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Logarithmically regularized inviscid models in borderline sobolev spaces

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Several inviscid models in hydrodynamics and geophysics such as the incompressible Euler vorticity equations, the surface quasi-geostrophic equation, and the Boussinesq equations are not known to have even local well-posedness in the corresponding borderline Sobolev spaces. Here H^s is referred to as a borderline Sobolev space if the L^∞-norm of the gradient of the velocity is not bounded by the H^s-norm of the solution but by the $H^{\widetilde{s}}$-norm for any $\widetilde{s} > s$. This paper establishes the local well-posedness of the logarithmically regularized counterparts of these inviscid models in the borderline Sobolev spaces. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4725531]

Dedicated to Professor Peter Constantin on the occasion of his sixtieth birthday.

I. INTRODUCTION

It is not clear if the 2D Euler vorticity equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^\perp \psi \equiv (-\partial_{x_2}, \partial_{x_1})\psi, \quad \Delta\psi = \omega \end{cases} \tag{1.1}$$

is locally well-posedness in the Sobolev space $H^1(\mathbb{R}^2)$. Since $H^1(\mathbb{R}^2)$ is not embedded in $L^\infty(\mathbb{R}^2)$, the classical Yudovich theory[13] does not apply. A simple energy estimate reveals that one may need to control the L^∞-norm of ∇u in order to obtain even a local bound for $\|\omega\|_{H^1}$, but unfortunately $\|\nabla u\|_{L^\infty}$ is not bounded by $\|\omega\|_{H^1}$. $H^1(\mathbb{R}^2)$ is at the borderline in the sense that $L^\infty(\mathbb{R}^2)$ is embedded in $H^s(\mathbb{R}^2)$ for any $s > 1$ and (1.1) is actually globally well-posed in H^s with $s > 1$. This phenomenon of lack of local well-posedness result in a corresponding borderline space appears to be universal for several other inviscid models. Among them are the 2D inviscid Boussinesq and the 2D ideal MHD equations. Another outstanding model with this property is the inviscid surface quasi-geostrophic (SQG) equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Lambda\psi = \theta, \end{cases} \tag{1.2}$$

where $\theta = \theta(x, t)$ is a scalar function of $x \in \mathbb{R}^2$ and $t \geq 0$, and $\Lambda = \sqrt{-\Delta}$. (1.2) models actual geophysical flows in the atmosphere and is useful in understanding certain weather phenomena such as the frontogenesis (see, e.g., Refs. 5, 7, and 10). (1.2) is locally well-posed in H^s with $s > 2$ (see Refs. 5 and 6), but the local existence in the borderline space H^2 remains unknown. The phenomena of lack of local well-posedness result also exists for the 3D inviscid models. For example, the 3D

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Euler vorticity equations and the 3D Boussinesq equations are not known to be locally well-posed in $H^{\frac{3}{2}}(\mathbb{R}^3)$.

This paper studies the local posedness of the logarithmically regularized counterparts of the aforementioned inviscid models, although a global result is also provided for the regularized 2D Euler equation. "Logarithmically" refers to the regularization at the level of logarithm of the Laplacian. This study is partially inspired by a recent work of Chae, Constantin, and Wu,[3] in which a general framework is laid out for dealing with inviscid models generalizing the 2D Euler and the SQG equations. In the following $P(\Lambda)$ denotes a Fourier multiplier operator, namely,

$$\widehat{P(\Lambda) f}(\xi) = P(|\xi|)\, \widehat{f}(\xi).$$

We assume that the symbol $P(|\xi|)$ satisfies

$$P \geq 0, \quad P \text{ is radially symmetric}, \quad P \in C(\mathbb{R}^d), \quad P \in C^\infty(\mathbb{R}^d \setminus \{0\}) \tag{1.3}$$

and, for any integer j and $n = 1, 2, \ldots, 1 + \left[\frac{d}{2}\right]$,

$$\sup_{2^{-1} \leq |\eta| \leq 2} \left|(I - \Delta_\eta)^n\, P(2^j|\eta|)\right| \leq C\, P(C_0\, 2^j), \tag{1.4}$$

where C and C_0 are two constants independent of j and n. As pointed out in Ref. 3, (1.4) is a very natural condition on symbols of Fourier multiplier operators and is similar to the main condition in the Mihlin-Hörmander multiplier theorem (see, e.g., Ref. [11, p. 96]). All the operators that we care about satisfy this condition. For the logarithmically regularized 2D Euler vorticity equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = P(\Lambda)\omega, \\ \omega(x.0) = \omega_0(x), \end{cases} \tag{1.5}$$

we are able to show that any $\omega_0 \in H^1$ leads to a unique local solution when $P(|\xi|)$ obeys an explicit integral condition as stated in Theorem 1.1. In particular, the result holds if

$$P(|\xi|) \leq (\ln(e + |\xi|^2))^{-\gamma} \quad \text{for any} \quad \gamma > \frac{1}{2}.$$

