on the 2D MHD equations with mixed kinematic dissipation and magnetic diffusion, namely

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + \nu u_{yy} + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = \eta b_{xx} + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0 \end{cases}$$

or

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + \nu u_{xx} + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = \eta b_{yy} + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0. \end{cases}$$

Section 7 is devoted to the partial dissipation case when the 2D MHD equations involve only the horizontal dissipation and the horizontal magnetic diffusion,

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \partial_{xx} u + b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b = \partial_{xx} b + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0. \end{cases}$$

We describes the results of C. Cao, D. Regmi and J. Wu [8] as well as the results of a followup preprint of C. Cao, D. Regmi, J. Wu and X. Zheng [9].

The last section summarizes the results on the global regularity problem concerning the incompressible MHD equations with fractional dissipation and proposes an open problem.

There is large literature on the MHD equations. This short survey focuses on the 2D incompressible MHD equations with partial or fractional dissipation. Due to the page constraints, we are not able to cover many significant results on the MHD equations, especially those on the 3D MHD equations (see, e.g., [7, 10, 14, 15, 16, 22, 23, 24, 25, 26, 29, 31, 33, 34, 37, 47, 49, 50, 52, 56, 58, 59, 61, 63]).

2 The inviscid MHD equations

This section is devoted to the initial-value problem for the inviscid MHD equations

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x) \end{cases} \tag{2.1}$$

where u_0 and b_0 satisfy $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$. Whether or not a reasonably regular initial datum (u_0, b_0) always leads to a globally regular solution of (2.1) remains an outstanding open problem. Besides the standard local existence and uniqueness result in the Sobolev setting, this section also provides two alternative formulations of the inviscid MHD equations: a purely Lagrangian approach and the Lagrangian-Eulerian formulation. These different formulations have their own advantages and may aid in the understanding of the global regularity issue. The rest of this section is divided into three subsections.

2.1 A standard local existence and uniqueness result

This subsection provides the local existence and uniqueness result in the Sobolev setting and explains why it is difficult to prove global *a priori* bounds. The Beale-Kato-Majda type regularity criterion follows as a consequence of this explanation.

Theorem 2.1. *Let $s > 2$. Assume $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then (2.1) has a unique local classical solution $(u, b) \in C([0, T_0); H^s)$ for some $T_0 = T_0(\|(u_0, b_0)\|_{H^s}) > 0$. In addition, if, for $T > T_0$,*

$$\int_0^T \left(\|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} + \|j(t)\|_{\dot{B}^0_{\infty,\infty}} \right) dt < \infty, \tag{2.2}$$

then the solution remains in H^s for any $t \le T$. Here $\omega = \nabla \times u$ denotes the vorticity and $j = \nabla \times b$ denotes the current density.

Here $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space (see, e.g., [1, 2, 39, 44, 48]). The original Beale-Kato-Majda criterion on the 3D Euler equations involves the L^∞-norm of the vorticity [38]. Due to the embedding

$$L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty},$$

the assumption in (2.2) is weaker than the corresponding one with L^∞-norm.

The condition that $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$ may not be weakened to $(u_0, b_0) \in H^2(\mathbb{R}^2)$. The work of Bourgain and Li [5, 6] on the Euler equations may be extended to the inviscid MHD equations to indicate the ill-posedness of (2.1) in $H^2(\mathbb{R}^2)$.

The proof of the existence and uniqueness part in Theorem 2.1 is standard. One may follow the steps in the proof of Theorem 3.4 of the book by Majda and Bertozzi [38], even though Theorem 3.4 there requires more regularity on the initial data. Another approach is Friedrichs' method, a regularization approximation process by spectral cutoffs (see, e.g., [1]). Our intention here is to explain why it is difficult to obtain global *a priori* bounds on the Sobolev norms of the solutions. The regularity criterion follows as a consequence.

Proof of Theorem 2.1. As we just remarked above, we only supply the proof for the regularity criterion. In the process, we explain the difficulties associated with the global bounds. When $\nabla \cdot b_0 = 0$, the solution (u, b) of (2.1) preserves this property, namely $\nabla \cdot b = 0$. This, together with $\nabla \cdot u = 0$, allows us to obtain the L^2-bound easily,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

But due to the lack of dissipation and magnetic diffusion, global bounds for any Sobolev-norm appear to be impossible. We use the estimate of the H^1-norm of (u, b) as an example. Due to $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$,

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2} \quad \text{and} \quad \|\nabla b\|_{L^2} = \|j\|_{L^2},$$

and consequently it suffices to consider the L^2-norm of (ω, j), which satisfies

$$\begin{cases} \omega_t + u \cdot \nabla \omega = b \cdot \nabla j, \\ j_t + u \cdot \nabla j = b \cdot \nabla \omega + 2\partial_x b_1(\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1(\partial_y b_1 + \partial_x b_2). \end{cases} \tag{2.3}$$

Dotting the first equation by ω and the second by j, adding up the resulting equations and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) = 2\int j(\partial_x b_1(\partial_y u_1 + \partial_x u_2) - \partial_x u_1(\partial_y b_1 + \partial_x b_2)).$$

We note that (2.3) does have a structure that allows us to eliminate four of the nonlinear terms. The terms on the right-hand side are of the triple product form and can not be bounded suitably unless we make the assumption

$$\int_0^T \|\nabla u\|_\infty dt < \infty \quad \text{or} \quad \int_0^T \|j\|_\infty dt < \infty.$$

This explains why we need (2.2) in order to control the H^1-norm. More generally the H^s-norm of (u, b) obeys the differential inequality, for $Y(t) = \|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2$,

$$\frac{d}{dt}Y(t) \le C(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})Y(t). \tag{2.4}$$

We can further bound $\|\nabla u\|_{L^\infty}$ and $\|\nabla b\|_{L^\infty}$ in terms of the logarithmic interpolation inequalities, for $s > 2$,

$$\|\nabla u\|_{L^\infty} \le C\left(1 + \|u\|_{L^2} + \|\omega\|_{\dot{B}^0_{\infty,\infty}} \log(1 + \|u\|_{\dot{H}^s})\right),$$
$$\|\nabla b\|_{L^\infty} \le C\left(1 + \|b\|_{L^2} + \|j\|_{\dot{B}^0_{\infty,\infty}} \log(1 + \|b\|_{\dot{H}^s})\right).$$

Inserting the bounds above in (2.4) yields

$$\frac{d}{dt}Y(t) \le C\left(1 + \|\omega\|_{\dot{B}^0_{\infty,\infty}} + \|j\|_{\dot{B}^0_{\infty,\infty}}\right) Y(t) \log(e + Y(t)).$$

Osgood's inequality then yields the desired regularity criterion part. This completes the proof of Theorem 2.1. □

2.2 Lagrangian-Eulerian type formulation

The Lagrangian-Eulerian approach of P. Constantin has the advantage that the Lagrangian-Eulerian evolution system and the solution map are Lipschitz continuous in lower regularity path spaces and there is Lipschitz dependence of solutions on initial data in these spaces [17]. The inviscid MHD equations can not be written in the exact Lagrangian-Eulerian form of Constantin, but we can still recast the inviscid MHD equations as an evolution system that shares some of the fine characteristics of the original Lagrangian-Eulerian formulation.

We now detail this formulation and start with the particle trajectory. For a sufficiently smooth velocity field u, say $u \in L^1(0,T; W^{1,\infty})$, the particle trajectory (or flow map) $X(a,t)$ with $a \in \mathbb{R}^2$ and $t \geq 0$ obeys the ordinary differential equation

$$\begin{cases} \dfrac{dX(a,t)}{dt} = u(X(a,t),t), \\ X(a,0) = a. \end{cases}$$

In addition, $X(a,t)$ is invertible for any fixed $t \in [0,T]$, and we follow the notation of Constantin [17] to write the inverse as

$$X^{-1}(x,t) = A(x,t),$$

which will also be called the "back-to-labels" map. The identities

$$A(X(a,t),t) = a \quad \text{for any } a \in \mathbb{R}^2 \text{ and} \quad X(A(x,t),t) = x \quad \text{for any } x \in \mathbb{R}^2$$

allow us to derive the equation of A,

$$\begin{cases} \partial_t A + u \cdot \nabla A = 0, \\ A(x,0) = x. \end{cases}$$

To derive the Lagrangian-Eulerian form, we rewrite the equation of b into an equivalent form. It is clear from the equation of b that

$$\sigma = b \otimes b \equiv (b_i b_j)$$

satisfies

$$\partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^*, \tag{2.5}$$

where $(\nabla u)^*$ denotes the transpose of ∇u. Thus, we can write (2.1) as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nabla \cdot \sigma, \\ \nabla \cdot u = 0, \\ \partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^*, \end{cases} \tag{2.6}$$

which, in terms of the vorticity ω and σ, can be further written as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nabla \times \nabla \cdot \sigma, \\ u = \nabla^\perp \Delta^{-1} \omega, \\ \partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^*. \end{cases} \tag{2.7}$$

For notational convenience, we define the operators $\mathbb{U}$ and $\mathbb{G}$ as

$$u = \mathbb{U}(\omega) \equiv \nabla^\perp \Delta^{-1} \omega, \qquad \nabla u = \mathbb{G}(\omega) \equiv \nabla \nabla^\perp \Delta^{-1} \omega.$$

In terms of the Lagrangian variables

$$\zeta(a,t) = \omega \circ X \equiv \omega(X(a,t),t) \quad \text{or} \quad \omega(x,t) = \zeta \circ A,$$
$$\tau(a,t) = \sigma \circ X \quad \text{or} \quad \sigma = \tau \circ A,$$

(2.7) can be written as

$$\begin{cases} \dfrac{d}{dt}\zeta = (\nabla \times \nabla \cdot (\tau \circ A)) \circ X, \\ \dfrac{d}{dt}\tau = g\tau + \tau g^*, \end{cases} \tag{2.8}$$

where

$$g = \mathbb{G}(\zeta \circ A) \circ X.$$

Therefore, we have reduced the inviscid MHD equations in (2.1) to the following system in terms of the Lagrangian variables X, ζ and τ as

$$\begin{cases} \dfrac{d}{dt}X = \mathbb{U}(\zeta \circ A) \circ X, \\ \dfrac{d}{dt}\zeta = (\nabla \times \nabla \cdot (\tau \circ A)) \circ X, \\ \dfrac{d}{dt}\tau = g\tau + \tau g^*. \end{cases}$$

Integrating in time yields

$$\begin{cases} X(a,t) = a + \displaystyle\int_0^t \mathbb{U}(\zeta \circ A) \circ X(a,\tau)d\tau, \\ \zeta(a,t) = \omega_0(a) + \displaystyle\int_0^t (\nabla \times \nabla \cdot (\tau \circ A)) \circ X(a,\tau)d\tau, \\ \tau(a,t) = (b_0 \otimes b_0)(a) + \displaystyle\int_0^t (g\tau + \tau g^*)(\tau)d\tau. \end{cases}$$

We can summarize what we have derived above as the following theorem.

Theorem 2.2. *The 2D inviscid incompressible MHD equations in* (2.1) *are formally equivalent to the following Lagrangian-Eulerian type formulation*

$$\begin{cases} X(a,t) = a + \displaystyle\int_0^t \mathbb{U}(\zeta \circ A) \circ X(a,\tau)d\tau, \\ \zeta(a,t) = \omega_0(a) + \displaystyle\int_0^t (\nabla \times \nabla \cdot (\tau \circ A)) \circ X(a,\tau)d\tau, \\ \tau(a,t) = (b_0 \otimes b_0)(a) + \displaystyle\int_0^t (g\tau + \tau g^*)(\tau)d\tau, \end{cases}$$

where X denotes the particle trajectory (or flow path), A is the inverse of X, ζ and τ are the Lagrangian counterparts of the Eulerian variables ω and $b \otimes b$, respectively, and $\mathbb{U}(f) = \nabla^\perp \Delta^{-1} f$ corresponds to the 2D Biot-Savart kernel

$$\mathbb{U}(f) = \int_{\mathbb{R}^2} K_2(x-y)f(y)dy, \qquad K_2(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \tag{2.9}$$

$\mathbb{G}(f) \equiv \nabla\nabla^\perp\Delta^{-1}f$ *is given by the explicit representation*

$$\mathbb{G}(f)(x,t) = PV\int_{\mathbb{R}^2} \frac{P(x-y)}{|x-y|^2} f(y)dy + \frac{f(x)}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with

$$P(z) = \frac{1}{2\pi|z|^2}\begin{bmatrix} 2z_1z_2 & z_2^2 - z_1^2 \\ z_2^2 - z_1^2 & -2z_1z_2 \end{bmatrix},$$

and $g = \mathbb{G}(\zeta \circ A) \circ X$ *and* g^* *is the transpose of* g.

2.3 Purely Lagrangian formulation

It is possible to reformulate the inviscid MHD equations completely in terms of the particle trajectory and the initial data. This purely Lagrangian formulation allows us to represent all relevant physical quantities in terms of the particle trajectory.

Due to the divergence-free condition $\nabla \cdot b = 0$, we can write $b = \nabla^\perp\psi$ for a scalar function ψ and the induction equation for b,

$$\partial_t b + u \cdot \nabla b = b \cdot \nabla u, \qquad b(x,0) = b_0(x)$$

is then reduce to a transport equation for ψ,

$$\partial_t \psi + u \cdot \nabla\psi = 0, \qquad \psi(x,0) = \psi_0(x) \text{with } \nabla^\perp\psi_0 = b_0. \tag{2.10}$$

Equivalently, ψ can be represented by the back-to-labels map A (defined in the previous subsection), namely the inverse of X,

$$\psi(x,t) = \psi_0(A(x,t)). \tag{2.11}$$

Since $b = \nabla^\perp\psi$, we have $j = \nabla \times b = \Delta\psi$ and $b \cdot \nabla j$ can be written in terms of the Poisson bracket

$$b \cdot \nabla j = J(\psi, \Delta\psi),$$

where J is the usual Poisson bracket,

$$J(f,g) = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g. \tag{2.12}$$

Therefore, the vorticity equation

$$\partial_t \omega + u \cdot \nabla\omega = b \cdot \nabla j$$

can be represented in the Lagrangian variable by

$$\frac{d}{dt}(\omega \circ X) = J(\psi, \Delta\psi) \circ X. \tag{2.13}$$

$J(\psi, \Delta\psi) \circ X$ can be further represented in terms of ψ_0 and X. In fact, according to (2.11) and the identity $A(X(a,t),t) = a$, we have

$$(\partial_1\psi) \circ X = J(\psi_0, X_2), \qquad (\partial_2\psi) \circ X = J(X_1, \psi_0),$$

where J again is the Poisson bracket. $\partial_1 \Delta\psi$ and $\partial_2 \Delta\psi$ are more complex and we shall not presenting them. Integrating (2.13) in time yields

$$\omega(X(a,t),t) = \omega_0(a) + \int_0^t (J(\psi, \Delta\psi) \circ X)(a,\tau)d\tau.$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}X(a,t) \\
&= u(X(a,t),t) \\
&= \int_{\mathbb{R}^2} K_2(X(a,t) - X(\widetilde{a},t))\omega(X(\widetilde{a},t),t)d\widetilde{a} \\
&= \int_{\mathbb{R}^2} K_2(X(a,t) - X(\widetilde{a},t)) \left(\omega_0(\widetilde{a}) + \int_0^t (J(\psi, \Delta\psi) \circ X)(\widetilde{a},\tau)d\tau\right) d\widetilde{a}
\end{aligned}$$

and $X(a,0) = a$. We sum this up in the following theorem.

