International Conference on Applications of Mathematics to Nonlinear Sciences,
Electronic Journal of Differential Equations, Conference 24 (2017), pp. 35–46.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GLOBAL REGULARITY RESULTS FOR FOUR SYSTEMS OF 2D MHD EQUATIONS WITH PARTIAL DISSIPATION

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Abstract. This article examines the global well-posedness problem on four closely related systems of the 2D magnetohydrodynamic (MHD) equations with partial dissipation. They all share the same partial dissipation in the equation of the magnetic field **b**, only the vertical magnetic diffusion in the horizontal component and the horizontal magnetic diffusion in the vertical component. When the velocity equation has no fluid viscosity, the global regularity problem is an outstanding open problem. We prove a weak-sensed small data global existence result for the case when there is no fluid viscosity. When the velocity equation involves partial dissipation of the same structure as in the equation of **b**, we show that any L^2 initial datum leads to a unique global solution, which becomes smooth instantaneously. When the partial dissipation in the velocity equation is either in the horizontal or vertical direction, we prove that any H^1 initial datum generates a unique global solution.

1. Introduction

This article concerns the global regularity problem on four closely related systems of the 2D magnetohydrodynamic (MHD) equations with partial dissipation. The first one is the following MHD equation without fluid viscosity

$$\begin{aligned} &\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{b}\cdot\nabla\mathbf{b},\\ &\partial_t b_1 + \mathbf{u}\cdot\nabla b_1 = \eta\partial_{22} b_1 + \mathbf{b}\cdot\nabla u_1,\\ &\partial_t b_2 + \mathbf{u}\cdot\nabla b_2 = \eta\partial_{11} b_2 + \mathbf{b}\cdot\nabla u_2,\\ &\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0, \end{aligned} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2)$ denotes the velocity field, $\mathbf{b} = (b_1, b_2)$ the magnetic field, p the pressure and $\eta > 0$ the magnetic diffusivity. Here we have used ∂_{22} and ∂_{11} to denote the second order partial derivatives in the vertical and horizontal directions, respectively. The model equation in (1.1) is rooted in the standard 2D MHD equations with only magnetic resistivity, namely

$$\begin{aligned} &\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{b}\cdot\nabla\mathbf{b},\\ &\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \eta\Delta\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u},\\ &\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0. \end{aligned} \tag{1.2}$$

2010 *Mathematics Subject Classification.* 35Q35, 35B65, 76B03.
Key words and phrases. 2D MHD equations; global well-posedness; partial dissipation.

Published November 15, 2017.

Equation (1.1) differs from (1.2) in that the resistivity term in (1.1) is only partial $\eta(\partial_{22}b_1, \partial_{11}b_2)$, as opposed to the full resistivity $\eta\Delta\mathbf{b}$ in (1.2).

Equation (1.2) is applicable when the fluid viscosity can be ignored while the role of resistivity is important such as in magnetic reconnection and magnetic turbulence. Magnetic reconnection refers to the breaking and reconnecting of oppositely directed magnetic field lines in a plasma and is at the heart of many spectacular events in our solar system such as solar flares and northern lights (see, e.g., [1, 5, 9, 10]). The mathematical study of (1.2) may help understand the Sweet-Parker model arising in magnetic reconnection theory [9, 10]. The global regularity problem on (1.2) is not completely solved at this moment, although recent efforts on this problem have significantly advanced our understanding (see, e.g., [7, 14]). Global *a priori* bounds in very regular functional settings have been obtained. What is lacking is a bound for the vorticity ω in $L^\infty(0,T;\ L^\infty)$. More details can be found in [7] or a very recent review paper by one of the authors [13].

One goal here is to reduce the resistivity (dissipation) as much as possible and still establish the global existence and regularity of solutions. Resistivity regularizes solutions and helps facilitate the proof of global regularity. As aforementioned, the global regularity problem on (1.2) and (1.1) remains outstandingly open. Our first result shows that (1.1) does possess global small solutions in a weak sense. More precisely, we prove the following theorem.

Theorem 1.1. *Let $\eta > 0$. Consider (1.1) supplemented with the initial data $(\mathbf{u}_0, \mathbf{b}_0)$ satisfying $(\mathbf{u}_0, \mathbf{b}_0) \in H^s$ with $s > 2$ and $\nabla\cdot\mathbf{u}_0 = 0$ and $\nabla\cdot\mathbf{b}_0 = 0$. Then, for any $T > 0$, there exists $\delta = \delta(\eta, T) > 0$ such that, if*

$$\|\mathbf{b}_0\|_{H^s} \le \delta,$$

then (1.1) has a unique solution $(\mathbf{u}, \mathbf{b})$ on $[0,T]$. In addition, $(\mathbf{u}, \mathbf{b})$ satisfies $\mathbf{u} \in L^\infty([0,T]; H^s)$ and

$$\|\mathbf{b}\|_{L^\infty([0,T];H^s)} + \eta\Big(\int_0^T \|\nabla\mathbf{b}(\tau)\|_{H^s}^2\, d\tau\Big)^{1/2} \le C\,\delta$$

for a pure constant C.

Here we have written $(\mathbf{u}_0, \mathbf{b}_0) \in H^s \times H^s$ simply as $(\mathbf{u}_0, \mathbf{b}_0) \in H^s$ for the conciseness of notation. We remark that the smallness condition depends on T and is only imposed on $\mathbf{b}_0$ (not on $\mathbf{u}_0$). A similar result was shown in [7] for the 2D MHD equation with full resistivity, namely (1.2). In order to prove Theorem 1.1 with only partial resistivity, we make use of the special structure of (1.1). A full proof is given in Section 2.

