A CLASS OF LARGE SOLUTIONS TO THE SUPERCRITICAL SURFACE QUASI-GEOSTROPHIC EQUATION

JIANLI LIU[1], KEJIA PAN[2] AND JIAHONG WU[3]

Abstract. Whether or not classical solutions to the surface quasi-geostrophic (SQG) equation can develop finite time singularities remains an outstanding open problem. This paper constructs a class of large global-in-time classical solutions to the SQG equation with supercritical dissipation. The construction process presented here implies that any solution of the supercritical SQG equation must be globally regular if its initial data is sufficiently close to a function (measured in a Sobolev norm) whose Fourier transform is supported in a suitable region away from the origin.

1. Introduction

The goal of this paper is to construct a class of large solutions to the surface quasi-geostrophic (SQG) equation with supercritical dissipation. The SQG equation concerned here is given by

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^{\alpha} \theta = 0, & x \in \mathbb{R}^2,\, t > 0, \\ u = \nabla^{\perp} \psi := \left(-\partial_{x_2} \psi, \partial_{x_1} \psi\right), & (-\Delta)^{\frac{1}{2}} \psi = \theta, \end{cases} \tag{1.1}$$

where the scalar function θ represents the potential temperature, u the fluid velocity and $\kappa > 0$ and $\alpha \geq 0$ are real parameters. Here the nonlocal operator $(-\Delta)^{\alpha}$ is determined through the Fourier transform

$$\widehat{(-\Delta)^{\alpha} f}(\xi) = |\xi|^{2\alpha}\, \widehat{f}(\xi),$$

where $\widehat{f}$ denotes the Fourier transform of f,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

The SQG equation has attracted enormous attention recently due to its applications in modeling geophysical fluids and its significance in the theory of partial differential equations (see, e.g., [1, 16, 22]). As detailed in the paper of Constantin, Majda and Tabak [5], the behavior of its strongly nonlinear solutions are strikingly analogous to that of the potentially singular solutions of the 3D incompressible Navier-Stokes and Euler equations. The study of this 2D model may shed light on the mystery surrounding the 3D hydrodynamics equations. Significant progress has been made on the global well-posedness and related problems on the SQG equation.

2010 *Mathematics Subject Classification.* 35K55, 76C99, 76U05, 86A10 .
Key words and phrases. Surface quasi-geostrophic equation; supercritical dissipation; large solutions.

Our attention will be focused on the SQG equation with supercritical dissipation or damping.

The level of difficulty involved in the global existence and smoothness issue on the dissipative QG equation is dictated by the parameter α. In the subcritical case $\alpha > \frac{1}{2}$, the SQG equation has been shown to possess a unique global smooth solution for every sufficiently smooth initial data (see [7],[23]). Furthermore, smooth solutions are shown to be real analytic and their wave numbers are known to decay exponentially [27]. When $\alpha \leq \frac{1}{2}$, the issue of global existence and smoothness becomes extremely difficult. The investigation of the critical case $\alpha = \frac{1}{2}$ started with the paper of Constantin, Córdoba and Wu [3], in which they proved the global existence and uniqueness of classical solutions corresponding to any initial data with L^∞-norm comparable to or less than the diffusion coefficient κ. The critical case $\alpha = \frac{1}{2}$ was successfully solved by Kiselev, Nazarov and Volberg [19] and by Caffarelli and Vasseur [2]. Important nonlinear inequalities involving the fractional Laplace operator and several different proofs were also developed ([10, 6, 20]).

The global existence and smoothness issue for the supercritical case $\alpha < \frac{1}{2}$ remains open. The two papers of Constantin and Wu [8, 9] assert that any Leray-Hopf weak solution of the supercritical SQG equation is actually essentially bounded and any weak solution in the Hölder class $C^{1-2\alpha}$ is actually a smooth solution of the SQG equation. Dong and Pavlovic extended the Hölder space $C^{1-2\alpha}$ to more general Besov setting [15]. Therefore, to completely resolve the global existence and smoothness issue for the supercritical SQG equation, it remains to show that any L^∞-weak solution of the supercritical SQG equation is actually in the Hölder class $C^{1-2\alpha}$. There are substantial more recent developments on the slightly supercritical SQG equation and the supercritical SQG equation (see, e.g., [11, 12, 13, 24, 25]).