Theorem 1.1: *Let $\omega_0 \in H^1(\mathbb{R}^2)$ and consider the initial-value problem (IVP) (1.5). Assume the symbol $P(r)$ of the operator $P(\Lambda)$ satisfies*

$$\int_1^\infty \frac{P^2(r)}{r}\, dr < \infty. \tag{1.6}$$

Then, there is $T = T(\|\omega_0\|_{H^1}) > 0$ such that (1.5) has a unique solution ω on $[0, T]$ satisfying

$$\theta \in C([0, T]; H^1(\mathbb{R}^2)).$$

In particular, if we take

$$P(\Lambda) = (\ln(e - \Delta))^{-\gamma}, \quad \gamma > \frac{1}{2},$$

then (1.6) is fulfilled and (1.5) has a unique local solution.

A key point in the proof of this theorem is that the nonlinear part can now be bounded in terms of $\|\omega\|_{H^1}$. Similar local results hold for logarithmically regularized 2D Boussinesq and the 2D MHD equations. The details are provided in Sec. II.

Attention is also paid to a family of regularized SQG equations

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Lambda^\beta \psi = P(\Lambda)\theta, \\ \theta(x, 0) = \theta_0(x), \end{cases} \tag{1.7}$$

where $1 \leq \beta \leq 2$. When $\beta = 2$, (1.7) reduces to (1.5) while (1.7) with $\beta = 1$ is a regularized version of (1.2). When $P(\Lambda)$ represents a logarithmic regularization, (1.7) possesses a unique local solution in $H^{3-\beta}(\mathbb{R}^2)$, as stated in the following theorem.

Theorem 1.2: *Let $\theta_0 \in H^{3-\beta}(\mathbb{R}^2)$. Assume that the symbol of the operator $P(\Lambda)$ satisfies the integral condition in (1.6). Then, there exists a $T = T(\|\theta_0\|_{H^{3-\beta}})$ such that (1.7) possesses a unique solution $\theta \in C([0, T]; H^{3-\beta})$. Especially the logarithmically regularized inviscid SQG equation, namely, (1.7) with $\beta = 1$, is locally well-posed in H^2.*

To prove this theorem, we identify $H^{3-\beta}$ with the Besov space $B^{3-\beta}_{2,2}$ and estimate the norm $\|\theta\|_{B^{3-\beta}_{2,2}}$ through the Besov space techniques. For the convenience of the readers, the definition of Besov spaces and some of its properties are provided in Appendix. This theorem is proved in Sec. III.

Even though the vorticity formulation of the 3D Euler equations involves the vortex stretching term, the local well-posedness theory can still be established for the logarithmically regularized 3D Euler equations and for the logarithmically regularized 3D Boussinesq equations. Here the regularized 3D Euler vorticity equations and the Boussinesq equations assume the form

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ u = \nabla \times \psi, \quad \Delta \psi = P(\Lambda)\omega \end{cases} \tag{1.8}$$

and

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u + \nabla \times (\theta \mathbf{e}_3), \\ u = \nabla \times \psi, \quad \Delta \psi = P(\Lambda)\omega, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \end{cases} \tag{1.9}$$

respectively, where $\mathbf{e}_3$ denotes the unit vector in the z-direction. The local theory in the space $H^{\frac{3}{2}}$ can be stated as follows.

Theorem 1.3: *Consider (1.8) with an initial vorticity $\omega_0 \in H^{\frac{3}{2}}(\mathbb{R}^3)$. If the symbol of the operator $P(\Lambda)$ satisfies (1.6), then (1.8) has a unique local solution $\omega \in C([0, T]; H^{\frac{3}{2}}(\mathbb{R}^3))$ for some $T = T\left(\|\omega_0\|_{H^{\frac{3}{2}}}\right) > 0$.*

