LOW REGULARITY SOLUTIONS OF TWO FIFTH-ORDER KDV TYPE EQUATIONS

By

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Abstract. The Kawahara and modified Kawahara equations are fifth-order KdV type equations that have been derived to model many physical phenomena such as gravity-capillary waves and magneto-sound propagation in plasmas. This paper establishes the local well-posedness of the initial-value problem for the Kawahara equation in $H^s(\mathbf{R})$ with $s > -7/4$ and the local well-posedness for the modified Kawahara equation in $H^s(\mathbf{R})$ with $s \geq -1/4$. To prove these results, we derive a fundamental estimate on dyadic blocks for the Kawahara equation through the $[k; Z]$ multiplier norm method of Tao [14] and use this to obtain new bilinear and trilinear estimates in suitable Bourgain spaces.

1 Introduction

This paper is mainly concerned with the local well-posedness of the initial-value problems (IVP) for the Kawahara equation

$$(1.1)\qquad \begin{cases} u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0, & x, t \in \mathbf{R}, \\ u(x,0) = u_0(x). \end{cases}$$

and the modified Kawahara equation

$$(1.2)\qquad \begin{cases} u_t + u^2 u_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0, & x, t \in \mathbf{R}, \\ u(x,0) = u_0(x), \end{cases}$$

where α and β are real constants and $\beta \neq 0$. Attention is focused on solutions in Sobolev spaces of negative indices. These fifth-order KdV type equations arise in modeling gravity-capillary waves on a shallow layer and magneto-sound propagation in plasmas (see, e.g., [3], [10]).

The well-posedness issue on these fifth-order KdV type equations has previously been studied by several authors. In [11], Ponce considered a general

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fifth-order KdV equation

$$u_t + u_x + c_1 u u_x + c_2 u_{xxx} + c_3 u_x u_{xx} + c_4 u u_{xxx} + c_5 u_{xxxxx} = 0, \quad x,\, t \in \mathbf{R}$$

and established the global well-posedness of the corresponding IVP for any initial data in $H^4(\mathbf{R})$. In [7] and [8], Kenig, Ponce and Vega studied the local well-posedness of the IVP for the odd-order equation

$$u_t + \partial_x^{2j+1} u + P(u, \partial_x u, \ldots, \partial_x^{2j} u) = 0,$$

where P is a polynomial having no constant or linear terms. They obtained local well-posedness for

$$u_0 \in H^s(\mathbf{R}) \cap L^2(|x|^m dx),$$

where $s, m \in \mathbf{Z}^+$. Cui, Deng and Tao [2] established local well-posedness in H^s with $s > -1$ for the Kawahara equation. More recently, Wang, Cui and Deng [15] obtained local well-posedness in H^s with $s \geq -7/5$ for the Kawahara equation by the same method as in [2]. Their method is derived from that of Kenig, Ponce and Vega [9] for the cubic KdV equation. In [13], Tao and Cui studied the low regularity solutions of the modified Kawahara equation (1.2) and proved the local well-posedness of the IVP in any Sobolev space $H^s(\mathbf{R})$ with $s \geq 1/4$ by employing an approach of Kenig-Ponce-Vega for the generalized KdV equations [5].

Our goal here is to improve the existing low regularity well-posedness results. To this end, we first derive a fundamental estimate on dyadic blocks (see Lemma 3.2 below) for the Kawahara equation by following the idea in the $[k;\, Z]$-multiplier norm method introduced by Tao [14]. We then apply this fundamental estimate to establish new bilinear and trilinear estimates in Bourgain spaces. Combining these estimates with a contraction mapping argument, we are able to prove the following two theorems.

Theorem 1.1. *Let* $s > -7/4$ *and* $u_0 \in H^s(\mathbf{R})$*. Then there exist* $b = b(s) \in (1/2,\, 1)$ *and* $T = T(\|u_0\|_{H^s}) > 0$ *such that the IVP(1.1) has a unique solution on* $[0, T]$ *satisfying*

$$u \in C([0,T]; H^s(\mathbf{R})) \quad \textit{and} \quad u \in X_{s,b},$$

where $X_{s,b}$ *is a Bourgain type space (defined in the next section). In addition, the dependence of* u *on* u_0 *is Lipschitz.*

Theorem 1.2. *Let* $s \geq -1/4$ *and* $u_0 \in H^s(\mathbf{R})$*. Then there exist* $b = b(s) \in (1/2,\, 1)$ *and* $T = T(\|u_0\|_{H^s}) > 0$ *such that the IVP for the modified Kawahara equation* (1.2) *has a unique solution on* $[0, T]$ *satisfying*

$$u \in C([0,T]; H^s(\mathbf{R})) \quad \textit{and} \quad u \in X_{s,b},$$

and the dependence of u on u_0 is Lipschitz.

The proofs of Theorems 1.1 and 1.2 are given in the subsequent sections.

2 Linear and bilinear estimates for the Kawahara equation

This section provides the linear and bilinear estimates for the Kawahara equation. We start with some notation. Denote by $W(t)$ the unitary group generating the solution of the IVP for the linear equation

$$\begin{cases} v_t + \alpha v_{xxx} + \beta v_{xxxxx} = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R}, \\ v(x,0) = v_0(x). \end{cases} \tag{2.1}$$

That is,

$$v(x,t) = W(t)v_0(x) = S_t * v_0(x),$$

where $\widehat{S}_t = e^{itp(\xi)}$ with $p(\xi) = -\beta\xi^5 + \alpha\xi^3$, or

$$S_t(x) = \int e^{i(x\xi + tp(\xi))}\, d\xi.$$

For $s, b \in \mathbf{R}$, let $X_{s,\,b}$ denote the completion of the functions in C_0^∞ with respect to the norm

$$\|f\|^2_{X_{s,b}} = \int \langle\xi\rangle^{2s} \langle\tau - p(\xi)\rangle^{2b}\, |\widehat{f}(\xi,\tau)|^2\, d\xi\, d\tau,$$

where $\langle\xi\rangle = 1 + |\xi|$. It is easy to verify that

$$\|f\|_{X_{s,\,b}} = \|J^s \Lambda^b W(-t) f\|_{L^2_{x,\,t}},$$

where

$$\widehat{Jg}(\xi) = (1+|\xi|) \quad \widehat{g}(\xi), \qquad \widehat{\Lambda h}(\tau) = (1+|\tau|)\widehat{h}(\tau).$$