It is currently unknown that (1.1) with a general initial data always possesses a unique global solution. This is an extremely difficult problem. To help understand this intriguing problem, we explore the existence and regularity of three systems that are closely related to (1.1),

$$\begin{aligned}
\partial_t u_1 + \mathbf{u}\cdot\nabla u_1 &= -\partial_1 p + \nu\partial_{22}u_1 + \mathbf{b}\cdot\nabla b_1,\\
\partial_t u_2 + \mathbf{u}\cdot\nabla u_2 &= -\partial_2 p + \nu\partial_{11}u_2 + \mathbf{b}\cdot\nabla b_2,\\
\partial_t b_1 + \mathbf{u}\cdot\nabla b_1 &= \eta\partial_{22}b_1 + \mathbf{b}\cdot\nabla u_1,\\
\partial_t b_2 + \mathbf{u}\cdot\nabla b_2 &= \eta\partial_{11}b_2 + \mathbf{b}\cdot\nabla u_2,\\
\nabla\cdot\mathbf{u} = 0, &\quad \nabla\cdot\mathbf{b} = 0;
\end{aligned} \tag{1.3}$$

$$
\begin{aligned}
&\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\partial_{11}\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b},\\
&\partial_t b_1 + \mathbf{u}\cdot\nabla b_1 = \eta\partial_{22} b_1 + \mathbf{b}\cdot\nabla u_1,\\
&\partial_t b_2 + \mathbf{u}\cdot\nabla b_2 = \eta\partial_{11} b_2 + \mathbf{b}\cdot\nabla u_2,\\
&\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0
\end{aligned}
\tag{1.4}
$$

and

$$
\begin{aligned}
&\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\partial_{22}\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{b},\\
&\partial_t b_1 + \mathbf{u}\cdot\nabla b_1 = \eta\,\partial_{22} b_1 + \mathbf{b}\cdot\nabla u_1,\\
&\partial_t b_2 + \mathbf{u}\cdot\nabla b_2 = \eta\partial_{11} b_2 + \mathbf{b}\cdot\nabla u_2,\\
&\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{b} = 0,
\end{aligned}
\tag{1.5}
$$

where $\nu > 0$ is a real parameter. We show that (1.3) supplemented with any L^2-initial data $(\mathbf{u}_0, \mathbf{b}_0)$ always possesses a unique global strong solution.

Theorem 1.2. *Let $\nu > 0$ and $\eta > 0$. Consider (1.3) with the initial data $(\mathbf{u}_0, \mathbf{b}_0) \in L^2$, and $\nabla\cdot\mathbf{u}_0 = 0$ and $\nabla\cdot\mathbf{b}_0 = 0$. Then, (1.3) has a unique global strong solution $(\mathbf{u}, \mathbf{b})$ satisfying*

$$
(\mathbf{u}, \mathbf{b}) \in L^\infty(0,\infty; L^2), \quad \partial_2 u_1, \partial_1 u_2, \partial_2 b_1, \partial_1 b_2 \in L^2(0,\infty; L^2). \tag{1.6}
$$

In addition, for any $t_0 > 0$, the solution $(\mathbf{u}, \mathbf{b})$ becomes infinitely smooth on $[t_0, \infty)$, namely

$$
(\mathbf{u}, \mathbf{b}) \in C^\infty(\mathbb{R}^2 \times [t_0, \infty)). \tag{1.7}
$$

The key point of Theorem 1.2 is that $(\mathbf{u}_0, \mathbf{b}_0)$ is merely required to be in L^2 and the solution of (1.3) emanating from this data is unique and becomes infinitely smooth instantaneously.

We are also able to establish the global existence and regularity for both (1.4) and (1.5) when the initial data $(\mathbf{u}_0, \mathbf{b}_0) \in H^1$. In addition, the H^1-level solutions are unique.

Theorem 1.3. *Let $\nu > 0$ and $\eta > 0$. Consider (1.4) or (1.5) with the initial data $(\mathbf{u}_0, \mathbf{b}_0) \in H^1$, and $\nabla\cdot\mathbf{u}_0 = 0$ and $\nabla\cdot\mathbf{b}_0 = 0$. Then, (1.4) has a unique global strong solution (u, b) satisfying*

$$
(\mathbf{u}, \mathbf{b}) \in L^\infty(0,\infty;\, H^1), \quad \partial_1\nabla\mathbf{u},\ \Delta\mathbf{b} \in L^2(0,\infty;\, L^2) \tag{1.8}
$$

and (1.5) has a unique global strong solution $(\mathbf{u}, \mathbf{b})$ satisfying

$$
(\mathbf{u}, \mathbf{b}) \in L^\infty(0,\infty;\, H^1), \quad \partial_2\nabla\mathbf{u},\ \Delta\mathbf{b} \in L^2(0,\infty;\, L^2).
$$

Theorems 1.2 and 1.3 contribute to the global well-posedness theory on the MHD equations with partial dissipation. In the last few years there have been substantial developments on the global regularity problem concerning the hydrodynamic equations with partial dissipation. These partially dissipative systems are physically relevant and mathematically important. The MHD equations with partial dissipation have attracted considerable interests and significant progress has been made (see, e.g., [2, 3, 4, 6, 7, 11, 13, 14, 15, 16, 17]). We apologize for not being able to cite all related references simply due to the sheer number of papers available. A more complete list of references can be found in the review paper [13]. Several previous results are especially relevant to what we obtain in this paper. Cao and Wu in [4] established the global regularity for the 2D MHD equations with the mixed partial dissipation given by $\partial_{11}\mathbf{u}$ and $\partial_{22}\mathbf{b}$ (or $\partial_{22}\mathbf{u}$ and $\partial_{11}\mathbf{b}$). Cao, Regmi, Wu and Zheng ([2, 3]) examined the case when the partial dissipation in the 2D MHD equations is in the same direction, namely $\partial_{11}\mathbf{u}$ and $\partial_{11}\mathbf{b}$ (or $\partial_{22}\mathbf{u}$ and $\partial_{22}\mathbf{b}$) and obtained global bounds for high regularity of the solutions, even though a complete solution to the same directional partial dissipation case is lacking. Later Du and Zhou obtained global well-posedness and blowup criteria results for some other partial dissipation cases [6]. Theorem 1.3 proves the global existence and uniqueness at the H^1-level for the partial dissipation case when the dissipation in one component of $\mathbf{b}$ is in one direction while the rest is in the other direction.