This paper examines the open global regularity problem on the supercritical SQG equation from a different perspective. Our goal here is to construct a class of large solutions of (1.1) with α in the supercritical regime $0 \leq \alpha < \frac{1}{2}$. The solutions constructed here belong to the Sobolev space $H^s(\mathbb{R}^2)$ with $s + \alpha > 2$, a natural setup for the SQG equation. The requirement $s > 2 - \alpha$ appears to be the minimal in order to insure the local wellposedness. When $\alpha = 0$, the term $\kappa(-\Delta)^\alpha\theta$ is reduced to the damping term $\kappa\theta$. We remark that this process is also valid for the cases when $\alpha \geq \frac{1}{2}$, even though our focus is on the case $\alpha < \frac{1}{2}$. As can be seen from the description below, the process of construction presented here actually states that any solution of the supercritical SQG equation close to that of the corresponding fractional heat equation with Fourier transform supported away from the origin must be globally regular in time. The closeness is measured in the norm of the Sobolev space $H^s(\mathbb{R}^2)$ with $s + \alpha > 2$.

The large solution of (1.1) we are seeking has the form

$$\theta := \Theta + h \tag{1.2}$$

with Θ solving the linear part of (1.1), namely

$$\begin{cases} \partial_t \Theta + \kappa(-\Delta)^\alpha \Theta = 0, \\ \Theta(x,0) = \Theta_0(x). \end{cases} \tag{1.3}$$

The initial data Θ_0 is in the Schwartz class $\mathcal{S}$ and has the following properties

$$\widehat{\Theta}_0(\xi) \in C_0^\infty(\mathbb{R}^2), \qquad \operatorname{supp} \widehat{\Theta}_0 \subset \left\{ \xi \in \mathbb{R}^2, 1-\delta \le |\xi| \le 1+\delta \right\},$$

where $0 < \delta < \frac{1}{2}$ is a small parameter depending on κ only. More precise information on δ will be specified later. A particular example of $\widehat{\Theta}_0$ is given by

$$\widehat{\Theta}_0(\xi) = \left(\delta^{-\frac{1}{2}} \log \delta \right) \gamma(\xi) \tag{1.4}$$

with $\gamma \in C_0^\infty(\mathbb{R}^2)$ being a smooth cutoff, namely

$$\gamma(\xi) = \begin{cases} 0 & \text{if } |\xi| \le 1-\delta \text{ or } |\xi| \ge 1+\delta, \\ 1 & \text{if } 1 - \frac{3}{2}\delta \le |\xi| \le 1 + \frac{1}{2}\delta. \end{cases}$$

The norm $\|\Theta_0\|_{H^s}$ is not small. In fact,

$$\begin{aligned} \|\Theta_0\|_{H^s} &:= \left[\int_{\mathbb{R}^2} (1+|\xi|^2)^s |\Theta_0(\xi)|^2 \, d\xi \right]^{\frac{1}{2}} \\ &\ge \delta^{-\frac{1}{2}} \, |\log \delta| \left[\int_{1-\frac{3}{2}\delta \le |\xi| \le 1 + \frac{1}{2}\delta} (1+|\xi|^2)^s \, d\xi \right]^{\frac{1}{2}} \\ &\ge \delta^{-\frac{1}{2}} \, |\log \delta| \, [(2\pi\delta(2-\delta)]^{\frac{1}{2}} \\ &\ge C \, |\log \delta| \, , \end{aligned}$$

which can be really big when $\delta > 0$ is small. The solution Θ of (1.3) is given by

$$\widehat{\Theta}(\xi, t) = e^{-\kappa t |\xi|^{2\alpha}} \, \widehat{\Theta_0}(\xi), \tag{1.5}$$

which, due to the support of $\widehat{\Theta_0}$, satisfies

$$|\widehat{\Theta}(\xi,t)| \le e^{-C_0 \, t} |\widehat{\Theta_0}(\xi)|, \quad C_0 := \kappa 4^{-\alpha}.$$

We can easily find the equation of h by substituting (1.2) in (1.1),

$$\begin{cases} \partial_t h + v \cdot \nabla h + \kappa(-\Delta)^\alpha h = -U \cdot \nabla h - v \cdot \nabla \Theta - U \cdot \nabla \Theta, \\ v = \nabla^\perp \Lambda^{-1} h, \\ h(x,0) = h_0(x), \end{cases} \tag{1.6}$$

where $\Lambda = \sqrt{-\Delta}$ and U is the corresponding velocity associated with Θ, namely

$$U = \nabla^\perp \Lambda^{-1} \Theta. \tag{1.7}$$

The main effort is devoted to establishing the small data global well-posedness for (1.6) in the aforementioned functional setting $H^s(\mathbb{R}^2)$. We are able to prove the following theorem.