Let $\psi \in C_0^\infty$ be a standard bump function, and consider the integral equation

$$u(t) = \psi(\delta^{-1}t)W(t)u_0 - \psi(\delta^{-1}t)\int_0^t W(t-t')u(t')\partial_x u(t')\, dt'.$$

Denote the right-hand side by $\mathcal{T}(u)$. The goal is to show that $\mathcal{T}(u)$ is a contraction on the complete metric space

$$Y = \{u \in X_{s,\,b} : \|u\|_{X_{s,\,b}} \le 2c_0\delta^{(1-2b)/2}\|u_0\|_{H^s}\}$$

with metric

$$d(u,v) = \|u - v\|_{X_{s,\,b}}, \qquad u, v \in Y,$$

where c_0 is the constant in Proposition 2.1 below. For this purpose, we need two linear estimates and one bilinear estimate stated in the following propositions.

Proposition 2.1. *For* $s \in \mathbf{R}$ *and* $b > 1/2$,

$$\|\psi(\delta^{-1}t)W(t)u_0\|_{X_{s,\,b}} \le c_0\delta^{(1-2b)/2}\|u_0\|_{H^s},$$
$$\left\|\psi(\delta^{-1}t)\int_0^t W(t-t')f(t')\,dt'\right\|_{X_{s,\,b}} \le c_0\delta^{(1-2b)/2}\|f\|_{X_{s,\,b-1}}.$$

The proof of these estimates follows directly from Kenig, Ponce and Vega [6].

Proposition 2.2. *For any* $s > -7/4$, *there is* b *satisfying* $1/2 < b < 1$ *such that*

$$\|\partial_x(uv)\|_{X_{s,\,b-1}} \le c_1\|u\|_{X_{s,\,b}}\|v\|_{X_{s,\,b}}, \tag{2.2}$$

where c_1 *is a constant depending only on* s *and* b.

Proposition 2.2 is proved in Section 4. In the next section, we introduce Tao's $[k;\,Z]$-multiplier norm method and prove a fundamental estimate on dyadic blocks for the Kawahara equation, from which a variety of bilinear estimates can be derived. Once the estimates in Propositions 2.1 and 2.2 are available, a standard argument then yields that $\mathcal{T}(u)$ is a contraction on Y.

3 Fundamental estimate on dyadic blocks for the Kawahara equation

In this section, we introduce Tao's $[k;\,Z]$-multiplier norm method and establish the fundamental estimate on dyadic blocks, viz. Lemma 3.2 for the Kawahara equation, from which Proposition 2.2 and other bilinear estimates (see Lemma 5.2 below) can be derived.

Let Z be any abelian additive group with an invariant measure $d\xi$. For any integer $k \ge 2$, let $\Gamma_k(Z)$ denote the hyperplane

$$\Gamma_k(Z) := \{(\xi_1, \dots, \xi_k) \in Z^k : \xi_1 + \cdots + \xi_k = 0\}$$

endowed with the measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \dots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1})d\xi_1 \cdots d\xi_{k-1}.$$

Following Tao [14], we define a $[k;\,Z]$-multiplier to be a function $m : \Gamma_k(Z) \to \mathbb{C}$. The multiplier norm $\|m\|_{[k;\,Z]}$ is the smallest constant such that

$$\left|\int_{\Gamma_k(Z)} m(\xi)\prod_{j=1}^k f_j(\xi_j)\right| \le c\prod_{j=1}^k \|f_j\|_{L^2(Z)}, \tag{3.1}$$

for all test functions f_j on Z. In [14], Tao systematically studied this kind of weighted convolution estimate on L^2. To establish the fundamental estimate on dyadic blocks for the Kawahara equation, we use the following notation.

We write $A \lesssim B$ to mean that $A \leq CB$ for some large constant C which may vary from line to line and depend on various parameters, and similarly write $A \ll B$ to mean that $A \leq C^{-1}B$. We write $A \sim B$ when $A \lesssim B \lesssim A$.

All summations over capitalized variables such as N_j, L_j, H are presumed to be dyadic, i.e., these variables range over numbers of the form 2^k for $k \in \mathbb{Z}$. Let $N_1,\, N_2,\, N_3 > 0$. It is convenient to define the quantities $N_{max} \geq N_{med} \geq N_{min}$ to be the maximum, median, and minimum of $N_1,\, N_2,\, N_3$ respectively. Similarly define $L_{max} \geq L_{med} \geq L_{min}$ whenever $L_1,\, L_2,\, L_3 > 0$. We also adopt the following summation convention: any summation of the form $L_{max} \sim \cdots$ is a sum over the three dyadic variables $L_1,\, L_2,\, L_3 \gtrsim 1$; thus, for instance,

$$\sum_{L_{max}\sim H} := \sum_{L_1,\, L_2,\, L_3 \gtrsim 1:\, L_{max}\sim H} .$$

Similarly, any summation of the form $N_{max} \sim \cdots$ is a sum over the three dyadic variables $N_1,\, N_2,\, N_3 > 0$; thus, for instance,

$$\sum_{N_{max}\sim N_{med}\sim N} := \sum_{N_1,\, N_2,\, N_3 > 0:\, N_{max}\sim N_{med}\sim N} .$$

If τ, ξ and $p(\xi)$ are given, we define

$$\lambda := \tau - p(\xi).$$

Similarly,

$$\lambda_j := \tau_j - p(\xi_j), \quad j = 1,\, 2,\, 3.$$

In this paper, we do not go further on the general framework of Tao's weighted convolution estimates. We focus our attention on the $[3;\, Z]$-multiplier norm estimate for the Kawahara equation. In the course of the estimate, we need the resonance function

$$h(\xi) = p(\xi_1) + p(\xi_2) + p(\xi_3) = -\lambda_1 - \lambda_2 - \lambda_3, \tag{3.2}$$

which measures the extent to which the spatial frequencies $\xi_1,\, \xi_2,\, \xi_3$ can resonate with each other.