Theorem 2.3. *The 2D inviscid incompressible MHD equations in* (2.1) *are formally equivalent to the following purely Lagrangian formulation*

$$\begin{aligned}
\frac{d}{dt}X(a,t) = &\int_{\mathbb{R}^2} K_2(X(a,t) - X(\widetilde{a},t)) \\
&\cdot \left(\omega_0(\widetilde{a}) + \int_0^t (J(\psi, \Delta\psi) \circ X)(\widetilde{a},\tau)d\tau\right) d\widetilde{a}, \\
X(a,0) = a,&
\end{aligned}$$

where K_2 is the Biot-Savart kernel defined in (2.9), *J is the Poisson bracket defined in* (2.12) *and $\nabla^\perp \psi = b$.*

3 The fully dissipative MHD equations

This section turns to another extreme case of the 2D incompressible MHD equations, the fully dissipative MHD equations

$$\begin{cases}
u_t + u \cdot \nabla u = -\nabla p + \nu\Delta u + b \cdot \nabla b, \\
b_t + u \cdot \nabla b = \eta\Delta b + b \cdot \nabla u, \\
\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\
u(x,0) = u_0(x), \quad b(x,0) = b_0(x)
\end{cases} \tag{3.1}$$

where u_0 and b_0 satisfy $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. The global regularity problem on (3.1) can be solved following the similar approaches as those for the 2D incompressible Navier-Stokes equations. The initial datum (u_0, b_0) does not have to be smooth. In fact, any $(u_0, b_0) \in L^2(\mathbb{R}^2)$ leads to a unique global solution that becomes smooth instantaneously. More precisely, we have the following theorem.

Theorem 3.1. *Let $(u_0, b_0) \in L^2(\mathbb{R}^2)$. Then (3.1) has a unique global solution (u, b) satisfying,*

$$u, b \in L^\infty(0,\infty; L^2(\mathbb{R}^2)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^2)). \tag{3.2}$$

In addition, for any $t_0 > 0$, (u, b) is smooth,

$$(u, b) \in C^\infty(\mathbb{R}^2 \times (t_0,\infty)). \tag{3.3}$$

The approaches for the 2D incompressible Navier-Stokes equations can be extended to (3.1). One approach is to use the method of Friedriches, namely regularization approximation via cutoffs in the Fourier space. The second approach is to use the classical fixed point theorem involving bilinear functions (see, e.g., [1, p. 207]). We shall not provide the details here. In the following proof, we emphasize that solutions of the fully dissipative MHD equations in the natural energy space (3.2) are unique. In addition, we explain the mechanism why (3.3) is true.

Proof. As aforementioned, we only prove the uniqueness part and (3.3). Assume that $(u^{(1)}, b^{(1)})$ and $(u^{(2)}, b^{(2)})$ are two solutions of (3.1). Then the difference

$$(\widetilde{u}, \widetilde{b}) = (u^{(1)} - u^{(2)}, b^{(1)} - b^{(2)})$$

satisfies

$$\begin{aligned}
\partial_t \widetilde{u} + u^{(1)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(2)} &= \nu \Delta \widetilde{u} + b^{(1)} \cdot \nabla \widetilde{b} + \widetilde{b} \cdot \nabla b^{(2)}, \\
\partial_t \widetilde{b} + u^{(1)} \cdot \nabla \widetilde{b} + \widetilde{u} \cdot \nabla b^{(2)} &= \eta \Delta \widetilde{b} + b^{(1)} \cdot \nabla \widetilde{u} + \widetilde{b} \cdot \nabla u^{(2)}.
\end{aligned}$$

Taking the inner product with $(\widetilde{u}, \widetilde{b})$, we obtain after integration by parts,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2\right) + \nu\|\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\nabla\widetilde{b}\|_{L^2}^2 \\
&= -\int \widetilde{u}\cdot\nabla u^{(2)}\cdot\widetilde{u} - \int \widetilde{u}\cdot\nabla b^{(2)}\cdot\widetilde{b} \\
&\quad + \int \widetilde{b}\cdot\nabla b^{(2)}\cdot\widetilde{u} + \int \widetilde{b}\cdot\nabla u^{(2)}\cdot\widetilde{b}.
\end{aligned} \tag{3.4}$$

The four terms have similar structure and can be bounded similarly. We provide the bound for one of them. By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
\left|\int \widetilde{u}\cdot\nabla b^{(2)}\cdot\widetilde{b}\right| &\le \|\widetilde{u}\|_{L^4}\|\widetilde{b}\|_{L^4}\|\nabla b^{(2)}\|_{L^2} \\
&\le C\|\nabla b^{(2)}\|_{L^2}\|\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\widetilde{b}\|_{L^2}^{\frac{1}{2}}\|\nabla\widetilde{b}\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{1}{8}\nu\|\nabla\widetilde{u}\|_{L^2}^2 + \frac{1}{8}\eta\|\nabla\widetilde{b}\|_{L^2}^2 + C\|\nabla b^{(2)}\|_{L^2}^2\|(\widetilde{u},\widetilde{b})\|_{L^2}^2.
\end{aligned}$$

After inserting these bounds in (3.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2\right) + \nu\|\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\nabla\widetilde{b}\|_{L^2}^2 \\
&\le C\left(\|\nabla u^{(2)}\|_{L^2}^2 + \|\nabla b^{(2)}\|_{L^2}^2\right)\|(\widetilde{u},\widetilde{b})\|_{L^2}^2.
\end{aligned}$$

Applying Gronwall's inequality and invoking the fact that $u^{(2)}, b^{(2)} \in L^2(0,\infty;\dot{H}^1)$ yield the desired uniqueness.

We now explain why (3.3) is true. Since $(u,b) \in L^2(0,\infty;\dot{H}^1)$, then (u,b) is in $\dot{H}^1$ for almost every $t \in (0,\infty)$. For any $t_0 > 0$, there is $0 < t_1 < t_0$ such that $(u(x,t_1), b(x,t_1)) \in H^1(\mathbb{R}^2)$. Starting with $(u(x,t_1), b(x,t_1))$, we then solve (3.1). The solution (u,b) satisfies

$$(u,b) \in L^\infty(t_1,\infty;H^1) \cap L^2(t_1,\infty;\dot{H}^2),$$

which allows us to further choose $t_2 \in (t_1,t_0)$ such that

$$(u(x,t_2), b(x,t_2)) \in H^2(\mathbb{R}^2).$$

We then solve (3.1) with more regular initial datum and repeating the process leads to the desired smoothness. This completes the proof of Theorem 3.1. □

4 The MHD equations with only magnetic diffusion

This section addresses the global regularity problem on the partial dissipation case when the 2D MHD equations involve only the magnetic diffusion,

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = \eta\Delta b + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{4.1}$$

where u_0 and b_0 satisfy $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$. The global regularity problem is not completely solved at this moment, but recent efforts have significantly advanced our understanding. Global *a priori* bounds in very regular functional settings have been obtained, for example, for any $T > 0$ and, for any $p \in [2,\infty)$ and $q \in (1,\infty)$,

$$(u,b) \in L^\infty(0,\infty;H^1), \quad u \in L^q(0,T;W^{1,p}), \quad b \in L^q(0,T;W^{2,p}),$$

$$\omega \in L^\infty(0,T;L^p), \quad j \in L^\infty(0,T;L^p)$$

where $\omega = \nabla\times u$ is the vorticity and $j = \nabla\times b$ denotes the current density. Unfortunately the global bound for $\omega \in L^\infty(0,T;L^\infty)$ is lacking. As a consequence, the uniqueness and the higher regularity can not be achieved. Various attempts have been made to prove the global L^∞ bound for ω. In particular, we will present the work of Q. Jiu, D. Niu, J. Wu, X. Xu and H. Yu [30] and some *a priori* estimates obtained in a work in progress with P. Zhang [54]. In addition, several combined quantities have been discovered to be globally bounded in highly regular functional settings.

This section reviews several recent results, explains the difficulty involved, describes some of the attempts and discusses paths that may potentially lead to the solution to this intriguing problem. We divide the rest of this section into four subsections.

4.1 Global *a priori* bounds

This subsection presents the global *a priori* bounds that have been established for solutions (u, b) of (4.1). Most of the materials presented in this subsection are taken from [30]. The first is a global uniform (in time) bound on (u, b) in the Sobolev space H^1 (see, e.g., [11, 30, 34]).

Proposition 4.1. *If (u, b) solves (4.1), then, for any $t > 0$,*

$$\begin{aligned}
&\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \|\nabla j(s)\|_{L^2}^2 ds \\
&\leq (\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2) e^{C(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)}
\end{aligned}$$

and consequently

$$\begin{aligned}
&\|u(t)\|_{H^1}^2 + \|b(t)\|_{H^1}^2 + \int_0^t \|b(s)\|_{H^2}^2 ds \\
&\leq C(\|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2) e^{C(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)} + \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.
\end{aligned}$$

A special consequence of this global H^1 bound is the global existence of weak solutions of (4.1) in the standard distributional sense. Higher regularity estimates can be established by making use of the regularizing effect of the heat kernel, namely

$$e^{t\Delta} f = K_t(x) * f, \qquad K_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}.$$

One frequently used tool is the following $L^p - L^q$ type estimate.

Lemma 4.2. *Let $1 \leq p \leq q \leq \infty$. Let β be a multi-index. For any $t > 0$, the heat operator $e^{\Delta t}$ and $\partial_x^\beta e^{\Delta t}$ are bounded from L^p to L^q. Further, for any $f \in L^p(\mathbb{R}^d)$,*

$$\|e^{\Delta t} f\|_{L^q(\mathbb{R}^d)} \leq C_1 t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}$$

and

$$\|\partial_x^\beta e^{\Delta t} f\|_{L^q(\mathbb{R}^d)} \leq C_2 t^{-\frac{|\beta|}{2} - \frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where $C_1 = C_1(p, q)$ and $C_2 = C_2(\beta, p, q)$ are constants.

Proposition 4.3. *If (u, b) solves (4.1), then, for any $T > 0$ and for any $2 \leq p < \infty$, $1 < q < \infty$,*

$$u \in L^q(0, T; W^{1,p}(\mathbb{R}^2)), \quad b \in L^q(0, T; W^{2,p}(\mathbb{R}^2)), \tag{4.2}$$

$$\omega \in L^\infty(0, T; L^p(\mathbb{R}^2)), \quad j \in L^\infty(0, T; L^p(\mathbb{R}^2)). \tag{4.3}$$

Especially, for any $r \in (1, \infty)$,

$$u \in L^\infty([0, T]; L^\infty), \qquad j \in L^r([0, T]; L^\infty).$$

Proof. The proof makes use of the maximal $L^q_t L^p_x$ regularity for the heat kernel (see, e.g., [35]). That is, the operator A defined by

$$Af \equiv \int_0^t e^{(t-s)\Delta} \Delta f(s) ds$$

is bounded from $L^p(0,T; L^q(\mathbb{R}^d))$ to $L^p(0,T; L^q(\mathbb{R}^d))$ for every fixed $T \in (0,\infty]$. Resorting to the heat kernel, we write the equation of b in the integral form

$$b(x,t) = e^{t\Delta} b_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot f(s,\cdot) ds, \tag{4.4}$$

with $f = (f_i)$ and $f_i = b_i u - u_i b$ $(i = 1, 2)$. The global bound in Proposition 4.1 and Sobolev's inequality imply that, for any $p \in [2,\infty)$,

$$f_i \in L^\infty(0,T; L^p).$$

(4.4), combined with Lemma 4.2, leads to a global L^∞ bound for b,

$$\begin{aligned}
&\|b\|_{L^\infty(0,T;L^\infty)} \\
&\qquad \le C(\|K_t\|_{L^\infty(0,T;L^1)} \|b_0\|_{L^\infty} + \|\nabla K_t\|_{L^1(0,T;L^{p'})} \|f\|_{L^\infty(0,T;L^p)}) \\
&\qquad \le C(\|b_0\|_{L^\infty} + \|u\|_{L^\infty(0,T;H^1)} \|b\|_{L^\infty(0,T;H^1)}).
\end{aligned}$$

(4.4), together with the maximal $L^q_t L^p_x$ regularity, yields

$$\begin{aligned}
\|\nabla b\|_{L^q(0,T;L^p)} \le& C(\|K_t\|_{L^q(0,T;L^1)} \|\nabla b_0\|_{L^p} + \|f\|_{L^q(0,T;L^p)}) \\
\le& C(\|b_0\|_{H^2} + \|u\|_{L^\infty(0,T;H^1)} \|b\|_{L^\infty(0,T;H^1)}).
\end{aligned}$$

The global bounds for $\|\Delta b\|_{L^q(0,T;L^p)}$ and $\|\omega\|_{L^q(0,T;L^p)}$ are obtained simultaneously,

$$\begin{aligned}
&\|\omega(t)\|_{L^p} \le C(\|\omega_0\|_{L^p} + \|\Delta b\|_{L^q_t L^p}), \\
&\|\Delta b\|_{L^q_t L^p} \le C(\|\Delta b_0\|_{L^p} + \|b\|_{L^\infty_t L^\infty} \|\omega\|_{L^q_t L^p} + \|u\|_{L^2_t H^1} \|\nabla b\|_{L_t^{\frac{2q}{2-q}} L^{2p}}).
\end{aligned}$$

The two estimates above and Gronwall's inequality yield the desired bounds in (4.3). The global bound for $j \in L^\infty(0,T; L^p)$ in (4.3) follows from energy estimates involving the equation of j and the global bound for $\omega \in L^\infty(0,T; L^p)$ follows from energy estimates on the vorticity equation. This completes the proof of Proposition 4.3. $\square$

4.2 An attempt to bound $\|\omega\|_{L^\infty(0,T;L^\infty)}$ and an equation for a combined quantity $G = \omega + curl \nabla \cdot (b \otimes b)$

It appears that the global *a priori* bounds in the previous subsection are not sufficient to hammer out the problem. The missing piece in the puzzle is a global bound for $\|\omega\|_{L^\infty(0,T;L^\infty)}$. Various attempts have been made to establish this global bound. This subsection provides two of them. The first one is to work with a combined quantity while the second one is a work in progress with P. Zhang [54].

In the paper of Q. Jiu, D. Niu, J. Wu, X. Xu and H. Yu [30], we attempted to solve the global regularity problem by working with a combined quantity. We are close to solving this problem and [30] provides a regularity criterion.

Theorem 4.4. *Let $s > 2$. Assume $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, b) be the local (in time) solution of (1.1) on $[0, T^*)$. Let $T_0 > T^*$. If there is $\sigma > 0$ and an integer $k_0 > 0$ such that b satisfies*

$$M(T_0) \equiv \int_0^{T_0} \sum_{k \geq k_0} 2^{\sigma k} \|S_{k-1}(b \otimes b)\|_{L^\infty} dt < \infty, \tag{4.5}$$

then the local solution can be extended to $[0, T_0]$. Here $b \otimes b$ denotes the tensor product and S_j denotes the identity approximator defined via the Littlewood-Paley decomposition.

We explain the proof of Theorem 4.4. Due to the lack of a global bound on ∇j in L^∞, the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j \tag{4.6}$$

does not allow us to extract a global bound for $\|\omega\|_{L^\infty}$. A natural idea is to eliminate this bad term. To this end, we rewrite the vorticity equation as

$$\partial_t \omega + u \cdot \nabla \omega = curl \nabla \cdot (b \otimes b) \tag{4.7}$$

and recast the equation of b as

$$\begin{aligned}(b \otimes b)_t + u \cdot \nabla (b \otimes b) &= \nabla u (b \otimes b) + (b \otimes b)(\nabla u)^* \\ &\quad + \Delta (b \otimes b) - 2 \sum_{k=1}^{2} (\partial_k b \otimes \partial_k b).\end{aligned} \tag{4.8}$$

Applying $\mathcal{R} \equiv (-\Delta)^{-1} curl \nabla \cdot$ to (4.8) yields to

$$\begin{aligned}&\big((\mathcal{R}(b \otimes b)\big)_t + u \cdot \nabla \mathcal{R}(b \otimes b) \\ &= -[\mathcal{R}, u \cdot \nabla](b \otimes b) + \mathcal{R}\big(\nabla u (b \otimes b) + (b \otimes b)(\nabla u)^*\big) \\ &\quad - curl \nabla \cdot (b \otimes b) - 2 \sum_{k=1}^{2} \mathcal{R}(\partial_k b \otimes \partial_k b).\end{aligned} \tag{4.9}$$

Adding (4.9) to (4.7) and setting $G = \omega + \mathcal{R}(b \otimes b)$, we get

$$\begin{aligned}\partial_t G + u \cdot \nabla G &= -[\mathcal{R}, u \cdot \nabla](b \otimes b) - 2 \sum_{k=1}^{2} \mathcal{R}(\partial_k b \otimes \partial_k b) \\ &\quad + \mathcal{R}(\nabla u (b \otimes b)) + \mathcal{R}((b \otimes b)(\nabla u)^*).\end{aligned} \tag{4.10}$$

The right-hand side of the equation of G looks more complex than $b \cdot \nabla j$, but it does not involve two derivatives of b and this gives us hope. We note that $\mathcal{R}$ is a Calderon-Zygmund singular integral operator (homogeneous of degree zero). Since $\mathcal{R}$ is not bounded on L^∞, it may not be a good idea to estimate $\|G\|_{L^\infty}$. Instead we estimate G in the Besov space $B^0_{\infty,1}$. Singular integral operators are bounded on $B^0_{\infty,1}$, but the trade-off is that the bound for the solution of a transport equation relies on the L^∞-norm of the gradient of the velocity, as stated in the following lemma.

