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Global regularity for 2D fractional magneto-micropolar equations

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Abstract
The magneto-micropolar equations are important models in fluid mechanics and material sciences. This paper focuses on the global regularity problem on the 2D magneto-micropolar equations with fractional dissipation. We establish the global regularity for three important fractional dissipation cases. Direct energy estimates are not sufficient to obtain the desired global a priori bounds in each case. To overcome the difficulties, we employ various technics including the regularization of generalized heat operators on the Fourier frequency localized functions, logarithmic Sobolev interpolation inequalities and the maximal regularity property of the heat operator.

Keywords Magneto-micropolar equations · Fractional dissipation · Global regularity

Mathematics Subject Classification 35Q35 · 35B65 · 76A10 · 76B03

1 Introduction

This paper studies the global (in time) regularity of solutions to the two-dimensional (2D) magneto-micropolar equations with fractional dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + (\nu + \kappa)(-\Delta)^{\alpha} u = -\nabla p + 2\kappa \nabla \times \Omega + b \cdot \nabla b, \\ \partial_t \Omega + u \cdot \nabla \Omega + 4\kappa \Omega + \mu\, (-\Delta)^{\gamma} \Omega = 2\kappa \nabla \times u, \\ \partial_t b + u \cdot \nabla b + \eta (-\Delta)^{\beta} b = b \cdot \nabla u, \\ \nabla \cdot u = 0,\ \nabla \cdot b = 0, \\ (u(x,0), \Omega(x,0), b(x,0)) = (u_0(x), \Omega_0(x), b_0(x)). \end{cases} \tag{1.1}$$

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Here the fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\, \widehat{f}(\xi).$$

When $\alpha = \beta = \gamma = 1$, (1.1) becomes the magneto-micropolar equations with standard Laplacian operator dissipation. Micropolar fluids represent a class of fluids with nonsymmetric stress tensor (called polar fluids) such as fluids consisting of suspending particles, dumbbell molecules, etc. The magneto-micropolar equations model the motion of electrically conducting micropolar fluids in the presence of a magnetic field. They govern a wide range of fluids such as the motion of aggregates of small solid ferromagnetic particles in viscous magnetic fluids. The magneto-micropolar equations are derived by combining the equations of continuity, momentum, Maxwell and angular momentum (see, e.g., [4,14]). The magneto-micropolar equations have recently attracted considerable attention in the community of mathematical fluids (see, e.g. [5–11,13,22,25,26,28,31,32,34,35]). Physically u denotes the velocity of the fluid, p the pressure, Ω denotes the micro-rotation velocity, b the magnetic field, ν the kinematic viscosity, κ the vortex viscosity, μ the angular viscosity, and η the magnetic diffusivity. In the 2D case, Ω is a scalar function and $\nabla \times \Omega$ really means $\nabla \times (0, 0, \Omega) = (\partial_{x_2}\Omega, -\partial_{x_1}\Omega)$, which represents a 2D vector. For the 2D velocity u, the vorticity $\omega = \nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1$ is a scalar function.

We remark that (1.1) with the fractional Laplacian operators is physically relevant. Replacing the standard Laplacian operators, these fractional diffusion operators model the so-called anomalous diffusion, a much studied topic in physics, probability and finance (see, e.g., [1,16,23]). Especially, (1.1) allows us to study long-range diffusive interactions. In addition, (1.1) with hyperviscosity ($\alpha > 1$) is used in turbulence modeling to control the effective range of the non-local dissipation and to make numerical resolutions more efficient (see, e.g., [12]). Two of our main theorems stated below, Theorems 1.1 and 1.2, deal with exactly the hyperviscosity case. The results presented here assess the validity of employing (1.1) with hyperviscosity in turbulence modeling.

Mathematically, by considering (1.1), we can examine a family of systems simultaneously. Our aim here is to establish the global regularity for (1.1) with the smallest amount of dissipation and broaden the current global well-posedness results on magneto-micropolar equations. The general approach to establish the global existence and regularity results consists of two main steps. The first step assesses the local (in time) well-posedness while the second extends the local solution into a global one by obtaining global (in time) a priori bounds. For the systems of equations concerned here, the local well-posedness follows from standard approach such as successive approximations. We are able to obtain the global regularity for three important fractional dissipation cases.

In the first case, we set $\nu > 0$, $\kappa > 0$, $\eta > 0$ and $\mu = 0$ in (1.1). No dissipation due to the angular viscosity is involved. More precisely, we consider

$$\begin{cases} \partial_t u + u \cdot \nabla u + (\nu + \kappa)(-\Delta)^\alpha u = -\nabla p + 2\kappa \nabla \times \Omega + b \cdot \nabla b, \\ \partial_t \Omega + u \cdot \nabla \Omega + 4\kappa \Omega = 2\kappa \nabla \times u, \\ \partial_t b + u \cdot \nabla b + \eta(-\Delta)^\beta b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ (u(x, 0), \Omega(x, 0), b(x, 0)) = (u_0(x), \Omega_0(x), b_0(x)). \end{cases} \tag{1.2}$$

Our regularity result can be stated as follows.

Theorem 1.1 *Consider (1.2) with $\nu > 0$, $\kappa > 0$, $\eta > 0$, and*

$$1 < \alpha < 2, \quad 0 < \beta < 1, \quad \alpha + \beta \geq 2. \tag{1.3}$$

Assume $(u_0, \Omega_0, b_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then* (1.2) *has a unique global solution* (u, Ω, b) *satisfying, for any* $T > 0$,

$$(u, \Omega, b) \in L^\infty([0, T]; H^s(\mathbb{R}^2)).$$

The proof of Theorem 1.1 relies on the global a priori bound for $\|(u, \Omega, b)\|_{H^s}$. When α and β satisfy (1.3), the global L^2-bound follows directly from the equations. However, for $0 < \beta < 1$, it is not easy to obtain the global bounds for the derivatives of (u, Ω, b). Due to the lack of dissipation in the equation of Ω, we need a global bound for $\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$ or for a quantity that is close to the regularity level of $\|\nabla u\|_{L^\infty}$. In fact, our main effort is devoted to the global bound, for some $q \in (1, \infty)$ and for any $t > 0$,

$$\int_0^t \|\Lambda^{\frac{2}{q}} \omega\|_{L^q}\, dt < \infty, \tag{1.4}$$

where $\omega = \nabla \times u$ represents the vorticity. $\|\Lambda^{\frac{2}{q}} \omega\|_{L^q}$ is close to $\|\nabla u(\tau)\|_{L^\infty}$ in the sense that, for $\sigma > 2$,

$$\|\nabla u\|_{L^\infty} \le C_1(1 + \|u\|_{L^2}) + C_2 \|\Lambda^{\frac{2}{q}} \omega\|_{L^q} \log(e + \|\Lambda^\sigma u\|_{L^2}).$$

It is easy to check that the vorticity ω obeys

$$\partial_t \omega + u \cdot \nabla \omega + (\nu + \kappa)(-\Delta)^\alpha \omega = b \cdot \nabla j - 2\kappa\, \Delta\Omega, \tag{1.5}$$

where $j = \nabla \times b$ denotes the current density. However, direct energy estimates involving the vorticity equation in (1.5) do not yield (1.4) due to the presence of the term $\Delta\Omega$ and the lack of global bound on the derivatives of Ω. We developed two different approaches to establish (1.4). The first is to hide the term $\Delta\Omega$ in (1.5) and work with the combined quantity

$$G = \omega + \frac{2\kappa}{\nu + \kappa} \mathcal{R}_\alpha \Omega \quad \text{with} \quad \mathcal{R}_\alpha = (-\Delta)^{-\alpha} \Delta.$$

G satisfies

$$\begin{aligned} &\partial_t G + u \cdot \nabla G + (\nu + \kappa)(-\Delta)^\alpha G \\ &\quad = \frac{4\kappa^2}{\nu + \kappa} \mathcal{R}_\alpha \omega - \frac{2\kappa}{\nu + \kappa} [\mathcal{R}_\alpha, u \cdot \nabla]\Omega + b \cdot \nabla j - \frac{4\kappa^2}{\nu + \kappa} \mathcal{R}_\alpha \Omega. \end{aligned}$$

By considering the equations of G and of j, we are then able to obtain a global L^2-bound for G and j. By showing a global bound for Ω in H^{-1}, we are able to obtain a global for ω in L^2 or u in H^1. Through an iterative process, we establish (1.4). The details are given Sect. 2. This approach is useful in handling coupled systems of equations with high derivative coupling terms. We remark that the practice of working with combined quantities has been exercised in the study of several other equations (see, e.g., [8,15,17]).

An alternative approach for proving (1.4) is to first obtain extra regularity on b (beyond what the basic L^2 estimate provides) and then rewrite the vorticity equation in an integral form via the fractional Laplacian operator. Taking full advantage of the generalized heat operator $e^{-(-\Delta)^\alpha t}$ and controlling $\|\Omega(t)\|_{L^q}$ in terms of $\int_0^t \|\omega(\tau)\|_{L^q} d\tau$, we are able to prove (1.4). This alternative approach is provided in the appendix.

The global regularity for (1.2) with $\alpha + \beta < 2$ with $0 < \beta < 1$ is currently open. It appears extremely difficult to establish the global H^1-bound on the solutions when $\alpha + \beta < 2$ with $\beta < 1$. In fact, even in the special case when $\Omega \equiv 0$ [(1.2) then becomes the magneto-hydrodynamics equations], the global well-posedness result still requires dissipation only logarithmically weaker than the dissipation level $\alpha + \beta = 2$ (see [30]).

Yamazaki has previously established the global regularity of (1.2) when $\alpha = 1$ and $\beta = 1$ [34]. Even this case is not trivial. The global regularity was achieved in [34] by fully exploiting the structure of the system and bounding the Lebesgue norm of the first derivatives of the solution.

The second fractional dissipation case examined here involves no angular viscosity and no magnetic diffusivity, namely $\mu = 0$ and $\eta = 0$ in (1.1). More precisely, we consider

$$\begin{cases} \partial_t u + u \cdot \nabla u + (\nu + \kappa)(-\Delta)^\alpha u = -\nabla p + 2\kappa \nabla \times \Omega + b \cdot \nabla b, \\ \partial_t \Omega + u \cdot \nabla \Omega + 4\kappa \Omega = 2\kappa \nabla \times u, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ (u(x,0), \Omega(x,0), b(x,0)) = (u_0(x), \Omega_0(x), b_0(x)). \end{cases} \tag{1.6}$$

We establish that (1.6) with $\alpha = 2$ possesses a unique global solution when (u_0, Ω_0, b_0) is sufficiently smooth, as stated in the following theorem.

Theorem 1.2 *Consider (1.6) with $\nu > 0$, $\kappa > 0$ and $\alpha = 2$. Assume $(u_0, \Omega_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then (1.6) has a unique global solution (u, Ω, b) satisfying, for any $T > 0$,*

$$(u, \Omega, b) \in L^\infty([0, T]; H^s(\mathbb{R}^2)).$$

The effort of proving Theorem 1.2 is devoted to the global bounds for $\|(u, \Omega, b)\|_{H^s}$. The global L^2 bound follows from (1.6) directly, but the global H^1-bound relies on a logarithmic interpolation inequality. Due to the lack of dissipation in the equations of Ω and b, it appears to be difficult to establish the global H^1 bound for $\alpha < 2$. We remark that Theorem 1.2 with $\Omega \equiv 0$ reduces to a global result on the 2D magneto-hydrodynamic equation [33].

The third fractional dissipation case dealt with here is given by

$$\begin{cases} \partial_t u + u \cdot \nabla u + (\nu + \kappa)(-\Delta)^\alpha u = -\nabla p + 2\kappa \nabla \times \Omega + b \cdot \nabla b, \\ \partial_t \Omega + u \cdot \nabla \Omega + 4\kappa \Omega - \mu \Delta \Omega = 2\kappa \nabla \times u, \\ \partial_t b + u \cdot \nabla b - \eta \Delta b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ (u(x,0), \Omega(x,0), b(x,0)) = (u_0(x), \Omega_0(x), b_0(x)). \end{cases} \tag{1.7}$$

We show that (1.7) with any $\alpha > 0$ always has a unique global solution.