By dyadic decomposition of the variables ξ_j, λ_j, as well as the function $h(\xi)$, one is led to consider

$$\|X_{N_1,\, N_2,\, N_3;\, H;\, L_1,\, L_2,\, L_3}\|_{[3,\, \mathbb{R}\times\mathbb{R}]}, \tag{3.3}$$

where $X_{N_1,\,N_2,\,N_3;\,H;\,L_1,\,L_2,\,L_3}$ is the multiplier

$$X_{N_1,\,N_2,\,N_3;\,H;\,L_1,\,L_2,\,L_3}(\xi,\,\tau) := \chi_{|h(\xi)|\sim H} \prod_{j=1}^{3} \chi_{|\xi_j|\sim N_j}\chi_{|\lambda_j|\sim L_j}. \tag{3.4}$$

From the identities

$$\xi_1+\xi_2+\xi_3=0$$

and

$$\lambda_1+\lambda_2+\lambda_3+h(\xi)=0$$

on the support of the multiplier, we see that $X_{N_1,\,N_2,\,N_3;\,H;\,L_1,\,L_2,\,L_3}$ vanishes unless

$$N_{max}\sim N_{med}, \tag{3.5}$$

and

$$L_{max}\sim \max(H,\,L_{med}). \tag{3.6}$$

From the definition of the resonance function (3.2), we obtain the following algebraic smoothing relation.

Lemma 3.1. *If* $N_{max}\sim N_{med}\gtrsim 1$*, then*

$$\max\{|\lambda_1|,|\lambda_2|,|\lambda_3|\}\gtrsim N_{\max}^4 N_{\min}. \tag{3.7}$$

Proof. Noticing that $p(\xi_j)=-\beta\xi_j^5+\alpha\xi_j^3$, $j=1,\,2,\,3$, we have

$$\begin{aligned} h(\xi) &= -\lambda_1-\lambda_2-\lambda_3 = p(\xi_1)+p(\xi_2)+p(\xi_3)\\ &= \xi_1\xi_2\xi_3(3\alpha-5\beta(\xi_1^2+\xi_1\xi_2+\xi_2^2)). \end{aligned}$$

Now $\xi_1^2+\xi_1\xi_2+\xi_2^2\sim\max\{\xi_1^2,\,\xi_2^2\}$, and if $N_{max}\sim N_{med}\gtrsim 1$ and $\beta\neq 0$, we obtain that

$$\max\{|\lambda_1|,|\lambda_2|,|\lambda_3|\}\geq \frac{1}{3}(|\lambda_1+\lambda_2+\lambda_3|)\gtrsim N_{\max}^4 N_{\min}.$$ □

Under the condition of Lemma 3.1, we see that we may assume that

$$H\sim N_{\max}^4 N_{\min}, \tag{3.8}$$

since the multiplier in (3.4) vanishes otherwise.

Now we are in the position to state the fundamental estimate on dyadic blocks for the Kawahara equation.

Lemma 3.2. *Let* H, N_1, N_2, N_3, L_1, L_2, $L_3 > 0$ *obey* (3.5), (3.6), (3.8).

$\diamond$*((++)Coherence) If* $N_{max} \sim N_{min}$ *and* $L_{max} \sim H$, *then*

$$(3.3) \lesssim L_{min}^{1/2} N_{max}^{-2} L_{med}^{1/2}. \tag{3.9}$$

$\diamond$*((+-)Coherence) If* $N_2 \sim N_3 \gg N_1$ *and* $H \sim L_1 \gtrsim L_2$, L_3, *then*

$$(3.3) \lesssim L_{min}^{1/2} N_{max}^{-2} \min(H, \frac{N_{max}}{N_{min}} L_{med})^{1/2}. \tag{3.10}$$

Similarly for permutations.

$\diamond$ *In all other cases, we have*

$$(3.3) \lesssim L_{min}^{1/2} N_{max}^{-2} \min(H, L_{med})^{1/2}. \tag{3.11}$$

Proof. The fundamental estimate on dyadic blocks for the Kawahara equation is new. We prove it by using the tools developed by Tao in [14].

In the high modulation case $L_{max} \sim L_{med} \gg H$, we have by an elementary estimate employed by Tao (see (37) on [14, p. 861])

$$(3.3) \lesssim L_{min}^{1/2} N_{min}^{1/2} \lesssim L_{min}^{1/2} N_{max}^{-2} N_{min}^{1/2} N_{max}^{2} \lesssim L_{min}^{1/2} N_{max}^{-2} H^{1/2}.$$

For the low modulation case $L_{max} \sim H$, by symmetry we may assume that $L_1 \geq L_2 \geq L_3$.

By Corollary 4.2 in [14], we have

$$(3.3) \lesssim L_3^{1/2} \Big| \{\xi_2 : |\xi_2 - \xi_2^0| \ll N_{min};\ |\xi - \xi_2 - \xi_3^0| \ll N_{min}; \\ p(\xi_2) + p(\xi - \xi_2) = \tau + O(L_2)\} \Big|^{1/2} \tag{3.12}$$

for some $\tau \in \mathbf{R}$, ξ, ξ_1^0, ξ_2^0, ξ_3^0 satisfying

$$|\xi_j^0| \sim N_j\ (j = 1, 2, 3);\ |\xi_1^0 + \xi_2^0 + \xi_3^0| \ll N_{min};\ |\xi + \xi_1^0| \ll N_{min}.$$

To estimate the right-hand side of the expression (3.12), we use the identity

$$p(\xi_2) + p(\xi - \xi_2) = p(\xi) + q(\xi, \xi_2), \tag{3.13}$$

where

$$q(\xi, \eta) = 5\beta\xi\eta(\xi - \eta)(\xi^2 - \xi\eta + \eta^2) - 3\alpha\xi\eta(\xi - \eta).$$

We need to consider three cases: $N_1 \sim N_2 \sim N_3$, $N_1 \sim N_2 \gg N_3$ and $N_2 \sim N_3 \gg N_1$. The case $N_1 \sim N_3 \gg N_2$ follows by symmetry. By (3.13) and (3.12), we have

$$p(\xi) + (\xi - \xi_2)(5\beta\xi\xi_2(\xi^2 - \xi\xi_2 + \xi_2^2) - 3\alpha\xi\xi_2) = \tau + O(L_2). \tag{3.14}$$

(i) If $N_1 \sim N_2 \sim N_3$, we see from (3.14) that the variable ξ_2 is contained in an interval of length $O(L_2 N_{max}^{-4})$; and then

$$(3.3) \lesssim L_3^{1/2} L_2^{1/2} N_{max}^{-2} = L_{min}^{1/2} L_{med}^{1/2} N_{max}^{-2},$$

so (3.9) follows.