Lemma 4.5. *Consider the linear equation*

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = f, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{4.11}$$

Then, there exists $C > 0$ such that

$$\|\theta\|_{L_t^\infty B^0_{q,1}} \le C(\|\theta_0\|_{B^0_{q,1}} + \|f\|_{L_t^1 B^0_{q,1}}) \left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau\right),$$

where $q \in [1, \infty]$.

If we apply Lemma 4.5, then

$$\|G\|_{L_t^\infty B^0_{\infty,1}} \le C(\|G_0\|_{B^0_{\infty,1}} + \|f\|_{L_t^1 B^0_{\infty,1}}) \left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau\right), \tag{4.12}$$

where we have written the right-hand side of (4.10) as f,

$$\begin{aligned} f = -[\mathcal{R}, u \cdot \nabla](b \otimes b) - 2\sum_{k=1}^{2} \mathcal{R}(\partial_k b \otimes \partial_k b) \\ +\mathcal{R}((b \otimes b)(\nabla u)^\intercal) + \mathcal{R}(\nabla u(b \otimes b)). \end{aligned}$$

If we can show

$$\|f\|_{L_t^1 B^0_{\infty,1}} < \infty, \tag{4.13}$$

then we would be able to obtain a global bound for $\|\omega\|_{B^0_{\infty,1}}$, which would imply global regularity. In fact, if we have (4.12) with (4.13), then

$$\begin{aligned} \|\omega\|_{B^0_{\infty,1}} &\le \|G\|_{B^0_{\infty,1}} + \|b \otimes b\|_{B^0_{\infty,1}} \\ &\le \|b \otimes b\|_{B^0_{\infty,1}} + C\left(1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau\right) \\ &\le \|b \otimes b\|_{B^0_{\infty,1}} + C\left(1 + \int_0^t (\|u\|_{L^2} + \|\omega\|_{B^0_{\infty,1}}) d\tau\right). \end{aligned}$$

Gronwall's inequality and $\|b \otimes b\|_{B^0_{\infty,1}} \le \|b\|^2_{B^\epsilon_{\infty,1}} < \infty$ $(0 < \epsilon < 1)$ then imply that

$$\|\omega\|_{B^0_{\infty,1}} < \infty.$$

Especially, $\|\omega\|_{L^\infty} < \infty$. Then higher regularities follow.

It then remains to check (4.13). f involves a commutator and the following lemma provides a bound of such a commutator in Besov spaces. This type of estimates can be found in [28, 45].

Lemma 4.6. *Let $\mathcal{R}$ denote a standard singular integral operator, say Riesz transform or $\mathcal{R} = (-\Delta)^{-1} curl \nabla \cdot$. Let $1 < p \le \infty$. For any integer k, for $0 \le s_1, s_2 \le 1$ and $s_1 + s_2 \le 1$, we have*

$$\|\Delta_k([\mathcal{R}, u\cdot\nabla]\theta)\|_{L^p} \le C_s 2^{(1-s_1-s_2)k} \|\Lambda^{s_1} u\|_{L^{p_1}} \|\Lambda^{s_2}\theta\|_{L^{p_2}} \tag{4.14}$$

where $1 < p_1, p_2 \le \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

The global *a priori* bounds obtained in the previous subsection allow us to bound the first two terms in f without any problem. The last two terms in f are similar and each one of them is split into three parts by the notion of paraproducts. It is only one of these parts that need the assumption in (4.5) to be bounded. Here are more details. By Lemma 4.6, for $s \in (0,1)$,

$$\|[\mathcal{R}, u\cdot\nabla](b\otimes b)\|_{B^0_{\infty,1}} \le C\|\Lambda^s u\|_{B^0_{\infty,1}} \|\Lambda^{1-s}(b\otimes b)\|_{B^0_{\infty,1}}.$$

For any $s \in (0,1)$, $\|\Lambda^s u\|_{B^0_{\infty,1}}$ can be bounded by $\|\omega\|_{L^q}$ for some large $q \in (2,\infty)$. In fact, by Bernstein's inequality,

$$\begin{aligned}
\|\Lambda^s u\|_{B^0_{\infty,1}} &\le \|\Delta_{-1}\Lambda^s u\|_{L^\infty} + \sum_{k=0}^{\infty} \|\Delta_k \Lambda^s u\|_{L^\infty} \\
&\le C\|u\|_{L^2} + C\sum_{k=0}^{\infty} 2^{(s-1)k} \|\Delta_k \omega\|_{L^\infty} \\
&\le C\|u\|_{L^2} + C\sum_{k=0}^{\infty} 2^{(s-1+\frac{2}{q})k} \|\Delta_k \omega\|_{L^q} \\
&\le C\|u\|_{L^2} + C\|\omega\|_{L^q} \sum_{k=0}^{\infty} 2^{(s-1+\frac{2}{q})k}.
\end{aligned}$$

Therefore, if we choose $q \in (2,\infty)$ such that $s - 1 + \frac{2}{q} < 0$, then

$$\|\Lambda^s u\|_{B^0_{\infty,1}} \le C(\|u\|_{L^2} + \|\omega\|_{L^q}) < \infty.$$

The regularity of b also implies that, for $s \in (0,1)$ close to 1,

$$\|\Lambda^{1-s}(b\otimes b)\|_{B^0_{\infty,1}} < \infty.$$

In fact, in a similar fashion as above, if $\widetilde{q} \in (2,\infty)$ such that $-s + \epsilon + \frac{2}{\widetilde{q}} < 0$,

$$\|\Lambda^{1-s}(b\otimes b)\|_{B^0_{\infty,1}} \le C\|\Lambda^{1-s} b\|^2_{B^\epsilon_{\infty,1}} \le C(\|b\|^2_{L^2} + \|\nabla b\|^2_{L^{\widetilde{q}}}).$$

Therefore, Proposition 4.3 implies

$$\begin{aligned}
&\|[\mathcal{R}, u\cdot\nabla](b\otimes b)\|_{L^1_t B^0_{\infty,1}} \\
&\le C(\|u\|_{L^\infty_t L^2} + \|\omega\|_{L^\infty_t L^q})(\|b\|^2_{L^2_t L^2} + \|\nabla b\|^2_{L^2_t L^{\tilde q}}) < \infty.
\end{aligned}$$

We now estimate the second term in f. For any $\epsilon > 0$,

$$\left\|\sum_{k=1}^{2} \mathcal{R}(\partial_k b \otimes \partial_k b)\right\|_{B^0_{\infty,1}} \le \|\nabla b\|^2_{B^\epsilon_{\infty,1}}.$$

By Bernstein's inequality,

$$\begin{aligned}
\|\nabla b\|_{B^\epsilon_{\infty,1}} &\le C\|b\|_{L^2} + C\sum_{k=0}^{\infty} 2^{\epsilon k}\|\Delta_k \nabla b\|_{L^\infty} \\
&\le C\|b\|_{L^2} + C\sum_{k=0}^{\infty} 2^{k(\epsilon+1+\frac{2}{q})}\|\Delta_k b\|_{L^q} \\
&\le C\|b\|_{L^2} + C\sum_{k=0}^{\infty} 2^{k(\epsilon+1+\frac{2}{q}-\gamma)}\|\Lambda^\gamma \Delta_k b\|_{L^q} \\
&\le C\|b\|_{L^2} + C\|\Lambda^\gamma b\|_{L^q},
\end{aligned}$$

where $\epsilon + 1 + \frac{2}{q} < \gamma < 2$. Therefore,

$$\begin{aligned}
\left\|\sum_{k=1}^{2} \mathcal{R}(\partial_k b \otimes \partial_k b)\right\|_{L^1_t B^0_{\infty,1}} &\le \|\nabla b\|^2_{L^2_t B^\epsilon_{\infty,1}} \\
&\le C\|b\|^2_{L^2_t L^2} + C\|\Lambda^\gamma b\|^2_{L^2_t L^q} < \infty.
\end{aligned}$$

We bound the last two terms in f. Their estimates are similar and we shall handle one of them. By Bernstein's inequality,

$$\begin{aligned}
\|\mathcal{R}(\nabla u(b\otimes b))\|_{B^0_{\infty,1}} \le& C\|\Delta_{-1}(\nabla u(b\otimes b))\|_{L^2} + \sum_{k=0}^{\infty}\|\Delta_k(\nabla u(b\otimes b))\|_{L^\infty} \\
\le& C\|\omega\|_{L^2}\|b\|^2_{L^\infty} + \sum_{k=0}^{\infty}\|\Delta_k(\nabla u(b\otimes b))\|_{L^\infty}.
\end{aligned}$$

Following the notion of paraproducts, we write

$$\begin{aligned}
\Delta_k(\nabla u(b\otimes b)) = &\sum_{|k-m|\le 2} \Delta_k(S_{m-1}\nabla u \Delta_m(b\otimes b)) \\
&+ \sum_{|k-m|\le 2} \Delta_k(\Delta_m \nabla u S_{m-1}(b\otimes b)) \\
&+ \sum_{m\ge k-1} \Delta_k(\Delta_m \nabla u \widetilde{\Delta}_m(b\otimes b)), \qquad (4.15)
\end{aligned}$$

where $\widetilde{\Delta}_m = \Delta_{m+1} + \Delta_m + \Delta_{m-1}$. By Bernstein's inequality,

$$
\begin{aligned}
&\sum_{|k-m|\le 2} \|\Delta_k (S_{m-1}\nabla u \Delta_m (b\otimes b))\|_{L^\infty} \\
&\le \sum_{|k-m|\le 2} \|S_{m-1}\nabla u\|_{L^\infty} \|\Delta_m (b\otimes b)\|_{L^\infty} \\
&\le C \sum_{|k-m|\le 2} 2^{\frac{2}{q}m} \|S_{m-1}\omega\|_{L^q} \|\Delta_m (b\otimes b)\|_{L^\infty} \\
&\le C\|\omega\|_{L^q} \sum_{|k-m|\le 2} 2^{\frac{2}{q}m} \|\Delta_m (b\otimes b)\|_{L^\infty}.
\end{aligned}
$$

By Bernstein's inequality and the Hardy-Littlewood-Sobolev inequality, the third term in (4.15) can be bounded by

$$
\begin{aligned}
&\sum_{m\ge k-1} \|\Delta_k (\Delta_m \nabla u \widetilde{\Delta}_m (b\otimes b))\|_{L^\infty} \\
&= \sum_{m\ge k-1} \|\Delta_k \Lambda^{\frac{2}{q}} \Lambda^{-\frac{2}{q}} (\Delta_m \nabla u \widetilde{\Delta}_m (b\otimes b))\|_{L^\infty} \\
&\le \sum_{m\ge k-1} 2^{\frac{2}{q}k} \|\Lambda^{-\frac{2}{q}} (\Delta_m \nabla u \widetilde{\Delta}_m (b\otimes b))\|_{L^\infty} \\
&\le C \sum_{m\ge k-1} 2^{\frac{2}{q}k} \|\Delta_m \nabla u\|_{L^q} \|\widetilde{\Delta}_m (b\otimes b)\|_{L^\infty} \\
&\le C\|\omega\|_{L^q} \sum_{m\ge k-1} 2^{\frac{2}{q}(k-m)} 2^{\frac{2}{q}m} \|\widetilde{\Delta}_m (b\otimes b)\|_{L^\infty}.
\end{aligned}
$$

The condition (4.5) is needed to handle the second term in (4.15). As in the estimate of the first term, we have

$$
\begin{aligned}
&\sum_{|k-m|\le 2} \|\Delta_k (\Delta_m \nabla u S_{m-1} (b\otimes b))\|_{L^\infty} \\
&\le C\|\omega\|_{L^q} \sum_{|k-m|\le 2} 2^{\frac{2}{q}m} \|S_{m-1} (b\otimes b)\|_{L^\infty}.
\end{aligned}
$$

Combining the estimates above, we have

$$
\begin{aligned}
&\|\mathcal{R}(\nabla u (b\otimes b))\|_{B^0_{\infty,1}} \\
&\le C\|\omega\|_{L^2} \|b\|_{L^\infty}^2 + C\|\omega\|_{L^q} \sum_{k\ge 0} \sum_{|k-m|\le 2} 2^{\frac{2}{q}m} \|\Delta_m (b\otimes b)\|_{L^\infty} \\
&\quad + C\|\omega\|_{L^q} \sum_{k\ge 0} \sum_{m\ge k-1} 2^{\frac{2}{q}(k-m)} 2^{\frac{2}{q}m} \|\widetilde{\Delta}_m (b\otimes b)\|_{L^\infty} \\
&\quad + C\|\omega\|_{L^q} \sum_{k\ge 0} \sum_{|k-m|\le 2} 2^{\frac{2}{q}m} \|S_{m-1} (b\otimes b)\|_{L^\infty}
\end{aligned}
$$

$$\leq C\|\omega\|_{L^2}\|b\|_{L^\infty}^2 + C\|\omega\|_{L^q}\|b\|_{L^\infty}\|b\|_{B^{\frac{2}{q}}_{\infty,1}}$$
$$+C\|\omega\|_{L^q}\sum_{k\geq 0} 2^{\frac{2}{q}k}\|S_{k-1}(b\otimes b)\|_{L^\infty}. \tag{4.16}$$

We provide some details for the last inequality, namely

$$\sum_{k\geq 0}\sum_{|k-m|\leq 2} 2^{\frac{2}{q}m}\|\Delta_m(b\otimes b)\|_{L^\infty} \leq C\|b\|_{L^\infty}\|b\|_{B^{\frac{2}{q}}_{\infty,1}}, \tag{4.17}$$

$$\sum_{k\geq 0}\sum_{m\geq k-1} 2^{\frac{2}{q}(k-m)}2^{\frac{2}{q}m}\|\widetilde{\Delta}_m(b\otimes b)\|_{L^\infty} \leq C\|b\|_{L^\infty}\|b\|_{B^{\frac{2}{q}}_{\infty,1}}. \tag{4.18}$$

In fact, by the paraproduct decomposition,

$$\begin{aligned}
&\sum_{k\geq 0}\sum_{|k-m|\leq 2} 2^{\frac{2}{q}m}\|\Delta_m(b\otimes b)\|_{L^\infty} \\
&\leq C\sum_{k\geq 0} 2^{\frac{2}{q}k}\sum_{|k-l|\leq 2}\|\Delta_k(S_{l-1}b\otimes\Delta_l b)\|_{L^\infty} \\
&\quad +C\sum_{k\geq 0} 2^{\frac{2}{q}k}\sum_{|k-l|\leq 2}\|\Delta_k(\Delta_l b\otimes S_{l-1}b)\|_{L^\infty} \\
&\quad +C\sum_{k\geq 0} 2^{\frac{2}{q}k}\sum_{l\geq k-1}\|\Delta_k(\Delta_l b\otimes\widetilde{\Delta}_l b)\|_{L^\infty} \\
&\leq C\sum_{k\geq 0} 2^{\frac{2}{q}k}\|b\|_{L^\infty}\|\Delta_k b\|_{L^\infty} \\
&\quad +C\sum_{k\geq 0}\sum_{l\geq k-1} 2^{\frac{2}{q}(k-l)}\|b\|_{L^\infty}2^{\frac{2}{q}l}\|\Delta_l b\|_{L^\infty} \\
&\leq C\|b\|_{L^\infty}\|b\|_{B^{\frac{2}{q}}_{\infty,1}}.
\end{aligned}$$

This proves (4.17). The proof of (4.18) is similar. Due to (4.5), the third term is time integrable if we choose q large enough, say $\frac{2}{q} < \sigma$. Therefore we have proven (4.13). This completes the proof of Theorem 4.4.