The rest of this paper is divided into three sections. Section 2 proves Theorem 1.1, Section 3 proves Theorem 1.2 while Section 4 proves Theorem 1.3.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We make several preparations. First we state the bootstrap argument (see, e.g., Tao [12]).

Lemma 2.1. *Let $T > 0$ and $I = [0, T]$. Let $H(t)$ and $C(t)$ with $t \in I$ be two statements satisfying the following conditions:*

(a) *$C(t)$ holds for at least some $t_0 \in I$;*
(b) *If $C(t)$ holds for some $t_1 \in I$, then $H(t)$ also holds for t_1;*
(c) *If $C(t)$ holds for $t_m \in I$ and $t_m \to t$, then $C(t)$ holds;*
(d) *If $H(t)$ holds for $t \in I$, then $C(t)$ also holds for $t \in I$.*

Then $C(t)$ holds for all $t \in I$.

The continuity argument is a special consequence of Lemma 2.1.

Corollary 2.2. *Let $T > 0$ and $I = [0, T]$. Let $f_0 \geq 0$. Assume $f = f(t)$ is a nonnegative continuous function on I satisfying, for some $C_0 > 0$ and $\beta > 1$,*

$$f(t) \leq f_0 + C_0\,(f(t))^{\beta}.$$

Then, there exists $A = A(C_0, \beta)$ such that, if $f_0 < A$, then $f(t) \leq 2A$ for all $t \in I$.

The following simple Osgood type inequality is used in the proof of Theorem 1.1.

Lemma 2.3. *Let $T > 0$. Let ρ_1 and ρ_2 be non-negative integrable functions on $[0, T]$. Let f be a non-negative measurable function on $[0, T]$ satisfying, for a.e. $t \in [0, T]$,*

$$f(t) \leq \int_0^t \rho_1(\tau)\, f(\tau)\, \ln(1 + f(\tau))\, d\tau + \rho_2(t).$$

Then, for a.e. $t \in [0, T]$,

$$f(t) \leq (1 + f(0))^{e^{G_1(t)}}\, e^{G_2(t)\, e^{G_1(t)}},$$

where

$$G_1(t) = \int_0^t \rho_1(\tau)\, d\tau \quad \textit{and} \quad G_2(t) = \int_0^t \rho_2(\tau)\, d\tau.$$

Let $J = (I - \Delta)^{1/2}$ denote the inhomogeneous differentiation operator. We recall two well-known calculus inequalities. (see, e.g., [8, p.334]).

Lemma 2.4. *Let $s > 0$. Let $p, p_1, p_3 \in (1, \infty)$ and $p_2, p_4 \in [1, \infty]$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then, for two constants C_1 and C_2,

$$\|J^s(f\,g)\|_{L^p} \le C_1\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|J^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}\right),$$
$$\|J^s(f\,g) - fJ^s g\|_{L^p} \le C_2\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|J^{s-1} g\|_{L^{p_3}}\|\nabla f\|_{L^{p_4}}\right).$$

Proof of Theorem 1.1. As we know, the local-in-time existence and uniqueness of solutions follows from a standard approximation process and local energy estimates. Our focus here is on the global existence and regularity and we use the bootstrap argument in Lemma 2.1. Let $T > 0$ be fixed. Let $\gamma > 0$ be suitably chosen (to be specified later). For $t \in [0, T]$, let $H(t)$ and $C(t)$ denote the following statements

$$\|b\|_{L^\infty(0,t;H^s)} + \eta\|b\|_{L^2(0,t;H^{s+1})} \le \gamma, \tag{2.1}$$
$$\|b\|_{L^\infty(0,t;H^s)} + \eta\|b\|_{L^2(0,t;H^{s+1})} \le \frac{\gamma}{2}. \tag{2.2}$$

It is clear that (a), (b) and (c) in Lemma 2.1 hold. It remains to verify (d), that is, to prove (2.2) under the assumption (2.1). When (2.1) holds, we show that ω and $\mathbf{u}$ are regular on $[0, T]$. It follows from the equation of ω, namely

$$\partial_t\omega + \mathbf{u}\cdot\nabla\omega = \mathbf{b}\cdot\nabla j$$

that, for $s > 2$,

$$\begin{aligned}
\|\omega(\cdot,t)\|_{L^\infty} &\le \|\omega_0\|_{L^\infty} + \int_0^t \|\mathbf{b}\cdot\nabla j\|_{L^\infty}\,d\tau \\
&\le \|\mathbf{u}_0\|_{H^s} + \|\mathbf{b}\|_{L^\infty(0,t);H^s)}\int_0^t \|\nabla j\|_{L^\infty}\,d\tau \\
&\le \|\mathbf{u}_0\|_{H^s} + \|\mathbf{b}\|_{L^\infty(0,t);H^s)}\sqrt{t}\Big(\int_0^t \|\nabla \mathbf{b}\|_{H^s}^2\,d\tau\Big)^{1/2} \\
&\le \|\mathbf{u}_0\|_{H^s} + \frac{1}{\eta}\sqrt{t}\,\gamma^2 \equiv C_0(\mathbf{u}_0, t, \gamma),
\end{aligned}$$

where we have invoked (2.1) to obtain the last inequality. As a consequence, $\|\mathbf{u}\|_{H^s}$ is also globally bounded. It follows from the velocity equation that

$$\frac{d}{dt}\|\mathbf{u}\|_{H^s} \le C\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{H^s} + \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{H^s}. \tag{2.3}$$