(ii) If $N_1 \sim N_2 \gg N_3$, the same computation as in the case (i) gives

$$(3.3) \lesssim L_{min}^{1/2} L_{med}^{1/2} N_{max}^{-2}.$$

(iii) If $N_2 \sim N_3 \gg N_1$, we see from (3.14) that the variable ξ_2 is contained in an interval of length $O(L_2 N_{max}^{-3} N_{min}^{-1})$; and then

$$(3.3) \lesssim L_3^{1/2} L_2^{1/2} N_{min}^{-1/2} N_{max}^{-3/2} = L_{min}^{1/2} L_{med}^{1/2} N_{min}^{-1/2} N_{max}^{-3/2}.$$

But ξ_2 is also contained in an interval of length $\ll N_{min}$. The claim (3.10) follows. □

4 Proof of Proposition 2.2

This section is devoted to the proof of Proposition 2.2 with the fundamental estimate on dyadic blocks in Lemma 3.2.

Proof. By Plancherel, it suffices to show that

$$(4.1) \qquad \left\| \frac{(\xi_1 + \xi_2)\langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-s} \langle \xi_3 \rangle^{s}}{\langle \tau_1 - p(\xi_1) \rangle^{b} \langle \tau_2 - p(\xi_2) \rangle^{b} \langle \tau_3 - p(\xi_3) \rangle^{1-b}} \right\|_{[3,\, \mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

By dyadic decomposition of the variables ξ_j, λ_j $(j = 1, 2, 3)$, $h(\xi)$, we may assume that $|\xi_j| \sim N_j$, $|\lambda_j| \sim L_j$ $(j = 1, 2, 3)$, $|h(\xi)| \sim H$. By the translation invariance of the $[k; Z]$-multiplier norm, we can always restrict our estimate on $L_j \gtrsim 1$ $(j = 1,\, 2,\, 3)$ and $\max(N_1, N_2, N_3) \gtrsim 1$. The comparison principle and orthogonality (see Schur's test in [14, p. 851]) reduce the multiplier norm estimate (4.1) to showing that

$$(4.2) \qquad \sum_{N_{max} \sim N_{med} \sim N} \sum_{L_1,\, L_2,\, L_3 \gtrsim 1} \frac{N_3 \langle N_3 \rangle^s}{\langle N_1 \rangle^s \langle N_2 \rangle^s L_1^b L_2^b L_3^{1-b}} \times \| X_{N_1,\, N_2,\, N_3;\, L_{max};\, L_1,\, L_2,\, L_3} \|_{[3;\, \mathbb{R}\times\mathbb{R}]} \lesssim 1$$

and

$$(4.3) \qquad \sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim L_{med}} \sum_{H \ll L_{max}} \frac{N_3 \langle N_3 \rangle^s}{\langle N_1 \rangle^s \langle N_2 \rangle^s L_1^b L_2^b L_3^{1-b}} \times \| X_{N_1,\, N_2,\, N_3;\, H;\, L_1,\, L_2,\, L_3} \|_{[3;\, \mathbb{R}\times\mathbb{R}]} \lesssim 1$$

for all $N \gtrsim 1$. Estimates (4.2) and (4.3) are obtained from the fundamental estimate Lemma 3.2 and some delicate summation.

Fix $N \gtrsim 1$. This implies (3.8). We first prove (4.3). By (3.11), we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{N_3\langle N_3\rangle^s}{\langle N_1\rangle^s\langle N_2\rangle^s L_1^b L_2^b L_3^{1-b}} L_{min}^{1/2} N_{min}^{1/2} \lesssim 1. \tag{4.4}$$

By symmetry, we only need to consider two cases: $N_1 \sim N_2 \sim N$, $N_3 = N_{min}$ and $N_1 \sim N_3 \sim N$, $N_2 = N_{min}$.

(i) In the first case $N_1 \sim N_2 \sim N$, $N_3 = N_{min}$, the estimate (4.4) can be further reduced to

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{N^{-2s} N_{min}\langle N_{min}\rangle^s}{L_{min}^b L_{med}^b L_{max}^{1-b}} L_{min}^{1/2} N_{min}^{1/2} \lesssim 1.$$

Then, performing the L summations, we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N} \frac{N^{-2s} N_{min}^{3/2}\langle N_{min}\rangle^s}{N^4 N_{min}} \lesssim 1,$$

which holds if $4 + 2s > 0$. So, (4.4) holds if $s > -2$.

(ii) In the second case $N_1 \sim N_3 \sim N$, $N_2 = N_{min}$, the estimate (4.4) can be reduced to

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{N}{\langle N_{min}\rangle^s L_{min}^b L_{med}^b L_{max}^{1-b}} L_{min}^{1/2} N_{min}^{1/2} \lesssim 1.$$

Before performing the L summations, we need to pay a little more attention to the summation of N_{min}. So we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N,\, N_{min}\le 1}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{N N_{min}^{1/2}}{L_{min}^{b-1/2} L_{max}^{1/2}} + \sum_{N_{max}\sim N_{med}\sim N,\, 1\le N_{min}\le N}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{N N_{min}^{1/2-s}}{L_{min}^{b-1/2} L_{max}} \lesssim 1,$$

which obviously holds if $s > -7/2$. So (4.4) holds if $s > -7/2$.