4.3 Another attempt to bound $\|\omega\|_{L^\infty(0,T;L^\infty)}$

This subsection describes an attempt to bound $\|\omega\|_{L^\infty(0,T;L^\infty)}$ via a more direct approach. This is part of the work in progress with P. Zhang [54]. It follows from the vorticity equation (4.6) that

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + C\|b\|_{L^\infty_{t,x}}\|\nabla j\|_{L^1_t L^\infty_x}. \tag{4.19}$$

We know that $\|b\|_{L^\infty_{t,x}}$ is bounded according to the *a priori* bounds of the previous subsection. A natural idea would be to bound $\|\nabla j\|_{L^1_t L^\infty_x}$ directly. We obtain the following proposition.

Proposition 4.7. *Assume that* (u, b) *is classical solution of* (4.1)*. Then,*

$$\|\nabla j\|_{L^1_t B^0_{\infty,1}} \le \|j_0\|_{B^{-1}_{\infty,1}} + C\|\omega\|_{L^1_t B^0_{\infty,1}} + C,$$

where C *depends on the initial data and* t *only.*

We remark that, if we could bound $\|\nabla j\|_{L^1_t L^\infty_x}$ in terms of $\|\omega\|_{L^1_t L^\infty_x}$, the global regularity problem would be solved. In fact, it follows from (4.19) that

$$\begin{aligned}\|\omega(t)\|_{L^\infty} &\le \|\omega_0\|_{L^\infty} + C\|b\|_{L^\infty_{t,x}}\|\nabla j\|_{L^1_t B^0_{\infty,1}}\\ &\le \|\omega_0\|_{L^\infty} + C\|b\|_{L^\infty_{t,x}}\|\omega\|_{L^1_t L^\infty}.\end{aligned}$$

Gronwall's inequality then yields the global bound on $\|\omega(t)\|_{L^\infty}$. As we shall see in the proof of Proposition 4.7, it is just one paraproduct part of one term that prevents us from bounding $\|\nabla j\|_{L^1_t L^\infty_x}$ in terms of $\|\omega\|_{L^1_t L^\infty_x}$.

In order to prove Proposition 4.7, we need a simple fact on the smoothing effort of the heat operator $e^{\Delta t}$ on distributions with Fourier transform supported on annulus (see, e.g., [1]).

Lemma 4.8. *There exist two constants* $C_1 > 0$ *and* $C_2 > 0$ *such that, for any* $q \in [1, \infty]$,

$$\|e^{\nu\Delta}\dot{\Delta}_k f\|_{L^q(\mathbb{R}^d)} \le C_1 e^{-C_2 \nu 2^{2j}}\|\dot{\Delta}_k f\|_{L^q(\mathbb{R}^d)},$$

where $\nu > 0$ *is a parameter and* $\dot{\Delta}_k$ *denotes the homogeneous Littlewood-Paley blocks.*

We can now prove Proposition 4.7.

Proof of Proposition 4.7. We know that $j = \nabla \times b$ satisfies

$$\partial_t j + u \cdot \nabla j = \Delta j + b \cdot \nabla \omega + Q,$$

where

$$Q = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2). \tag{4.20}$$

The corresponding integral form is given by

$$\begin{aligned}j(t) = e^{t\Delta} j_0 &- \int_0^t e^{(t-\tau)\Delta}(u \cdot \nabla j)(\tau)d\tau\\ &+ \int_0^t e^{(t-\tau)\Delta}(b \cdot \nabla \omega)(\tau)d\tau + \int_0^t e^{(t-\tau)\Delta} Q(\tau)d\tau.\end{aligned} \tag{4.21}$$

Applying Δ_k to (4.21) yields

$$\begin{aligned}\Delta_k j(t) = e^{t\Delta}\Delta_k j_0 &- \int_0^t e^{(t-\tau)\Delta}\Delta_k(u \cdot \nabla j)(\tau)d\tau\\ &+ \int_0^t e^{(t-\tau)\Delta}\Delta_k(b \cdot \nabla \omega)(\tau)d\tau + \int_0^t e^{(t-\tau)\Delta}\Delta_k Q(\tau)d\tau.\end{aligned}$$

According to Lemma 4.8, we have

$$\|\Delta_k j(t)\|_{L^\infty} \leq Ce^{-C2^{2k}t}\|\Delta_k j_0\|_{L^\infty} + J_1 + J_2 + J_3, \tag{4.22}$$

where

$$\begin{aligned}
J_1 &= \left\| \int_0^t e^{(t-\tau)\Delta} \Delta_k (u\cdot\nabla j)(\tau) d\tau \right\|_{L^\infty} \\
&\leq C\int_0^t e^{-C(t-\tau)2^{2k}} \|\Delta_k(u\cdot\nabla j)(\tau)\|_{L^\infty} d\tau, \\
J_2 &= \left\| \int_0^t e^{(t-\tau)\Delta} \Delta_k (b\cdot\nabla \omega)(\tau) d\tau \right\|_{L^\infty} \\
&\leq C\int_0^t e^{-C(t-\tau)2^{2k}} \|\Delta_k(b\cdot\nabla \omega)(\tau)\|_{L^\infty} d\tau, \\
J_3 &= \left\| \int_0^t e^{(t-\tau)\Delta} \Delta_k Q(\tau) d\tau \right\|_{L^\infty} \\
&\leq C\int_0^t e^{-C(t-\tau)2^{2k}} \|\Delta_k Q(\tau)\|_{L^\infty} d\tau.
\end{aligned}$$

To estimate J_1, we invoke the notion of paraproduct to write

$$\begin{aligned}
\Delta_k(u\cdot\nabla j) = &\sum_{|j-k|\leq 2} \Delta_k(S_{l-1}u\cdot\nabla\Delta_l j) \\
&+ \sum_{|j-k|\leq 2} \Delta_k(\Delta_l u\cdot\nabla S_{l-1} j) + \sum_{l\geq k-1} \Delta_k(\Delta_l u\cdot\nabla\widetilde{\Delta}_l j).
\end{aligned}$$

Employing the bound on u in Proposition 4.3, we have

$$\begin{aligned}
\|\Delta_k(u\cdot\nabla j)\|_{L^\infty} \leq C2^k\|u\|_{L^\infty}\|\Delta_k j\|_{L^\infty} + C\|u\|_{L^\infty} \sum_{m\leq k-1} 2^m\|\Delta_m j\|_{L^\infty} \\
+C\|u\|_{L^\infty} 2^k \sum_{l\geq k-1} \|\Delta_l j\|_{L^\infty}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_1 \leq C\|u\|_{L^\infty}\bigg[\int_0^t e^{-C(t-\tau)2^{2k}} \Big(2^k\|\Delta_k j\|_{L^\infty} + \sum_{m\leq k-1} 2^m \|\Delta_m j\|_{L^\infty} \\
+2^k \sum_{l\geq k-1} \|\Delta_l j\|_{L^\infty} \Big) d\tau \bigg].
\end{aligned} \tag{4.23}$$

We now turn to J_2. To estimate J_2, we write

$$\begin{aligned}
\Delta_k(b\cdot\nabla\omega) = &\sum_{|l-k|\leq 2} \Delta_k(S_{l-1}b\cdot\Delta_l\nabla\omega) + \sum_{|l-k|\leq 2} \Delta_k(\Delta_l b\cdot S_{l-1}\nabla\omega) \\
&+ \sum_{l\geq k-1} \Delta_k(\Delta_l b\cdot\widetilde{\Delta}_l\nabla\omega).
\end{aligned}$$

Thus,

$$\|\Delta_k(b\cdot\nabla\omega)\|_{L^\infty} \le C2^k\|S_{k-1}b\|_{L^\infty}\|\Delta_k\omega\|_{L^\infty} + C2^k\|\Delta_k b\|_{L^\infty}\|\omega\|_{L^\infty} + \sum_{l\ge k-1} 2^k\|\Delta_l b\|_{L^\infty}\|\omega\|_{L^\infty}.$$

Therefore,

$$J_2 \le C\int_0^t e^{-C(t-\tau)2^{2k}}\Big(2^k\|S_{k-1}b\|_{L^\infty}\|\Delta_k\omega\|_{L^\infty} + 2^k\|\Delta_k b\|_{L^\infty}\|\omega\|_{L^\infty} + \sum_{l\ge k-1} 2^k\|\Delta_l b\|_{L^\infty}\|\omega\|_{L^\infty}\Big)d\tau. \tag{4.24}$$

The difficult term is $C2^k\|S_{k-1}b\|_{L^\infty}\|\Delta_k\omega\|_{L^\infty}$. The other terms are all right. We now turn to J_3. Recalling the definition of Q in (4.20), it suffices to deal with the typical term

$$\partial_1 b_1\partial_2 u_1.$$

To estimate $\partial_1 b_1\partial_2 u_1$, we write $\Delta_k(\partial_1 b_1\partial_2 u_1)$ into paraproducts

$$\begin{aligned}\Delta_k(\partial_1 b_1\partial_2 u_1) = &\sum_{|l-k|\le 2}\Delta_k(S_{l-1}\partial_1 b_1\Delta_l\partial_2 u_1)\\ &+\sum_{|l-k|\le 2}\Delta_k(\Delta_l\partial_1 b_1 S_{l-1}\partial_2 u_1)\\ &+\sum_{l\ge k-1}\Delta_k(\Delta_l\partial_1 b_1\widetilde{\Delta}_l\partial_2 u_1).\end{aligned}$$

By Bernstein's inequality,

$$\begin{aligned}\|\Delta_k(\partial_1 b_1\partial_2 u_1)\|_{L^\infty} &\le C\|S_{k-1}\partial_1 b_1\|_{L^\infty}\|\Delta_k\partial_2 u_1\|_{L^\infty}\\ &\quad +C\|S_{k-1}\partial_2 u_1\|_{L^\infty}\|\Delta_k\partial_1 b_1\|_{L^\infty}\\ &\quad +C\sum_{l\ge k-1}\|\Delta_l\partial_1 b_1\|_{L^\infty}\|\widetilde{\Delta}_l\partial_2 u_1\|_{L^\infty}\\ &\le C2^k\|\nabla b\|_{L^\infty}\|\Delta_k u\|_{L^\infty}\\ &\quad +C2^k\|u\|_{L^\infty}\|\Delta_k\nabla b\|_{L^\infty}\\ &\quad +C\|u\|_{L^\infty}\sum_{l\ge k-1}2^{k-l}\|\Delta\Delta_l b\|_{L^\infty}.\end{aligned}$$

Now we then multiply (4.22) by 2^k and integrate in time to obtain

$$2^k\|\Delta_k j\|_{L^1_tL^\infty} \le 2^{-k}\|\Delta_k j_0\|_{L^\infty} + 2^k\|J_1\|_{L^1_t} + 2^k\|J_2\|_{L^1_t} + 2^k\|J_3\|_{L^1_t}. \tag{4.25}$$

Invoking the bound in (4.23) and applying Young's inequality for convolution yield

$$\begin{aligned}2^k\|J_1\|_{L^1_t} &\le C2^{-k}\|u\|_{L^\infty}2^k\|\Delta_k j\|_{L^1_tL^\infty}\\ &\quad +C2^{-k}\|u\|_{L^\infty}\sum_{m\le k-1}2^m\|\Delta_m j\|_{L^1_tL^\infty}\\ &\quad +C2^{-k}\|u\|_{L^\infty}\sum_{l\ge k-1}2^{k-l}2^l\|\Delta_l j\|_{L^1_tL^\infty}.\end{aligned}$$

Similarly,

$$2^k \|J_2\|_{L^1_t} \le C\|b\|_{L^\infty_{t,x}} \|\Delta_k \omega\|_{L^1_t L^\infty} \\ +C\|\omega\|_{L^1_t L^\infty_x} \|\Delta_k b\|_{L^\infty_{t,x}} + C\|\omega\|_{L^1_t L^\infty_x} \sum_{l\ge k-1} \|\Delta_l b\|_{L^\infty_{t,x}}$$

and

$$2^k \|J_3\|_{L^1_t} \le C\|\Delta_k u\|_{L^\infty_t L^\infty_x} \|\nabla b\|_{L^1_t L^\infty_x} \\ +C\|u\|_{L^\infty_t L^\infty_x} \|\Delta_k \nabla b\|_{L^1_t L^\infty_x} \\ +C2^{-k}\|u\|_{L^\infty_t L^\infty_x} \sum_{l\ge k-1} 2^{k-l} \|\Delta\Delta_l b\|_{L^\infty}.$$

We choose $N > 0$ sufficiently large such that

$$2^{-N}\|u\|_{L^\infty} < \frac{1}{16}.$$

Summing over $k \ge N$ in (4.25) and invoking the global bounds in Proposition 4.3, we have

$$\sum_{k=N}^{\infty} 2^k \|\Delta_k j\|_{L^1_t L^\infty} \le \sum_{k=N}^{\infty} 2^{-k}\|\Delta_k j_0\|_{L^\infty} + \frac{1}{4}\sum_{k=0}^{\infty} 2^k \|\Delta_k j\|_{L^1_t L^\infty} \\ +C\sum_{k=N}^{\infty} \|\Delta_k \omega\|_{L^1_t L^\infty} + C(1+\|\omega\|_{L^1_t L^\infty}). \tag{4.26}$$

In addition, by Proposition 4.3,

$$\sum_{k=0}^{N-1} 2^k \|\Delta_k j\|_{L^1_t L^\infty} < \infty.$$

Inserting this bound in (4.26) yields

$$\|j\|_{L^1_t B^1_{\infty,1}} \le \|j_0\|_{B^{-1}_{\infty,1}} + \frac{1}{4}\|j\|_{L^1_t B^1_{\infty,1}} \\ + \sum_{k=0}^{N-1} 2^k \|\Delta_k j\|_{L^1_t L^\infty} + C\|\omega\|_{L^1_t B^0_{\infty,1}} + C(1+\|\omega\|_{L^1_t L^\infty}).$$

We thus have obtained that

$$\|\nabla j\|_{L^1_t B^0_{\infty,1}} \le \|j_0\|_{B^{-1}_{\infty,1}} + C\|\omega\|_{L^1_t B^0_{\infty,1}} + C.$$

This completes the proof of Proposition 4.7. □

4.4 The criticality

The global regularity problem on the 2D MHD equations with only magnetic diffusion (4.1) is critical in two senses. The first is that solutions of (4.1) admit global bounds in very regular functional settings such as

$$\|\omega\|_{L^\infty(0,T;L^p)} \quad \text{for any } 1 < p < \infty,$$

but the crucial global bound on $\|\omega\|_{L^\infty(0,T;L^\infty)}$ is missing. The second is that, if we replace Δb by $(-\Delta)^\beta b$ with any $\beta > 1$, the resulting system then admits a unique global solution. More precisely, we have the following theorem.

Theorem 4.9. *Consider the following 2D MHD equations with fractional magnetic diffusion*

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \quad x\in\mathbb{R}^2, t>0,\\ \partial_t b + u\cdot\nabla b + (-\Delta)^\beta b = b\cdot\nabla u, \quad x\in\mathbb{R}^2, t>0,\\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \quad x\in\mathbb{R}^2, t>0,\\ u(x,0)=u_0(x), \quad b(x,0)=b_0(x), \quad x\in\mathbb{R}^2. \end{cases} \tag{4.27}$$

Let $\beta > 1$. Assume that $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, $\nabla\cdot u_0 = 0$, $\nabla\cdot b_0 = 0$ and $j_0 = \nabla\times b_0$ satisfying

$$\|\nabla j_0\|_{L^\infty} < \infty.$$

Then (4.27) has a unique global solution (u, b) satisfying, for any $T > 0$,

$$(u,b)\in L^\infty([0,T]; H^s(\mathbb{R}^2)), \quad \nabla j \in L^1([0,T]; L^\infty(\mathbb{R}^2))$$

where $j = \nabla\times b$.