We bound $\|\nabla\mathbf{u}\|_{L^\infty}$ in terms of the logarithmic Sobolev inequality

$$\|\nabla\mathbf{u}\|_{L^\infty} \le C(1 + \|\mathbf{u}\|_{L^2} + \|\omega\|_{L^\infty}\ln(1 + \|\mathbf{u}\|_{H^s})).$$

Clearly, $\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{H^s}$ is locally time integrable,

$$\int_0^t \|\mathbf{b}\cdot\nabla\mathbf{b}\|_{H^s}\,d\tau \le \frac{1}{\eta}\sqrt{t}\,\gamma^2.$$

Applying Lemma 2.3 yields a global bound on $\|\mathbf{u}\|_{H^s}$,

$$\|\mathbf{u}(t)\|_{H^s} \le (1 + \|\mathbf{u}_0\|_{H^s})^{e^{tC_0}} e^{\left(Ct(1+\|\mathbf{u}_0\|_{L^2}) + \frac{1}{\eta}\,t\,\sqrt{t}\,\gamma^2\right)e^{tC_0}}. \tag{2.4}$$

Taking the curl of the equation of $\mathbf{b}$, we find that $j = \nabla\times\mathbf{b}$ obeys

$$\partial_t j + \mathbf{u}\cdot\nabla j = \eta\,\partial_{111}b_2 - \eta\,\partial_{222}b_1 + \mathbf{b}\cdot\nabla\omega + Q(\nabla\mathbf{u}, \nabla\mathbf{b}), \tag{2.5}$$

where

$$Q(\nabla\mathbf{u}, \nabla\mathbf{b}) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

Applying the differential operator J^{s-1} to (2.5) and then dotting with $J^{s-1}j$, we have

$$\frac{1}{2}\frac{d}{dt}\|j\|_{H^{s-1}}^2 = K_1 + K_2 + K_3 + K_4,$$

where

$$K_1 = \eta \int J^{s-1}(\partial_{111}b_2 - \partial_{222}b_1)\, J^{s-1}j, \quad K_2 = -\int J^{s-1}(\mathbf{u}\cdot\nabla j)\, J^{s-1}j,$$

$$K_3 = \int J^{s-1}(\mathbf{b}\cdot\nabla\omega)\, J^{s-1}j, \quad K_4 = \int J^{s-1}Q(\nabla\mathbf{u},\nabla\mathbf{b})\, J^{s-1}j.$$

Writing $j = \partial_1 b_2 - \partial_2 b_1$ and integrating by parts lead to

$$\begin{aligned} K_1 &= -\eta \int \big((\partial_{11}J^{s-1}b_1)^2 + (\partial_{22}J^{s-1}b_1)^2 + (\partial_{11}J^{s-1}b_2)^2 + (\partial_{22}J^{s-1}b_2)^2\big) \\ &\equiv -\eta\, H(J^{s-1}\mathbf{b}). \end{aligned}$$

Because $\nabla\cdot\mathbf{u} = 0$, we have

$$K_2 = -\int \big(J^{s-1}(\mathbf{u}\cdot\nabla j) - \mathbf{u}\cdot\nabla J^{s-1}j\big)\ J^{s-1}j.$$

By Lemma 2.4,

$$|K_2| \le C\|\mathbf{u}\|_{H^s}\|j\|_{H^{s-1}}^2.$$

Integrating by parts and Hölder's inequality,

$$\begin{gathered} |K_3| \le \|J^{s-1}(\mathbf{b}\omega)\|_{L^2}\|j\|_{\dot H^s} \le C\|\mathbf{u}\|_{H^s}\|b\|_{H^{s-1}}\|j\|_{H^s}, \\ |K_4| \le C\|j\|_{H^{s-1}}\|\mathbf{u}\|_{H^s}\|b\|_{H^s} = C\|\mathbf{u}\|_{H^s}\|j\|_{H^{s-1}}^2. \end{gathered}$$

Furthermore, by Young's inequality,

$$|K_3| \le \frac{\eta}{64}\|j\|_{H^s}^2 + C\|\mathbf{u}\|_{H^s}^2\|b\|_{H^{s-1}}^2.$$

Noticing that, due to $\nabla\cdot\mathbf{b} = 0$,

$$\nabla j = \begin{pmatrix} \Delta b_2 \\ -\Delta b_1 \end{pmatrix},$$

we have

$$\|j\|_{H^s}^2 = \|\nabla j\|_{H^{s-1}}^2 = \|\Delta\mathbf{b}\|_{H^{s-1}}^2 \le 2H(J^{s-1}\mathbf{b}).$$

Therefore,

$$|K_3| \le \frac{\eta}{32}\, H(J^{s-1}\mathbf{b}) + C\|\mathbf{u}\|_{H^s}^2\|b\|_{H^{s-1}}^2.$$

Combining the estimates above and noticing that $\|j\|_{\dot H^{s-1}} = \|\mathbf{b}\|_{H^s}$, we obtain

$$\frac{d}{dt}\|\mathbf{b}\|_{H^s}^2 + \eta\, H(J^{s-1}\mathbf{b}) \le C(1 + \|\mathbf{u}\|_{H^s}^2)\|\mathbf{b}\|_{H^s}^2.$$

Therefore,

$$\|\mathbf{b}\|_{H^s}^2 + \eta\int_0^t H(J^{s-1}\mathbf{b})\, d\tau \le \|\mathbf{b}_0\|_{H^s}^2 e^{\int_0^t (1+\|\mathbf{u}\|_{H^s}^2)\, d\tau}.$$