Theorem 1.3 *Consider (1.7) with $\alpha > 0$, $\nu > 0$, $\kappa > 0$, $\mu > 0$ and $\eta > 0$. Assume $(u_0, \Omega_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then (1.7) has a unique global solution (u, Ω, b) satisfying, for any $T > 0$,*

$$(u, \Omega, b) \in L^\infty([0, T]; H^s(\mathbb{R}^2)).$$

To prove Theorem 1.3, our focus is again on the global *a priori* bound for $\|(u, \Omega, b)\|_{H^s}$. In this case, there is no essential difficulty to obtain the global L^2 and global H^1-bounds, but we need to resort to the maximal regularity principle of the heat operator in order to obtain global space-time $L^q_t L^p_x$ bounds for the vorticity ω and the current density j. These bounds allow us to further obtain the global H^2 and H^s bounds.

The rest of this paper is organized as follows. Sections 2, 3 and 4 are devoted to the proofs of Theorems 1.1, 1.2 and 1.3, respectively. The last part of this paper is an appendix, which presents an alternative approach to the global H^1-bound for solutions of (1.2).

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, stating the global existence and uniqueness of solutions to the 2D magneto-micropolar equations (1.2). The key step is to establish the global a priori bound on the solution (u, Ω, b) in H^s. More precisely, we prove the following main proposition.

Proposition 2.1 *Consider (1.2) with $\nu > 0, \kappa > 0, \eta > 0$, and α and β satisfying*

$$1 < \alpha < 2, \quad 0 < \beta < 1, \quad \alpha + \beta \geq 2.$$

Assume $(u_0, \Omega_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, and $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. Then the corresponding solution of (1.2) is globally bounded in $H^s(\mathbb{R}^2)$.

The proof of Proposition 2.1 requires several steps. We also need some preparatory facts. For the sake of clarity, we divide the rest of this section into four subsections. The first subsection provides several calculus inequalities and logarithmic interpolation inequalities involving fractional Laplacian operators.

2.1 Inequalities involving fractional Laplacian operators

This subsection prepares several tools needed for the proofs of the theorems. The first contains two calculus inequalities involving fractional differential operators. They can be found in many references, e.g., [18], [19, p. 334]. We recall that $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator.

Lemma 2.2 *Let $s > 0$. Let $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then,*

$$\|[\Lambda^s, f]g\|_{L^r} \leq C \left(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right)$$

and

$$\|\Lambda^s (f\, g)\|_{L^r} \leq C \left(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^s g\|_{L^{q_2}}\right),$$

where C's are constants depending on the indices s, r, p_1, q_1, p_2 and q_2.

The second tool provides an upper bound for the action of the generalized heat operator on functions with Fourier transform supported on an annulus. The bound for the standard heat operator can be found in [2]. This generalization can be established by modifying the original proof.

Lemma 2.3 *Let $\mathcal{A} = \{\xi \in \mathbb{R}^d : r_1 \leq |\xi| \leq r_2\}$ with $0 < r_1 < r_2$ being constants. Let $\alpha > 0$. Then, there exist $c > 0$ and $C > 0$ such that, for $p \in [1, \infty]$ and f satisfying*

$$f \in L^p(\mathbb{R}^d), \quad supp \widehat{f} \subset \lambda \mathcal{A},$$

we have, for $t > 0$ and $\lambda > 0$,

$$\|e^{-(-\Delta)^\alpha t} f\|_{L^p(\mathbb{R}^d)} \leq C\, e^{-ct\lambda^{2\alpha}} \|f\|_{L^p(\mathbb{R}^d)}.$$

Another very important class of tools are the logarithmic Besov interpolation inequalities. The version presented here is different from the standard ones and we provide a proof. In the following lemma, $B^0_{q,\infty}$ denotes a Besov space.

Lemma 2.4 *For $s > 2$ and $q \in (1, \infty)$, there exists two constants C_1 and C_2 such that*

$$\|\nabla u\|_{L^\infty} \le C_1(1 + \|u\|_{L^2}) + C_2 \, \|\Lambda^{\frac{2}{q}} \nabla u\|_{B^0_{q,\infty}} \; \log(e + \|\Lambda^s u\|_{L^2}). \tag{2.1}$$

Especially,

$$\|\nabla u\|_{L^\infty} \le C_1(1 + \|u\|_{L^2}) + C_2 \, \|\Lambda^{\frac{2}{q}} \nabla u\|_{L^q} \; \log(e + \|\Lambda^s u\|_{L^2}).$$

In addition, a variant of (2.1) *is to replace $\|\Lambda^{\frac{2}{q}} \nabla u\|_{B^0_{q,\infty}}$ by the H^1-norm of $\omega = \nabla \times u$,*

$$\|\nabla u\|_{L^\infty} \le C_1(1 + \|u\|_{L^2}) + C_2 \, \|\omega\|_{H^1} \; \log(e + \|\Lambda^s u\|_{L^2}). \tag{2.2}$$

To help understand this lemma and its proof, we recall the definitions of the Fourier localization operators and the Besov spaces, and provide some closely related facts below. The materials presented here can be found in several books and many papers (see, e.g., [2,3,24,27,29]).

We start with several notational conventions. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{ \xi \in \mathbb{R}^d : \; 2^{j-1} \le |\xi| < 2^{j+1} \right\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1\,, & \text{if } \xi \in \mathbb{R}^d \backslash \{0\}, \\ 0\,, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \backslash \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{2.3}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \dots. \end{cases} \tag{2.4}$$

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. It is clear from (2.3) that $S_j \to Id$ as $j \to \infty$ in the distributional sense. In addition, the notation $\widetilde{\Delta}_k$, defined by

$$\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1},$$

is also useful and has been used in the previous sections.

Definition 2.5 The inhomogeneous Besov space $B^s_{p,q}$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty,$$

where $\Delta_j f$ is as defined in (2.4).

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations. For any $s \in \mathbb{R}$,

$$H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$B^s_{q,\min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,\max\{q,2\}}.$$

For any non-integer $s > 0$, the Hölder space C^s is equivalent to $B^s_{\infty,\infty}$.

Bernstein's inequalities are very useful in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Lemma 2.6 *Let* $\alpha \ge 0$. *Let* $1 \le p \le q \le \infty$.

1. *If f satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \le K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2. *If f satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and constants $0 < K_1 \le K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

We are now ready to prove Lemma 2.4.

Proof of Lemma 2.4 For an integer $N > 0$ to be determined later, we write

$$\|\nabla u\|_{L^\infty} \le \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{0\le m\le N} \|\Delta_m \nabla u\|_{L^\infty} + \sum_{m\ge N+1} \|\Delta_m \nabla u\|_{L^\infty}.$$

It then follows from Bernstein's inequality that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C\,\|u\|_{L^2} + \sum_{0\le m\le N} 2^{\frac{2m}{q}} \|\nabla \Delta_m u\|_{L^q} + \sum_{m\ge N+1} 2^{2m} \|\Delta_m u\|_{L^2} \\ &\le C\,\|u\|_{L^2} + \sum_{0\le m\le N} \|\Lambda^{\frac{2}{q}} \nabla \Delta_m u\|_{L^q} + C \sum_{m\ge N+1} 2^{m(2-s)} \|\Lambda^s \Delta_m u\|_{L^2} \\ &\le C\,\|u\|_{L^2} + C\,N \sup_{0\le m\le N} \|\Lambda^{\frac{2}{q}} \nabla \Delta_m u\|_{L^q} + C\,2^{N(2-s)} \|\Lambda^s u\|_{L^2}. \end{aligned} \tag{2.5}$$

Since $s > 2$, we can take N such that $2^{N(2-s)}\|\Lambda^s u\|_{L^2} \le C$, namely

$$N = 1 + \mathrm{INT}\left[\frac{1}{(s-2)\ln 2} \log(e + \|\Lambda^s u\|_{L^2})\right],$$

where INT denotes the integer part of a real number. Inserting N in (2.5) and noticing that

$$\sup_{0\le m\le N} \|\Lambda^{\frac{2}{q}} \nabla \Delta_m u\|_{L^q} \le \|\Lambda^{\frac{2}{q}} \nabla u\|_{B^0_{q,\infty}},$$

we obtain (2.1). (2.2) is obtained by setting $q = 2$ in (2.1). This completes the proof of Lemma 2.4. □

2.2 Global L^2-bound for (u, Ω, b) and global bound for $\|\Lambda^\sigma b\|_{L^2}$

This subsection provides the global L^2-bound and a global bound for $\|\Lambda^\sigma b\|_{L^2}$ for any $0 < \sigma < \alpha - 1$. To obtain the global L^2-bound, we dot (1.2) with (u, Ω, b) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,\Omega,b)\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^\alpha u\|_{L^2}^2 + \eta\|\Lambda^\beta b\|_{L^2}^2 \\ &= 2\kappa\int(\nabla\times\Omega)\cdot u + 2\kappa\int(\nabla\times u)\cdot\Omega = 4\kappa\int(\nabla\times u)\cdot\Omega \\ &\le 4\kappa\|\nabla u\|_{L^2}\|\Omega\|_{L^2} \le 4\kappa\|u\|_{L^2}^{1-\frac{1}{\alpha}}\|\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Omega\|_{L^2} \\ &\le \frac{\nu+\kappa}{2}\|\Lambda^\alpha u\|_{L^2}^2 + C\left(\|u\|_{L^2}^2 + \|\Omega\|_{L^2}^2\right).\end{aligned}$$

Gronwall's inequality then leads to

$$\begin{aligned}&\|(u(t),\Omega(t),b(t))\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\Lambda^\alpha u\|_{L^2}^2 d\tau \\ &\quad + \eta\int_0^t \|\Lambda^\beta b\|_{L^2}^2 d\tau \le C\left(\|u_0,\Omega_0,b_0)\|_{L^2}, t\right) < \infty.\end{aligned} \tag{2.6}$$

A better regularity can be further obtained for b by estimates involving the equation of b only. This improved regularity for b is crucial for the global time integrability obtained in the following subsection.

Lemma 2.7 *Assume that* (u, Ω, b) *solves* (1.2). *Assume* α *and* β *satisfy*

$$1 < \alpha < 2, \quad 0 < \beta < 1, \quad \alpha + \beta \geq 2.$$

Then b *obeys the following global bound, for any* $t > 0$,

$$\|\Lambda^\sigma b(t)\|_{L^2}^2 + \eta \int_0^t \|\Lambda^{\beta+\sigma} b(\tau)\|_{L^2}^2 \, d\tau \leq C(t, u_0, b_0, \Omega_0) < \infty \quad \text{for any } 0 < \sigma < \alpha - 1.$$

Proof of Lemma 2.7 We focus on the case when $\alpha + \beta = 2$ since the case $\alpha + \beta > 2$ is even simpler. Assume $0 < \sigma < \alpha - 1$. Applying Λ^σ to the equation of b in (1.2) and dotting the resulting equation with $\Lambda^\sigma b$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\sigma b\|_{L^2}^2 + \eta\|\Lambda^{\beta+\sigma} b\|_{L^2}^2 = J_1 + J_2, \tag{2.7}$$

where

$$J_1 = -\int \Lambda^\sigma (u \cdot \nabla b) \cdot \Lambda^\sigma b, \quad J_2 = \int \Lambda^\sigma (b \cdot \nabla u) \cdot \Lambda^\sigma b.$$

Due to $\nabla \cdot u = 0$, we write J_1 in the commutator form

$$J_1 = -\int (\Lambda^\sigma (u \cdot \nabla b) - u \cdot \nabla \Lambda^\sigma b) \cdot \Lambda^\sigma b.$$

By Lemma 2.2, Sobolev's inequality and Young's inequality,

$$\begin{aligned} |J_1| &\leq C\, \|\Lambda^\sigma b\|_{L^2} \left(\|\Lambda^\sigma \nabla u\|_{L^{p_1}} \|b\|_{L^{p_2}} + \|\nabla u\|_{L^{p_3}} \|\Lambda^\sigma b\|_{L^{p_4}}\right) \\ &\leq C\, \|\Lambda^\sigma b\|_{L^2} \|\Lambda^\alpha u\|_{L^2} \|\Lambda^{\beta+\sigma} b\|_{L^2} \\ &\leq \frac{\eta}{4} \|\Lambda^{\beta+\sigma} b\|_{L^2}^2 + C\, \|\Lambda^\alpha u\|_{L^2}^2 \|\Lambda^\sigma b\|_{L^2}^2, \end{aligned}$$

where $2 < p_1, p_2, p_3, p_4 < \infty$ have been chosen to fit the Sobolev inequalities used above,

$$\begin{aligned} &\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}, \quad \frac{1}{p_1} - \frac{1+\sigma}{2} = \frac{1-\alpha}{2}, \quad \frac{1}{p_2} = \frac{1-(\beta+\sigma)}{2}, \\ &\frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{2}, \quad \frac{1}{p_3} = \frac{2-\alpha}{2}, \quad \frac{1}{p_4} = \frac{1-\beta}{2}. \end{aligned}$$

J_2 obeys exactly the same bound as J_1. Inserting these bounds in (2.7) and applying Gronwall's inequality, we obtain the desired global bound after invoking the global L^2 bound for u in (2.6). This completes the proof of Lemma 2.7. □

2.3 Global time integrability of $\|\Lambda^{\frac{2}{q}} \omega\|_{L^q}$

The major result of this subsection is the global bound stated in the following proposition.