Now we show the low modulation case (4.2). In this case, $L_{max} \sim N_{max}^4 N_{min}$. We first deal with the contribution where (3.9) holds. In this case, we have $N_1, N_2, N_3 \sim N \gtrsim 1$, so we reduce to

$$\sum_{L_{max}\sim N^5} \frac{N^{-s} N}{L_{min}^b L_{med}^b L_{max}^{1-b}} L_{min}^{1/2} N^{-2} L_{med}^{1/2} \lesssim 1. \tag{4.5}$$

Performing the L summations, we reduce to

$$\frac{1}{N^{1+s}N^{5(1-b)}} \lesssim 1,$$

which holds if $1+s+5(1-b) > 0$. So (4.5) holds if $s > -7/2$ and $1/2 < b < (6+s)/5$.

Now we deal with the cases where (3.10) holds. By symmetry, we need only to consider two cases:

$$\begin{array}{ll} N \sim N_1 \sim N_2 \gg N_3; & H \sim L_3 \gtrsim L_1,\, L_2 \\ N \sim N_2 \sim N_3 \gg N_1; & H \sim L_1 \gtrsim L_2,\, L_3 \end{array}.$$

In the first case, we reduce by (3.10) to

(4.6)
$$\sum_{N_3 \ll N} \sum_{1 \lesssim L_1,\, L_2 \lesssim N^4 N_3} \frac{N_3 \langle N_3 \rangle^s}{N^{2s} L_1^b L_2^b L_3^{1-b}} L_{min}^{1/2} N^{-2} \min\left(N^4 N_3,\, \frac{N}{N_3} L_{med}\right)^{1/2} \lesssim 1.$$

Decompose the left-hand side of (4.6) into the following two terms:

$$\begin{aligned} &\sum_{N_3 \le 1} \sum_{1 \lesssim L_1,\, L_2 \lesssim N^4 N_3} \frac{N_3 \langle N_3 \rangle^s}{N^{2s} L_1^b L_2^b L_3^{1-b}} L_{min}^{1/2} N^{-2} \min\left(N^4 N_3,\, \frac{N}{N_3} L_{med}\right)^{1/2} \\ &+ \sum_{1 < N_3 \ll N} \sum_{1 \lesssim L_1,\, L_2 \lesssim N^4 N_3} \frac{N_3 \langle N_3 \rangle^s}{N^{2s} L_1^b L_2^b L_3^{1-b}} L_{min}^{1/2} N^{-2} \min\left(N^4 N_3,\, \frac{N}{N_3} L_{med}\right)^{1/2} \\ &\hspace{9cm} =: I_1 + I_2. \end{aligned}$$

We estimate the above two terms separately.

We first consider the estimate of I_1. If $N^4 N_3 = \frac{N}{N_3} L_{med}$, then $N_3 = \left(\frac{L_{med}}{N^3}\right)^{1/2}$. We divide two cases: $\left(\frac{L_{med}}{N^3}\right)^{1/2} \ge 1$ and $\left(\frac{L_{med}}{N^3}\right)^{1/2} < 1$ to estimate I_1. When $\left(\frac{L_{med}}{N^3}\right)^{1/2} \ge 1$,

$$I_1 \le \sum_{N_3 \le 1} \sum_{1 \lesssim L_1,\, L_2 \lesssim N^4 N_3} \frac{N_3}{N^{2s} L_{min}^{b-1/2} L_{med}^b (N^4 N_3)^{1-b}} N^{-2} N^2 N_3^{1/2}. \tag{4.7}$$

Performing the N_3 summation in (4.7), we have

$$I_1 \lesssim \sum_{\substack{1 \lesssim L_1,\, L_2 \lesssim N^4 \\ L_{med} \ge N^3}} \frac{1}{N^{2s} L_{min}^{b-1/2} L_{med}^b N^{4(1-b)}} \lesssim 1$$

if $2s + 4 - b > 0$. That means that $I_1 \lesssim 1$ if $s > -7/4$ and $1/2 < b < 4 + 2s$.

When $\left(\frac{L_{med}}{N^3}\right)^{1/2} < 1$,

$$I_1 \le \sum_{N_3<(\frac{L_{med}}{N^3})^{1/2}} \sum_{1\lesssim L_1,\, L_2\lesssim N^4N_3} \frac{N_3}{N^{2s}L_{min}^{b-1/2}L_{med}^{b}(N^4N_3)^{1-b}} N^{-2}N^2N_3^{1/2}$$
$$+ \sum_{(\frac{L_{med}}{N^3})^{1/2}<N_3<1} \sum_{1\lesssim L_1,\, L_2\lesssim N^4N_3} \frac{N_3}{N^{2s}L_{min}^{b-1/2}L_{med}^{b}(N^4N_3)^{1-b}} \times N^{-2}N^{1/2}L_{med}^{1/2}N_3^{-1/2}.$$

Performing the N_3 summation, we have

$$\begin{aligned}
I_1 &\lesssim \sum_{1\lesssim L_1,\, L_2\lesssim N^4} \frac{\left(\frac{L_{med}}{N^3}\right)^{\frac{1}{2}(\frac{1}{2}+b)}}{N^{2s}L_{min}^{b-1/2}L_{med}^{b}N^{4(1-b)}} + \sum_{1\lesssim L_1,\, L_2\lesssim N^4} \frac{N^{-\frac{3}{2}}}{N^{2s}L_{min}^{b-1/2}L_{med}^{b-1/2}N^{4(1-b)}}\\
&\lesssim \sum_{1\lesssim L_1,\, L_2\lesssim N^4} \frac{1}{N^{2s}L_{min}^{b-1/2}L_{med}^{\frac{b}{2}-\frac{1}{4}}N^{4(1-b)}N^{\frac{3}{2}(\frac{1}{2}+b)}}\\
&\quad + \sum_{1\lesssim L_1,\, L_2\lesssim N^4} \frac{N^{-\frac{3}{2}}}{N^{2s}L_{min}^{b-1/2}L_{med}^{b-1/2}N^{4(1-b)}}\\
&=: I_{11} + I_{12}.
\end{aligned}$$

Performing the L summations, we see that $I_{11} \lesssim 1$ if $2s + 4(1-b) + \frac{3}{2}(\frac{1}{2}+b) > 0$ and $I_{12} \lesssim 1$ if $2s + 4(1-b) + 3/2 > 0$. This implies that $I_1 \lesssim 1$ if $s > -7/4$ and $1/2 < b < \frac{4s+11}{8}$.