Recalling the global bound for $\|\mathbf{u}\|_{H^s}$ in (2.4), we can certainly choose $\delta = \delta(T,\eta) > 0$ sufficiently small such that

$$\sup_{0\le\tau\le t}\|\mathbf{b}(\tau)\|_{H^s}^2 \le \frac{\gamma^2}{16} \quad \text{and} \quad \eta\int_0^t H(J^{s-1}\mathbf{b})\, d\tau \le \frac{\gamma^2}{32\eta}.$$

when γ is sufficiently large, say $\gamma > 10\|\mathbf{b}_0\|_{H^s}$. Therefore,

$$\sup_{0\le\tau\le t}\|\mathbf{b}(\tau)\|_{H^s} + \eta\Big(\int_0^t \|\mathbf{b}\|_{H^{s+1}}^2\,d\tau\Big)^{1/2} \le \frac{\gamma}{2}.$$

Therefore, we have verified all conditions of Lemma 2.1. It then follows that, for any $t\in[0,T]$,

$$\sup_{0\le\tau\le t}\|\mathbf{b}(\tau)\|_{H^s} + \eta\left(\int_0^t \|\mathbf{b}\|_{H^{s+1}}^2\,d\tau\right)^{1/2} \le \frac{\gamma}{2},$$

which is the desired global bound that ensures the global existence and regularity. This completes the proof. □

3. Proof of Theorem 1.2

Proof. The proof for the global L^2-bound is easy. Taking the inner product of $(\mathbf{u},\mathbf{b})$ with (1.3), we obtain, after integration by parts and applying the divergence-free condition,

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_{L^2}^2 + \nu\|(\partial_2 u_1,\partial_1 u_2)\|_{L^2}^2 + 2\eta\|(\partial_2 b_1,\partial_1 b_2)\|_{L^2}^2 = 0$$

or

$$\begin{aligned}
&\|(\mathbf{u},\mathbf{b})(t)\|_{L^2}^2 + 2\nu\int_0^t \|(\partial_2 u_1,\partial_1 u_2)\|_{L^2}^2\,d\tau + 2\eta\int_0^t \|(\partial_2 b_1,\partial_1 b_2)\|_{L^2}^2\,d\tau\\
&= \|(\mathbf{u}_0,\mathbf{b}_0)\|_{L^2}^2.
\end{aligned}$$

The uniqueness part is more delicate. Let $(\mathbf{u}^{(1)},\mathbf{b}^{(1)})$ and $(\mathbf{u}^{(2)},\mathbf{b}^{(2)})$ be two solutions of (1.3) satisfying (1.6). Because of $\nabla\cdot\mathbf{u}^{(1)}=0$,

$$\|\nabla\mathbf{u}^{(1)}\|_{L^2}^2 = \|\nabla\times\mathbf{u}^{(1)}\|_{L^2}^2 \le 2\|(\partial_2 u_1^{(1)},\partial_1 u_2^{(1)})\|_{L^2}^2. \tag{3.1}$$

Similarly,

$$\|\nabla\mathbf{u}^{(2)}\|_{L^2}^2 \le 2\|(\partial_2 u_1^{(2)},\partial_1 u_2^{(2)})\|_{L^2}^2, \tag{3.2}$$

$$\|\nabla\mathbf{b}^{(1)}\|_{L^2}^2 \le 2\|(\partial_2 b_1^{(1)},\partial_1 b_2^{(1)})\|_{L^2}^2, \tag{3.3}$$

$$\|\nabla\mathbf{b}^{(2)}\|_{L^2}^2 \le 2\|(\partial_2 b_1^{(2)},\partial_1 b_2^{(2)})\|_{L^2}^2. \tag{3.4}$$

Consider the difference $(\mathbf{u},\mathbf{b})$ between $(\mathbf{u}^{(1)},\mathbf{b}^{(1)})$ and $(\mathbf{u}^{(2)},\mathbf{b}^{(2)})$,

$$\mathbf{u}=\mathbf{u}^{(1)}-\mathbf{u}^{(2)},\quad \mathbf{b}=\mathbf{b}^{(1)}-\mathbf{b}^{(2)},$$

which satisfies

$$\begin{gathered}
\partial_t u_1 + \mathbf{u}^{(1)}\cdot\nabla u_1 + \mathbf{u}\cdot\nabla u_1^{(2)} = -\partial_1 p + \nu\partial_{22}u_1 + \mathbf{b}^{(1)}\cdot\nabla b_1 + \mathbf{b}\cdot\nabla b_1^{(2)},\\
\partial_t u_2 + \mathbf{u}^{(1)}\cdot\nabla u_2 + \mathbf{u}\cdot\nabla u_2^{(2)} = -\partial_2 p + \nu\partial_{11}u_2 + \mathbf{b}^{(1)}\cdot\nabla b_2 + \mathbf{b}\cdot\nabla b_2^{(2)},\\
\partial_t b_1 + \mathbf{u}^{(1)}\cdot\nabla b_1 + \mathbf{u}\cdot\nabla b_1^{(2)} = \eta\partial_{22}b_1 + \mathbf{b}^{(1)}\cdot\nabla u_1 + \mathbf{b}\cdot\nabla u_1^{(2)},\\
\partial_t b_2 + \mathbf{u}^{(1)}\cdot\nabla b_2 + \mathbf{u}\cdot\nabla b_2^{(2)} = \eta\partial_{11}b_2 + \mathbf{b}^{(1)}\cdot\nabla u_2 + \mathbf{b}\cdot\nabla u_2^{(2)},\\
\mathbf{u}(x,0)=0,\quad \mathbf{b}(x,0)=0,
\end{gathered} \tag{3.5}$$

where p represents the difference between the associated pressures. Taking the inner products of $(\mathbf{u},\mathbf{b})$ with (3.5) and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_{L^2}^2 + \nu\|(\partial_2 u_1,\partial_1 u_2)\|_{L^2}^2 + \eta\|(\partial_2 b_1,\partial_1 b_2)\|_{L^2}^2 = I_1+I_2+I_3+I_4,$$

where

$$I_1 = -\int (\mathbf{u}\cdot\nabla)\mathbf{u}^{(2)}\cdot\mathbf{u}, \quad I_2 = \int (\mathbf{b}\cdot\nabla)\mathbf{b}^{(2)}\cdot\mathbf{u},$$
$$I_3 = -\int (\mathbf{u}\cdot\nabla)\mathbf{b}^{(2)}\cdot\mathbf{b}, \quad I_4 = \int (\mathbf{b}\cdot\nabla)\mathbf{u}^{(2)}\cdot\mathbf{b}.$$