Proposition 2.8 *Assume that* (u, Ω, b) *solves* (1.2). *Assume* α *and* β *satisfy*

$$1 < \alpha < 2, \quad 0 < \beta < 1, \quad \alpha + \beta \geq 2.$$

Then, (u, Ω, b) *obeys the following global a priori bounds,*

$$\|\omega(t)\|_{L^2} \le C(t, u_0, \Omega_0, b_0) < \infty;$$
$$\|\Omega(t)\|_{L^r} \le C(t, u_0, \Omega_0, b_0) < \infty \quad \text{for any } 2 \le r < \infty;$$
$$\int_0^t \|\Lambda^{2\alpha-2}\omega(\tau)\|_{L^q}\, d\tau \le C(t, u_0, \Omega_0, b_0) < \infty \quad \text{for any } 2 \le q \le \frac{2}{\alpha-1}. \tag{2.8}$$

As a special consequence, there is $1 < q < \infty$ *such that*

$$\int_0^t \|\Lambda^{\frac{2}{q}}\omega(\tau)\|_{L^q}\, d\tau \le C(t, u_0, \Omega_0, b_0) < \infty.$$

Proof of Proposition 2.8 To obtain the global H^1-bound and the time integrability of $\|\Lambda^{\frac{2}{q}}\omega\|_{L^q}$, we make use of the equations of $\omega = \nabla \times u$ and of $j = \nabla \times b$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla\omega + (\nu+\kappa)(-\Delta)^\alpha \omega = b \cdot \nabla j - 2\kappa\, \Delta\Omega, \\ \partial_t j + u \cdot \nabla j + \eta(-\Delta)^\beta j = b \cdot \nabla\omega + Q(\nabla u, \nabla b), \end{cases} \tag{2.9}$$

where $Q(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)$. We will just focus on the case when $1 < \alpha \le \frac{3}{2}$. The case $\frac{3}{2} < \alpha < 2$ can be treated similarly. It appears that direct energy estimates on (2.9) would not lead to the desired global bound, due to the presence of the bad term $-\Delta\Omega$. The idea here is to hide this term by working with a combined quantity. More precisely, we set

$$\mathcal{R}_\alpha = (-\Delta)^{-\alpha}\Delta, \quad G = \omega + \frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha\Omega.$$

The vorticity equation becomes

$$\partial_t\omega + u \cdot \nabla\omega + (\nu+\kappa)(-\Delta)^\alpha\, G = b \cdot \nabla j.$$

To obtain an equation for G, we apply $\frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha$ to the second equation in (1.2) to get

$$\partial_t\left(\frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha\Omega\right) + u\cdot\nabla\left(\frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha\Omega\right) + \frac{4\kappa^2}{\nu+\kappa}\mathcal{R}_\alpha\Omega = -\frac{2\kappa}{\nu+\kappa}[\mathcal{R}_\alpha, u\cdot\nabla]\Omega + \frac{4\kappa^2}{\nu+\kappa}\mathcal{R}_\alpha\omega.$$

Adding them up yields

$$\begin{aligned} &\partial_t G + u\cdot\nabla G + (\nu+\kappa)(-\Delta)^\alpha G \\ &\quad = \frac{4\kappa^2}{\nu+\kappa}\mathcal{R}_\alpha\omega - \frac{2\kappa}{\nu+\kappa}[\mathcal{R}_\alpha, u\cdot\nabla]\Omega + b\cdot\nabla j - \frac{4\kappa^2}{\nu+\kappa}\mathcal{R}_\alpha\Omega. \end{aligned} \tag{2.10}$$

We estimate $\|G\|_{L^2}$ and $\|j\|_{L^2}$. Taking the inner product of (2.10) with G and the equation of j with j, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^\alpha G\|_{L^2}^2 &= \frac{4\kappa^2}{\nu+\kappa}\int \mathcal{R}_\alpha\omega\, G - \frac{2\kappa}{\nu+\kappa}\int [\mathcal{R}_\alpha, u\cdot\nabla]\Omega\, G \\ &\quad + \int b\cdot\nabla j(\omega + \frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha\Omega) - \frac{4\kappa^2}{\nu+\kappa}\int \mathcal{R}_\alpha\Omega\, G, \\ \frac{1}{2}\frac{d}{dt}\|j\|_{L^2}^2 + \eta\|\Lambda^\beta j\|_{L^2}^2 &= \int b\cdot\nabla\omega j + \int Q(\nabla u, \nabla b)\, j. \end{aligned}$$

Adding them up yields

$$\frac{1}{2}\frac{d}{dt}\|(G, j)\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^\alpha G\|_{L^2}^2 + \eta\|\Lambda^\beta j\|_{L^2}^2 = L_1 + L_2 + L_3 + L_4 + L_5, \tag{2.11}$$

where

$$L_1 = \frac{4\kappa^2}{\nu+\kappa}\int \mathcal{R}_\alpha \omega\, G, \quad L_2 = -\frac{4\kappa^2}{\nu+\kappa}\int \mathcal{R}_\alpha \Omega\, G, \quad L_3 = \frac{2\kappa}{\nu+\kappa}\int b\cdot\nabla j\, R_\alpha\Omega,$$
$$L_4 = -\frac{2\kappa}{\nu+\kappa}\int [\mathcal{R}_\alpha, u\cdot\nabla]\Omega\, G, \quad L_5 = -\int Q(\nabla u, \nabla b)\, j.$$

The terms on the right can be estimated as follows. For $\alpha \le \frac{3}{2}$,

$$\begin{aligned}|L_1| &\le \frac{4\kappa^2}{\nu+\kappa}\|\mathcal{R}_\alpha\omega\|_{L^2}\|G\|_{L^2} = \frac{4\kappa^2}{\nu+\kappa}\|\Lambda^{3-2\alpha}u\|_{L^2}\|G\|_{L^2}\\ &\le C\,\|u\|_{L^2}^{\frac{3\alpha-3}{\alpha}}\,\|\Lambda^\alpha u\|_{L^2}^{\frac{3-2\alpha}{\alpha}}\,\|G\|_{L^2},\end{aligned}$$

where we note that $\frac{3-2\alpha}{\alpha} < 1$. Recalling $G = \omega + \frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha\Omega$, we have

$$\begin{aligned}|L_2| &\le 2\kappa\,\|G\|_{L^2}^2 + 2\kappa\,\|u\|_{L^2}\,\|\nabla G\|_{L^2}\\ &\le 2\kappa\,\|G\|_{L^2}^2 + 2\kappa\,\|u\|_{L^2}\,\|G\|_{L^2}^{1-\frac{1}{\alpha}}\,\|\Lambda^\alpha G\|_{L^2}^{\frac{1}{\alpha}}\\ &\le \frac{1}{2}(\nu+\kappa)\|\Lambda^\alpha G\|_{L^2}^2 + C\left(1+\|u\|_{L^2}^{\frac{2\alpha}{\alpha-1}}\right)\|G\|_{L^2}^{\frac{2\alpha-2}{2\alpha-1}}.\end{aligned}$$

For $\alpha \le \frac{3}{2}$ and $\alpha+\beta = 2$, by Sobolev's inequality,

$$\begin{aligned}|L_3| &\le \frac{2\kappa}{\nu+\kappa}\left|\int \Lambda^{3-2\alpha}(bj)\,\Omega\right| \le \frac{2\kappa}{\nu+\kappa}\|\Lambda^{3-2\alpha}(bj)\|_{L^2}\|\Omega\|_{L^2}\\ &\le C\,\|\Lambda^{3-2\alpha}b\|_{L^{\frac{2}{2-\alpha}}}\,\|j\|_{L^{\frac{2}{1-\beta}}}\,\|\Omega\|_{L^2} + C\,\|b\|_{L^{\frac{2}{1-\beta}}}\,\|\Lambda^{3-2\alpha}j\|_{\frac{2}{2-\alpha}}\,\|\Omega\|_{L^2}\\ &\le C\,\|\Lambda^\beta b\|_{L^2}\|\Lambda^\beta j\|_{L^2}\,\|\Omega\|_{L^2}\\ &\le \frac{\eta}{4}\|\Lambda^\beta j\|_{L^2}^2 + C\,\|\Lambda^\beta b\|_{L^2}^2\,\|\Omega\|_{L^2}^2.\end{aligned}$$

L_4 can be controlled without appealing to the commutator structure. We split L_4 into two terms,

$$L_4 = \frac{2\kappa}{\nu+\kappa}\int \mathcal{R}_\alpha\nabla\cdot(u\Omega)\,G - \frac{2\kappa}{\nu+\kappa}\int u\cdot\nabla\mathcal{R}_\alpha\Omega\,G.$$

After shifting all derivatives from Ω and applying Lemma 2.2, we obtain

$$|L_4| \le \frac{2\kappa}{\nu+\kappa}\|\Omega\|_{L^2}\,\|u\|_{L^{q_1}}\,\|\Lambda^{3-2\alpha}G\|_{L^{q_2}} + \frac{2\kappa}{\nu+\kappa}\|\Omega\|_{L^2}\,\|\Lambda^{3-2\alpha}u\|_{L^{q_2}}\,\|G\|_{L^{q_1}}, \quad (2.12)$$

where $q_1, q_2 \in [2,\infty]$ and $\frac{1}{q_1}+\frac{1}{q_2} = \frac{1}{2}$. The terms on the right of (2.12) can be bounded via the dissipation term $\|\Lambda^\alpha G\|_{L^2}$ and the time integral of $\|\Lambda^\alpha u\|_{L^2}$. Intuitively, for $1 < \alpha \le \frac{3}{2}$, $\|u\|_{L^{q_1}}$ can be bounded by $\|\Lambda^\alpha u\|_{L^2}$ and $\|\Lambda^{3-2\alpha}u\|_{L^{q_2}}$ by $\|\Lambda^\alpha G\|_{L^2}$. More careful estimates are given below. Due to $3-2\alpha < \alpha$, by Sobolev's inequality,

$$\|u\|_{L^{q_1}} \le C\,\|u\|_{L^2}^{\sigma_1}\,\|\Lambda^\alpha u\|_{L^2}^{1-\sigma_1}, \quad \|\Lambda^{3-2\alpha}G\|_{L^{q_2}} \le C\,\|G\|_{L^2}^{\sigma_2}\,\|\Lambda^\alpha G\|_{L^2}^{1-\sigma_2},$$

where $\sigma_1, \sigma_2 \in [0, 1]$ and

$$\begin{aligned}
&\frac{1}{q_1} = \frac{1}{2}\sigma_1 + (1-\sigma_1)\left(\frac{1}{2} - \frac{\alpha}{2}\right) = \frac{1}{2} - (1-\sigma_1)\frac{\alpha}{2},\\
&\frac{1}{q_2} - \frac{3-2\alpha}{2} = \frac{1}{2}\sigma_2 + (1-\sigma_2)\left(\frac{1}{2} - \frac{\alpha}{2}\right) = \frac{1}{2} - (1-\sigma_2)\frac{\alpha}{2},\\
&3 - 2\alpha \le \alpha(1-\sigma_2).
\end{aligned}$$

Therefore, the first term in (2.12) is bounded by

$$\begin{aligned}
&\frac{2\kappa}{\nu+\kappa}\|\Omega\|_{L^2}\,\|u\|_{L^{q_1}}\,\|\Lambda^{3-2\alpha}G\|_{L^{q_2}}\\
&\le \frac{\nu+\kappa}{4}\|\Lambda^\alpha G\|_{L^2}^2 + C\,\|\Omega\|_{L^2}^{\frac{2}{1+\sigma_2}}\,\|u\|_{L^2}^{\frac{2\sigma_1}{1+\sigma_2}}\,\|\Lambda^\alpha u\|_{L^2}^{\frac{2(1-\sigma_1)}{1+\sigma_2}}\,\|G\|_{L^2}^{\frac{2\sigma_2}{1+\sigma_2}}.
\end{aligned}$$