Theorem 1.1. *Assume Θ_0, Θ and U are given by (1.4), (1.5) and (1.7), respectively. Let $0 \le \alpha < \frac{1}{2}$. Let $s > 2-\alpha$. Assume $h_0 \in H^s(\mathbb{R}^2)$. Then there exists a pure constant C_1 depending on α only such that, if*

$$\|h_0\|_{H^s} \le C_1 \kappa,$$

then (1.6) has a unique global solution h satisfying

$$h \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,\infty; H^{s+\alpha}(\mathbb{R}^2)),$$
$$\|h(t)\|_{H^s} \le C_2 \kappa,$$

where C_2 is a pure constant independent of κ.

We make a remark on the possibility of replacing the functional space H^s with $s > 2-\alpha$ in our Theorem 1.1 by the scaling invariant space $\dot{H}^{2-2\alpha}$. The Sobolev space H^s with $s > 2-\alpha$ arises naturally from the energy estimates. When we perform the L^2-based energy estimates on H^s-norm of h (as in the proof of Theorem 1.1), the dissipative term becomes $\|\Lambda^{\alpha+s} h\|^2_{L^2(\mathbb{R}^2)}$ and the bound for the nonlinear term in general contains $\|\nabla h\|_{L^\infty(\mathbb{R}^2)}$. To bound the nonlinear term suitably by the dissipative term, we need $\alpha + s > 2$ or $s > 2-\alpha$. This is why we need H^s with $s > 2-\alpha$. In order to lower the functional setting to the critical space $\dot{H}^{2-2\alpha}(\mathbb{R}^2)$, we need to employ some heavy machinery such as the Littlewood-Paley decomposition and Besov type space techniques (see [14]). We may have difficulty in implementing this method here due to the presence of three extra terms involving Θ or U, in addition to the standard nonlinear term $v \cdot \nabla h$. These terms do not share the same properties as $v \cdot \nabla h$.

Coti Zelati and Vicol in [11] were able to establish the continuity of the solution map of the supercritical SQG equation (1.1) with respect to α as $\alpha \to \frac{1}{2}$. Their result involves a scaling invariant quantity and a natural issue is whether or not we could replace the norm $\|h\|_{H^s}$ in Theorem 1.1 with the corresponding scaling invariant quantity $\|h\|^a_{L^2} \|h\|^{1-a}_{\dot{H}^2}$? After reviewing the estimates on the terms in (2.2), we find that it would be extremely difficult to do so. The reason is that three terms in (2.2) each contains U or Θ and it appears hard to bound them by the combined quantity $\|h\|^a_{L^2} \|h\|^{1-a}_{\dot{H}^2}$.

In the proof of Theorem 1.1, the support of the Fourier transform of Θ is taken to be near the unit circle. The construction process still works even when the unit circle is changed to any curve away from the origin. Therefore a slight modification of the proof presented here allows us to reach the following conclusion.

Corollary 1.2. *Let $0 \le \alpha < \frac{1}{2}$. Let Θ denote a solution of the corresponding fractional heat equation of (1.1), namely $\partial_t \Theta + \kappa(-\Delta)^\alpha \Theta = 0$. Assume that the Fourier transform of Θ is supported in a suitable region away from the origin. Then any solution of (1.1) with $0 \le \alpha < \frac{1}{2}$ that is close to Θ in the space $H^s(\mathbb{R}^2)$ with $s > 2-\alpha$ is globally regular in time.*

We remark that Theorem 1.1 and Corollary 1.2 can be extended to the generalized SQG equation (see, e.g., [4, 21])

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \kappa(-\Delta)^{\alpha}\theta = 0, \quad x\in\mathbb{R}^2,\, t>0, \\ u=(u_1,u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi)\,, \qquad (-\Delta)^{\beta}\psi = \theta, \end{cases}$$

where $\alpha \geq 0$ and $\beta \in [1,2]$ are real parameters. We omit further details due to the similarities. The rest of this paper proves Theorem 1.1.

2. Proof of Theorem 1.1

This section proves Theorem 1.1. To prepare for the proof, we state several bounds for Θ and U.