Now we consider the estimate of the second term I_2. The estimate is a little simpler than the estimate of I_1. We do not need to distinguish the cases $\left(\frac{L_{med}}{N^3}\right)^{1/2} \ge 1$ and $\left(\frac{L_{med}}{N^3}\right)^{1/2} < 1$. We obtain the following estimate for I_2 in a unified way:

(4.8)
$$I_2 \le \sum_{1<N_3\ll N} \sum_{1\lesssim L_1,\, L_2\lesssim N^4N_3} \frac{N_3N_3^s}{N^{2s}L_{min}^{b-1/2}L_{med}^{b}(N^4N_3)^{1-b}} N^{-2}N^{1/2}L_{med}^{1/2}N_3^{-1/2}.$$

Performing the N_3 summation in (4.8) and noticing that if $s - \frac{1}{2} + b < 0$, we have

$$I_2 \lesssim \sum_{1\lesssim L_1,\, L_2\lesssim N^5} \frac{1}{N^{2s}L_{min}^{b-1/2}L_{med}^{b-1/2}N^{4(1-b)}N^{3/2}} \lesssim 1$$

under the condition that $2s + 4(1-b) + 3/2 > 0$. If $s - 1/2 + b \ge 0$, we have

$$I_2 \lesssim \sum_{1\lesssim L_1,\, L_2\lesssim N^5} \frac{N^{s-1/2+b}}{N^{2s}L_{min}^{b-1/2}L_{med}^{b-1/2}N^{4(1-b)}N^{3/2}} \lesssim 1$$

under the condition that $2s + 4(1 - b) + 3/2 > s - 1/2 + b$. That means that $I_2 \lesssim 1$ if $s > -7/4$ and $1/2 < b < \min\left(\frac{4s+11}{8}, \frac{s+6}{5}\right)$. Combining the estimates for I_1 and I_2, we obtain the desired estimate (4.6).

Now we deal with the case $N \sim N_2 \sim N_3 \gg N_1$; $H \sim L_1 \gtrsim L_2, L_3$. In this case, we reduce by (3.10) to

(4.9)
$$\sum_{N_1 \ll N} \sum_{1 \lesssim L_2, L_3 \lesssim N^4 N_1} \frac{N^{1+s} L_{min}^{1/2}}{N^s < N_1 >^s L_2^b L_3^{1-b} (N^4 N_1)^b} \min\left(H, \frac{N}{N_1} L_{med}\right)^{1/2} \lesssim 1.$$

Decompose the left-hand side of (4.9) into the following two terms:

$$\sum_{N_1 \le 1} \sum_{1 \lesssim L_2, L_3 \lesssim N^4 N_1} \frac{N^{1+s}}{N^s L_2^b L_3^{1-b} (N^4 N_1)^b} L_{min}^{1/2} N_1^{1/2} + \sum_{1 < N_1 \ll N} \sum_{1 \lesssim L_2, L_3 \lesssim N^4 N_1} \frac{N^{1+s}}{N_1^s N^s L_2^b L_3^{1-b} (N^4 N_1)^b} L_{min}^{1/2} N^{-2} N^{5/4} L_{med}^{1/4} =: J_1 + J_2.$$

In J_1, we assume $N_1 \gtrsim N^{-4}$; otherwise, the summation of L vanishes. Performing the summation of L, we get

$$J_1 \lesssim \sum_{N^{-4} \lesssim N_1 \le 1} \frac{N N_1^{1/2-b}}{N^{4b}} \lesssim \frac{N N^{(-4)(1/2-b)}}{N^{4b}} \lesssim 1.$$

If we take $1/2 < b < 3/4$ in J_2, then performing the summation of L yields

$$J_2 \lesssim \sum_{1 \le N_1 \ll N} \frac{N^{1/4} N_1^{-s-b}}{N^{4b}} \lesssim \frac{N^{1/4} N^{-s-b}}{N^{4b}} \lesssim 1,$$

if $4b + s + b - 1/4 > 0$. The condition $4b + s + b - 1/4 > 0$ holds if $s > -7/4$ and $1/2 < b \le 1$. So $J_2 \lesssim 1$ if $s > -7/4$ and $1/2 < b < 3/4$. Combining the estimates for J_1 and J_2, we get the needed estimate (4.9).

To finish the proof of (4.2), it remains to deal with the cases where (3.11) holds. This reduces to

(4.10)
$$\sum_{N_{max} \sim N_{med} \sim N} \sum_{L_{max} \sim N^4 N_{min}} \frac{N_3 \langle N_3 \rangle^s}{\langle N_1 \rangle^s \langle N_2 \rangle^s L_1^b L_2^b L_3^{1-b}} \times L_{min}^{1/2} N^{-2} \min\left(H, L_{med}\right)^{1/2} \lesssim 1.$$

To estimate (4.10), by symmetry we need to consider two cases: $N_1 \sim N_2 \sim N$, $N_3 = N_{min}$ and $N_1 \sim N_3 \sim N$, $N_2 = N_{min}$.

(i) When $N_1 \sim N_2 \sim N$, $N_3 = N_{min}$, the estimate (4.10) further reduces to

$$\sum_{\substack{N_1 \sim N_2 \sim N \\ N_3 \ll N}} \sum_{L_{max} \sim N^4 N_3} \frac{N_3 \langle N_3 \rangle^s}{N^{2s} L_{min}^b L_{med}^b (N^4 N_3)^{1-b}} L_{min}^{1/2} N^{-2} L_{med}^{1/2} \lesssim 1.$$

Then, performing the L summations, we reduce to

$$\sum_{N_3 \ll N} \frac{N_3 \langle N_3 \rangle^s}{N^{2+2s} N^{4(1-b)} N_3^{1-b}} \lesssim 1,$$

which holds if $2 + 2s + 4(1-b) > 0$. So (4.10) holds if $s > -2$ and $1/2 < b < \frac{s+3}{2}$.