It is not hard to check that, for $1 < \alpha \le \frac{3}{2}$, the indices $q_1, q_2, \sigma_1, \sigma_2$ can be selected. The second term in (2.12) can be similarly bounded. Finally we turn to L_5, which can be bounded by

$$\begin{aligned}
|L_5| &\le C\,\|\nabla u\|_{L^{\frac{2}{\beta}}}\,\|\nabla b\|_{L^{\frac{2}{1-\beta}}}\,\|j\|_{L^2}\\
&\le C\|u\|_{L^2}^{\frac{\alpha+\beta-2}{\alpha}}\,\|\Lambda^\alpha u\|_{L^2}^{\frac{2-\beta}{\alpha}}\,\|\Lambda^\beta j\|_{L^2}\|j\|_{L^2}\\
&\le \frac{\eta}{4}\|\Lambda^\beta j\|_{L^2}^2 + C\|u\|_{L^2}^{\frac{2(\alpha+\beta-2)}{\alpha}}\,\|\Lambda^\alpha u\|_{L^2}^{\frac{2(2-\beta)}{\alpha}}\,\|j\|_{L^2}^2.
\end{aligned}$$

Here we note that $\frac{2(2-\beta)}{\alpha} \le 2$ due to $\alpha + \beta \ge 2$. Inserting the bounds for L_1, L_2, L_3, L_4 and L_5 in (2.11) and applying Gronwall's inequality yields the following global bound,

$$\|(G, j)(t)\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\Lambda^\alpha G\|_{L^2}^2\,d\tau + \eta\int_0^t \|\Lambda^\beta j\|_{L^2}^2\,d\tau \le C. \tag{2.13}$$

Due to the relation $G = \omega + \frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha\Omega$, we would like to obtain a global bound on $\|\omega\|_{L^2}$. We need to bound $\mathcal{R}_\alpha\Omega$, but we have difficulty bounding it directly. Instead we bound $\|\mathcal{R}_\alpha\Omega\|_{H^{-1}}$ first. Applying Λ^{-1} to the Ω-equation in (1.2) and dotting by $\Lambda^{-1}\Omega$ yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^{-1}\Omega\|_{L^2}^2 + 2\kappa\|\Lambda^{-1}\Omega\|_{L^2}^2 &= -\int \Lambda^{-1}(u\cdot\nabla\Omega)\,\Lambda^{-1}\Omega + 2\kappa\int \Lambda^{-1}\Omega\,\Lambda^{-1}\omega\\
&\le \|u\,\Omega\|_{L^2}\,\|\Lambda^{-1}\Omega\|_{L^2} + 2\kappa\,\|u\|_{L^2}\,\|\Lambda^{-1}\Omega\|_{L^2}\\
&\le \left(\|u\|_{L^\infty}\,\|\Omega\|_{L^2} + 2\kappa\|u\|_{L^2}\right)\,\|\Lambda^{-1}\Omega\|_{L^2}.
\end{aligned}$$

Due to the Sobolev inequality

$$\|u\|_{L^\infty} \le C\,\|u\|_{L^2}^{1-\frac{1}{\alpha}}\,\|\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}},$$

$\|u\|_{L^\infty}$ is time integrable. We then obtain a global bound for $\|\Lambda^{-1}\Omega\|_{L^2}$. As a consequence, for $1 < \alpha \le \frac{3}{2}$,

$$\|\mathcal{R}_\alpha\Omega\|_{L^2} \le \|\Omega\|_{L^2}^{3-2\alpha}\,\|\Lambda^{-1}\Omega\|_{L^2}^{2\alpha-2} < \infty \tag{2.14}$$

for any $t > 0$. Combining (2.13) and (2.14) yields a global bound on ω,

$$\|\omega(t)\|_{L^2} < \infty. \tag{2.15}$$

We further show the following global time integrability,

$$\int_0^t \|\Lambda^{2\alpha-2}\omega(\tau)\|_{L^q}\, d\tau < \infty \quad \text{for any } 2 \le q \le \tfrac{2}{\alpha-1}. \tag{2.16}$$

To do so, we first show that

$$\|\Omega(t)\|_{L^r} < \infty \quad \text{for any } 2 \le r < \infty. \tag{2.17}$$

(2.16) and (2.17) are obtained via an iterative process. First, we use the facts

$$\int_0^t \|\Lambda^\alpha G(\tau)\|_{L^2}^2\, d\tau < \infty, \quad G = \omega + \frac{2\kappa}{\nu+\kappa}\mathcal{R}_\alpha \Omega, \quad \|\Omega\|_{L^2} < \infty,$$

to obtain, for $\alpha \le \frac{3}{2}$,

$$\|\Lambda^{2\alpha-2}\omega\|_{L^2} \le \|\Lambda^{2\alpha-2}G\|_{L^2} + \|\Omega\|_{L^2} \le C\,\|G\|_{L^2} + C\,\|\Lambda^\alpha G\|_{L^2} + \|\Omega\|_{L^2}.$$

Consequently,

$$\int_0^t \|\Lambda^{2\alpha-2}\omega\|_{L^2}^2\, d\tau < \infty.$$

This information allows to show that, for $r_1 = \frac{2}{3-2\alpha}$

$$\|\Omega\|_{L^{r_1}} < \infty. \tag{2.18}$$

Taking L^{r_1} norm yields

$$\frac{d}{dt}\|\Omega\|_{L^{r_1}}^{r_1} + 2\kappa\|\Omega\|_{L^{r_1}}^{r_1} = 2\kappa\int \omega\,\Omega|\Omega|^{r_1-2} \le 2\kappa\|\omega\|_{L^{r_1}}\,\|\Omega\|_{L^{r_1}}^{r_1-1}.$$

Since $\|\omega\|_{L^{r_1}} \le C\,\|\Lambda^{2\alpha-2}\omega\|_{L^2}$, we then obtain (2.18). We can then use (2.18) to further show that

$$\int_0^t \|\Lambda^{2\alpha-2}\omega\|_{L^{r_1}}\, d\tau \le \int_0^t \|\Lambda^{2\alpha-2}G\|_{L^{r_1}}\, d\tau + \frac{2\kappa}{\nu+\kappa}\int_0^t \|\Omega\|_{L^{r_1}} d\tau < \infty.$$

This information can then be further used to show that, if $\alpha < \frac{5}{4}$,

$$\|\Omega\|_{L^{r_2}} < \infty, \quad r_2 = \frac{2}{5-4\alpha}.$$

This process can be repeated until q is sufficiently large as long as

$$\int_0^t \|\Lambda^{2\alpha-2}G\|_{L^q}^2 dt < \infty.$$

This is exactly how q in (2.16) is restricted. This completes the proof for (2.15) and (2.16), and thus the proof of Proposition 2.8. □

2.4 Global H^s bound

This subsection completes the proof of Proposition 2.1 by establishing the global H^s bound.

Proof of Proposition 2.1 We first note the global bound, for any $q \in (1, \infty)$ and for any fixed $t > 0$,

$$\|\Omega(t)\|_{L^q} < \infty, \tag{2.19}$$

according to (2.8). Applying Λ^s to the equations in (1.2) and dotting the resulting equations with $(\Lambda^s u, \Lambda^s \Omega, \Lambda^s b)$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\Lambda^s u, \Lambda^s \Omega, \Lambda^s b)\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \eta\|\Lambda^{s+\beta} b\|_{L^2}^2 + 4\kappa\|\Lambda^s \Omega\|_{L^2}^2 \\ = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7,\end{aligned} \tag{2.20}$$

where

$$\begin{aligned}
&K_1 = -\int \Lambda^s u \cdot [\Lambda^s, u\cdot\nabla]u, \quad K_2 = 2\kappa \int \Lambda^s \nabla\times\Omega \cdot \Lambda^s u,\\
&K_3 = \int \Lambda^s u \cdot \Lambda^s(b\cdot\nabla b), \quad K_4 = -\int \Lambda^s\Omega \cdot [\Lambda^s, u\cdot\nabla]\Omega,\\
&K_5 = 2\kappa \int \Lambda^s\omega\, \Lambda^s\Omega, \quad K_6 = -\int \Lambda^s b\cdot [\Lambda^s, u\cdot\nabla]b,\\
&K_7 = \int \Lambda^s b\cdot \Lambda^s(b\cdot\nabla u).
\end{aligned}$$

The terms on the right-hand side can be estimated as follows. By Lemma 2.2,

$$|K_1| \le C\,\|\nabla u\|_{L^\infty}\,\|\Lambda^s u\|_{L^2}^2.$$

By Hölder's inequality and the relation $\omega = \nabla\times u$,

$$|K_2|,\ |K_5| \le C\,\|\Lambda^s\Omega\|_{L^2}\|\Lambda^{s+1}u\|_{L^2}.$$

Using the divergence-free condition $\nabla\cdot b = 0$ to shift one derivative to u, we have

$$\begin{aligned}
|K_3| &\le C\,\|\Lambda^{s+1}u\|_{L^{\frac{2}{2-\alpha}}}\,\|\Lambda^s b\|_{L^2}\,\|b\|_{L^{\frac{2}{1-\beta}}}\\
&\le C\,\|\Lambda^{s+\alpha}u\|_{L^2}\,\|\Lambda^s b\|_{L^2}\,\|\Lambda^\beta b\|_{L^2}\\
&\le \frac{\nu+\kappa}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2 + C\,\|\Lambda^\beta b\|_{L^2}^2\,\|\Lambda^s b\|_{L^2}^2.
\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}
|K_4| &\le C\,\|\nabla u\|_{L^\infty}\,\|\Lambda^s\Omega\|_{L^2}^2 + C\,\|\Lambda^s\Omega\|_{L^2}\,\|\Lambda^{1+s}u\|_{L^{\frac{2}{2-\alpha}}}\,\|\Omega\|_{L^{\frac{2}{\alpha-1}}}\\
&\le C\,\|\nabla u\|_{L^\infty}\,\|\Lambda^s\Omega\|_{L^2}^2 + C\,\|\Lambda^s\Omega\|_{L^2}\,\|\Lambda^{s+\alpha}u\|_{L^2}\,\|\Omega\|_{L^{\frac{2}{\alpha-1}}}\\
&\le \frac{\nu+\kappa}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2 + C\,\|\nabla u\|_{L^\infty}\,\|\Lambda^s\Omega\|_{L^2}^2 + C\,\|\Omega\|_{L^{\frac{2}{\alpha-1}}}^2\,\|\Lambda^s\Omega\|_{L^2}^2.
\end{aligned}$$

As in the estimate for $|K_3|$, by Lemma 2.2, K_6 and K_7 are bounded by

$$\begin{aligned}
|K_6|,\ |K_7| &\le C\,\|\nabla u\|_{L^\infty}\,\|\Lambda^s b\|_{L^2}^2 + C\,\|\Lambda^s b\|_{L^2}\,\|\Lambda^{s+1}u\|_{L^{\frac{2}{2-\alpha}}}\,\|b\|_{L^{\frac{2}{1-\beta}}}\\
&\le \frac{\nu+\kappa}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2 + C\,\|\nabla u\|_{L^\infty}\,\|\Lambda^s b\|_{L^2}^2 + C\,\|\Lambda^\beta b\|_{L^2}^2\,\|\Lambda^s b\|_{L^2}^2.
\end{aligned}$$

Inserting the bounds above in (2.20), we have

$$\frac{d}{dt}\|(\Lambda^s u, \Lambda^s\Omega, \Lambda^s b)\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^{s+\alpha}u\|_{L^2}^2 + \eta\|\Lambda^{s+\beta}b\|_{L^2}^2 + 4\kappa\|\Lambda^s\Omega\|_{L^2}^2$$
$$\leq C\,\|\nabla u\|_{L^\infty}\,\|(\Lambda^s u, \Lambda^s\Omega, \Lambda^s b)\|_{L^2}^2 + C\,\|\Lambda^s\Omega\|_{L^2}\|\Lambda^{s+1}u\|_{L^2}$$
$$+ C\,\|\Lambda^\beta b\|_{L^2}^2\,\|\Lambda^s b\|_{L^2}^2 + C\,\|\Omega\|_{L^{\frac{2}{\alpha-1}}}^2\,\|\Lambda^s\Omega\|_{L^2}^2.$$

We bound $\|\nabla u\|_{L^\infty}$ by Lemma 2.4,

$$\|\nabla u\|_{L^\infty} \leq C_1(1+\|u\|_{L^2}) + C_2\,\|\Lambda^{\frac{2}{q}}\omega\|_{B^0_{q,\infty}}\,\log(e+\|\Lambda^s u\|_{L^2}).$$

By Proposition 2.8,

$$\int_0^t \|\Lambda^{\frac{2}{q}}\omega(\tau)\|_{B^0_{q,\infty}}\,d\tau < \infty.$$

In addition, due to the global bounds in (2.6) and (2.19), or

$$\int_0^t \|\Lambda^\beta b\|_{L^2}^2\,d\tau < \infty, \quad \int_0^t \|\Omega\|_{L^{\frac{2}{\alpha-1}}}^2\,d\tau < \infty,$$

We obtain the desired global H^s-bound after applying Osgood's inequality, which states that a differential inequality on a nonnegative function f,

$$\frac{df}{dt} \leq a(t)\,f(t)\,\log(e+f)$$

implies that $f = f(t)$ is bounded on $[0, T]$ if the nonnegative function $a = a(t)$ is integrable on $[0, T]$. This completes the proof of Proposition 2.1. □

3 Proof of Theorem 1.2

This section proves Theorem 1.2. As we know, it suffices to establish the global *a priori* estimates on the H^s-norm of the solution (u, Ω, b) to (1.6), which is obtained in the following proposition.