Lemma 2.1. *Assume Θ and U are given by (1.5) and (1.7) with Θ_0 defined by (1.4). Then, for any $b>0$ and any $0<\sigma<\frac{1}{2}$,*

$$\begin{aligned} &\|\Lambda^b\Theta(t)\|_{L^\infty} \leq C\,\delta^{\frac12-\sigma}\,e^{-C_0t} \\ &\|\Lambda^b\Theta(t)\|_{L^2} \leq C\,\delta^{-\sigma}\,e^{-C_0t}, \\ &\|\Lambda^b(U-\nabla^\perp\Theta)\|_{L^2} \leq\, C\,\delta^{1-\sigma}\,e^{-C_0t}. \end{aligned}$$

Proof of Lemma 2.1. By (1.4) and (1.5),

$$\begin{aligned} \|\Lambda^b\Theta\|_{L^\infty} &\leq \int_{\mathbb{R}^2} |\xi|^b|\Theta(\xi)|\,d\xi \\ &\leq \delta^{-\frac12}\,|\log\delta|\,e^{-C_0t}\int_{1-\delta\leq|\xi|\leq 1+\delta}|\xi|^b\xi \\ &\leq \delta^{-\frac12}\,|\log\delta|\,e^{-C_0t}\,4\pi\delta(1+\delta)^b \\ &\leq C\,\delta^{\frac12}\,|\log\delta|\,e^{-C_0t} \\ &\leq C\,\delta^{\frac12-\sigma}\,e^{-C_0t}, \end{aligned}$$

where we have used the fact that $\delta^\sigma|\log\delta|\leq C$. The proof of the second bound is similar. In fact, we can show that

$$\|\Lambda^b\Theta\|_{L^2} \leq C\,|\log\delta|\,e^{-C_0t} \leq C\,\delta^{-\sigma}\,e^{-C_0t}.$$

By (1.4), (1.5) and (1.7),

$$\begin{aligned} \|\Lambda^b(U-\nabla^\perp\Theta)\|_{L^2}^2 &= \|\Lambda^b(\nabla^\perp\Lambda^{-1}\Theta-\nabla^\perp\Theta)\|_{L^2}^2 \\ &= \int_{\mathbb{R}^2}|\xi|^{2b}|\xi^\perp|\xi|^{-1}\widehat{\Theta}(\xi)|^2(1-|\xi|)^2\,d\xi \\ &\leq (1+\delta)^{2b}\,\delta^2\,\delta^{-1}|\log\delta|^2\,e^{-2C_0t}\,4\pi\delta \\ &= C\,\delta^2\,|\log\delta|^2\,e^{-2C_0t} \\ &\leq C\,\delta^{2-2\sigma}\,e^{-2C_0t}. \end{aligned}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Assume Θ and U are given by (1.5) and (1.7) with Θ_0 defined by (1.4). Let $b \geq 0$. Let $2 \leq q \leq \infty$. Then, for any $0 < \sigma < \frac{1}{2}$,*

$$\|\Lambda^b \Theta\|_{L^q} \leq C\, \delta^{\frac{1}{2}-\frac{1}{q}-\sigma}\, e^{-C_0 t}, \quad \|\Lambda^b U\|_{L^q} \leq C\, \delta^{\frac{1}{2}-\frac{1}{q}-\sigma}\, e^{-C_0 t}.$$

The proof of Lemma 2.2 is very similar to that of Lemma 2.1. In addition, the following commutator and bilinear estimates involving fractional derivatives will be used (see, e.g., [17, 18]).

Lemma 2.3. *Let $s > 0$. Let $p, p_1, p_3 \in (1, \infty)$ and $p_2, p_4 \in [1, \infty]$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then there exist two constants C_1 and C_2,

$$\begin{aligned} \|[\Lambda^s, f]g\|_{L^p} &\leq C_1 \left(\|\Lambda^s f\|_{L^{p_1}}\, \|g\|_{L^{p_2}} + \|\Lambda^{s-1} g\|_{L^{p_3}}\, \|\nabla f\|_{L^{p_4}}\right), \\ \|\Lambda^s (f\, g)\|_{L^p} &\leq C_2\, \left(\|\Lambda^s f\|_{L^{p_1}}\, \|g\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}}\, \|f\|_{L^{p_4}}\right). \end{aligned}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The local-in-time well-posedness of (1.6) can be obtained by following a standard procedure (see, e.g., [28]). This proof focuses on the global uniform bound via a bootstrap argument.