Theorem 1.4: *Consider (1.9) with $\omega_0 \in H^{\frac{3}{2}}(\mathbb{R}^3)$ and $\theta_0 \in H^{\frac{5}{2}}(\mathbb{R}^3)$. If the symbol of the operator $P(\Lambda)$ satisfies (1.6), then (1.9) has a unique local solution $\omega \in C([0, T]; H^{\frac{3}{2}}(\mathbb{R}^3))$ and $\theta \in C([0, T]; H^{\frac{5}{2}}(\mathbb{R}^3))$ for some $T = T\left(\|\omega_0\|_{H^{\frac{3}{2}}}, \|\theta_0\|_{H^{\frac{5}{2}}}\right) > 0$.*

The proofs of the two theorems above involve Besov spaces techniques. The vortex stretching term $\omega \cdot \nabla u$ is handled differently from the convection term $u \cdot \nabla \omega$. The details are given in Sec. IV. Finally we remark that it appears to be very difficult to extend the local well-posedness result for the slightly regularized 2D Euler equation into a global solution in H^1. Nevertheless, we are able to obtain the global existence in $W^{1,p}(\mathbb{R}^2)$ for any $p > 2$. The precise result is stated in Theorem 5.1 of Sec. V.

II. LOGARITHMICALLY REGULARIZED 2D EULER AND RELATED EQUATIONS

This section is devoted to proving Theorem 1.1. In addition, we also obtain parallel local well-posedness theory for the logarithmically regularized 2D inviscid Boussinesq equations and the 2D ideal MHD equations. First we prove Theorem 1.1.

Proof of Theorem 1.1: The key component of the proof is a local *a priori* H^1-bound for ω. Once the bound for $\|\omega\|_{H^1}$ is established, the local well-posedness follows from a standard Picard fixed-point theorem (see Ref. 8). It is clear that, for any $1 \leq q \leq \infty$,

$$\|\omega(\cdot, t)\|_{L^q} \leq \|\omega_0\|_{L^q}.$$

We now estimate $\|\nabla\omega\|_{L^2}$. Due to $\nabla \cdot u = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega \le \|\nabla u\|_{L^\infty}\,\|\nabla\omega\|_{L^2}^2. \tag{2.1}$$

In the Fourier space, u is related to ω by

$$\widehat{u}(\xi) = \xi^\perp|\xi|^{-2}\,P(|\xi|)\,\widehat{\omega}(\xi) \tag{2.2}$$

and thus, thanks to (1.6),

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le \int_{\mathbb{R}^2} |P(|\xi|)\,|\widehat{\omega}(\xi)|\,d\xi\\ &\le \left[\int_{\mathbb{R}^2}|P(|\xi|)|^2\,|\xi|^{-2}\,d\xi\right]^{\frac{1}{2}}\left[\int_{\mathbb{R}^2}|\xi|^2\,|\widehat{\omega}(\xi)|^2\,d\xi\right]^{\frac{1}{2}}\\ &\le C\,\|\nabla\omega\|_{L^2}.\end{aligned} \tag{2.3}$$

Inserting the bound above in (2.1) and combining with the L^2-bound yields

$$\frac{d}{dt}\|\omega\|_{H^1} \le C\,\|\omega\|_{H^1}^2,$$

which implies

$$\|\omega(\cdot,t)\|_{H^1} \le \frac{\|\omega_0\|_{H^1}}{1 - C\,t\|\omega_0\|_{H^1}}.$$

To complete the proof for the H^1-local well-posedness, a Picard type theorem on a Banach space suffices (see Ref. [8, pp. 100–112]). One starts with the mollified equation

$$\begin{cases}\partial_t\omega^\epsilon + J_\epsilon(J_\epsilon u^\epsilon\cdot\nabla J_\epsilon\omega^\epsilon) = 0,\\ u^\epsilon = \nabla^\perp\psi^\epsilon, \quad \Delta\psi^\epsilon = P(\Lambda)\omega^\epsilon\end{cases}$$

and treats it as an ordinary differential equation on H^1. One then verifies that the nonlinear part defines a locally Lipschitz map on H^1 and the Picard theorem assesses the existence of a local solution ω^ϵ. A limiting process then yields a desired local solution. This completes the proof of Theorem 1.1. □

Similar local well-posedness can be established for other logarithmically regularized 2D inviscid models that share similar structures with the 2D Euler. Especially, the H^1 local well-posedness holds for the logarithmically regularized 2D Boussinesq and the 2D ideal MHD equations.