These terms can be bounded as follows. By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}|I_1| &\le \|\nabla\mathbf{u}^{(2)}\|_{L^2}\|\mathbf{u}\|_{L^4}^2 \le C\|\nabla\mathbf{u}^{(2)}\|_{L^2}\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\\ &\le \frac{\nu}{64}\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\nabla\mathbf{u}^{(2)}\|_{L^2}^2\|\mathbf{u}\|_{L^2}^2.\end{aligned}$$

The other three terms can be bound similarly, for example,

$$|I_2| \le \frac{\nu}{64}\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{\eta}{64}\|\nabla\mathbf{b}\|_{L^2}^2 + C\|\nabla\mathbf{b}^{(2)}\|_{L^2}^2\|(\mathbf{u},\mathbf{b})\|_{L^2}^2.$$

Invoking (3.1) through (3.4), we obtain

$$\begin{aligned}&\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_{L^2}^2 + \nu\|(\partial_2 u_1, \partial_1 u_2)\|_{L^2}^2 + \eta\|(\partial_2 b_1, \partial_1 b_2)\|_{L^2}^2\\ &\le C(\|\nabla\mathbf{u}^{(2)}\|_{L^2}^2 + \|\nabla\mathbf{b}^{(2)}\|_{L^2}^2)\|(\mathbf{u},\mathbf{b})\|_{L^2}^2.\end{aligned}$$

Gronwall's inequality then implies the desired uniqueness.

Finally we show that, for any $t_0 > 0$, any solution $(\mathbf{u},\mathbf{b})$ of (1.3) satisfying (1.6) is infinitely smooth. As we explained above,

$$\|\nabla\mathbf{u}\|_{L^2}^2 \le 2\|(\partial_2 u_1, \partial_1 u_2)\|_{L^2}^2, \quad \|\nabla\mathbf{b}\|_{L^2}^2 \le 2\|(\partial_2 b_1, \partial_1 b_2)\|_{L^2}^2,$$

or $(\mathbf{u},\mathbf{b}) \in L^2(0,\infty;\dot H^1)$. Then $(\mathbf{u},\mathbf{b})$ is in $\dot H^1$ for almost every $t \in (0,\infty)$. For any $t_0 > 0$, there is $0 < t_1 < t_0$ such that $(u(x,t_1), b(x,t_1)) \in H^1(\mathbb{R}^2)$. Starting with $(u(x,t_1), b(x,t_1))$, we then solve (1.3). The solution (u,b) satisfies

$$(u,b) \in L^\infty(t_1,\infty;H^1)\cap L^2(t_1,\infty;\dot H^2), \tag{3.6}$$

which can be easily verified via energy estimates. (3.6) allows us to further choose $t_2 \in (t_1,t_0)$ such that

$$(u(x,t_2), b(x,t_2)) \in H^2(\mathbb{R}^2).$$

We then solve (1.3) with this H^2 initial datum and repeating the process leads to the desired smoothness. This completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

We need the following anisotropic Sobolev inequality for a triple product (see [4]).

Lemma 4.1. *There exists a constant C such that, for any $f, g, \partial_2 g, h$ and $\partial_1 h$ in $L^2(\mathbb{R}^2)$,*

$$\int |f\,g\,h|\,dx \le C\|f\|_{L^2}\|g\|_{L^2}^{1/2}\|\partial_2 g\|_{L^2}^{1/2}\|h\|_{L^2}^{1/2}\|\partial_1 h\|_{L^2}^{1/2}.$$

Proof of Theorem 1.3. We shall only provide the proof for (1.4) since the proof for (1.5) is very similar. The global H^1 bound follows from energy estimates. The global L^2-bound reads, for any $t > 0$,

$$\|(\mathbf{u}(t),\mathbf{b}(t))\|_{L^2}^2 + 2\nu\int_0^t \|\partial_1\mathbf{u}\|_{L^2}^2\,d\tau + 2\eta\int_0^t \|(\partial_2 b_1, \partial_1 b_2)\|_{L^2}^2\,d\tau = \|(\mathbf{u}_0,\mathbf{b}_0)\|_{L^2}^2.$$

As a special consequence, due to $\nabla\cdot\mathbf{b}=0$, we have the global uniform bound

$$\int_0^t \|\nabla \mathbf{b}\|_{L^2}^2\,d\tau = \int_0^t \|j\|_{L^2}^2\,d\tau \le 2\int_0^t \|(\partial_2 b_1, \partial_1 b_2)\|_{L^2}^2\,d\tau \le C\|(\mathbf{u}_0,\mathbf{b}_0)\|_{L^2}^2. \tag{4.1}$$

To prove the global $\dot H^1$-bound, we invoke the equations of ω and j,

$$\begin{gathered}\partial_t\omega + \mathbf{u}\cdot\nabla\omega = \nu\partial_{11}\omega + \mathbf{b}\cdot\nabla j,\\ \partial_t j + \mathbf{u}\cdot\nabla j = \eta\,\partial_{111}b_2 - \eta\,\partial_{222}b_1 + \mathbf{b}\cdot\nabla\omega + Q(\nabla u,\nabla b).\end{gathered}$$

Dotting with (ω, j) and integrating by parts yields

$$\frac12\frac{d}{dt}\|(\omega,j)\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2 = \eta\int j\,(\partial_{111}b_2 - \partial_{222}b_1) + \int jQ(\nabla u,\nabla b).$$