We estimate the H^s-norm in two steps. The first step estimates the L^2-norm while the second step bounds the homogeneous $\dot{H}^s$-norm. Taking the inner product of the first equation in (1.6) with h and integrating by parts, we have

$$\begin{aligned} \frac{1}{2}\, \frac{d}{dt} \|h\|_{L^2}^2 + \kappa \|\Lambda^\alpha h\|_{L^2}^2 &= -\int v \cdot \nabla\Theta\, h\, dx - \int U \cdot \nabla\Theta\, h\, dx \\ &:= I_1 + I_2. \end{aligned}$$

By Hölder's inequality and Lemma 2.1,

$$|I_1| \leq \|\nabla\Theta\|_{L^\infty}\, \|v\|_{L^2} \|h\|_{L^2} \leq C\, \delta^{\frac{1}{2}-\sigma}\, e^{-C_0 t}\, \|h\|_{L^2}^2,$$

where we have used the fact that

$$\|v\|_{L^2} = \|\nabla^\perp \Lambda^{-1} h\|_{L^2} = \|h\|_{L^2}.$$

Due to $\nabla\Theta \cdot \nabla^\perp\Theta = 0$,

$$I_2 = -\int (U - \nabla^\perp\Theta) \cdot \nabla\Theta\, h\, dx.$$

By Lemma 2.1,

$$\begin{aligned} |I_2| &\leq \|U - \nabla^\perp\Theta\|_{L^2}\, \|\nabla\Theta\|_{L^\infty}\, \|h\|_{L^2} \\ &\leq C\, \delta^{\frac{3}{2}-2\sigma}\, e^{-2C_0 t}\, \|h\|_{L^2}. \end{aligned}$$

By taking $\sigma = \frac{1}{4}$, we find

$$\frac{d}{dt} \|h\|_{L^2}^2 + 2\kappa \|\Lambda^\alpha h\|_{L^2}^2 \leq C\, \delta^{\frac{1}{4}}\, e^{-C_0 t}\, \|h\|_{L^2}^2 + C\, \delta\, e^{-2C_0 t}\, \|h\|_{L^2}. \tag{2.1}$$

We now estimate the homogeneous $\dot{H}^s$-norm of h. Applying Λ^s to the equation of h in (1.6) and then taking the inner product with $\Lambda^s h$, we find

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s h\|_{L^2}^2 + \kappa\|\Lambda^{\alpha+s}h\|_{L^2}^2 = J_1 + J_2 + J_3 + J_4, \tag{2.2}$$

where

$$\begin{aligned} J_1 &= -\int \Lambda^s h\, \Lambda^s(v\cdot\nabla h)\,dx,\\ J_2 &= -\int \Lambda^s h\, \Lambda^s(v\cdot\nabla \Theta)\,dx,\\ J_3 &= -\int \Lambda^s h\, \Lambda^s(U\cdot\nabla h)\,dx,\\ J_4 &= -\int \Lambda^s h\, \Lambda^s(U\cdot\nabla \Theta)\,dx. \end{aligned}$$

J_1, J_2, J_3 and J_4 can be estimated as follows. Using the fact that $\nabla\cdot v = 0$, we rewrite the integral in the form of a commutator,

$$J_1 = -\int \Lambda^s h\,[\Lambda^s, v\cdot\nabla]h\,dx,$$

where $[\Lambda^s, v\cdot\nabla]h = \Lambda^s(v\cdot\nabla h) - v\cdot\nabla\Lambda^s h$. By Lemma 2.3,

$$|J_1| \le C\,\|\Lambda^s h\|_{L^2}\left(\|\Lambda^s v\|_{L^2}\|\nabla h\|_{L^\infty} + \|\nabla v\|_{L^\infty}\,\|\Lambda^s h\|_{L^2}\right).$$

By Gagliardo-Nirenberg's inequality,

$$\begin{aligned} \|\nabla h\|_{L^\infty} &\le C\,\|\Lambda^\alpha h\|_{L^2}^{\frac{\alpha+s-2}{s}}\,\|\Lambda^{\alpha+s}h\|_{L^2}^{\frac{2-\alpha}{s}},\\ \|\Lambda^s h\|_{L^2} &\le C\,\|\Lambda^\alpha h\|_{L^2}^{\frac{\alpha}{s}}\,\|\Lambda^{\alpha+s}h\|_{L^2}^{1-\frac{\alpha}{s}}. \end{aligned}$$