Proposition 3.1 *Consider* (1.6) *with* $\nu > 0$, $\kappa > 0$ *and* $\alpha = 2$. *Assume the initial data* $(u_0, \Omega_0, b_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then, the corresponding solution* (u, Ω, b) *are globally bounded in* H^s.

The proof of (3.1) is achieved via three lemmas, which consecutively establish the L^2, H^1 and H^s global bounds. The first one is the global L^2-bound.

Lemma 3.2 (u, Ω, b) *obeys the following global* L^2 *bound, for any* $t > 0$,

$$\|(u, \Omega, b)(t)\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\Delta u(\tau)\|_{L^2}^2\,d\tau \leq C\,\left(\|(u_0, \Omega_0, b_0)\|_{L^2}, t\right), \tag{3.1}$$

where C *depends on the initial* L^2*-norm and* t.

Proof Dotting (1.1) by (u, Ω, b) and using $\nabla \cdot u = \nabla \cdot b = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u, \Omega, b)\|_{L^2}^2 + (\nu+\kappa)\|\Delta u\|_{L^2}^2 \\
&= 2\kappa\int(\nabla\times\Omega)\cdot u + 2\kappa\int(\nabla\times u)\cdot\Omega = 4\kappa\int(\nabla\times u)\cdot\Omega \\
&\le 4\kappa\|\nabla u\|_{L^2}\|\Omega\|_{L^2} \le 4\kappa\|u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\Omega\|_{L^2} \\
&\le \frac{\nu+\kappa}{2}\|\Delta u\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\Omega\|_{L^2}^2),
\end{aligned}$$

where we have used Ladyzhenskaya's inequality. Gronwall's inequality then implies (3.1). □

To prove the global H^1-bound, we need a lemma generalizing the standard Osgood type lemma. Li and Titi has previously stated an inequality of the Gronwall type with double logarithms [21].

Lemma 3.3 *Assume that Y, Z, A and B are non-negative functions satisfying*

$$\frac{d}{dt}Y(t) + Z(t) \le A(t)\,Y(t) + B(t)\,Y(t)\,\ln(1+Z(t)), \tag{3.2}$$

Let $T > 0$. Assume $A \in L^1(0,T)$ and $B \in L^2(0,T)$. Then, for any $t \in [0,T]$,

$$Y(t) \le (1+Y(0))^{e^{\int_0^t B(\tau)\,d\tau}}\, e^{\int_0^t e^{\int_s^t B(\tau)\,d\tau}(A(s)+B^2(s))\,ds} \tag{3.3}$$

and

$$\int_0^t Z(\tau)\,d\tau \le Y(t)\int_0^t A(\tau)\,d\tau + Y^2(t)\int_0^t B^2(\tau)\,d\tau < \infty. \tag{3.4}$$

Proof of Lemma 3.3 Setting

$$Y_1(t) = \ln(1+Y(t)), \quad Z_1(t) = Z(t)/(1+Y(t)),$$

we have

$$\begin{aligned}
\frac{d}{dt}Y_1(t) + Z_1(t) &\le A(t) + B(t)\,\ln(1+Z(t)) \\
&\le A(t) + B(t)\,\ln(1+(1+Y(t))\,Z_1(t)) \\
&\le A(t) + B(t)\,\ln(1+Y(t))(1+Z_1(t)) \\
&\le A(t) + B(t)\,Y_1(t) + B(t)\,\ln(1+Z_1(t)).
\end{aligned}$$

Using the simple fact that, for $f \ge 0$,

$$\ln(1+f(t)) \le f^{\frac{1}{2}}(t), \tag{3.5}$$

we obtain

$$\frac{d}{dt}Y_1(t) + Z_1(t) \le A(t) + B(t)\,Y_1(t) + B^2(t) + \frac{1}{4}Z_1(t).$$

Gronwall's inequality then implies

$$Y_1(t) \le Y_1(0)\,e^{\int_0^t B(\tau)\,d\tau} + \int_0^t e^{\int_s^t B(\tau)\,d\tau}(A(s)+B^2(s))\,ds,$$

which yields (3.3). In addition, (3.3) allows us to obtain (3.4) by using the inequality (3.5) in (3.2) and integrating in time. This completes the proof of Lemma 3.3. □

We now state and prove the global H^1 bound.

Lemma 3.4 *(u, Ω, b) obeys the following global H^1 bound, for any $t > 0$,*

$$\|(\omega, \nabla\Omega, j)\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\Delta\omega\|_{L^2}^2 d\tau \le C(\|(u_0, \Omega_0, b_0)\|_{H^1}, t), \tag{3.6}$$

where C depends on the H^1-norm of the initial data and t.

Proof To prove (3.6), we make use of the equations of the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$,

$$\begin{cases} \partial_t \omega + u\cdot\nabla\omega + (\nu+\kappa)(-\Delta)^2\omega = b\cdot\nabla j - 2\kappa\Delta\Omega, \\ \partial_t \nabla\Omega + u\cdot\nabla(\nabla\Omega) + 4\kappa(\nabla\Omega) = -(\nabla u)^T\nabla\Omega + 2\kappa\nabla\omega, \\ \partial_t j + u\cdot\nabla j = b\cdot\nabla\omega + Q(\nabla u, \nabla b), \end{cases} \tag{3.7}$$

where $(\nabla u)^T$ denotes the transpose of ∇u and

$$Q(\nabla u, \nabla b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

Dotting (3.7) with $(\omega, \nabla\Omega, j)$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + 4\kappa\|\nabla\Omega\|_{L^2}^2 + (\nu+\kappa)\|\Delta\omega\|_{L^2}^2 \\ &= -2\kappa\int \Delta\Omega\,\omega - \int (\nabla u)^T\,\nabla\Omega\cdot\nabla\Omega + 2\kappa\int \nabla\omega\cdot\nabla\Omega + \int j\,Q(\nabla u, \nabla b) \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

where, due to $\nabla\cdot b = 0$, we have used the fact

$$\int b\cdot\nabla j\,\omega + \int b\cdot\nabla\omega j = 0.$$

I_1, I_2, I_3 and I_4 can be bounded as follows.

$$\begin{aligned} &I_1 = I_3 \le 2\kappa\|\Delta\omega\|_{L^2}\|\Omega\|_{L^2} \le \frac{\nu+\kappa}{2}\|\Delta\omega\|_{L^2}^2 + C\,\|\Omega\|_{L^2}^2, \\ &I_2 \le \|\nabla u\|_{L^\infty}\|\nabla\Omega\|_{L^2}^2, \quad I_4 \le 2\,\|\nabla u\|_{L^\infty}\|j\|_{L^2}^2. \end{aligned}$$

By Lemma 2.4, we have

$$\|\nabla u\|_{L^\infty} \le C_1(1 + \|u\|_{L^2}) + C\,\|\omega\|_{H^1}\,\log(e + \|\Delta\omega\|_{L^2}).$$

Therefore,

$$\begin{aligned} &\frac{d}{dt}\|(\omega, \nabla\Omega, j)\|_{L^2}^2 + (\nu+\kappa)\|\Delta\omega\|_{L^2}^2 \\ &\quad\le C + C\,\|\omega\|_{H^1}\,\log(e + \|\Delta\omega\|_{L^2})(\|\omega\|_{L^2}^2 + \|\nabla\Omega\|_{L^2}^2 + \|j\|_{L^2}^2). \end{aligned}$$

Lemma 3.3 and the fact that $\int_0^t \|\Omega(\tau)\|_{L^2}^2\,d\tau < \infty$ then implies (3.6). □

With the global H^1-bound at our disposal, we now prove the global H^s bound.

Lemma 3.5 (u, Ω, b) *obeys the following global bound in* H^s, *for any* $t > 0$,

$$\|(u, \Omega, b)(t)\|_{H^s}^2 + (\nu+\kappa)\int_0^t \|\Delta u(\tau)\|_{H^2}\,d\tau \le C(\|(u_0, \Omega_0, b_0)\|_{H^s}, t).$$

Proof Applying Λ^s to the equations of ω and j in (3.7) and Λ^{s+1} to the equation of Ω in (1.6) and then dotting the resulting equations with $\Lambda^s\omega$, $\Lambda^s j$ and $\Lambda^{s+1}w$, respectively, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\Lambda^s\omega, \Lambda^{s+1}\Omega, \Lambda^s j)\|_{L^2}^2 + 4\kappa\|\Lambda^{s+1}\Omega\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^{s+2}\omega\|_{L^2}^2\\ &\quad = K_1 + K_2 + K_3 + K_4 + K_5 + K_6,\end{aligned}$$

where

$$\begin{aligned}
K_1 &= -\int \Lambda^s\omega\,[\Lambda^s, u\cdot\nabla]\omega,\\
K_2 &= -\int \Lambda^s j\,[\Lambda^s, u\cdot\nabla] j,\\
K_3 &= \int \Lambda^s\omega\,\Lambda^s(b\cdot\nabla j) + \Lambda^s j\,\Lambda^s(b\cdot\nabla\omega),\\
K_4 &= \int \Lambda^s j\,\Lambda^s Q(\nabla u, \nabla b),\\
K_5 &= -2\kappa\int \Lambda^s\Delta\Omega\,\Lambda^s\omega + 2\kappa\int \Lambda^{s+1}\omega\,\Lambda^{s+1}\Omega,\\
K_6 &= -\int [\Lambda^{s+1}, u\cdot\nabla]\Omega\,\Lambda^{s+1}\Omega.
\end{aligned}$$

These terms can be bounded as follows. By Lemma 2.2,

$$|K_1| \le C\,\|\nabla u\|_{L^\infty}\,\|\Lambda^s\omega\|_{L^2}^2.$$

By Hölder's inequality and the standard commutator estimates,

$$|K_2| \le C\,\|\Lambda^s j\|_{L^2}\left(\|\nabla u\|_{L^\infty}\|\Lambda^s j\|_{L^2} + \|\Lambda^s\nabla u\|_{L^\infty}\|j\|_{L^2}\right).$$

By the Gagliardo–Nirenberg inequality and the Calderon–Zygmund inequality,

$$\begin{aligned}\|\Lambda^s\nabla u\|_{L^\infty} &\le C\,\|\Lambda^s\nabla u\|_{L^2}^{\frac12}\,\|\Lambda^{s+2}\nabla u\|_{L^2}^{\frac12}\\ &\le C\,\|\Lambda^s\omega\|_{L^2}^{\frac12}\,\|\Lambda^{s+2}\omega\|_{L^2}^{\frac12}.\end{aligned}$$