(ii) When $N_1 \sim N_3 \sim N$, $N_2 = N_{min}$, the estimate (4.10) can be reduced to

$$\sum_{\substack{N_1 \sim N_3 \sim N \\ N_2 \ll N}} \sum_{L_{max} \sim N^4 N_2} \frac{N^{1+s} L_{min}^{1/2} N^{-2}}{N^s \langle N_2 \rangle^s L_{min}^b L_{med}^b L_{max}^{1-b}} \min(H,\, L_{med})^{1/2} \lesssim 1. \tag{4.11}$$

Before performing the L summations, as before we need to pay a little more attention to the summation of N_2. Decompose the left-hand side of (4.11) into the following two terms:

$$\sum_{N_2 \leq 1} \sum_{L_{max} \sim N^4 N_2} \frac{N}{L_{min}^{b-1/2} L_{med}^b L_{max}^{1-b}} N^{-2} L_{med}^{1/4} (N^4 N_2)^{1/4}$$
$$+ \sum_{1 \leq N_2 \leq N} \sum_{L_{max} \sim N^4 N_2} \frac{N}{N_2^s L_{min}^{b-1/2} L_{med}^b L_{max}^{(1-b)}} N^{-2} L_{med}^{1/2}$$
$$=: J_3 + J_4.$$

It is easily seen that $J_3 \lesssim 1$ for any $1/2 < b \leq 1$. For J_4, if $s + 1 - b \geq 0$, we always have $J_4 \lesssim 1$ for any $1/2 < b \leq 1$. If $s + 1 - b < 0$, we have $J_4 \lesssim 1$ under the condition that $4(1-b) + s + 1 + (1-b) > 0$. So, (4.11) holds if $s > -7/2$ and $1/2 < b < (s+6)/5$. This completes the proof of Proposition (2.2). □

5 A trilinear estimate and local well-posedness of the modified Kawah ara equation

In this section, we prove a trilinear estimate in Bourgain spaces, which together with the linear estimates presented in Section 2, enables us to derive the local well-posedness of the initial-value problem for the modified Kawahara equation (Theorem1.2).

Lemma 5.1. *Let $s \geq -1/4$. For all u_1, u_2, u_3 on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $1/2 < b \leq 1$, we have*

$$\|\partial_x(u_1 u_2 u_3)\|_{X_{s,\,b-1}} \lesssim \|u_1\|_{X_{s,\,b}} \|u_2\|_{X_{s,\,b}} \|u_3\|_{X_{s,\,b}}. \tag{5.1}$$

This is the first trilinear estimate in Bourgain spaces associated to the class of Kawahara equations. It seems difficult to obtain this kind of trilinear estimate by the method first presented by Bourgain, Kenig-Ponce-Vega for KdV. We reduce the trilinear estimate by the TT^* identity Tao developed in [14] to a bilinear estimate and then prove the bilinear estimate by the fundamental estimate on dyadic blocks in Lemma 3.2.

Proof. By duality and Plancherel, it suffices to show that

$$\left\| \frac{(\xi_1+\xi_2+\xi_3)\langle\xi_4\rangle^s}{\langle\tau_4-p(\xi_4)\rangle^{1-b}\prod_{j=1}^3\langle\xi_j\rangle^s\langle\tau_j-p(\xi_j)\rangle^b} \right\|_{[4,\,\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

We estimate $|\xi_1+\xi_2+\xi_3|$ by $\langle\xi_4\rangle$. Applying the fractional Leibnitz rule, we have

$$\langle\xi_4\rangle^{s+1} \lesssim \langle\xi_4\rangle^{1/2}\sum_{j=1}^3\langle\xi_j\rangle^{s+1/2},$$

where we assume $s > -1/2$, and symmetry to reduce to

$$\left\| \frac{\langle\xi_1\rangle^{-s}\langle\xi_3\rangle^{-s}\langle\xi_2\rangle^{1/2}\langle\xi_4\rangle^{1/2}}{\langle\tau_4-p(\xi_4)\rangle^{1-b}\prod_{j=1}^3\langle\tau_j-p(\xi_j)\rangle^b} \right\|_{[4,\,\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

We may replace $\langle\tau_2-p(\xi_2)\rangle^b$ by $\langle\tau_2-p(\xi_2)\rangle >^{1-b}$ (this is true for any $b \geq 1/2$). By the TT^* identity (see Lemma 3.7 in [14, p. 847]), the estimate is reduced to the following bilinear estimate. □

Lemma 5.2 (Bilinear estimate)**.** *Let* $s \geq -1/4$. *For all* u, v *on* $\mathbb{R}\times\mathbb{R}$ *and* $0 < \epsilon \ll 1$,

$$\|uv\|_{L^2(\mathbb{R}\times\mathbb{R})} \lesssim \|u\|_{X_{-1/2,\,1/2-\epsilon}(\mathbb{R}\times\mathbb{R})}\|v\|_{X_{s,\,1/2+\epsilon}(\mathbb{R}\times\mathbb{R})}. \tag{5.2}$$

This lemma can be proved in the same way as Proposition 2.2 by using the fundamental estimate on dyadic blocks in Lemma 3.2. However, there are some differences between this lemma and Proposition 2.2. Proposition 2.2 is a symmetric bilinear estimate, while Lemma 5.2 is an asymmetric bilinear estimate, which leads to a certain lack of symmetry in the proof of Lemma 5.2. On the other hand, since there is no derivative in the left-hand side of (5.2), the proof of Lemma 5.2 is rather simpler than that of Proposition 2.2.

Proof. By Plancherel, it suffices to prove that

$$\left\| \frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{1/2}}{\langle\tau_1-p(\xi_1)\rangle^{1/2+\epsilon}\langle\tau_2-p(\xi_2)\rangle^{1/2-\epsilon}} \right\|_{[3,\,\mathbb{R}\times\mathbb{R}]} \lesssim 1. \tag{5.3}$$

By dyadic decomposition and orthogonality as in the proof of Proposition 2.2, we reduce the multiplier norm estimate (5.3) to showing that

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_1,\,L_2,\,L_3\gtrsim 1} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{1/2}}{L_1^{1/2+\epsilon}L_2^{1/2-\epsilon}} \times \|X_{N_1,\,N_2,\,N_3;\,L_{max};\,L_1,\,L_2,\,L_3}\|_{[3;\,\mathbb{R}\times\mathbb{R}]} \lesssim 1 \tag{5.4}$$

and

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}}\ \sum_{H\ll L_{max}} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{1/2}}{L_1^{1/2+\epsilon}L_2^{1/2-\epsilon}} \times \|X_{N_1,\,N_2,\,N_3;\,H;\,L_1,\,L_2,\,L_3}\|_{[3;\,\mathbb{R}\times\mathbb{R}]} \lesssim 1 \tag{5.5}$$

for all $N \gtrsim 1$.