Similar inequalities hold for v. Therefore,

$$\begin{aligned} |J_1| &\le C\,\|\Lambda^s h\|_{L^2}\,\|\Lambda^\alpha h\|_{L^2}^{\frac{\alpha+s-2}{s}+\frac{\alpha}{s}}\,\|\Lambda^{\alpha+s}h\|_{L^2}^{\frac{2-\alpha}{s}+1-\frac{\alpha}{s}}\\ &\le C\,\|\Lambda^s h\|_{L^2}\left(\|\Lambda^\alpha h\|_{L^2}^2 + \|\Lambda^{\alpha+s}h\|_{L^2}^2\right), \end{aligned}$$

where we have used the simple facts that $\|\Lambda^\alpha v\|_{L^2} = \|\Lambda^\alpha h\|_{L^2}$ and $\|\Lambda^{\alpha+s}v\|_{L^2} = \|\Lambda^{\alpha+s}h\|_{L^2}$. By Lemmas 2.1, 2.2 and 2.3, for $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ with $p>2$ but close to 2,

$$\begin{aligned} |J_2| &\le \|\Lambda^s h\|_{L^2}\left(\|\Lambda^s v\|_{L^2}\|\nabla\Theta\|_{L^\infty} + \|v\|_{L^p}\,\|\Lambda^s\nabla\Theta\|_{L^q}\right)\\ &\le C\,\|\Lambda^s h\|_{L^2}^2\,\delta^{\frac{1}{2}-\sigma}\,e^{-C_0 t} + C\,\|\Lambda^s h\|_{L^2}\,\|v\|_{H^s}\,\delta^{\frac{1}{2}-\frac{1}{q}-\sigma}\,e^{-C_0 t} \end{aligned}$$

where we have used the embedding inequality $\|v\|_{L^p} \le C\,\|v\|_{H^s}$. By setting $q=8$ and $\sigma=\frac{1}{4}$, we find

$$|J_2| \le C\,\delta^{\frac{1}{8}}\,e^{-C_0 t}\,\|h\|_{H^s}^2.$$

Due to $\nabla\cdot U = 0$,

$$J_3 = -\int \Lambda^s h\,[\Lambda^s, U\cdot\nabla]h\,dx.$$

As in the estimate of J_2, by Lemmas 2.1, 2.2 and 2.3,

$$|J_3| \le C\,\|\Lambda^s h\|_{L^2}^2\,\|\nabla U\|_{L^\infty} + C\,\|\Lambda^s h\|_{L^2}\,\|\nabla h\|_{L^p}\,\|\Lambda^s U\|_{L^q}$$

$$\leq \quad C\,\delta^{\frac{1}{8}}\,e^{-C_0 t}\,\|h\|_{H^s}^2.$$

By Lemma 2.1, 2.2 and 2.3,

$$\begin{aligned}
|J_4| &\leq \|\Lambda^s h\|_{L^2}\,\|\Lambda^s((U-\nabla^\perp\Theta)\cdot\nabla\Theta)\|_{L^2}\\
&\leq C\,\|\Lambda^s h\|_{L^2}\,\|\Lambda^s(U-\nabla^\perp\Theta)\|_{L^2}\,\|\nabla\Theta\|_{L^\infty}\\
&\quad +C\,\|\Lambda^s h\|_{L^2}\,\|\nabla(U-\nabla^\perp\Theta)\|_{L^p}\|\Lambda^s\nabla\Theta\|_{L^q}\\
&\leq C\,\|\Lambda^s h\|_{L^2}\,\delta^{\frac{3}{2}-2\sigma}\,e^{-2C_0t}+C\,\|\Lambda^s h\|_{L^2}\,\|\nabla(U-\nabla^\perp\Theta)\|_{H^s}\,\delta^{\frac{1}{2}-\frac{1}{q}-\sigma}\,e^{-C_0t}\\
&\leq C\,\|\Lambda^s h\|_{L^2}\,\delta^{\frac{3}{2}-2\sigma}\,e^{-2C_0t}+C\,\|\Lambda^s h\|_{L^2}\,\delta^{2-\sigma-\frac{1}{q}-\frac{1}{2}-\sigma}\,e^{-2C_0t}
\end{aligned}$$