Theorem 2.1: *Consider the generalized inviscid 2D Boussinesq equations in vorticity formulation*

$$\begin{cases}\partial_t\omega + u\cdot\nabla\omega = \partial_{x_1}\rho,\\ \partial_t\rho + u\cdot\nabla\rho = 0,\\ u = \nabla^\perp\psi, \quad \Delta\psi = P(\Lambda)\omega,\\ \omega(x,0) = \omega_0(x), \quad \rho(x,0) = \rho_0(x).\end{cases} \tag{2.4}$$

Assume that $\omega_0 \in H^1(\mathbb{R}^2)$ and $\rho_0 \in H^2(\mathbb{R}^2)$. If the operator $P(\Lambda)$ obeys the condition in (1.6), then (2.4) has a unique local solution $(\omega,\rho) \in C([0,T];H^1) \times C([0,T];H^2)$.

We now turn to the generalized 2D ideal MHD equations. We use the formulation in terms of the vorticity ω and the current density j. It is easy to check that this formulation is formally equivalent to the standard 2D MHD equations of the velocity and the magnetic field (see, e.g., Ref. 2).

Theorem 2.2: *Consider the generalized ideal 2D MHD equations*

$$\begin{cases} \partial_t \omega + u\cdot\nabla\omega = b\cdot\nabla j, \\ \partial_t j + u\cdot\nabla j = b\cdot\nabla\omega + 2\partial_{x_1} b(\partial_{x_1}u_2 + \partial_{x_2}u_1) - 2\partial_{x_1}u(\partial_{x_1}b_2 + \partial_{x_2}b_1), \\ u = \nabla^\perp\psi, \quad \Delta\psi = P(\Lambda)\omega, \\ b = \nabla^\perp\phi, \quad \Delta\phi = P(\Lambda)j, \\ \omega(x,0) = \omega_0(x), \quad j(x,0) = j_0(x). \end{cases} \tag{2.5}$$

Assume that $\omega_0 \in H^1(\mathbb{R}^2)$ and $j_0 \in H^1(\mathbb{R}^1)$. If the operator $P(\Lambda)$ obeys the condition in (1.6), then (2.5) has a unique local solution $(\omega, j) \in C([0,T]; H^1(\mathbb{R}^2)) \times C([0,T]; H^1(\mathbb{R}^2))$.

III. LOGARITHMICALLY REGULARIZED INVISCID SQG TYPE EQUATION

This section proves Theorem 1.2. The approach is to identify the Sobolev space H^σ with the Besov space $B^\sigma_{2,2}$ defined the Littlewood-Paley theory (see Appendix). This allows us to employ the techniques associated with the estimates of Besov norms. An important ingredient in the proof involves bounding ∇u in terms of θ and we need a proposition from Ref. [3, p. 41]. In this proposition, Δ_j with $j = -1, 0, 1, 2, \ldots$ denotes the Fourier localization operators as defined in Appendix.

Proposition 3.1. Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by

$$(\nabla u)_{ik} = \mathcal{R}_l\mathcal{R}_m\, P(\Lambda)\,\theta,$$

where $1 \le i, k, l, m \le d$, $(\nabla u)_{ik}$ denotes the (i,k)th entry of ∇u and $\mathcal{R}_l$ denotes the Riesz transform. Assume the symbol $P(|\xi|)$ satisfies (1.3) and (1.4). Then, for any integer $j \ge 0$,

$$\|\Delta_j \nabla u\|_{L^q} \le C_d\, P(C_0 2^j)\, \|\Delta_j\theta\|_{L^q}, \quad 1 \le q \le \infty, \tag{3.1}$$

where C_d is a constant depending on d only.

Proof of Theorem 1.2: As we have explained in the proof of Theorem 1.1, it suffices to establish a local *a priori* bound for $\|\theta\|_{H^{3-\beta}}$. For this purpose, we write $\sigma = 3-\beta$ (purely for notational convenience) and identify H^σ with the Besov space $B^\sigma_{2,2}$. Applying Δ_j to the first equation in (1.7), taking the inner product with $\Delta_j\theta$, multiplying by $2^{2\sigma j}$ and summing over $j = -1, 0, \ldots$, we find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|_{H^\sigma} &= -\sum_{j=-1}^{\infty} 2^{2\sigma j}\int \Delta_j\theta\, \Delta_j(u\cdot\nabla\theta)\,dx \\ &= I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where, by the notion of paraproducts,