Therefore,

$$\begin{aligned}
|K_2| &\le C\,\|\nabla u\|_{L^\infty}\|\Lambda^s j\|_{L^2}^2 + C\,\|j\|_{L^2}\,\|\Lambda^s j\|_{L^2}\|\Lambda^s\omega\|_{L^2}^{\frac12}\,\|\Lambda^{s+2}\omega\|_{L^2}^{\frac12}\\
&\le C\,\|\nabla u\|_{L^\infty}\|\Lambda^s j\|_{L^2}^2 + \frac{\nu+\kappa}{8}\|\Lambda^{s+2}\omega\|_{L^2}^2 + C\,\|j\|_{L^2}^{\frac43}\,\|\Lambda^s j\|_{L^2}^{\frac43}\,\|\Lambda^s\omega\|_{L^2}^{\frac23}\\
&\le C\,\|\nabla u\|_{L^\infty}\|\Lambda^s j\|_{L^2}^2 + \frac{\nu+\kappa}{8}\|\Lambda^{s+2}\omega\|_{L^2}^2 + C\,\|j\|_{L^2}^{\frac43}\,\left(\|\Lambda^s j\|_{L^2}^2 + \|\Lambda^s\omega\|_{L^2}^2\right).
\end{aligned}$$

Thanks to $\nabla\cdot b = 0$, we can write K_3 as

$$K_3 = \int \Lambda^s\omega\,[\Lambda^s, b\cdot\nabla] j + \Lambda^s j\,[\Lambda^s, b\cdot\nabla]\omega.$$

Without loss of generality, we can assume $s \le 2$. Since higher regularity bounds for $s > 2$ follow easily once the bound for $s \le 2$ is obtained. By the commutator estimate and the Sobolev embedding,

$$\begin{aligned}\left|\int \Lambda^s \omega\, [\Lambda^s, b\cdot\nabla] j\right| &\le \|\Lambda^s\omega\|_{L^2}\, \|[\Lambda^s, b\cdot\nabla]j\|_{L^2}\\ &\le C\,(\|\omega\|_{L^2} + \|\Delta\omega\|_{L^2})\, \|\nabla b\|_{L^\infty}\, \|\Lambda^s j\|_{L^2}\\ &\le C\,(\|\omega\|_{L^2} + \|\Delta\omega\|_{L^2})\, \|\Lambda^s j\|_{L^2}^2.\end{aligned}$$

Similarly, for $s \le 2$,

$$\begin{aligned}\left|\int \Lambda^s j\, [\Lambda^s, b\cdot\nabla]\omega\right| &\le \|\Lambda^s j\|_{L^2}\, \|[\Lambda^s, b\cdot\nabla]\omega\|_{L^2}\\ &\le C\,\|\Lambda^s j\|_{L^2}\, \left(\|\Lambda^s j\|_{L^2}\|\omega\|_{L^\infty} + \|\nabla b\|_{L^\infty}\|\Lambda^s\omega\|_{L^2}\right)\\ &\le C\,\|\nabla u\|_{L^\infty}\, \|\Lambda^s j\|_{L^2}^2 + C\,\left(\|\omega\|_{L^2} + \|\Delta\omega\|_{L^2}\right)\, \|\Lambda^s j\|_{L^2}^2.\end{aligned}$$

To bound K_4, it suffices to consider the typical term

$$\begin{aligned}\left|\int \Lambda^s j\, \Lambda^s(\partial_1 b_1\, \partial_2 u_1)\right| &\le C\,\|\Lambda^s j\|_{L^2}\, \left(\|\Lambda^s \partial_1 b_1\|_{L^2}\, \|\nabla u\|_{L^\infty} + \|\Lambda^s\partial_2 u_1\|_{L^2}\, \|\nabla b\|_{L^\infty}\right)\\ &\le C\,\|\nabla u\|_{L^\infty}\, \|\Lambda^s j\|_{L^2}^2 + C\,\|\nabla b\|_{L^\infty}\, \|\Lambda^s\omega\|_{L^2}\, \|\Lambda^s j\|_{L^2}\\ &\le C\,\|\nabla u\|_{L^\infty}\, \|\Lambda^s j\|_{L^2}^2 + C\,(\|\omega\|_{L^2} + \|\Delta\omega\|_{L^2})\, \|\Lambda^s j\|_{L^2}^2.\end{aligned}$$

Now we turn to the estimates of K_5 and K_6.

$$\begin{aligned}|K_5| &\le \|\Lambda^{s+1}\omega\|_{L^2}\, \|\Lambda^{s+1}\Omega\|_{L^2}\\ &\le C\|\omega\|_{L^2}^{\frac{1}{s+2}}\, \|\Lambda^{s+2}\omega\|_{L^2}^{\frac{s+1}{s+2}}\, \|\Lambda^{s+1}\Omega\|_{L^2}\\ &\le \frac{\nu+\kappa}{8}\|\Lambda^{s+2}\omega\|_{L^2}^2 + C\,\|\omega\|_{L^2}^{\frac{2}{s+3}}\, \|\Lambda^{s+1}\Omega\|_{L^2}^{\frac{2(s+2)}{s+3}}\end{aligned}$$

and

$$\begin{aligned}|K_6| &\le \|[\Lambda^{s+1}, u\cdot\nabla]\Omega\|_{L^2}\, \|\Lambda^{s+1}\Omega\|_{L^2}\\ &\le C\left(\|\Lambda^{s+2}u\|_{L^4}\|\Omega\|_{L^4} + \|\nabla u\|_{L^\infty}\|\Lambda^{s+1}\Omega\|_{L^2}\right)\, \|\Lambda^{s+1}\Omega\|_{L^2}\\ &\le C\left(\|\Lambda^s\omega\|_{L^2}^{\frac14}\|\Lambda^{s+2}\omega\|_{L^2}^{\frac34}\|\Omega\|_{L^2}^{\frac12}\|\nabla\Omega\|_{L^2}^{\frac12} + \|\nabla u\|_{L^\infty}\|\Lambda^{s+1}\Omega\|_{L^2}\right)\, \|\Lambda^{s+1}\Omega\|_{L^2}\\ &\le \frac{\nu+\kappa}{8}\|\Lambda^{s+2}\omega\|_{L^2}^2 + C\,\|\Lambda^s\omega\|_{L^2}^{\frac25}\|\Omega\|_{L^2}^{\frac45}\|\nabla\Omega\|_{L^2}^{\frac45}\|\Lambda^{s+1}\Omega\|_{L^2} + C\|\nabla u\|_{L^\infty}\|\Lambda^{s+1}\Omega\|_{L^2}^2\\ &\le \frac{\nu+\kappa}{8}\|\Lambda^{s+2}\omega\|_{L^2}^2 + C\,\left(\|\Omega\|_{L^2}^{\frac45}\|\nabla\Omega\|_{L^2}^{\frac45} + \|\nabla u\|_{L^\infty}\right)\left(\|\Lambda^{s+1}\omega\|_{L^2}^2 + \|\Lambda^{s+1}\Omega\|_{L^2}^2\right),\end{aligned}$$

where we have used the classical commutator estimates, Young's inequality and the following Gagliardo–Nirenberg inequality

$$\begin{aligned}\|\Lambda^{s+1}\omega\|_{L^2} &\le C\|\omega\|_{L^2}^{\frac{1}{s+2}}\, \|\Lambda^{s+2}\omega\|_{L^2}^{\frac{s+1}{s+2}}\\ \|\Lambda^{s+2}u\|_{L^4} &\le \|\Lambda^{s+1}\omega\|_{L^4} \le C\|\Lambda^s\omega\|_{L^2}^{\frac14}\|\Lambda^{s+2}\omega\|_{L^2}^{\frac34}\end{aligned}$$

and

$$\|\Omega\|_{L^4} \le C\|\Omega\|_{L^2}^{\frac12}\|\nabla\Omega\|_{L^2}^{\frac12}.$$

Therefore, if we invoke the embedding inequality

$$\|\nabla u\|_{L^\infty} \le \|\Delta\omega\|_{L^2},$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\| \left(\Lambda^s\omega, \Lambda^{s+1}\Omega, \Lambda^s j\right) \|_{L^2}^2 + \|\Lambda^{s+2}\omega\|_{L^2}^2 \\
&\quad \le C\left(\|j\|_{L^2}^{\frac{4}{3}} + \|\Delta\omega\|_{L^2} + \|\Omega\|_{L^2}^{\frac{4}{5}}\|\nabla\Omega\|_{L^2}^{\frac{4}{5}}\right) \| \left(\Lambda^s\omega, \Lambda^{s+1}\Omega, \Lambda^s j\right) \|_{L^2}^2 \\
&\qquad + C\left(\|\omega\|_{L^2} + \|\Delta\omega\|_{L^2}\right) \|\Lambda^s j\|_{L^2}^2 + C\,\|\omega\|_{L^2}^{\frac{2}{s+3}}\,\|\Lambda^{s+1}\Omega\|_{L^2}^{\frac{2(s+2)}{s+3}}.
\end{aligned}$$

The desired global H^s bound then follows from the inequality above via Gronwall's inequality combined with the global H^1 bound in Lemma 3.4. □

4 Proof of Theorem 1.3

This section proves Theorem 1.3. Again we prove the global a priori bounds in H^s. More precisely, we prove the following proposition.

Proposition 4.1 *Consider (1.7) with $\alpha > 0$, $\nu > 0$, $\kappa > 0$, $\mu > 0$ and $\eta > 0$. Assume $(u_0, \Omega_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, the corresponding solution (u, Ω, b) are globally bounded in H^s.*

In order to prove Proposition 4.1, we make use of the maximal regularity and related properties of the heat operator,

$$e^{\Delta t} f = K(\cdot, t) * f, \quad K(x,t) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^d,\ t > 0.$$

We recall the following simple facts.

Lemma 4.2 *Let $1 \le p \le q \le \infty$. Let β be a multi-index. For any $t > 0$, the heat operator $e^{\Delta t}$ and $\partial_x^\beta e^{\Delta t}$ are bounded from L^p to L^q. Furthermore, for any $f \in L^p(\mathbb{R}^d)$,*

$$\|\partial_x^\beta e^{\Delta t} f\|_{L^q(\mathbb{R}^d)} \le C_1\, t^{-\frac{|\beta|}{2} - \frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\, \|f\|_{L^p(\mathbb{R}^d)},$$

where $C_1 = C_1(\beta, p, q)$ are constants.

The following maximal regularity property for the heat kernel can be found in many references (see, e.g., [20, p. 64]).

Lemma 4.3 *The operator A defined by*

$$Af(t) \equiv \int_0^t \Delta\, e^{\Delta(t-\tau)}\, f(\tau)\, d\tau$$

maps $L^p(0, T; L^q(\mathbb{R}^d))$ to $L^p(0, T; L^q(\mathbb{R}^d))$ for any $T \in (0, \infty]$ and $p, q \in (1, \infty)$.