The theorem stated above is taken from a recent paper of C. Cao, J. Wu and B. Yuan [12]. Q. Jiu and J. Zhao was able to give a different proof of this result [32].

4.5 Discussions

The recent efforts have advanced the course on the global regularity problem on the 2D MHD equations with only magnetic diffusion (4.1), but so far this problem has resisted a complete resolution. This subsection discusses some ideas that may be useful.

One thought is to seek suitable functional spaces that are close to L^∞ and have the desired properties described below. One type of functional spaces X would allow us to obtain the following estimates

$$\|\omega(t)\|_X \le \|\omega_0\|_X + C\|\nabla j\|_{L^1_t X}, \tag{4.28}$$

$$\|\nabla j\|_{L^1_t X} \le C + C\|\omega\|_{L^1_t X}. \tag{4.29}$$

Obviously, (4.28) and (4.29), together with Gronwall's inequality, would yield a global bound for $\|\omega\|_X$. The Lebesgue space L^∞ would certainly satisfy (4.28), but we do not know how to prove (4.29). It appears that we need a stronger norm than L^∞ for the term on the right of (4.29). Another type of functional spaces Y would have the following desired properties

$$\|\omega(t)\|_Y \le \left(\|\omega_0\|_Y + C\|\nabla j\|_{L^1_t Y}\right)\left(1+\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right), \tag{4.30}$$

$$\|\nabla u(t)\|_{L^\infty} \le C + C\|\omega(t)\|_Y, \tag{4.31}$$

$$\|\nabla j\|_{L^1_t Y} \le C. \tag{4.32}$$

Obviously, (4.30), (4.31) and (4.32), together with Gronwall's inequality, would allow us to derive a global bound for $\|\omega\|_Y$. The Besov space $B^0_{\infty,1}$ certainly satisfies (4.30) and (4.31), but unfortunately, as we have seen in the previous subsection, we do not know how to prove (4.32). Instead we only know how to prove

$$\|\nabla j\|_{L^1_t Y} \le C + C\|\omega\|_{L^1_t Y},$$

which is not good enough. Another hopeful example of Y is the space

$$Y = L^p \cap LBMO, \qquad p \in (1,2),$$

where $LBMO$ is the logarithmic BMO space defined by F. Bernicot and S. Keraani [3]. $L^p \cap LBMO$ can be shown to satisfy (4.30) and (4.31). We have not been able to prove (4.32) for this space. We will continue to search for function spaces that satisfy either (4.28) and (4.29) or (4.30), (4.31) and (4.32).

Another thought is to make use of some of quantities that admit global bounds in very regular spaces. There are many such quantities. One of them, as kindly mentioned to the author by H. Dong, is

$$A \equiv j + u\cdot b^\perp, \qquad b^\perp = (-b_2, b_1).$$

It is easy to verify that A satisfies

$$\partial_t A = \Delta A + \partial_t(u\cdot b^\perp).$$

The global bounds stated in Proposition 4.3 implies that, for any $p \in [2,\infty)$ and $q \in [1,\infty)$, and for any $T > 0$,

$$\partial_t(u\cdot b^\perp) \in L^q(0,T; L^p(\mathbb{R}^2)).$$

As a consequence of the maximal regularity of the heat operator, we have

$$\partial_t A \in\in L^q(0,T; L^p(\mathbb{R}^2)), \quad \Delta A \in L^q(0,T; L^p(\mathbb{R}^2)). \tag{4.33}$$

In order to make use of this global bound, we substitute $j = A - u\cdot b^\perp$ in the vorticity equation,

$$\partial_t\omega + u\cdot\nabla\omega = b\cdot\nabla(A - u\cdot b^\perp). \tag{4.34}$$

The global bound in (4.33) guarantees that $b\cdot\nabla A$ can be bounded suitably. The trouble is the term involving $b\cdot\nabla u\cdot b^\perp$. Recalling that $\nabla u = \mathcal{P}\omega$ for a singular integral operator $\mathcal{P}$, (4.34) essentially reduces to the model proposed by P. Constantin,

$$\partial_t\omega + u\cdot\nabla\omega = \mathcal{P}\omega, \qquad u = K_2 * \omega, \tag{4.35}$$

where $\mathcal{P}$ is a singular integral operator such as the 2D Riesz transform and K_2 is the 2D Biot-Savart kernel (2.9). The global regularity problem on (4.35) remains open. Small data global solutions have been obtained (see, e.g., [20, 42]).

5 The MHD equations with only kinematic dissipation

This section is devoted to the case when only the kinematic dissipation is present in the 2D MHD equations, namely

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + \nu\Delta u + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{5.1}$$

where u_0 and b_0 satisfy $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$. Even though the global regularity problem on (5.1) remains an outstanding open problem, there are very exciting new developments. This section review some of these recent results.

The kinematic dissipation alone is not sufficient for proving the desired global bounds to ensure the global regularity. Even the global existence of weak solutions in this partial dissipation case remains open and the main difficulty is how to pass the limit in the term $b\cdot\nabla b$. Due to the lack of magnetic diffusion, we do not even know how to establish the standard small data global well-posedness.

Recent strategy has been to seek global solutions near an equilibrium. Since the pioneering work of F. Lin, L. Xu and P. Zhang [36], this direction has flourished. This section reviews several recent results in this direction. Attention is focused on the whole space case. For the sake of clarity, the rest of this section is divided into five subsections. The first subsection describes a local well-posedness result of P. Constantin [17]. The second subsection states and describes the global result (near an equilibrium) of Lin, Xu and Zhang [36] and their Lagrangian approach. The third subsection presents the results of X. Ren, J. Wu, Z. Xiang and Z. Zhang [43] and outlines the proof. The fourth subsection gives an account of the result of J. Wu, Y. Wu and X. Xu [55]. The last subsection briefly describes the results of X. Hu and F. Lin [27], and of T. Zhang [62].

5.1 A local well-posedness of P. Constantin

It is not difficult to see that the standard local existence and uniqueness result in the Sobolev space setting like Theorem 2.1 remains true for (5.1). This subsection presents a different local well-posedness result, obtained by P. Constantin [17]. Constantin introduced the Lagrangian-Eulerian approach on hydrodynamic equations and was able to establish the existence and uniqueness in a very weak functional setting. Consider the initial value problem for the system

$$
\begin{cases}
\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \nabla \cdot \sigma, \\
\nabla \cdot u = 0, \\
\partial_t \sigma + u \cdot \nabla \sigma = F(\nabla u, \sigma), \\
u(x,0) = u_0(x), \qquad \sigma(x,0) = \sigma_0(x),
\end{cases}
\tag{5.2}
$$

where F is a quadratic function of its variables.

Theorem 5.1. *Let $0 < \alpha < 1$, $1 < p < \infty$, let $u_0 \in C^{1+\alpha} \cap L^p$ be divergence-free and $\sigma_0 \in C^\alpha \cap L^p$.*

(A) *There exists $T > 0$ and a solution (u, σ) of (5.2) with $u \in L^\infty(0,T; C^{1+\alpha} \cap L^p)$ and with $\sigma \in Lip(0,T; C^\alpha \cap L^p)$.*

(B) *Two solutions $u_j \in L^\infty(0,T; C^{1+\alpha} \cap L^p)$ and $\sigma_j \in Lip(0,T; C^\alpha \cap L^p)$, $j = 1,2$ obey the strong Lipschitz bound*

$$
\begin{aligned}
&\|\partial_t X_2 - \partial_t X_1\|_{L^\infty(0,T;C^{1+\alpha}\cap L^p)} + \|\partial_t \tau_2 - \partial_t \tau_1\|_{L^\infty(0,T;C^\alpha \cap L^p)} \\
&\le C(T) \left(\|u_2(0) - u_1(0)\|_{C^{1+\alpha}\cap L^p} + \|\sigma_2(0) - \sigma_1(0)\|_{C^\alpha \cap L^p} \right).
\end{aligned}
$$

where X_j, $j = 1,2$ denote the particle trajectories corresponding to u_j, and τ_j, $j = 1,2$ denote the Lagrangian counterparts of σ_j, or $\tau_j = \sigma \circ X_j$. In particular, two such solutions with the same initial data must coincide.

When (u, b) solves (5.1), (u, σ) with $\sigma = b \otimes b$ solves (5.2), as explained in (2.5). Theorem 5.1 provides a local well-posedness for the MHD equations (5.1).

5.2 The work of F. Lin, L. Xu and P. Zhang

The work of F. Lin, L. Xu and P. Zhang pioneered the study on the global well-posedness of (5.1) with smooth initial data which is close to some non-trivial steady state. We first explain the mechanism on why the solutions of the perturbation equation near an equilibrium may exist for all time. As in (2.10), setting $b = \nabla^\perp \phi$ allows us to write (5.1) as

$$
\begin{cases}
u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + \nabla^\perp \phi \cdot \nabla \nabla^\perp \phi, \\
\nabla \cdot u = 0, \\
\phi_t + u \cdot \nabla \phi = 0.
\end{cases}
\tag{5.3}
$$

Clearly, $(u, \phi) = (0, x_2)$ is a steady solution. Setting $\phi = x_2 + \psi$ yields

$$
\begin{cases}
\partial_t \psi + u \cdot \nabla \psi + u_2 = 0, \\
\partial_t u_1 + u \cdot \nabla u_1 - \nu \Delta u_1 + \partial_1 \partial_2 \psi = -\partial_1 p - \nabla \cdot (\partial_1 \psi \nabla \psi), \\
\partial_t u_2 + u \cdot \nabla u_2 - \nu \Delta u_2 + \partial_1^2 \psi = -\partial_2 p - \nabla \cdot (\partial_2 \psi \nabla \psi), \\
(\psi, u_1, u_2)(x,0) = (\psi_0(x), u_{10}(x), u_{20}(x)).
\end{cases}
\tag{5.4}
$$

The aim is then to show that (5.4) possesses a unique global solution when the initial data is sufficiently small. The most significant difference between (5.3) and

(5.4) is that (5.4) contains an extra term u_2 in the equation of ψ. This term appears to play a role in the process.

The approach of Lin, Xu and Zhang is Lagrangian. They reformulated (5.4) in Lagrangian coordinates. More precisely, they work with the displacement

$$Y(x,t) = X(x,t) - x$$

where $X = X(x,t)$ be the particle trajectory determined by u. They then derive the equation for Y, which satisfies

$$\begin{cases} Y_{tt} - \Delta_y Y_t - \partial_{y_1}^2 Y = f(Y,q), \\ \nabla_y \cdot Y = \rho(Y), \\ Y(y,0) = Y_0(y), \qquad Y_t(y,0) = Y_1(y), \end{cases}$$

where $q(y,t) = (p + |\nabla\psi|^2) \circ X(y,t)$, $f(Y,q)$ denotes a functions of Y and q, and $\rho(Y) = J(Y_2, Y_1)$ with J being the Poisson bracket defined in (2.12). They then estimate the Lagrangian velocity Y_t in $L^1_t Lip_x$, using the anisotropic Littlewood-Paley theory and anisotropic Besov space techniques. The estimates involved are very delicate.

Due to their use of the Lagrangian coordinates, they need to impose a compatibility condition on the initial data ψ_0, more precisely, $\partial_y\psi_0$ and $(1+\partial_y\psi_0, -\partial_x\psi_0)$ are admissible on $0 \times \mathbb{R}$ and supp $\partial_y\psi_0(\cdot,y) \subset [-K,K]$ for some K. Here $\partial_y\psi_0$ and $(1+\partial_y\psi_0, -\partial_x\psi_0)$ are admissible on $0 \times \mathbb{R}$ if

$$\int_{\mathbb{R}} \partial_y\psi_0(Z(a,t))dt = 0 \quad \text{for all } a \in 0 \times \mathbb{R},$$

where Z is the particle trajectory defined by $(1+\partial_y\psi_0, -\partial_x\psi_0)$. Their main result can be stated as follows.

Theorem 5.2. *Given u_0 and ψ_0 satisfying $(u_0, \nabla\psi_0) \in H^s \cap \dot{H}^{s_2}$ with $s_1 > 1$, $s_2 \in (-1, -\frac{1}{2})$ and $s > s_1 + 2$, and*

$$\|\nabla\psi_0\|_{H^{s_1+2}} \le \epsilon_0, \quad \|(\nabla\psi_0, u_0)\|_{\dot{H}^{s_1+1}\cap\dot{H}^{s_2}} + \|\partial_y\psi_0\|_{H^{s_1+2}} \le \epsilon_0$$

for some ϵ_0 small. Assume that $\partial_y\psi_0$ and $(1+\partial_y\psi_0, -\partial_x\psi_0)$ are admissible on $0 \times \mathbb{R}$ and $\partial_y\psi_0(\cdot,y) \subset [-K,K]$ for some K. Then (5.4) has a unique global solution (ψ, u, p).

5.3 The work of X. Ren, J. Wu, Z. Xiang and Z. Zhang

Motivated by the work of Lin, Xu and Zhang [36], Ren, Wu, Xiang and Zhang [43] examined the same issue via a different approach, namely the global well-posedness on the perturbed system

$$\begin{cases} \partial_t\psi + u\cdot\nabla\psi + u_2 = 0, \\ \partial_t u_1 + u\cdot\nabla u_1 - \nu\Delta u_1 + \partial_1\partial_2\psi = -\partial_1 p - \nabla\cdot(\partial_1\psi\nabla\psi), \\ \partial_t u_2 + u\cdot\nabla u_2 - \nu\Delta u_2 + \partial_1^2\psi = -\partial_2 p - \nabla\cdot(\partial_2\psi\nabla\psi). \end{cases} \tag{5.5}$$

The paper of Ren, Wu, Xiang and Zhang [43] is based on the Eulerian approach and employs direct energy estimates. In addition, [43] also investigates the large-time behavior of the solutions and verifies a numerical observation that the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity.

The functional setting is the anisotropic Sobolev spaces defined below.

Definition 5.3. *Let $\sigma, s \in \mathbb{R}$. The anisotropic Sobolev space $\mathring{H}^{\sigma,s}(\mathbb{R}^2)$ is defined by*

$$\mathring{H}^{\sigma,s}(\mathbb{R}^2) = \left\{ f \in S'(\mathbb{R}^2) : \|f\|_{\mathring{H}^{\sigma,s}} < +\infty \right\},$$

where

$$\|f\|_{\mathring{H}^{\sigma,s}} = \left\| \left\{ 2^{js} 2^{\sigma k} \|\Delta_j \Delta_k^h f\|_{L^2} \right\}_{j,k} \right\|_{\ell^2},$$

or, in terms of the Fourier transforms,

$$\|f\|_{\mathring{H}^{\sigma,s}} = \left[\int_{\mathbb{R}^2} |\xi|^{2s} \xi_1^{2\sigma} |\widehat{f}(\xi)|^2 d\xi \right]^{\frac{1}{2}}.$$

We need the anisotropic Sobolev spaces to deal with the anisotropic nature of the equations here. It is not very difficult to show that (u_1, u_2, ψ) satisfies a degenerate damped wave equation that exhibits anisotropicity. As derived in detail in the next subsection, the linear part of the equation is given by

$$u_{tt} - \Delta u_t - \partial_1^2 u = 0.$$

The characteristic equation satisfies

$$\lambda^2 + |\xi|^2 \lambda + \xi_1^2 = 0,$$

which has two roots

$$\lambda_\pm = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4\xi_1^2}}{2}.$$

As $|\xi| \to \infty$,

$$\lambda_-(\xi) \to -\frac{\xi_1^2}{|\xi|^2} \sim \begin{cases} -1, & |\xi| \sim |\xi_1|, \\ 0, & |\xi| \gg |\xi_1|. \end{cases}$$

Therefore, the dissipative effect is weak in the case of $|\xi| \gg |\xi_1|$.

The main result of [43] can be stated as in the following theorem.