$$\begin{aligned} I_1 &= -\sum_{j=-1}^{\infty} 2^{2\sigma j}\int \Delta_j\theta \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta, \\ I_2 &= -\sum_{j=-1}^{\infty} 2^{2\sigma j}\int \Delta_j\theta \cdot \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k\theta, \\ I_3 &= -\sum_{j=-1}^{\infty} 2^{2\sigma j}\int \Delta_j\theta\cdot(S_j u\cdot\nabla)\Delta_j\theta, \\ I_4 &= -\sum_{j=-1}^{\infty} 2^{2\sigma j}\int \Delta_j\theta \cdot \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta), \\ I_5 &= -\sum_{j=-1}^{\infty} 2^{2\sigma j}\int \Delta_j\theta \cdot \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta) \end{aligned}$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Thanks to $\nabla \cdot u = 0$, we have $I_3 = 0$. We now bound I_1. Noticing that the summation over k contains only a finite number of terms, namely, $|k - j| \le 2$, it suffices to bound the typical term with $k = j$. By Hölder's inequality and a standard commutator estimate,

$$|I_1| \le C \sum_{j=-1}^{\infty} 2^{2\sigma j} \|\Delta_j \theta\|_2^2 \, \|\nabla S_{j-1} u\|_\infty \le C \, \|\nabla u\|_\infty \|\theta\|_{H^\sigma}^2. \tag{3.2}$$

The second equation in (1.7) implies that u is related to θ by

$$\widehat{u}(\xi) = \xi^\perp |\xi|^{-\beta} P(|\xi|)\, \widehat{\theta}(\xi)$$

and thus, thanks to (1.6),

$$\begin{aligned}
\|\nabla u\|_\infty &\le \int_{\mathbb{R}^2} |\xi|^{2-\beta} P(|\xi|)\, |\widehat{\theta}(\xi)|\, d\xi \\
&\le \left[\int_{\mathbb{R}^2} (|\xi|^{-1} P(|\xi|))^2\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^2} (|\xi|^{3-\beta} |\widehat{\theta}(\xi)|)^2 \, d\xi\right]^{\frac{1}{2}} \\
&\le C \, \|\theta\|_{H^\sigma}.
\end{aligned} \tag{3.3}$$

Inserting this bound in (3.2) yields

$$|I_1| \le C \, \|\theta\|_{H^\sigma}^3. \tag{3.4}$$

Noticing that $S_{k-1}u - S_j u$ is a sum of a finite number of terms Δ_l with l between $k - 1$ and j and that $|k - j| \le 2$, we apply Hölder's inequality to obtain

$$|I_2| \le \sum_{j=-1}^{\infty} 2^{2\sigma j} \, \|\Delta_j \theta\|_2 \|\Delta_j u\|_\infty \, \|\nabla \Delta_j \theta\|_2.$$

To further estimate, we shift the derivative from $\nabla \Delta_j \theta$ to $\Delta_j u$. For this purpose, we divide the sum into two parts $j \le 2$ and $j \ge 3$,

$$|I_2| \le \left(\sum_{j=-1}^{2} + \sum_{j=3}^{\infty} \right) 2^{2\sigma j} \, \|\Delta_j \theta\|_2 \|\Delta_j u\|_\infty \, \|\nabla \Delta_j \theta\|_2. \tag{3.5}$$

For the part $j \le 2$, we apply Bernstein's inequality of Appendix, Proposition 3.1 and the Hardy-Littlewood-Sobolev inequality to obtain

$$\|\Delta_j u\|_\infty \, \|\nabla \Delta_j \theta\|_2 \le C \, \|\Delta_j \nabla^\perp \Lambda^{-\beta} P(\Lambda)\theta\|_\infty \, \|\Delta_j \theta\|_2 \le C \, \|\Delta_j \theta\|_2^2.$$

For the large mode part $j \ge 3$, the lower bound part of Bernstein's inequality then applies and yields

$$\|\Delta_j u\|_\infty \, \|\nabla \Delta_j \theta\|_2 \le C \, \|\Delta_j u\|_\infty \, 2^j \, \|\Delta_j \theta\|_2 \le C \|\nabla \Delta_j u\|_\infty \, \|\Delta_j \theta\|_2.$$

Bounding $\|\nabla \Delta_j u\|_\infty$ by (3.3) and inserting these estimates in (3.5) lead to

$$|I_2| \le C \, \|\theta\|_{H^\sigma}^3. \tag{3.6}$$

By Hölder's and Bernstein's inequalities,

$$\begin{aligned}
|I_4| &\le C \sum_{j=-1}^{\infty} 2^{2\sigma j} \|\Delta_j \theta\|_2 \, \|\Delta_j u \cdot \nabla S_{j-1} \theta\|_2 \\
&\le \sum_{j=-1}^{\infty} 2^{2\sigma j} \|\Delta_j \theta\|_2 \, \|\Delta_j u\|_2 2^j \|S_{j-1}\theta\|_\infty.
\end{aligned} \tag{3.7}$$