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1 We start with the global L^2 bound. Dotting (1.7) by (u, Ω, b) yields

$$\frac{1}{2}\frac{d}{dt}\|(u,\Omega,b)\|_{L^2}^2 + 4\kappa\|\Omega\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^\alpha u\|_{L^2}^2 + \mu\|\Lambda\Omega\|_{L^2}^2 + \eta\|\Lambda b\|_{L^2}^2$$
$$= 2\kappa\int(\nabla\times\Omega)\cdot u + 2\kappa\int(\nabla\times u)\cdot\Omega = 4\kappa\int(\nabla\times\Omega)\cdot u$$
$$\leq 4\kappa\|u\|_{L^2}\|\Lambda\Omega\|_{L^2} \leq \frac{\mu}{2}\|\Lambda\Omega\|_{L^2}^2 + C\|u\|_{L^2}^2.$$

Gronwall's inequality then implies

$$\|(u,\Omega,b)\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\Lambda^\alpha u\|_{L^2}^2\,d\tau + \mu\int_0^t\|\Lambda\Omega\|_{L^2}^2\,d\tau$$
$$+\eta\int_0^t\|\Lambda b\|_{L^2}^2\,d\tau \leq C\left(\|(u_0,\Omega_0,b_0)\|_{L^2}, t\right).$$

To estimate the H^1-norm, we start with the equations of ω, $\nabla\Omega$ and j,

$$\begin{cases}\partial_t\omega + u\cdot\nabla\omega + (\nu+\kappa)(-\Delta)^\alpha\omega = b\cdot\nabla j - 2\kappa\Delta\Omega,\\ \partial_t\nabla\Omega + u\cdot\nabla(\nabla\Omega) + 4\kappa(\nabla\Omega) - \mu\Delta\nabla\Omega = -(\nabla u)^T\nabla\Omega + 2\kappa\nabla\omega,\\ \partial_t j + u\cdot\nabla j - \eta\Delta j = b\cdot\nabla\omega + Q(\nabla u,\nabla b).\end{cases} \tag{4.1}$$

Dotting (4.1) with $(\omega, \nabla\Omega, j)$ and using $\nabla\cdot u = 0$ and $\nabla\cdot b = 0$, we have

$$\frac{1}{2}\frac{d}{dt}(\|(\omega,\nabla\Omega,j)\|_{L^2}^2 + 4\kappa\|\nabla\Omega\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^\alpha\omega\|_{L^2}^2 + \mu\|\Delta\Omega\|_{L^2}^2 + \eta\|\nabla j\|_{L^2}^2$$
$$= -2\kappa\int\Delta\Omega\omega + \int(\nabla u)^T\nabla\Omega\cdot\nabla\Omega + \int\nabla\omega\cdot\nabla\Omega + \int j\,Q(\nabla u,\nabla b)$$
$$:= J_1 + J_2 + J_3 + J_4. \tag{4.2}$$

The terms on the right of (4.2) can be bounded by Hölder's inequality, Young's inequality and the Gagliardo–Nirenberg inequality as follows,

$$J_1, J_3 \leq 2\kappa\left|\int\omega\Delta\Omega\right| \leq \frac{\mu}{4}\|\Delta\Omega\|_{L^2}^2 + C\,\|\omega\|_{L^2}^2,$$
$$J_2 \leq \|\nabla u\|_{L^2}\|\nabla\Omega\|_{L^4}^2 \leq C\,\|\nabla u\|_{L^2}\|\Omega\|_{L^2}\|\Delta\Omega\|_{L^2}$$
$$\leq \frac{\mu}{4}\|\Delta\Omega\|_{L^2}^2 + C\|\Omega\|_{L^2}^2\|\nabla u\|_{L^2}^2,$$
$$J_4 \leq \|\nabla u\|_{L^2}\|j\|_{L^4}^2 \leq C\,\|\nabla u\|_{L^2}\|j\|_{L^2}\|\nabla j\|_{L^2}$$
$$\leq \frac{\eta}{2}\|\nabla j\|_{L^2}^2 + C\|j\|_{L^2}^2\|\nabla u\|_{L^2}^2.$$

Combining these estimates, we obtain

$$\frac{d}{dt}\|(\omega,\nabla\Omega,j)\|_{L^2}^2 + 2(\nu+\kappa)\|\Lambda^\alpha\omega\|_{L^2}^2 + \frac{\mu}{2}\|\Delta\Omega\|_{L^2}^2 + \eta\|\nabla j\|_{L^2}^2$$
$$\leq C\,(1 + \|\Omega\|_{L^2}^2 + \|j\|_{L^2}^2)\|\omega\|_{L^2}^2.$$

Gronwall's inequality then implies the global H^1 bound,

$$\|(\omega, \nabla\Omega, j)\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\Lambda^\alpha \omega\|_{L^2}^2\,d\tau + \mu\int_0^t \|\Delta\Omega\|_{L^2}^2\,d\tau + \eta\int_0^t \|\nabla j\|_{L^2}^2\,d\tau \le C(\|(u_0, \Omega_0, b_0)\|_{H^1}, t).$$

We now prove the global H^2 bounds. Writing the equation of b in the integral form and applying Lemmas 4.2 and 4.3, we have, for any $p \in (1,2)$ and $\frac{1}{p}+\frac{1}{q}=1$, and for any $T>0$,

$$\begin{aligned}\|b\|_{L^\infty(0,T;L^\infty)} &\le \|b_0\|_{L^\infty} + \left\|\int_0^t e^{\Delta(t-\tau)}(b\cdot\nabla u - u\cdot\nabla b)(\tau)\,d\tau\right\|_{L^\infty(0,T;L^\infty)}\\ &\le \|b_0\|_{L^\infty} + C\,\|\nabla K\|_{L^1(0,T;L^p)}\,\|u\|_{L^\infty(0,T;L^{2q})}\,\|b\|_{L^\infty(0,T;L^{2q})}\\ &\le \|b_0\|_{L^\infty} + C(T)\,\|(u,b)\|_{L^\infty(0,T;H^1)} < \infty.\end{aligned}$$

By Lemma 4.3, for any $p, q \in (1,\infty)$ and for any $T>0$,

$$\|\nabla b\|_{L^q(0,T;L^p(\mathbb{R}^2))} \le \|K\|_{L^q(0,T;L^1)}\,\|\nabla b_0\|_{L^p} + C(T)\,\|(u,b)\|_{L^\infty(0,T;H^1)} < \infty.$$

Similar estimates involving the equation of Ω lead to

$$\|\Omega\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))},\quad \|\Omega\|_{L^q(0,T;L^p(\mathbb{R}^2))},\quad \|\nabla\Omega\|_{L^q(0,T;L^p(\mathbb{R}^2))} < \infty.$$

Furthermore, due to Lemmas 4.2 and 4.3,

$$\begin{aligned}\|\Delta b\|_{L_t^q L^p} &\le C\left(\|\Delta b_0\|_{L^p} + \|b\cdot\nabla u - u\cdot\nabla b\|_{L_t^q L^p}\right)\\ &\le C\left(\|\Delta b_0\|_{L^p} + \|b\|_{L_t^\infty L^\infty}\|\omega\|_{L_t^q L^p} + \left(\int_0^t \|u\|_{L_x^\infty}^q \|\nabla b\|_{L_x^p}^q\right)^{\frac{1}{q}}\right)\\ &\le C\left(\|\Delta b_0\|_{L^p} + \|b\|_{L_t^\infty L^\infty}\|\omega\|_{L_t^q L^p} + \left(\int_0^t \|u\|_{L_x^2}^{\frac{(p-2)q}{2(p-1)}} \|\omega\|_{L_x^p}^{\frac{pq}{2(p-1)}} \|\nabla b\|_{L_x^p}^q\right)^{\frac{1}{q}}\right)\\ &\le C\left(\|\Delta b_0\|_{L^p} + \|\omega\|_{L_t^q L^p} + \left(\int_0^t (1+\|\omega\|_{L_x^p}^q)\|\nabla b\|_{L_x^p}^q\right)^{\frac{1}{q}}\right),\end{aligned} \tag{4.3}$$

where we have used Hölder's inequality and Sobolev's inequality. Similarly,

$$\|\Delta\Omega\|_{L_t^q L^p} \le C\left(1 + \|\Delta\Omega_0\|_{L^p} + \|\omega\|_{L_t^q L^p} + \left(\int_0^t (1+\|\omega\|_{L_x^p}^q)\|\nabla\Omega\|_{L_x^p}^q\right)^{\frac{1}{q}}\right). \tag{4.4}$$

Multiplying the vorticity equation by $\omega|\omega|^{p-2}$ with $p \in [2,\infty)$, and integrating in space, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p &\le \left|\int b\cdot\nabla j\cdot\omega|\omega|^{p-2}\right| + \left|\int \Delta\Omega\cdot\omega|\omega|^{p-2}\right|\\ &\le \|b\|_{L^\infty}\|\nabla j\|_{L^p}\|\omega\|_{L^p}^{p-1} + \|\Delta\Omega\|_{L^p}\|\omega\|_{L^p}^{p-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\omega\|_{L^p} &\le \|\omega_0\|_{L^p} + \int_0^t \|b(\tau)\|_{L^\infty}\|\nabla j(\tau)\|_{L^p}d\tau + \int_0^t \|\Delta\Omega(\tau)\|_{L^p}d\tau \\ &\le \|\omega_0\|_{L^p} + C\left(\|b\|_{L_t^{\frac{q-1}{q}}L^\infty}\|\nabla j\|_{L_t^qL^p} + \|\Delta\Omega\|_{L_t^qL^p}\right) \\ &\le \|\omega_0\|_{L^p} + C\left(\|\nabla j\|_{L_t^qL^p} + \|\Delta\Omega\|_{L_t^qL^p}\right).\end{aligned}$$

Invoking (4.3) and (4.4), and applying Gronwall's inequality, we obtain

$$\|\omega\|_{L_t^\infty L^p} < \infty, \quad \|\nabla j\|_{L_t^qL^p} < \infty, \quad \|\Delta\Omega\|_{L_t^qL^p} < \infty.$$

With these global bounds at our disposal, we now prove the global H^2- bound. Multiplying the equations of ω, j and Ω by $-\Delta\omega$, $-\Delta j$ and $(-\Delta)^2\Omega$, respectively, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|(\nabla\omega, \Delta\Omega, \nabla j)\|_{L^2}^2 + 4\kappa\|\Delta\Omega\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \mu\|\Delta\nabla\Omega\|_{L^2}^2 + \nu\|\Delta j\|_{L^2}^2 \\ &= -\int \nabla(u\cdot\nabla\omega)\cdot\nabla\omega + \int \nabla(b\cdot\nabla j)\cdot\nabla\omega + 2\kappa\int \Delta\Omega\cdot\Delta\omega + 2\kappa\int \omega\cdot(-\Delta)^2\Omega \\ &\quad - \int u\cdot\nabla\Omega(-\Delta)^2\Omega - \int \nabla(u\cdot\nabla j)\cdot\nabla j + \int \nabla(b\cdot\nabla\omega)\cdot\nabla j - \int Q(\nabla u, \nabla b)\Delta j \\ &:= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8. \qquad (4.5)\end{aligned}$$

We estimate the terms on the right as follows.

$$\begin{aligned}|K_1| &\le \|\nabla u\|_{L^p}\|\nabla\omega\|^2_{L^{\frac{2p}{p-1}}} \\ &\le \|\omega\|_{L^p}^{1+\frac{2(p\alpha-1)}{2+p\alpha}}\|\Lambda^\alpha\nabla\omega\|_{L^2}^{\frac{6}{2+p\alpha}} \\ &\le \frac{\nu+\kappa}{2}\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + C\|\omega\|_{L^p}^{2+\frac{2+p\alpha}{p\alpha-1}},\end{aligned}$$

where we have used the Gagliardo–Nirenberg inequality, for $p\alpha > 1$,

$$\|\nabla\omega\|_{L^{\frac{2p}{p-1}}} \le C\|\omega\|_{L^p}^{\frac{p\alpha-1}{2+p\alpha}}\|\Lambda^\alpha\nabla\omega\|_{L^2}^{\frac{3}{2+p\alpha}}.$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}|K_2+K_7| &\le \|\nabla b\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\nabla j\|_{L^2} \\ &\le \|\nabla b\|_{L^2}^{\frac{2p-2}{3p-2}}\|\Delta b\|_{L^p}^{\frac{p}{3p-2}}\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right). \\ |K_3+K_4| &\le 4\kappa\left|\int \Delta\nabla\Omega\cdot\nabla\omega\right| \le \frac{\mu}{4}\|\Delta\nabla\Omega\|_{L^2}^2 + C\|\nabla\omega\|_{L^2}^2. \\ |K_5| &= \left|\int u\cdot\nabla\Omega(-\Delta)^2\Omega\right| \\ &= \left|\int \nabla u\cdot\nabla\Omega\cdot\Delta\nabla\Omega + \int u\cdot\nabla^2\Omega\cdot\Delta\nabla\Omega\right| \\ &\le \|\nabla u\|_{L^4}\|\nabla\Omega\|_{L^4}\|\Delta\nabla\Omega\|_{L^2} + \|u\|_{L^\infty}\|\nabla^2\Omega\|_{L^2}\|\Delta\nabla\Omega\|_{L^2} \\ &\le C\|\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\Omega\|_{L^2}^{\frac{1}{2}}\|\Delta\Omega\|_{L^2}^{\frac{1}{2}}\|\Delta\nabla\Omega\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+ C\|u\|_{L^2}^{\frac{p-2}{2(p-1)}} \|\omega\|_{L^2}^{\frac{p}{2(p-1)}} \|\nabla^2\Omega\|_{L^2} \|\Delta\nabla\Omega\|_{L^2} \\
&\le \frac{\mu}{4}\|\Delta\nabla\Omega\|_{L^2}^2 + C\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}\|\nabla\Omega\|_{L^2}\|\Delta\Omega\|_{L^2} \\
&+ C\|u\|_{L^2}^{\frac{p-2}{p-1}} \|\omega\|_{L^2}^{\frac{p}{p-1}} \|\nabla^2\Omega\|_{L^2}^2 . \\
|K_6| &\le \|\nabla u\|_{L^2}\|\nabla j\|_{L^4}^2 \\
&\le C\|\omega\|_{L^2}\|\nabla j\|_{L^2}\|\Delta j\|_{L^2} \\
&\le \frac{\nu}{4}\|\Delta j\|_{L^2}^2 + C\|\omega\|_{L^2}^2\|\nabla j\|_{L^2}^2 .
\end{aligned}
$$