Fix $N \gtrsim 1$. We first prove (5.5). By (3.11), we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{1/2}}{L_1^{1/2+\epsilon}L_2^{1/2-\epsilon}} L_{min}^{1/2}N_{min}^{1/2} \lesssim 1. \tag{5.6}$$

We consider two cases: $s \geq 0$ and $s < 0$.

(i) In the first case $s \geq 0$, the estimate (5.6) can be further reduced to

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{N^{1/2}\langle N_{min}\rangle^{-s}}{L_{min}^{1/2+\epsilon}L_{med}^{1/2-\ epsilon}} L_{min}^{1/2}N_{min}^{1/2} \lesssim 1.$$

Then, performing the L summations, we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N} \frac{N^{1/2}N_{min}^{1/2}\langle N_{min}\rangle^{-s}}{(N^4 N_{min})^{1/2-\epsilon}} \lesssim 1,$$

which is always true for $s \geq 0$.

(ii) In the second case $s < 0$, the estimate (5.6) can be reduced to

$$\sum_{N_{max}\sim N_{med}\sim N}\ \sum_{L_{max}\sim L_{med}\gtrsim N^4 N_{min}} \frac{N^{1/2-s}}{L_{min}^{1/2+\epsilon}(N^4 N_{min})^{1/2-\epsilon}} L_{min}^{1/2}N_{min}^{1/2} \lesssim 1.$$

Performing the L summations, we reduce to

$$\sum_{N_{max}\sim N_{med}\sim N} \frac{N^{1/2-s}N_{min}^{\epsilon}}{N^{2-4\epsilon}} \lesssim 1,$$

which holds if $s > -3/2$. So (5.6) holds if $s > -3/2$.

Now we show the low modulation case (5.4). In this case, we may assume $L_{max} \sim N_{max}^4 N_{min}$. We first deal with the contribution where (3.9) holds. In this case, $N_1,\, N_2,\, N_3 \sim N \gtrsim 1$, so we reduce to

$$\sum_{L_{max} \sim N^5} \frac{N^{-s} N^{1/2}}{L_{min}^{1/2+\epsilon} L_{med}^{1/2-\epsilon}} L_{min}^{1/2} N^{-2} L_{med}^{1/2} \lesssim 1. \tag{5.7}$$

Performing the L summations, we reduce to

$$\frac{N^{5\epsilon}}{N^{3/2+s}} \lesssim 1,$$

which holds if $s > -3/2$.

Now we deal with the cases where (3.10) holds. Because of the lack of symmetry, we need to consider three cases:

$$\begin{aligned} &N \sim N_1 \sim N_2 \gg N_3; \quad H \sim L_3 \gtrsim L_1,\, L_2 \\ &N \sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2,\, L_3 \\ &N \sim N_1 \sim N_3 \gg N_2; \quad H \sim L_2 \gtrsim L_1,\, L_3. \end{aligned}$$

In the first case, we reduce by (3.10) to

$$\sum_{N_3 \ll N} \sum_{1 \lesssim L_1,\, L_2 \lesssim N^4 N_3} \frac{N^{1/2-s}}{L_{min}^{1/2+\epsilon} L_{med}^{1/2-\epsilon}} L_{min}^{1/2} N^{-2} \min\Big(N^4 N_3,\, \frac{N}{N_3} L_{med}\Big)^{1/2} \lesssim 1. \tag{5.8}$$

Performing the N_3 summation, we reduce to

$$\sum_{1 \lesssim L_1,\, L_2 \lesssim N^4 N_3} \frac{N^{1/2-2-s+5/4}}{L_{min}^{1/2+\epsilon} L_{med}^{1/2-\epsilon}} L_{min}^{1/2} L_{med}^{1/4} \lesssim 1,$$

which holds if $s \geq -1/4$.

Next we deal with the second case $N \sim N_2 \sim N_3 \gg N_1$; $H \sim L_1 \gtrsim L_2,\, L_3$. In this case, we use the first half of (3.10) and reduce to

$$\sum_{N_1 \ll N} \sum_{1 \lesssim L_2,\, L_3 \lesssim N^4 N_1} \frac{N^{1/2}}{\langle N_1 \rangle^s L_2^{1/2-\epsilon} (N^4 N_1)^{1/2+\epsilon}} L_{min}^{1/2} N_1^{1/2} \lesssim 1, \tag{5.9}$$

which holds if $s > -1$.

In the third case $N \sim N_1 \sim N_3 \gg N_2$; $H \sim L_2 \gtrsim L_1,\, L_3$, we similarly reduce by using the first half of (3.10) to

$$\sum_{N_2 \ll N} \sum_{1 \lesssim L_1,\, L_3 \lesssim N^4 N_2} \frac{\langle N_2 \rangle^{1/2}}{N^s L_1^{1/2+\epsilon} (N^4 N_2)^{1/2-\epsilon}} L_{min}^{1/2} N_2^{1/2} \lesssim 1, \tag{5.10}$$

which holds if $s > -3/2$.

To finish the proof of (5.4), it remains to deal with the cases where (3.11) holds. This reduces to

$$\sum_{N_{max}\sim N_{med}\sim N}\sum_{L_{max}\sim N^4 N_{min}}\frac{\langle N_2\rangle^{1/2}}{\langle N_1\rangle^s L_1^{1/2+\epsilon}L_2^{1/2-\epsilon}}L_{min}^{1/2}N^{-2}L_{med}^{1/2}\lesssim 1, \tag{5.11}$$

which is true if $s > -3/2$. This finishes the proof of Lemma 5.2.

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