By setting $q=8$ and $\sigma=\frac{1}{4}$, we have

$$|J_4|\leq C\,\|\Lambda^s h\|_{L^2}\,\delta^{\frac{7}{8}}\,e^{-2C_0t}.$$

Inserting the bounds for J_1 through J_4 above in (2.2), we obtain

$$\begin{aligned}
\frac{d}{dt}\|\Lambda^s h\|_{L^2}^2+2\kappa\|\Lambda^{\alpha+s}h\|_{L^2}^2 &\leq C\,\|\Lambda^s h\|_{L^2}\,\left(\|\Lambda^\alpha h\|_{L^2}^2+\|\Lambda^{\alpha+s}h\|_{L^2}^2\right)\\
&\quad +C\,\delta^{\frac{1}{8}}\,e^{-C_0t}\,\|h\|_{H^s}^2+C\,\delta^{\frac{7}{8}}\,e^{-2C_0t}\|\Lambda^s h\|_{L^2}. \qquad (2.3)
\end{aligned}$$

Adding (2.1) and (2.3) leads to

$$\begin{aligned}
&\frac{d}{dt}\|h\|_{H^s}^2+(2\kappa-C_3\,\|\Lambda^s h\|_{L^2})\left(\|\Lambda^\alpha h\|_{L^2}^2+\|\Lambda^{\alpha+s}h\|_{L^2}^2\right)\\
&\leq C_4\,\delta^{\frac{1}{8}}\,e^{-C_0t}\,\|h\|_{H^s}^2+C_5\,\delta^{\frac{7}{8}}\,e^{-2C_0t}\|\Lambda^s h\|_{L^2}. \qquad (2.4)
\end{aligned}$$

We apply the bootstrap argument to (2.4) to establish that $\|h(t)\|_{H^s}$ remains uniform bounded if $\|h_0\|_{H^s}$ is taken to be sufficiently small. The bootstrap argument starts with an ansatz that $\|h(t)\|_{H^s}$ is bounded, say

$$\|h(t)\|_{H^s}\leq M$$

and shows that $\|h(t)\|_{H^s}$ actually admits a smaller bound, say

$$\|h(t)\|_{H^s}\leq\frac{1}{2}M$$

when $\|h_0\|_{H^s}$ is sufficiently small. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book (see [26, p.21]). To apply the bootstrap argument to (2.4), we assume that

$$\|h(t)\|_{H^s}\leq M:=\frac{2\kappa}{C_3}\quad\text{or}\quad 2\kappa-C_3\,\|\Lambda^s h\|_{L^2}\geq 0.$$

It then follows from (2.4) that

$$\frac{d}{dt}\|h\|_{H^s}^2\leq C_4\,\delta^{\frac{1}{8}}\,e^{-C_0t}\,\|h\|_{H^s}^2+C_5\,\delta^{\frac{7}{8}}\,e^{-2C_0t}\|\Lambda^s h\|_{L^2}.$$

By Gronwall's inequality,

$$\begin{aligned}
\|h(t)\|_{H^s} &\leq e^{C_4\,\delta^{\frac{1}{8}}\,\int_0^t e^{-C_0\tau}d\tau}\left(\|h_0\|_{H^s}+\int_0^t C_5\,\delta^{\frac{7}{8}}\,e^{-2C_0\tau}\,d\tau\right)\\
&\leq M_1\,\|h_0\|_{H^s}+M_1\,\delta^{\frac{7}{8}}, \qquad (2.5)
\end{aligned}$$

where

$$M_1 = \max\left\{ e^{C_4 C_0^{-1} \delta^{\frac{1}{8}}},\ \frac{1}{2} C_5 C_0^{-1} e^{C_4 C_0^{-1} \delta^{\frac{1}{8}}} \right\}.$$

If h_0 and δ satisfies

$$\|h_0\|_{H^s} \le \frac{\kappa}{2M_1 C_3}, \qquad \delta \le \left(\frac{\kappa}{2M_1 C_3} \right)^{\frac{8}{7}}, \tag{2.6}$$

then (2.5) implies

$$\|h(t)\|_{H^s} \le M_1 \frac{\kappa}{2M_1 C_3} + M_1 \frac{\kappa}{2M_1 C_3} = \frac{\kappa}{C_3} = \frac{M}{2}.$$

The bootstrap argument then implies that, for all $t > 0$,

$$\|h(t)\|_{H^s} \le \frac{\kappa}{C_3}$$

when h_0 and δ satisfy (2.6). This completes the proof of Theorem 1.1. □

Acknowledgments

The authors thank the referees for their constructive and insightful reports that have helped improved the manuscript. J. Liu was partially supported by the National Natural Science Foundation of China (No.11771274, 11401367). K. Pan was partially supported by the National Natural Science Foundation of China (No. 41474103), the Excellent Youth Foundation of Hunan Province of China (No. 2018JJ1042) and the Innovation-Driven Project of Central South University (No. 2018CX042). J. Wu was partially supported by the National Science Foundation of the United States under the grant DMS-1614246 and by the AT&T Foundation at the Oklahoma State University, USA.