Theorem 5.4. *Assume $(\nabla\psi_0, u_0) \in H^8(\mathbb{R}^2)$. Let $s \in (0, \frac{1}{2})$. There exists a small positive constant ε such that, if, $(\nabla\psi_0, u_0) \in \mathring{H}^{-s,-s} \cap \mathring{H}^{-s,8}(\mathbb{R}^2)$, and*

$$\|(\nabla\psi_0, u_0)\|_{H^8} + \|(\nabla\psi_0, u_0)\|_{\mathring{H}^{-s,-s}} + \||(\nabla\psi_0, u_0)\|_{\mathring{H}^{-s,8}} \le \varepsilon,$$

then (5.5) has a unique global solution (ψ, u) satisfying

$$(\nabla\psi, u) \in C([0, +\infty); H^8(\mathbb{R}^2)).$$

Moreover, the solution decays at the same rate as that for the linearized solutions,

$$\|\partial_x^k \nabla\psi\|_{L^2} + \|\partial_x^k u\|_{L^2} \le C\varepsilon(1+t)^{-\frac{s+k}{2}},$$

for any $t \in [0, +\infty)$ and $k = 0, 1, 2$.

To help understand the proof of Theorem 5.4 and extract the necessary decay rates, we first examine the linearized version of (5.5). The following proposition holds.

Proposition 5.5. *Consider the linearized equation*

$$\begin{cases} \partial_t u_1 - \Delta u_1 - \partial_{x_1 x_2}\psi = 0, \\ \partial_t u_2 - \Delta u_2 + \partial_{x_1 x_1}\psi = 0, \\ \partial_t \psi + u_2 = 0, \\ u(x,0) = u_0(x), \quad \psi(x,0) = \psi_0(x). \end{cases}$$

Assume $(u_0, \nabla\psi_0) \in H^4$ *and* $|D_1|^{-s} u_0 \in H^{1+s}$ *and* $|D_1|^{-s}\nabla\psi_0 \in H^{1+s}$ *for* $s > 0$, *then, for* $k = 0, 1, 2$,

$$\|\partial_{x_1}^k u\|_{L^2} + \|\partial_{x_1}^k \nabla\psi\|_{L^2} \le C(1+t)^{-\frac{k+s}{2}}.$$

To explain why we need the functional setting in the Sobolev space with negative indices, we provide the main lines of the proof for this proposition.

Proof. For small $\varepsilon_1 > 0$, define

$$\begin{aligned} D_0(t) &= \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + \|\nabla^2\psi\|_{L^2}^2 + 2\varepsilon_1\langle u_2, \nabla\psi\rangle, \\ H_0(t) &= \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \varepsilon_1\|\nabla\partial_1\psi\|_{L^2}^2 - \varepsilon_1\|\nabla u_2\|_{L^2}^2 - \varepsilon_1\langle\Delta u_2, \Delta\psi\rangle, \\ E_s(t) &= \||D_1|^{-s}u\|_{L^2}^2 + \||D_1|^{-s}\nabla\psi\|_{L^2}^2 \\ &\quad + \||D|^{1+s}|D_1|^{-s}u\|_{L^2}^2 + \||D|^{1+s}|D_1|^{-s}\nabla\psi\|_{L^2}^2. \end{aligned}$$

We can show

$$\frac{d}{dt}D_0(t) + CH_0(t) \le 0, \qquad \frac{d}{dt}E_s(t) \le 0.$$

By interpolation inequalities,

$$D_0(t) \le E_s(t)^{\frac{1}{1+s}} H_0(t)^{\frac{s}{1+s}}, \quad H_0(t) \ge E_s(0)^{-\frac{1}{s}} D_0(t)^{1+\frac{1}{s}}.$$

Thus,

$$\frac{d}{dt}D_0(t) + CE_s(0)^{-\frac{1}{s}} D_0(t)^{1+\frac{1}{s}} \le 0$$

and consequently

$$E(t) \le (E(0)^{-\frac{1}{s}} + C(s)t)^{-s} = E_0\left(E_0^{\frac{1}{s}} C(s)t + 1\right)^{-s}.$$

This completes the proof of Proposition 5.5. □

We now return to the full nonlinear system (5.5) and outline the proof of Theorem 5.4. The frame work to prove the global existence of small solutions is the Bootstrap Principle. The following abstract bootstrap principle is taken from the book of T. Tao [46, p. 21].

Lemma 5.6 (Abstract Bootstrap Principle)**.** *Let I be an interval. Let $C(t)$ and $H(t)$ be two statements related to $t \in I$. If $C(t)$ and $H(t)$ satisfy*

(a) *if $H(t)$ is true, then $C(t)$ is true for the same t,*
(b) *if $C(t_1)$ is true, then $H(t)$ is true for t in a neighborhood of t_1,*
(c) *if $C(t_k)$ is true for a sequence $t_k \to t$, then $C(t)$ is true,*
(d) *$C(t)$ is true for at least one $t_0 \in I$,*

then, $C(t)$ is true for all $t \in I$.

We only provide a sketch of the proof while the detailed proof can be found in [43].

Proof of Theorem 5.4. The proof is divided into two main steps:

(1) The first step is to obtain decay rates under the assumption that the solution is small.
(2) The second step is to show that the solution is even smaller if the initial data is small.

Then the Bootstrap Principle would imply that the solution remain small for all time. The initial step is to show that, if $(u, \nabla\psi)$ satisfies

$$\|(u(t), \nabla\psi(t))\|_{H^4} \leq \delta$$

for sufficiently small $\delta > 0$ and for $t \in [0, T]$, then we can show

$$\frac{d}{dt} D_0 + CH_0 \leq 0 \quad \text{for } t \in [0, T],$$

where D_0 and H_0 are defined as in the proof of Proposition 5.5. To further the estimate, we define the higher-order counterparts of D_0 and H_0. For $l = 1, 2$, we define

$$\begin{aligned} D_l(t) = \sum_{j,k} 2^{2lk} (&\|\Delta_j \Delta_k^h u\|_{L^2}^2 + \|\Delta_j \Delta_k^h \nabla u\|_{L^2}^2 + \|\Delta_j \Delta_k^h \nabla\psi\|_{L^2}^2 \\ &+ \|\Delta_j \Delta_k^h \nabla^2\psi\|_{L^2}^2 + 2\varepsilon_1 \langle \Delta_j \Delta_k^h u_2, \Delta_j \Delta_k^h \Delta\psi\rangle), \end{aligned}$$

$$\begin{aligned} H_l(t) = \sum_{j,k} 2^{2lk} (&\|\Delta_j \Delta_k^h \nabla u\|_{L^2}^2 + \|\Delta_j \Delta_k^h \nabla^2 u\|_{L^2}^2 + \varepsilon_1 \|\Delta_j \Delta_k^h \nabla\partial_1\psi\|_{L^2}^2 \\ &- \|\Delta_j \Delta_k^h \nabla u_2\|_{L^2}^2 - \varepsilon_1 \langle \Delta_j \Delta_k^h \Delta u_2, \Delta_j \Delta_k^h \Delta\psi\rangle). \end{aligned}$$

We then show that, for $e(t) = \|(u, \nabla\psi)\|_{H^8 \cap H^{-s,-s} \cap H^{-s,8}}$, if

$$\sup_{t \in [0,T]} e(t) \leq \delta$$

for some sufficiently small δ, then

$$\frac{d}{dt} D_l(t) + CH_l(t) \leq 0, \qquad t \in [0, T].$$

In order to extract the decay rates from these differential inequalities, we also need to include some intermediate norms in the estimates. We further define

$$E_{s,s_1} = \|(u,\nabla\psi)\|^2_{\dot{H}^{-s,s_1}} + \|(u,\nabla\psi)\|^2_{\dot{H}^{-s,s_1+1}},$$
$$\varepsilon_{s,k}(t) = E_{s,0}(t) + E_{s,s+k}(t).$$

We then prove that, for sufficiently small $\delta > 0$, if, for $k = 0, 1, 2$,

$$\sup_{t\in[0,T]} e(t) \le \delta, \qquad \sup_{t\in[0,T]} \varepsilon_{s,k}(t) \le \delta,$$

then

$$\|\partial^l_{x_1}(u,\nabla\psi)\|_{L^2} + \|\partial^l_{x_1}(\nabla u,\nabla^2\psi)\|_{L^2} \le C(1+t)^{-\frac{l+s}{2}}.$$

Finally we show that, for sufficiently small $r_0 > 0$, if

$$e(0) = \|(u_0,\nabla\psi_0)\|_{H^8\cap\dot{H}^{-s,8}\cap\dot{H}^{-s,-s}} \le r_0,$$

then $(u,\nabla\psi)$ satisfies

$$e(t) = \|(u,\nabla\psi)\|_{H^8\cap\dot{H}^{-s,8}\cap\dot{H}^{-s,-s}} \le 2r_0, \qquad \mathcal{E}_s(t) \le 2r_0,$$

where $\mathcal{E}_s(t) = E_{0,7}(t) + E_{s,-s}(t) + E_{7,0}(t)$. Therefore, if we choose $2r_0 < \frac{1}{2}\delta$, then our proof above implies $e(t) < \frac{1}{2}\delta$. This then fulfills the assumptions of the Bootstrap Principle, which then concludes that $e(t) < \frac{1}{2}\delta$ for all $t \in [0,T]$. This completes the proof of Theorem 5.4. $\square$

5.4 The work of J. Wu, Y. Wu and X. Xu

This subsection presents the work of J. Wu, Y. Wu and X. Xu [55], which studied the global well-posedness near an equilibrium for the 2D MHD type equations with a velocity damping term instead of the dissipation, namely

$$\begin{cases} \partial_t\vec{u} + \vec{u}\cdot\nabla\vec{u} + \vec{u} + \nabla P = -\mathrm{div}(\nabla\phi\otimes\nabla\phi), \quad (t,x,y)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \partial_t\phi + \vec{u}\cdot\nabla\phi = 0, \\ \nabla\cdot\vec{u} = 0, \\ \vec{u}|_{t=1} = \vec{u}_0(x,y), \quad \phi|_{t=1} = \phi_0(x,y), \end{cases} \tag{5.6}$$

where $\vec{u} = (u,v)$. The notation in this subsection is slightly different from the previous subsections. We use $\vec{u}$ for the 2D velocity, u and v for its components, and (x,y) for the coordinates of a 2D point. Their paper takes the dispersive nature of the perturbed equations into full consideration and makes use of the tools and techniques for dispersive type equations.

We again consider the perturbation near the equilibrium $u = 0, \phi = y$. Substituting $\phi = y + \psi$ in (5.6) yields

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + u + \partial_x\tilde{P} = -\Delta\psi\partial_x\psi, \\ \partial_t v + u\partial_x v + v\partial_y v + v + \partial_y\tilde{P} = -\Delta\psi - \Delta\psi\partial_y\psi, \\ \partial_t\psi + u\partial_x\psi + v\partial_y\psi + v = 0, \\ \partial_x u + \partial_y v = 0, \end{cases} \tag{5.7}$$

where $\tilde{P} = P + \frac{1}{2}|\nabla\phi|^2$. By $\nabla \cdot \vec{u} = 0$,

$$\Delta\widetilde{P} = -\nabla \cdot (\vec{u} \cdot \nabla\vec{u}) - \nabla \cdot (\Delta\psi\nabla\psi) - \Delta\partial_y\psi.$$

Moving all nonlinear terms in (5.7) to the right yields

$$\partial_t u + u - \partial_{xy}\psi = N_1, \tag{5.8}$$
$$\partial_t v + v + \partial_{xx}\psi = N_2, \tag{5.9}$$
$$\partial_t\psi + v = -u\partial_x\psi - v\partial_y\psi, \tag{5.10}$$

where

$$\begin{aligned} N_1 &= -\vec{u} \cdot \nabla u + \partial_x\Delta^{-1}\nabla \cdot (\vec{u} \cdot \nabla\vec{u}) - \Delta\psi\partial_x\psi + \partial_x\Delta^{-1}\nabla \cdot (\Delta\psi\nabla\psi), \\ N_2 &= -\vec{u} \cdot \nabla v + \partial_y\Delta^{-1}\nabla \cdot (\vec{u} \cdot \nabla\vec{u}) - \Delta\psi\partial_y\psi + \partial_y\Delta^{-1}\nabla \cdot (\Delta\psi\nabla\psi). \end{aligned}$$

Taking the time derivative leads to

$$\begin{cases} \partial_{tt}u + \partial_t u - \partial_{xx}u = F_1, \\ \partial_{tt}v + \partial_t v - \partial_{xx}v = F_2, \\ \partial_{tt}\psi + \partial_t\psi - \partial_{xx}\psi = F_0, \\ \vec{u}|_{t=1} = \vec{u}_0(x,y), \quad \vec{u}_t|_{t=1} = \vec{u}_1(x,y), \\ \psi|_{t=1} = \psi_0(x,y), \quad \psi_t|_{t=1} = \psi_1(x,y), \end{cases} \tag{5.11}$$

where $\vec{u}_1 = (u_1(x,y), v_1(x,y)), \psi_0 = \phi_0 - y$, and

$$\begin{aligned} u_1 &= (-u + \partial_{xy}\psi + N_1)|_{t=1}, \\ v_1 &= (-v - \partial_{xx}\psi + N_2)|_{t=1}, \\ \psi_1 &= (-u\partial_x\psi - v\partial_y\psi - v)|_{t=1}, \end{aligned}$$

and

$$\begin{aligned} F_0 &= -\vec{u} \cdot \nabla\psi - \partial_t(\vec{u} \cdot \nabla\psi) - N_2, \\ F_1 &= \partial_t N_1 - \partial_{xy}(\vec{u} \cdot \nabla\psi), \\ F_2 &= \partial_t N_2 + \partial_{xx}(\vec{u} \cdot \nabla\psi). \end{aligned}$$

Magically all the linear parts have the same structure, and u, v and ψ all satisfy a degenerate wave equation with a damping term.

Next we convert (5.11) into an integral form by inverting the linear part of this wave equation. To this end, we consider the linear equation

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = 0, \tag{5.12}$$

with the initial data

$$\Phi(0,x,y) = \Phi_0(x,y), \Phi_t(0,x,y) = \Phi_1(x,y).$$

Taking the Fourier transform on the equation (5.12), we have

$$\partial_{tt}\hat{\Phi} + \partial_t\hat{\Phi} + \xi^2\hat{\Phi} = 0, \tag{5.13}$$

where the Fourier transform $\hat{\Phi}$ is defined as

$$\hat{\Phi}(t,\xi,\eta) = \int_{\mathbb{R}^2} e^{ix\xi+iy\eta}\Phi(t,x,y)dxdy.$$

Solving (5.13) by a simple ordinary differential equation theory, we have

$$\begin{aligned}
&\hat{\Phi}(t,\xi,\eta)\\
&= \frac{1}{2}\Big(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} + e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\Big)\widehat{\Phi_0}(\xi,\eta)\\
&\quad + \frac{1}{2\sqrt{\frac{1}{4}-\xi^2}}\Big(e^{(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2})t} - e^{(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2})t}\Big)\Big(\frac{1}{2}\widehat{\Phi_0}(\xi,\eta) + \widehat{\Phi_1}(\xi,\eta)\Big).
\end{aligned}$$

If we define the operators $K_0(t,\partial_x), K_1(t,\partial_x)$ by

$$\widehat{K_0(t,\partial_x)}f(t,\xi,\eta) = \frac{1}{2}\Big(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} + e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\Big)\hat{f}(t,\xi,\eta) \tag{5.14}$$

and

$$\begin{aligned}
&\widehat{K_1(t,\partial_x)}f(t,\xi,\eta)\\
&= \frac{1}{2\sqrt{\frac{1}{4}-\xi^2}}\Big(e^{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\xi^2}\right)t} - e^{\left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\xi^2}\right)t}\Big)\hat{f}(t,\xi,\eta),
\end{aligned} \tag{5.15}$$

where $\sqrt{-1} = i$, then the solution Φ of the equation (5.12) can be written as

$$\Phi(t,x,y) = K_0(t,\partial_x)\Phi_0 + K_1(t,\partial_x)\Big(\frac{1}{2}\Phi_0 + \Phi_1\Big).$$

Moreover, for the inhomogeneous equation,

$$\partial_{tt}\Phi + \partial_t\Phi - \partial_{xx}\Phi = F,$$

with initial data $\Phi(x,1) = \Phi_0(x), \partial_t\Phi(x,1) = \Phi_1(x)$, the standard Duhamel formula implies

$$\begin{aligned}
\Phi(t,x,y) &= K_0(t,\partial_x)\Phi_0 + K_1(t,\partial_x)\Big(\frac{1}{2}\Phi_0 + \Phi_1\Big)\\
&\quad + \int_1^t K_1(t-s,\partial_x)F(s,x,y)ds.
\end{aligned} \tag{5.16}$$

We need to estimate the decay properties of these kernel functions in order to show that the terms in the integral representation remain small when the initial data is sufficiently small. This lemma also indicates the anisotropicity in the decay estimates.