Writing $j=\partial_1 b_2 - \partial_2 b_1$ and integrating by parts, we have

$$\int j(\partial_{111}b_2 - \partial_{222}b_1) = -\int \big((\partial_{11}b_1)^2 + (\partial_{22}b_1)^2 + (\partial_{11}b_2)^2 + (\partial_{22}b_2)^2\big) \equiv -H(\mathbf{b}).$$

The nonlinear term $\int jQ$ contains similar terms and we bound a typical one.

$$\begin{aligned}\Big|\int j\partial_1 b_1\partial_2 u_1\Big| &\le \|j\|_{L^4}\|\partial_1 b_1\|_{L^4}\|\partial_2 u_1\|_{L^2}\\ &\le C\|j\|_{L^2}\|\nabla j\|_{L^2}\|\omega\|_{L^2}\\ &\le \frac{\eta}{64}\|\nabla j\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^2\end{aligned}$$

Noticing that, due to $\nabla\cdot b=0$,

$$\nabla j = \begin{pmatrix}\Delta b_2\\ -\Delta b_1\end{pmatrix},$$

we have $\|\nabla j\|_{L^2}\le 2H(\mathbf{b})$. Combining the estimates above yields

$$\frac{d}{dt}\|(\omega,j)\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2 + \eta H(b) \le C\|j\|_{L^2}^2\|\omega\|_{L^2}^2.$$

Gronwall's inequality, together with (4.1), then yields the desired global uniform bound.

We now prove the uniqueness. Assume $(\mathbf{u}^{(1)},\mathbf{b}^{(1)})$ and $(\mathbf{u}^{(2)},\mathbf{b}^{(2)})$ are two solutions of (1.4) satisfying (1.8). Consider the difference $(\mathbf{u},\mathbf{b})$ between $(\mathbf{u}^{(1)},\mathbf{b}^{(1)})$ and $(\mathbf{u}^{(2)},\mathbf{b}^{(2)})$,

$$\mathbf{u} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)},\quad \mathbf{b} = \mathbf{b}^{(1)} - \mathbf{b}^{(2)},$$

which satisfies

$$\begin{gathered}\partial_t\mathbf{u} + \mathbf{u}^{(1)}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}^{(2)} = -\nabla p + \nu\,\partial_{11}\mathbf{u} + \mathbf{b}^{(1)}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{b}^{(2)},\\ \partial_t b_1 + \mathbf{u}^{(1)}\cdot\nabla b_1 + \mathbf{u}\cdot\nabla b_1^{(2)} = \eta\,\partial_{22}b_1 + \mathbf{b}^{(1)}\cdot\nabla u_1 + \mathbf{b}\cdot\nabla u_1^{(2)},\\ \partial_t b_2 + \mathbf{u}^{(1)}\cdot\nabla b_2 + \mathbf{u}\cdot\nabla b_2^{(2)} = \eta\,\partial_{11}b_2 + \mathbf{b}^{(1)}\cdot\nabla u_2 + \mathbf{b}\cdot\nabla u_2^{(2)},\\ \mathbf{u}(x,0)=\mathbf{u}_0=0,\quad \mathbf{b}(x,0)=\mathbf{b}_0=0,\end{gathered} \tag{4.2}$$

where p represents the difference between the associated pressures. Taking the inner products of $(\mathbf{u},\mathbf{b})$ with (4.2) and integrating by parts, we find

$$\frac12\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_{L^2}^2 + \nu\|\partial_1\mathbf{u}\|_{L^2}^2 + \eta\|(\partial_2 b_1,\partial_1 b_2)\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4, \tag{4.3}$$

where I_1 through I_4 are as before, namely

$$I_1 = -\int (\mathbf{u}\cdot\nabla)\mathbf{u}^{(2)}\cdot\mathbf{u}, \quad I_2 = \int (\mathbf{b}\cdot\nabla)\mathbf{b}^{(2)}\cdot\mathbf{u},$$
$$I_3 = -\int (\mathbf{u}\cdot\nabla)\mathbf{b}^{(2)}\cdot\mathbf{b}, \quad I_4 = \int (\mathbf{b}\cdot\nabla)\mathbf{u}^{(2)}\cdot\mathbf{b}.$$

I_1 is of a quadratic form and contains four terms

$$I_1 = \int \Big(\partial_1 u_1^{(2)} u_1 u_1 \, + \partial_1 u_2^{(2)} u_1 u_2 + \partial_2 u_1^{(2)} u_1 u_2 + \partial_2 u_2^{(2)} u_2 u_2\Big)\, dx.$$

When we estimate the terms in I_1, we keep in mind that the dissipation is only in the horizontal direction. By Lemma 4.1 and Young's inequality,

$$\begin{aligned}
\Big|\int \partial_1 u_1^{(2)}\, u_1\, u_1\, dx\Big| &\le C\|u_1\|_{L^2}\|u_1\|_{L^2}^{1/2}\|\partial_1 u_1\|_{L^2}^{1/2}\|\partial_1 u_1^{(2)}\|_{L^2}^{1/2}\|\partial_2\partial_1 u_1^{(2)}\|_{L^2}^{1/2}\\
&\le \frac{\nu}{64}\|\partial_1 u_1\|_{L^2}^2 + C\|u_1\|_{L^2}^2\|\partial_1 u_1^{(2)}\|_{L^2}^{2/3}\|\partial_2\partial_1 u_1^{(2)}\|_{L^2}^{2/3},
\end{aligned}$$

$$\begin{aligned}
\Big|\int \partial_1 u_2^{(2)}\, u_1\, u_2\, dx\Big| &\le C\|u_2\|_{L^2}\|u_1\|_{L^2}^{1/2}\|\partial_1 u_1\|_{L^2}^{1/2}\|\partial_1 u_2^{(2)}\|_{L^2}^{1/2}\|\partial_2\partial_1 u_2^{(2)}\|_{L^2}^{1/2}\\
&\le \frac{\nu}{64}\|\partial_1 u_1\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^2\|\partial_1 u_2^{(2)}\|_{L^2}^{2/3}\|\partial_2\partial_1 u_2^{(2)}\|_{L^2}^{2/3},
\end{aligned}$$