References

[1] W. Blumen, Uniform potential vorticity flow, Part I. Theory of wave interactions and two-dimensional turbulence, *J. Atmos. Sci.* **35** (1978), 774-783.

[2] L. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Annals of Math.* **171** (2010), 1903-1930.

[3] P. Constantin, D. Cordoba and J. Wu, On the critical dissipative quasi-geostrophic equation, *Indiana Univ. Math. J.* **50** (2001), 97-107.

[4] P. Constantin, G. Iyer and J. Wu, Global regularity for a modified critical dissipative quasi-geostrophic equation, *Indiana Univ. Math. J.* **57** (2008), 2681-2692.

[5] P. Constantin, A. Majda, and E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, *Nonlinearity* **7** (1994), 1495-1533.

[6] P. Constantin and V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, *Geom. Funct. Anal.* **22** (2012), 1289-1321.

[7] P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, *SIAM J. Math. Anal.* **30** (1999), 937-948.

[8] P. Constantin and J. Wu, Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), 1103-1110.

[9] P. Constantin and J. Wu, Hölder continuity of solutions of supercritical dissipative hydrodynamic transport equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 159-180.
[10] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, *Comm. Math. Phys.* **249** (2004), 511-528.
[11] M. Coti Zelati and V. Vicol, On the global regularity for the supercritical SQG equation, *Indiana Univ. Math. J.* **65** (2016), 535-552.
[12] M. Dabkowski, Eventual regularity of the solutions to the supercritical dissipative quasi-geostrophic equation, *Geom. Funct. Anal.* **21** (2011), 1-13.
[13] M. Dabkowski, A. Kiselev, L. Silvestre, and V. Vicol, Global well-posedness of slightly supercritical active scalar equations, *Analysis and PDE* **7** (2014), 43-72.
[14] H. Dong, Dissipative quasi-geostrophic equations in critical Sobolev spaces: smoothing effect and global well-posedness, *Discrete Contin. Dyn. Syst.* **26** (2010), 1197-1211.
[15] H. Dong and N. Pavlovic, A regularity criterion for the dissipation quasi-geostrophic equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 1607-1619.
[16] I. Held, R. Pierrehumbert, S. Garner, and K. Swanson, Surface quasi-geostrophic dynamics, *J. Fluid Mech.* **282** (1995), 1-20.
[17] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure. Appl. Math.* **41** (1988), 891-907.
[18] C. Kenig, G. Ponce and L. Vega, Well-posedness of the initial value problem for the Korteweg-de-Vries equation, *J. Amer. Math. Soc.* **4** (1991), 323-347.
[19] A. Kiselev, F. Nazarov and A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, *Invent. Math.* **167** (2007), 445-453.
[20] A. Kiselev and F. Nazarov, A variation on a theme of Caffarelli and Vasseur, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **370** (2009), 58-72.
[21] A. Kiselev, L. Ryzhik, Y. Yao and A. Zlatos, Finite time singularity for the modified SQG patch equation, *Ann. of Math.* **184** (2016), 909-948.
[22] J. Pedlosky, "Geophysical fluid dynamics", Springer, New York, 1987.
[23] S. Resnick, Dynamical problems in nonlinear advective partial differential equations, Ph.D. thesis, University of Chicago, 1995.
[24] L. Silvestre, Eventual regularization for the slightly supercritical quasi-geostrophic equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), 693-704.
[25] L. Silvestre, V. Vicol, and A. Zlatos, On the Loss of Continuity for Super-Critical Drift-Diffusion Equations, *Arch. Ration. Mech. Anal.* **207** (2013), 845-877.
[26] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Regional Conference Series in Mathematics, Providence, RI: American Mathematical Society, 2006.
[27] J. Wu, Inviscid limits and regularity estimates for the solutions of the 2-D dissipative quasi-geostrophic equations, *Indiana Univ. Math. J.* **46** (1997), 1113-1124.
[28] J. Wu, Quasi-geostrophic-type equations with initial data in Morrey spaces, *Nonlinearity* **10** (1997), 1409-1420.

[1] Department of Mathematics, Shanghai University, Shanghai 200444, China
E-mail address: jlliu@shu.edu.cn

[2] School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
E-mail address: pankejia@hotmail.com

[3] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States
E-mail address: jiahong.wu@okstate.edu