Lemma 5.7. *Let K_0, K_1 be defined in* (5.14) *and* (5.15). *Then*

(1) $\big\| |\xi|^\alpha \widehat{K_i}(t,\cdot) \big\|_{L^q_\xi(|\xi|\le \frac{1}{2})} \lesssim t^{-\frac{1}{2}(\frac{1}{q}+\alpha)}$, *for any* $\alpha \ge 0$, $1 \le q \le \infty, i = 0, 1$.

(2) $\big\| \partial_t \widehat{K_i}(t,\cdot) \big\|_{L^q_\xi(|\xi|\le \frac{1}{2})} \lesssim t^{-1-\frac{1}{2q}}, i = 0, 1$.

(3) $\big|\widehat{K_i}(t,\xi)\big| \lesssim e^{-\frac{1}{2}t}$, *for any* $|\xi| \ge \frac{1}{2}, i = 0, 1$.

(4) $\big|\langle\xi\rangle^{-1}\partial_t \widehat{K_0}(t,\xi)\big|, \big|\partial_t \widehat{K_1}(t,\xi)\big| \lesssim e^{-\frac{1}{2}t}$, *for any* $|\xi| \ge \frac{1}{2}$.

Due to the complexity of the nonlinear terms, we need to choose a suitable functional setting for the initial data and for the solutions. Let X_0 be the Banach space defined by the following norm

$$\begin{aligned} \|(\vec{u}_0, \psi_0)\|_{X_0} &= \|\langle\nabla\rangle^N (\vec{u}_0, \nabla\psi_0)\|_{L^2_{xy}} \\ &\quad + \|\langle\nabla\rangle^{6+}(\vec{u}_0, \psi_0)\|_{L^1_{xy}} + \|\langle\nabla\rangle^{6+}(\vec{u}_1, \psi_1)\|_{L^1_{xy}}, \end{aligned}$$

where $\langle\nabla\rangle = (I-\Delta)^{\frac{1}{2}}$, $N \gg 1$ and $a+$ denotes $a+\epsilon$ for small $\epsilon > 0$. The solution spaces X is defined by

$$\begin{aligned} \|(\vec{u},\psi)\|_X = \sup_{t\ge 1} \Big\{ & t^{-\varepsilon}\|\langle\nabla\rangle^N(\vec{u}(t), \nabla\psi(t))\|_2 + t^{\frac{1}{4}}\|\langle\nabla\rangle^3\psi\|_2 \\ & + t^{\frac{1}{4}}\|\langle\nabla\rangle^3\psi\|_2 + t^{\frac{3}{2}}\|\partial_{xx}\psi\|_\infty + t^{\frac{5}{4}}\|\langle\nabla\rangle^2\partial_{xx}\psi\|_2 + t^{\frac{3}{2}}\|\partial_{xxx}\psi\|_2 \\ & + t^{\frac{3}{2}}\|\partial_t\vec{u}\|_\infty + t^{\frac{5}{4}}\|\langle\nabla\rangle\partial_t\vec{u}\|_2 + t\|\langle\nabla\rangle\partial_x\vec{u}\|_\infty + t^{\frac{3}{2}}\|\partial_x\partial_t v\|_2 \Big\}. \end{aligned}$$

Our main result can be stated as follows:

Theorem 5.8. *There exists a small constant $\varepsilon > 0$ such that, if the initial data satisfies $\|(\vec{u}_0,\psi_0)\|_{X_0} \le \varepsilon$, then* (5.6) *possesses a unique global solution $(u, v, \psi) \in X$. Moreover, the following decay estimates hold*

$$\|u(t)\|_{L^\infty_x} \lesssim \varepsilon t^{-1}; \quad \|v(t)\|_{L^\infty_x} \lesssim \varepsilon t^{-\frac{3}{2}}; \quad \|\psi(t)\|_{L^\infty_x} \lesssim \varepsilon t^{-\frac{1}{2}}.$$

The proof of this theorem relies on the continuity argument, which is a consequence of the Bootstrap Principle.

Lemma 5.9 (Continuity Argument)**.** *Suppose that $(\vec{u},\psi)$ with the initial data $(\vec{u}_0,\psi_0)$, satisfies*

$$\|(\vec{u},\psi)\|_X \lesssim \|(\vec{u}_0,\psi_0)\|_{X_0} + C\|\vec{u},\psi)\|_X^\beta \tag{5.17}$$

with $\beta > 1$. Then, there exists r_0 such that, if

$$\|(\vec{u}_0,\psi_0)\|_{X_0} \lesssim r_0,$$

then $\|(\vec{u},\psi)\|_X \lesssim 2r_0$.

Various tool estimates are obtained in [55]. The following is one of them.

Proposition 5.10. *Let $K(t,\partial_x)$ be a Fourier multiplier operator satisfying*

$$\|\widehat{\partial_x^\alpha K}(t,\xi)\|_{L^1_\xi(|\xi|\le\frac12)}<\infty,\quad \|\widehat{K}(t,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac12)}<\infty,\quad \alpha\ge 0.$$

Then, for any space-time Schwartz function f,

$$\begin{aligned}\|\partial_x^\alpha K(t,\partial_x)f\|_{L^\infty_{xy}} \lesssim&\Big(\|\widehat{\partial_x^\alpha K}(t,\xi)\|_{L^1_\xi(|\xi|\le\frac12)}+\|\widehat{K}(t,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac12)}\Big)\\ &\times\|\langle\nabla\rangle^{\alpha+1+\epsilon}\partial_y f\|_{L^1_{xy}}.\end{aligned}\tag{5.18}$$

Many delicate estimates on the kernel functions K_0 and K_1 have also been established in [55]. To get a flavoring of how we proceed to bound the solution in the solution space, we provide a segment of the estimates for one of the norms defining the solution space. A lot more details can be found in [55]. By the Duhamel formula, namely (5.16),

$$\begin{aligned}\psi(t,x,y)=&K_0(t,\partial_x)\psi_0+K_1(t,\partial_x)\left(\frac12\psi_0+\psi_1\right)\\ &+\int_1^t K_1(t-s,\partial_x)F_0(s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\langle\nabla\rangle\partial_{xx}\psi\|_\infty\lesssim&\|\langle\nabla\rangle\partial_{xx}K_0(t)\psi_0\|_\infty+\left\|\langle\nabla\rangle\partial_{xx}K_1(t)\left(\frac12\psi_0+\psi_1\right)\right\|_\infty\\ &+\left\|\int_1^t\langle\nabla\rangle\partial_{xx}K_1(t-s)F_0(s)ds\right\|_\infty.\end{aligned}$$

By Proposition 5.10 and Lemma 5.7,

$$\begin{aligned}&\|\langle\nabla\rangle\partial_{xx}K_0(t)\psi_0\|_\infty\\ &\lesssim\big(\|\widehat{\partial_{xx}K_0}(t,\xi)\|_{L^1_\xi(|\xi|\le\frac12)}+\|\widehat{K_0}(t,\xi)\|_{L^\infty_\xi(|\xi|\ge\frac12)}\big)\|\langle\nabla\rangle^{2+\varepsilon}\partial_{xx}\partial_y\psi_0\|_{L^1_{xy}}\\ &\lesssim\big(t^{-\frac32}+e^{-t}\big)\|\langle\nabla\rangle^{5+\varepsilon}\psi_0\|_{L^1_{xy}}\lesssim t^{-\frac32}\|\langle\nabla\rangle^{5+\varepsilon}\psi_0\|_{X_0}.\end{aligned}$$

Similarly, we have

$$\left\|\langle\nabla\rangle\partial_{xx}K_1(t)\left(\frac12\psi_0+\psi_1\right)\right\|_\infty\lesssim t^{-\frac32}\left\|\langle\nabla\rangle^{5+\varepsilon}\left(\frac12\psi_0+\psi_1\right)\right\|_{X_0}.$$

Moreover,

$$\begin{aligned}&\left\|\int_1^t\langle\nabla\rangle\partial_{xx}K_1(t-s)F_0(s)ds\right\|_\infty\\ &\lesssim\int_1^t\|\partial_{xx}K_1(t-s)\langle\nabla\rangle F_0(s)\|_\infty ds\\ &\lesssim\int_1^{\frac t2}\|\partial_{xx}K_1(t-s)\langle\nabla\rangle F_0(s)\|_\infty ds\\ &\quad+\int_{\frac t2}^t\|\partial_xK_1(t-s)\langle\nabla\rangle\partial_xF_0(s)\|_\infty ds.\end{aligned}$$

The lines above illustrate how we bound the solution in one of the norms in the solution space. More details can be found in [55].

5.5 The results of X. Hu and F. Lin, and of T. Zhang

This subsection briefly describes two more papers on the global well-posedness near an equilibrium for the 2D MHD equations with kinematic dissipation only, the work of X. Hu and F. Lin [27], and the work of T. Zhang [62]. Both papers establish the global well-posedness but through somewhat different approaches. Their main results can be summarized as follows. We start with the result of T. Zhang [62].

Theorem 5.11 (T. Zhang). *Consider* (5.5) *with u_0 and ψ_0 satisfying*

$$u_0 \in H^2, \quad \nabla \cdot u_0 = 0, \quad \nabla\psi_0 \in H^1,$$

$$e^{-|\xi|^2 t}\widehat{u_0}, e^{-|\xi|^2 t}\widehat{\nabla\psi_0} \in L^2(0,\infty; L^1).$$

Then there exists a constant $c_0 > 0$ such that, if

$$\begin{aligned} A_0 \equiv \|u_0\|_{H^2} + \|\nabla\psi_0\|_{H^1} + \|e^{-|\xi|^2 t}\widehat{u_0}\|_{L^2(0,\infty;L^1)} \\ +\|e^{-|\xi|^2 t}\widehat{\nabla\psi_0}\|_{L^2(0,\infty;L^1)} \le c_0, \end{aligned}$$

then (5.5) *has a unique global solution (u, ψ, p) satisfying*

$$\begin{aligned} &u, \psi \in C([0,T]; H^2), \quad \nabla p \in C([0,T]; H^1), \\ &\nabla u \in L^2(0,T; H^2), \quad \partial_1\nabla\psi \in L^2(0,T;H^1), \quad \widehat{u}, \widehat{\partial_1\psi} \in L^2(0,T;L^1) \end{aligned}$$

and, for any $T > 0$,

$$A_T \le C A_0, \qquad \|\nabla p\|_{L^\infty_T H^1} \le C(1+c_0)A_0,$$

where A_T is given by

$$\begin{aligned} A_T \equiv \|u\|_{L^\infty_T H^2} + \|\nabla\psi\|_{L^\infty_T H^1} + \|\nabla u\|_{L^2_T H^2} \\ +\|\partial_1\nabla\psi\|_{L^2_T H^1} + \|\widehat{u}\|_{L^2_T L^1} + \|\widehat{\partial_1\psi}\|_{L^2_T L^1}. \end{aligned}$$

The proof uses extensively the divergence-free conditions $\nabla \cdot u = 0$, , the interpolation inequality and the first equation in (5.5), namely $u_2 = -(\partial_t\psi + u \cdot \nabla\psi)$.

The work of X. Hu and F. Lin [27] has also established an interesting result on the global existence and uniqueness of solutions to the perturbation equation (5.5). The functional spaces are hybrid Besov spaces. The initial perturbations u_0 and b_0 are required to be small and the initial inverse map of the particle trajectory is required to be close to the identity map in a Besov norm. We omit the precise statement of their main result, which can be found in [27].

6 The 2D MHD equation with mixed dissipation

This section briefly explains the global existence and uniqueness result of C. Cao and J. Wu [11] on the 2D MHD equations with mixed kinematic dissipation and magnetic diffusion, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu u_{yy} + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta b_{xx} + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \end{cases} \tag{6.1}$$

or

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu u_{xx} + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta b_{yy} + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0. \end{cases} \tag{6.2}$$

It is not very difficult to see that (6.2) can be converted to (6.1) by making suitable changes of variables. In fact, if we set

$$\begin{aligned} &U_1(x,y,t) = u_2(y,x,t), \quad U_2(x,y,t) = u_1(y,x,t), \\ &B_2(x,y,t) = b_1(y,x,t), \quad B_1(x,y,t) = b_2(y,x,t), \\ &P(x,y,t) = p(y,x,t), \end{aligned}$$

then $U = (U_1, U_2)$, P and $B = (B_1, B_2)$ satisfy

$$\begin{aligned} &U_t + U \cdot \nabla U = -\nabla P + \nu U_{yy} + B \cdot \nabla B, \\ &B_t + U \cdot \nabla B = \eta B_{xx} + B \cdot \nabla U, \\ &\nabla \cdot U = 0, \quad \nabla \cdot B = 0. \end{aligned}$$

Therefore, as far as the global regularity problem is concerned, it suffices to consider one of them, say (6.1).

C. Cao and J. Wu was able to show that (6.1) always possesses a unique global solution for any sufficiently smooth general initial data.

Theorem 6.1. *Assume $u_0 \in H^2(\mathbb{R}^2)$ and $b_0 \in H^2(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then (6.1) has a unique global classical solution (u,b). In addition, (u,b) satisfies*

$$\begin{aligned} &(u,b) \in L^\infty([0,\infty); H^2), \\ &\omega_y \in L^2([0,\infty); H^1), \quad j_x \in L^2([0,\infty); H^1). \end{aligned}$$

The core part in the proof of Theorem 6.1 is the global *a priori* bounds for H^1 and H^2 norms. We shall only sketch the proof for the global H^1-bound. One tool in the proof is the anisotropic Sobolev estimates for the triple product.

Lemma 6.2. *Assume that f, g, g_y, h and h_x are all in $L^2(\mathbb{R}^2)$. Then,*

$$\iint |fgh| dxdy \le C \|f\|_{L^2} \|g\|_{L^2}^{1/2} \|g_y\|_{L^2}^{1/2} \|h\|_{L^2}^{1/2} \|h_x\|_{L^2}^{1/2}.$$

Any classical solution of (6.1) admits a global H^1-bound.

Proposition 6.3. *If (u,b) is a classical solution of (6.1), then*

$$\|\omega(t)\|_2^2 + \|j(t)\|_2^2 + \nu \int_0^t \|\omega_y(\tau)\|_2^2 d\tau + \eta \int_0^t \|j_x(\tau)\|_2^2 d\tau$$
$$\leq C(\nu,\eta)\left(\|\omega_0\|_2^2 + \|j_0\|_2^2\right),$$

where $C(\nu,\eta)$ denotes a constant depending on ν and η only.