By the inequalities of Hölder, Sobolev and Ladyzhenskaya,

$$
\begin{aligned}
|K_8| &= \left| \int Q(\nabla\nabla u, \nabla b)\cdot\nabla j + \int Q(\nabla u, \nabla\nabla b)\cdot \nabla j \right| \\
&\le C\left(\|\nabla b\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\nabla j\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla j\|_{L^4}^2\right) \\
&\le C(\|\nabla b\|_{L^2}^{\frac{2p-2}{3p-2}} \|\Delta b\|_{L^p}^{\frac{p}{3p-2}} (\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \|\omega\|_{L^2}\|\nabla j\|_{L^2}\|\Delta j\|_{L^2}) \\
&\le \frac{1}{8}\|\Delta j\|_{L^2}^2 + C\left(\|\omega\|_{L^2} + \|\nabla b\|_{L^2}^{\frac{2p-2}{3p-2}} \|\Delta b\|_{L^p}^{\frac{p}{3p-2}} \right)\left(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right).
\end{aligned}
$$

Inserting these estimates into (4.5), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla\omega, \Delta\Omega, \nabla j)\|_{L^2}^2 + (\nu+\kappa)\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \mu\|\Delta\nabla\Omega\|_{L^2}^2 + \eta\|\Delta j\|_{L^2}^2 \\
&\le C\|\omega\|_{L^p}^{2+\frac{2+p\alpha}{p\alpha-1}} + C\Big(1 + \|\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\|\nabla\omega\|_{L^2}^2\|\nabla\Omega\|_{L^2}^2 + \|u\|_{L^2}^{\frac{p-2}{p-1}}\|\omega\|_{L^2}^{\frac{p}{p-1}} \\
&\quad + \|\nabla b\|_{L^2}^{\frac{2p-2}{3p-2}}\|\Delta b\|_{L^p}^{\frac{p}{3p-2}}\Big)\left(\|\nabla\omega\|_{L^2}^2 + \|\Delta\Omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right).
\end{aligned}
$$

Then Gronwall's inequality implies

$$
\begin{aligned}
&\|(\nabla\omega, \Delta\Omega, \nabla j)\|_{L^2}^2 + (\nu+\kappa)\int_0^t \|\Lambda^\alpha\nabla\omega(\tau)\|_{L^2}^2\, d\tau \\
&+ \mu\int_0^t \|\Delta\nabla\Omega(\tau)\|_{L^2}^2\, d\tau + \eta\int_0^t \|\Delta j\|_{L^2}^2\, d\tau \le C.
\end{aligned}
$$

This establishes the global H^2-bound. The global H^s-bound then follows via standard estimates. This completes the proof of Proposition 4.1. □

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Appendix A: An alternative approach to the global H^1-bound for (1.2)

A crucial component in the proof of the global bounds for Theorem 1.1 is the global (in time) H^1-bound and the global integrability of $\|\Lambda^{\frac{2}{q}}\omega(\tau)\|_{L^q}$. Sect. 2 provided one way to get these bounds. The purpose of this appendix is to provide an alternative approach to these bounds. More precisely, we prove the following result.

Proposition A.1 *Assume that* (u, Ω, b) *solves* (1.2). *Assume* α *and* β *satisfy*

$$1 < \alpha < 2, \quad 0 < \beta < 1, \quad \alpha + \beta \geq 2.$$

Then, for any $1 < q < \infty$ *and for any* $t > 0$,

$$\sup_{l \geq -1} 2^{(2\alpha-2)l} \int_0^t \|\Delta_l \omega(\tau)\|_{L^q} \, d\tau \leq C(t, u_0, \Omega_0, b_0) < \infty. \tag{A.1}$$

As a special consequence, for any $1 < q < \infty$ *and* $\rho < 2\alpha - 2$,

$$\int_0^t \|\omega(\tau)\|_{B^{\rho}_{q,1}} \, d\tau < \infty. \tag{A.2}$$

Especially, for $\rho < 2\alpha - 2$ *and* $q = \frac{2}{\rho}$,

$$\int_0^t \left(\|\omega(\tau)\|_{L^q} + \|\Lambda^{\frac{2}{q}} \omega\|_{L^q} \right) d\tau < \infty. \tag{A.3}$$

Proof of Proposition A.1 We start with the following fact about Ω, for any $r \in [1, \infty]$,

$$\|\Omega(t)\|_{L^r} \leq \|\Omega_0\|_{L^r} + 2\kappa \int_0^t \|\omega(\tau)\|_{L^r} \, d\tau, \tag{A.4}$$

which can be obtained by performing the standard Lebesgue norm estimate on the equation of Ω. Next, we write the equation of ω given by (1.5) in the integral form

$$\omega(t) = e^{-(\nu+\kappa)(-\Delta)^\alpha t} \omega_0 + \int_0^t e^{-(\nu+\kappa)(-\Delta)^\alpha (t-\tau)} (-\Delta\Omega(\tau)) \, d\tau$$
$$+ \int_0^t e^{-(\nu+\kappa)(-\Delta)^\alpha (t-\tau)} \nabla \times \nabla \cdot ((u \otimes u)(\tau) + \nabla \times \nabla \cdot (b \otimes b)(\tau)) \, d\tau.$$

We further localize it by applying Δ_l with $l \in \mathbb{Z}$ and $l \geq -1$,

$$\Delta_l \omega(t) = \Delta_l e^{-(\nu+\kappa)(-\Delta)^\alpha t} \omega_0 + \int_0^t \Delta_l e^{-(\nu+\kappa)(-\Delta)^\alpha (t-\tau)} (-\Delta\Omega(\tau)) \, d\tau$$
$$+ \int_0^t \Delta_l e^{-(\nu+\kappa)(-\Delta)^\alpha (t-\tau)} \nabla \times \nabla \cdot ((u \otimes u)(\tau) + (b \otimes b)(\tau)) \, d\tau.$$

For $q \in (1, \infty)$, taking the L^q-norm and applying Lemma 2.3 and Bernstein's inequality, we have

$$\|\Delta_l \omega(t)\|_{L^q} \leq C \, e^{-c_0(\nu+\kappa)t \, 2^{2\alpha l}} \|\Delta_l \omega_0\|_{L^q} + C \int_0^t 2^{2l} e^{-c_0(\nu+\kappa)(t-\tau) \, 2^{2\alpha l}} \|\Delta_l \Omega\|_{L^q} \, d\tau$$
$$+ C \int_0^t 2^{2l} e^{-c_0(\nu+\kappa)(t-\tau) \, 2^{2\alpha l}} (\|\Delta_l (u \otimes u)(\tau)\|_{L^q} + \|\Delta_l (b \otimes b)(\tau)\|_{L^q}) \, d\tau. \tag{A.5}$$

To further improve the estimates, we invoke (A.4) to obtain

$$\|\Delta_l \Omega(t)\|_{L^q} \leq \|\Omega(t)\|_{L^q} \leq \|\Omega_0\|_{L^q} + \int_0^t \|\omega(\tau)\|_{L^q} \, d\tau.$$

To bound $\|\Delta_l(b\otimes b)(\tau)\|_{L^q}$, noticing $\alpha+\beta=2$, we choose σ satisfying

$$0<\sigma<\alpha-1, \quad \sigma+\beta=1-\frac{1}{q}.$$

By Sobolev's inequality,

$$\|\Delta_l(b\otimes b)(\tau)\|_{L^q}\le \|b\otimes b(\tau)\|_{L^q}\le \|b\|^2_{L^{2q}}\le C\|\Lambda^{\sigma+\beta}b\|^2_{L^2}.$$

Similarly, we can also estimate $\|\Delta_l(u\otimes u)(\tau)\|_{L^q}$,

$$\|\Delta_l(u\otimes u)(\tau)\|_{L^q}\le C\left(\|u\|^2_{L^2}+\|\Lambda^\alpha u\|^2_{L^2}\right)=C\,\|u\|^2_{H^\alpha}.$$

Integrating (A.5) in time and applying Young's inequality for the time convolution, we obtain

$$\begin{aligned}\int_0^t \|\Delta_l\omega(\tau)\|_{L^q}\,d\tau \le{}& C\,2^{-2\alpha l}\,\|\Delta_l\omega_0\|_{L^q}\\ &+C\,2^{(2-2\alpha)l}\int_0^t\left(\|\Omega_0\|_{L^q}+\int_0^\tau\|\omega\|_{L^q}ds\right)d\tau\\ &+C\,2^{(2-2\alpha)l}\int_0^t\left(\|u\|^2_{H^\alpha}+\|\Lambda^{\sigma+\beta}b\|^2_{L^2}\right)d\tau,\end{aligned}$$

or

$$2^{(2\alpha-2)l}\int_0^t\|\Delta_l\omega(\tau)\|_{L^q}\,d\tau\le A(t)+C\int_0^t\int_0^\tau\|\omega\|_{L^q}ds\,d\tau, \tag{A.6}$$

where, due to Lemma 2.7,

$$A(t)=C\,2^{-2l}\,\|\Delta_l\omega_0\|_{L^q}+C\,\|\Omega_0\|_{L^q}t+C\int_0^t\left(\|u\|^2_{H^\alpha}+\|\Lambda^{\sigma+\beta}b\|^2_{L^2}\right)d\tau<\infty.$$

Noting that

$$\|\omega\|_{L^q}\le\sum_{m\ge-1}\|\Delta_m\omega\|_{L^q}\le\sum_{m\ge-1}2^{(2-2\alpha)m}\,2^{(2\alpha-2)m}\,\|\Delta_m\omega\|_{L^q},$$

we obtain, due to $\alpha>1$,

$$\begin{aligned}\int_0^t\int_0^\tau\|\omega\|_{L^q}ds\,d\tau &=\int_0^t\sum_{m\ge-1}2^{(2-2\alpha)m}\,2^{(2\alpha-2)m}\int_0^\tau\|\Delta_m\omega\|_{L^q}\,ds\,d\tau\\ &\le C\int_0^t\sup_{m\ge-1}2^{(2\alpha-2)m}\int_0^\tau\|\Delta_m\omega\|_{L^q}\,ds\,d\tau.\end{aligned}$$

Inserting this bound in (A.6) yields, for $Y(t)\equiv\sup_{l\ge-1}2^{(2\alpha-2)l}\int_0^t\|\Delta_l\omega(\tau)\|_{L^q}\,d\tau$,

$$Y(t)\le A(t)+C\int_0^t Y(\tau)\,d\tau.$$

Gronwall's inequality then leads to the global bound in (A.1). It is easy to see that (A.2) is a special consequence of (A.1). In fact, for any $\rho<2\alpha-2$,

$$\begin{aligned}\int_0^t\|\omega(\tau)\|_{B^\rho_{q,1}}\,d\tau &=\int_0^t\sum_{l\ge-1}2^{\rho l}\|\Delta_l\omega\|_{L^q}\,d\tau\\ &\le\int_0^t\sum_{l\ge-1}2^{(\rho-(2\alpha-2))l}\,2^{(2\alpha-2)l}\|\Delta_l\omega\|_{L^q}\,d\tau\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l \geq -1} 2^{(\rho-(2\alpha-2))l}\, 2^{(2\alpha-2)l} \int_0^t \|\Delta_l \omega\|_{L^q}\, d\tau \\
&\leq \sup_{l \geq -1} 2^{(2\alpha-2)l} \int_0^t \|\Delta_l \omega\|_{L^q}\, d\tau \sum_{l \geq -1} 2^{(\rho-(2\alpha-2))l} \\
&= C \sup_{l \geq -1} 2^{(2\alpha-2)l} \int_0^t \|\Delta_l \omega\|_{L^q}\, d\tau < \infty,
\end{aligned}
$$

which is (A.2). Finally (A.3) follows from (A.2) by taking $q = \frac{2}{\rho}$. This completes the proof of Proposition A.1. □

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