By Lemma 4.1, $\nabla\cdot u = 0$ and $\nabla\cdot u^{(2)} = 0$,

$$\begin{aligned}
\Big|\int \partial_2 u_1^{(2)}\, u_1\, u_2\, dx\Big| &\le C\|u_1\|_{L^2}\|u_2\|_{L^2}^{1/2}\|\partial_2 u_2\|_{L^2}^{1/2}\|\partial_2 u_1^{(2)}\|_{L^2}^{1/2}\|\partial_1\partial_2 u_1^{(2)}\|_{L^2}^{1/2}\\
&\le \frac{\nu}{64}\|\partial_1 u_1\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^2\|\partial_2 u_1^{(2)}\|_{L^2}^{2/3}\|\partial_1\partial_2 u_1^{(2)}\|_{L^2}^{2/3}\\
&\le \frac{\nu}{64}\|\partial_1 u_1\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^2\|\partial_1 u_2^{(2)} - \omega^{(2)}\|_{L^2}^{2/3}\|\partial_1\partial_2 u_1^{(2)}\|_{L^2}^{2/3},
\end{aligned}$$

$$\begin{aligned}
\Big|\int \partial_2 u_2^{(2)}\, u_2\, u_2\, dx\Big| &\le C\|u_2\|_{L^2}\|u_2\|_{L^2}^{1/2}\|\partial_1 u_2\|_{L^2}^{1/2}\|\partial_1 u_1^{(2)}\|_{L^2}^{1/2}\|\partial_2\partial_1 u_1^{(2)}\|_{L^2}^{1/2}\\
&\le \frac{\nu}{64}\|\partial_1 u_2\|_{L^2}^2 + C\|u_2\|_{L^2}^2\|\partial_1 u_1^{(2)}\|_{L^2}^{2/3}\|\partial_2\partial_1 u_1^{(2)}\|_{L^2}^{2/3}.
\end{aligned}$$

We now turn to I_2. Since the dissipation in the equation of $\mathbf{b}$ is effectively in both directions, there is no need to split I_2 into four terms, as we did in I_1. By Hölder's and Sobolev's inequalities,

$$\begin{aligned}
|I_2| &\le \|\mathbf{u}\|_{L^2}\|\mathbf{b}\|_{L^4}\|\nabla\mathbf{b}^{(2)}\|_{L^4}\\
&\le C\|\mathbf{u}\|_{L^2}\|\mathbf{b}\|_{L^2}^{1/2}\|\nabla\mathbf{b}\|_{L^2}^{1/2}\|\nabla\mathbf{b}^{(2)}\|_{L^2}^{1/2}\|\nabla\nabla\mathbf{b}^{(2)}\|_{L^2}^{1/2}\\
&\le \frac{\eta}{64}\|\nabla\mathbf{b}\|_{L^2}^2 + C\|(\mathbf{u},\mathbf{b})\|_{L^2}^2\|\nabla\mathbf{b}^{(2)}\|_{L^2}^{2/3}\|\nabla\nabla\mathbf{b}^{(2)}\|_{L^2}^{2/3}.
\end{aligned}$$

I_3 admits exactly the same bound. I_4 can be bounded in a similar fashion.

$$|I_4| \le \|\mathbf{b}\|_{L^4}^2\|\nabla\mathbf{u}^{(2)}\|_{L^2} \le \frac{\eta}{64}\|\nabla\mathbf{b}\|_{L^2}^2 + C\|\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{u}^{(2)}\|_{L^2}^2.$$

Inserting the estimates above in (4.3) and noticing the fact

$$\|(\partial_2 b_1, \partial_1 b_2)\|_{L^2}^2 \ge \frac{1}{2}\|j\|_{L^2}^2 = \frac{1}{2}\|\nabla\mathbf{b}\|_{L^2}^2,$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\|(\mathbf{u},\mathbf{b})\|_{L^2}^2 + \nu\|\partial_1\mathbf{u}\|_{L^2}^2 + \frac{1}{2}\eta\|\nabla\mathbf{b}\|_{L^2}^2 \\
&\le C\|\mathbf{u}\|_{L^2}^2 \left(\|\partial_1\mathbf{u}^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{L^2}\right)^{2/3} \|\partial_1\nabla\mathbf{u}^{(2)}\|_{L^2}^{2/3} \\
&\quad + C\|(\mathbf{u},\mathbf{b})\|_{L^2}^2\|\nabla\mathbf{b}^{(2)}\|_{L^2}^{2/3}\|\nabla\nabla\mathbf{b}^{(2)}\|_{L^2}^{2/3} + C\|\mathbf{b}\|_{L^2}^2\|\nabla\mathbf{u}^{(2)}\|_{L^2}^2.
\end{aligned}$$

Since $(\mathbf{u}^{(2)},\mathbf{b}^{(2)})$ is in the regularity class (1.8),

$$(\mathbf{u}^{(2)},\mathbf{b}^{(2)}) \in L^\infty(0,\infty;\, H^1), \quad \partial_1\nabla\mathbf{u}^{(2)},\, \nabla\nabla\mathbf{b}^{(2)} \in L^2(0,\infty;\, L^2),$$

Gronwall's inequality then implies the desired uniqueness. This completes the proof. □

Acknowledgments. B. Dong was partially supported by the NNSFC (Nos. 11271019, 11571240). J. Li was partially supported by the NNSFC (No. 11201181). J. Wu was supported by NSF grant DMS 1614246, by the AT&T Foundation at Oklahoma State University, and by NNSFC (No. 11471103, a grant awarded to B. Yuan).

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