We start with the L^2-bound:

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\nu \int_0^t \|u_y(\tau)\|_2^2 d\tau + 2\eta \int_0^t \|b_x(\tau)\|_2^2 d\tau \leq \|u_0\|_2^2 + \|b_0\|_2^2.$$

To get the global H^1-bound for (u,b), we consider ω and j satisfying

$$\begin{aligned}
&\omega_t + u\cdot\nabla\omega = \nu\omega_{yy} + b\cdot\nabla j,\\
&j_t + u\cdot\nabla j = \eta j_{xx} + b\cdot\nabla\omega\\
&\qquad\qquad +2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1).
\end{aligned}$$

A simple energy estimate indicates that $X(t) = \|\omega(t)\|_2^2 + \|j(t)\|_2^2$ obeys

$$\begin{aligned}
&\frac{1}{2}\frac{dX(t)}{dt} + \nu\|\omega_y\|_2^2 + \eta\|j_x\|_2^2\\
&\leq 2\int \left(\partial_x b_1(\partial_x u_2 + \partial_y u_1) - \partial_x u_1(\partial_x b_2 + \partial_y b_1)\right) j\,dxdy.
\end{aligned}$$

The terms on the right-hand side can be bounded as follows. By Lemma 6.2,

$$\begin{aligned}
&\int |\partial_x b_1||\partial_x u_2||j|dxdy\\
&\leq C\|\partial_x u_2\|_2^{1/2}\|\partial_{xy} u_2\|_2^{1/2}\|j\|_2^{1/2}\|j_x\|_2^{1/2}\|\partial_x b_1\|_2\\
&\leq \frac{\nu}{4}\|\partial_{xy} u_2\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C\|\partial_x u_2\|_2\|\partial_x b_1\|^2\|j\|_2\\
&\leq \frac{\nu}{4}\|\omega_y\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C\|\omega\|_2\|\partial_x b_1\|_2^2\|j\|_2\\
&\leq \frac{\nu}{4}\|\omega_y\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C\|\partial_x b_1\|_2^2 X(t).
\end{aligned}$$

Similarly,

$$\int |\partial_x b_1||\partial_y u_1||j|dxdy \leq \frac{\nu}{4}\|\omega_y\|_2^2 + \frac{\eta}{8}\|j_x\|_2^2 + C(\|\partial_x b_1\|_2^2 + \|\partial_y u_1\|_2^2)\|j\|_2^2.$$

By integration by parts and Lemma 6.2,

$$
\begin{aligned}
&\left|\int \partial_x u_1 \partial_x b_2 j\,dxdy\right| \\
&= \left|\int (u_1 \partial_{xx} b_2 j + u_1 \partial_x b_2 j_x)\,dxdy\right| \\
&\le C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|j\|_2^{\frac12}\|j_x\|_2^{\frac12}\|\partial_{xx} b_2\|_2 \\
&\quad + C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xx} b_2\|_2^{\frac12}\|j_x\|_2 \\
&\le C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|j\|_2^{\frac12}\|j_x\|_2^{\frac32} + C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|j_x\|_2^{\frac32} \\
&\le \frac{\eta}{8}\|j_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\partial_x b_2\|_2^2 \\
&\le \frac{\eta}{8}\|j_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2.
\end{aligned}
$$

Similarly,

$$
\left|\int \partial_x u_1 \partial_y b_1 j\,dxdy\right| \le \frac{\eta}{8}\|j_x\|_2^2 + C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2.
$$

Combining these estimates, we have

$$
\frac{dX(t)}{dt} + \nu\|\omega_y\|_2^2 + \eta\|j_x\|_2^2 \le C((1+\|u_1\|_2^2)\|\partial_y u_1\|_2^2 + \|\partial_x b_1\|_2^2)X(t),
$$

which, together with the time integrability of $(\|\partial_y u_1\|_2^2 + \|\partial_x b_1\|_2^2)$, yields the desired H^1-bound. The global H^2-bound and more details can be found in [11].

7 The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion

This section is devoted to the partial dissipation case when the 2D MHD equations involve only the horizontal dissipation and the horizontal magnetic diffusion,

$$
\begin{cases}
\partial_t u + u\cdot\nabla u = -\nabla p + \partial_{xx} u + b\cdot\nabla b,\\
\partial_t b + u\cdot\nabla b = \partial_{xx} b + b\cdot\nabla u,\\
\nabla\cdot u = 0, \quad \nabla\cdot b = 0.
\end{cases} \tag{7.1}
$$

The global regularity problem on (7.1) is close to a complete solution. C. Cao, D. Regmi and J. Wu examined this problem in [8] and obtained various global *a priori* bounds. C. Cao, D. Regmi, J. Wu and X. Zheng [9] made further investigations and a preprint was completed in 2015. We are currently clearing up some of the minor issues in [8] and [9] and hopefully we can put this difficult problem to rest soon.

To solve the global regularity problem on (7.1), the idea of [8] is to take advantage of the symmetric structure of the system for the combined quantities

$w^\pm = u \pm b$,

$$\begin{cases} \partial_t w^+ + (w^- \cdot \nabla) w^+ = -\nabla p + \partial_x^2 w^+, \\ \partial_t w^- + (w^+ \cdot \nabla) w^- = -\nabla p + \partial_x^2 w^-, \\ \nabla \cdot w^+ = 0, \quad \nabla \cdot w^- = 0. \end{cases} \tag{7.2}$$

The main difficulty to obtain a global bound for the H^1-norm of $w^\pm$ is the appearance of the L^∞-norm of the horizontal components of $w^\pm$, namely $\|w_1^\pm\|_{L^\infty}^2$. As a consequence, the global regularity problem boils down to control

$$\int_0^T \|w_1^\pm\|_{L^\infty}^2 dt \quad \text{or} \quad \int_0^T (\|b_1\|_{L^\infty}^2 + \|u_1\|_{L^\infty}^2) dt.$$

Motivated by the work of Cao and Wu on the 2D Boussinesq equation with partial dissipation [13], the program in [8] is to obtain sharp bounds for $\|w_1^\pm\|_{L^q}^2$ (in terms of q), where $1 < q < \infty$. The symmetric formulation in (7.2) is more complex than the 2D Boussinesq equations. (7.2) consists of a system of two vector equations and the interaction between them makes it more difficult mathematically. For example, the global H^1 bound on the pressure established for the 2D Boussinesq equations has to be replaced by a global bound in a weaker space, namely in the H^s-norm with $s \in (0,1)$. To deal with this more difficult situation, new tools such as the triple product estimate involving fractional derivatives (see Lemma 7.2) are needed to cope with the difficulty here.

If $\|w_1^\pm\|_{L^q}^2$ does not grow too fast in q, say

$$\|w_1^\pm\|_{L^q}^2 \le Cq\log(2+q) \quad \text{or} \quad \|(u_1,b_1)\|_{L^q}^2 \le Cq\log(2+q), \tag{7.3}$$

the preprint of Cao, Regmi, Wu and Zheng [9] was able to show that

$$Y(t) = \|\omega(\cdot,t)\|_{L^2}^2 + \|j(\cdot,t)\|_{L^2}^2$$

obeys the differential inequality

$$\begin{aligned} &\frac{d}{dt} Y(t) + \frac{1}{2}(\|\partial_x \omega\|_{L^2}^2 + \|\partial_x j\|_{L^2}^2) \\ &\le C(1 + \|(u_1,b_1)\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 + \|\partial_x b\|_{L^2}^2) Y(t) \\ &\quad + C\left[\sup_{q\in[2,\infty)} \frac{\|(u_1,b_1)\|_{L^q}}{\sqrt{q\log(q)}}\right]^2 Y(t)\log(e+Y(t))\log(e+\log(e+Y(t))). \end{aligned}$$

The Osgood inequality then would imply a global bound for Y. The global H^2-bound then follows as a consequence.

In order to prove (7.3), one needs a good global bound on the pressure p. The following global *a priori* bounds have been established in [8].

Proposition 7.1. *Assume that $(u_0,b_0) \in H^2$ and let (u,b) be the corresponding solution of* (7.1). *Then, the following global a priori bounds hold:*

(1) *For any $1 \le r < \infty$, the first component (u_1,b_1) admits the global bound*

$$\|(u_1,b_1)\|_{L^{2r}} \le C_1 e^{C_2 r^3},$$

where C_1 and C_2 are constants depending only on $\|(u_0,b_0)\|_{L^{2r}}$.

(2) *Let p be the corresponding pressure. Let $s \in (0,1)$. Then, for any $T>0$ and $t \le T$, the second component (u_2, b_2) and p admit the global bounds,*

$$\|(u_2, b_2)\|_{L^{2r}} \le C \quad r = 2, 3,$$

and, for any $1 < q \le 3$,

$$\|p\|_{L^q} \le C, \qquad \int_0^T \|p(\tau)\|_{H^s}^2 d\tau \le C,$$

where C is a constant depending on T and the initial data only.

In order to obtain sharper global bounds for $\|(u_1, b_1)\|_{L^{2r}}$, we need two tool estimates (see Lemma 4.1 and Lemma 4.2 in [8]).

Lemma 7.2. *Let $q \in [2,\infty)$ and $s \in (\frac{1}{2}, 1]$. Assume that $f, g, \partial_y g \in L^2(\mathbb{R}^2)$, $h \in L^{2(q-1)}(\mathbb{R}^2)$ and $\Lambda_x^s h \in L^2(\mathbb{R}^2)$. Then,*

$$\left| \iint_{R^2} fghdxdy \right| \le C\|f\|_2 \|g\|_2^{\rho} \|\partial_y g\|_2^{1-\rho} \|h\|_{2(q-1)}^{\vartheta} \|\Lambda_x^s h\|_2^{1-\vartheta}, \tag{7.4}$$

where ρ and ϑ are given by

$$\rho = \frac{1}{2} + \frac{(2s-1)(q-2)}{2(2s-1)(q-1)+2}, \quad \vartheta = \frac{(2s-1)(q-1)}{(2s-1)(q-1)+1},$$

and Λ_x^s denotes a fractional derivative with respect to x and is defined by

$$\Lambda_x^s h(x) = \int e^{ix\cdot\xi} |\xi_1|^s \widehat{h}(\xi) d\xi.$$

The following lemma allows us to bound the high frequency and low frequency parts of a function in H^s $(0 < s < 1)$ separately.

Lemma 7.3. *Let $f \in H^s(\mathbb{R}^2)$ with $s \in (0,1)$. Let $R \in (0,\infty)$. Denote by $B(0,R)$ the ball centered at zero with radius R and by $\chi_{B(0,R)}$ the characteristic function on $B(0,R)$. Write*

$$f = \overline{f} + \widetilde{f} \quad \text{with} \quad \overline{f} = \mathcal{F}^{-1}(\chi_{B(0,R)} \mathcal{F} f) \quad \text{and} \quad \widetilde{f} = \mathcal{F}^{-1}((1-\chi_{B(0,R)})\mathcal{F} f), \tag{7.5}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. Then we have the following estimates for $\overline{f}$ and $\widetilde{f}$.

(1) *For a pure constant C_0 (independent of s),*

$$\|\overline{f}\|_\infty \le \frac{C_0}{\sqrt{1-s}} R^{1-s} \|f\|_{H^s(\mathbb{R}^2)}, \tag{7.6}$$

(2) *For any $2 \le q < \infty$ satisfying $1 - s - \frac{2}{q} < 0$, there is a constant C_1 independent of s, q, R and f such that*

$$\|\widetilde{f}\|_q \le C_1 q R^{1-s-\frac{2}{q}} \|f\|_{H^s(\mathbb{R}^2)}. \tag{7.7}$$

The key to obtain a sharper bound for $\|(u_1, b_1)\|_{L^{2r}}$ is to control the pressure associated terms. The pressure is split into high frequency and low frequency parts according to (7.3) and bounded accordingly. More details can be found in [8].

8 The MHD equations with fractional dissipation

This section summarizes recent results on the incompressible MHD equations with fractional dissipation

$$\begin{cases} u_t + u \cdot \nabla u + \nu(-\Delta)^\alpha u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b + \eta(-\Delta)^\beta b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \end{cases} \tag{8.1}$$

The aim is the global regularity of (8.1) for smallest possible parameters $\alpha \geq 0$ and $\beta \geq 0$. Since (8.1) was proposed for study in [51], there have been considerable activities and the global well-posedness problem on (8.1) is now much better understood (see, e.g., [12, 21, 31, 32, 47, 52, 53, 58, 59, 61]).

We summarize some of the results on (8.1). First of all, (8.1) with any $\alpha > 0$ and $\beta > 0$ always possesses a global weak solution in both 2D and 3D cases.

Theorem 8.1. *Consider* **(8.1)** *with $\nu > 0$, $\eta > 0$, $\alpha > 0$ and $\beta > 0$. Let $(u_0, b_0) \in L^2(\mathbb{R}^d)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then* (8.1) *has a global Leray-Hopf weak solution (u, b) satisfying, for any $T > 0$,*

$$\begin{aligned} &u \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; \dot{H}^\alpha), \quad b \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; \dot{H}^\beta), \\ &\partial_t u \in L^{\frac{4\alpha}{d}}(0,T; H^{-1}), \quad \partial_t b \in L^{\frac{4\beta}{d}}(0,T; H^{-1}). \end{aligned}$$

A general result on the global existence and uniqueness of classical solutions that are valid for general d-dimensional MHD equations can be found in [51] and [53]. The result stated below is taken from [53].

Theorem 8.2. *Consider the generalized incompressible magnetohydrodynamic (GMHD) equations of the form*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mathcal{L}_1^2 u = -\nabla p + b \cdot \nabla b, \quad x \in \mathbb{R}^d, t > 0, \\ \partial_t b + u \cdot \nabla b + \mathcal{L}_2^2 b = b \cdot \nabla u, \quad x \in \mathbb{R}^d, t > 0, \end{cases} \tag{8.2}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are multiplier operators with symbols given by m_1 and m_2, namely

$$\widehat{\mathcal{L}_1 u}(\xi) = m_1(\xi)\widehat{u}(\xi), \quad \widehat{\mathcal{L}_2 b}(\xi) = m_2(\xi)\widehat{b}(\xi).$$

Assume the initial data $(u_0, b_0) \in H^s(\mathbb{R}^d)$ with $s > 1 + \frac{d}{2}$, and $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Assume the symbols m_1 and m_2 satisfy

$$m_1(\xi) \geq \frac{|\xi|^\alpha}{g_1(\xi)} \quad \text{and} \quad m_2(\xi) \geq \frac{|\xi|^\beta}{g_2(\xi)}, \tag{8.3}$$

where α and β satisfy

$$\alpha \geq \frac{1}{2} + \frac{d}{4}, \quad \beta > 0, \quad \alpha + \beta \geq 1 + \frac{d}{2}, \tag{8.4}$$

and $g_1 \geq 1$ and $g_2 \geq 1$ are radially symmetric, nondecreasing and satisfy

$$\int_1^\infty \frac{ds}{s\,(g_1^2(s)+g_2^2(s))^2} = +\infty. \tag{8.5}$$

Then (8.2) has a unique global classical solution (u, b).

A special consequence of Theorem 8.2 is the global regularity for the special dissipative operators

$$\mathcal{L}_1^2 u = \frac{(-\Delta)^\alpha}{\log^{1/2}(3-\Delta)}, \qquad \mathcal{L}_2^2 u = \frac{(-\Delta)^\beta}{\log^{1/2}(3-\Delta)}$$

with

$$\alpha \geq \frac{1}{2} + \frac{d}{4}, \quad \beta > 0, \quad \alpha + \beta \geq 1 + \frac{d}{2}.$$

We remark that this result for the GMHD equations is not completely parallel to that for the generalized Navier-Stokes equations. In fact, the condition that $\beta \geq \frac{1}{2} + \frac{d}{4}$ is not imposed and (8.4) implies that it suffices to assume $\beta > 0$ when α is sufficiently large.

The borderline case $\alpha > 0$ and $\beta = 0$ is studied by K. Yamazaki [60] and his main result can be stated as follows.

Theorem 8.3. *The 2D fractional MHD equations*

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p - \nu \dfrac{(-\Delta)^2}{\log^{\frac{1}{2}}(3-\Delta)} u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \end{cases} \tag{8.6}$$

always possess a unique global solutions when the initial data is sufficiently smooth.

Several papers have been exclusively devoted to the global regularity problem in the 2D case. As already described in Subsection 4, the fractional 2D MHD equations with $\alpha = 0$ (or $\nu = 0$) and $\beta > 1$ have been shown to possess a unique global solution for any sufficiently smooth data [12]. The paper of [21] has established the global regularity of (8.1) with any $\alpha > 0$ and $\beta = 1$.

The global regularity problem on (8.1) with $\beta < 1$ and $\alpha + \beta < 2$ is currently open.

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