As in the estimate of I_2, we split the sum into two parts: $j \le 2$ and $j \ge 3$. The low mode part $j \le 2$ can be bounded as before and the high mode part satisfies

$$\|\Delta_j u\|_2 2^j \le C \|\nabla \Delta_j u\|_2 \quad \text{for } j \ge 3.$$

By Proposition 3.1, we have

$$\|\nabla \Delta_j u\|_2 \leq C\, P(2^j)\, \|\Delta_j \theta\|_2. \tag{3.8}$$

In addition, by (A1), $\widehat{S_{j-1}\theta}$ is supported on the ball of radius 2^j and thus

$$\begin{aligned} \|S_{j-1}\theta\|_\infty &\leq \int_{\mathbb{R}^2} |\widehat{S_{j-1}\theta}(\xi)|\, d\xi = \int_{\mathbb{R}^2} \psi_j(\xi)\, |\theta(\xi)|\, d\xi \leq C \int_{|\xi|\leq 2^j} |\theta(\xi)|\, d\xi \\ &\leq C \left[\int_{|\xi|\leq 2^j} \langle\xi\rangle^{-2(3-\beta)}\, d\xi\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^2} (\langle\xi\rangle^{3-\beta}\, |\widehat{\theta}(\xi)|)^2\, d\xi\right]^{\frac{1}{2}}, \end{aligned} \tag{3.9}$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. Since $1 \leq \beta \leq 2$, we evaluate the first integral by polar coordinates and find

$$\|S_{j-1}\theta\|_\infty \leq C\, \left(\log(1+2^j)\right)^{\frac{1}{2}}\, \|\theta\|_{H^\sigma}. \tag{3.10}$$

Inserting (3.8) and (3.9) in (3.7), we find that

$$|I_4| \leq C \sup_{j\geq -1} P(2^j)\, \left(\log(1+2^j)\right)^{\frac{1}{2}}\, \|\theta\|_{H^\sigma}^3.$$

Clearly, (1.6) implies that, there is an integer $j_0 > 0$,

$$\sup_{j\geq j_0} P(2^j)\, \left(\log(1+2^j)\right)^{\frac{1}{2}} \leq C, \tag{3.11}$$

where C is a constant independent of j. Thus,

$$|I_4| \leq C\, \|\theta\|_{H^\sigma}^3. \tag{3.12}$$

To bound I_5, we first apply Hölder's and Bernstein's inequalities to obtain

$$|I_5| \leq \sum_{j=-1}^{\infty} 2^{2\sigma j} \|\Delta_j \theta\|_\infty \sum_{k\geq j-1} 2^j\, \|\Delta_k u\|_2\, \|\widetilde{\Delta}_k \theta\|_2.$$

Interchanging the order of the double summation, we have

$$|I_5| \leq \sum_{k\geq -1}^{\infty} 2^{2\sigma k} 2^k \|\Delta_k u\|_2\, \|\widetilde{\Delta}_k \theta\|_2 \sum_{j\leq k+1} 2^{(2\sigma+1)(j-k)} \|\Delta_j \theta\|_\infty.$$

We again split the summation over k into the low and high mode parts. The low mode part is easily handled and the high mode part obeys

$$2^k \|\Delta_k u\|_2\, \leq C\, \|\nabla \Delta_k u\|_2.$$

Invoking similar estimates as in (3.8) and (3.10), we obtain

$$|I_5| \leq C\, \|\theta\|_{H^\sigma} \sum_{k\geq -1}^{\infty} 2^{2\sigma k}\, \|\Delta_k \theta\|_2^2 \sum_{j\leq k+1} 2^{(2\sigma+1)(j-k)} P(2^k)\, \left(\log(1+2^j)\right)^{\frac{1}{2}}.$$

Thanks to (3.11), we have

$$|I_5| \leq C\, \|\theta\|_{H^\sigma}^3. \tag{3.13}$$

Combining (3.4), (3.6), (3.12), and (3.13), we find

$$\frac{d}{dt} \|\theta\|_{H^\sigma}^2 \leq C\, \|\theta\|_{H^\sigma}^3,$$

which yields a local bound for $\|\theta\|_{H^\sigma}$. This completes the proof of Theorem 1.2. □

IV. LOGARITHMICALLY REGULARIZED 3D INVISCID MODELS

This section provides the proofs of Theorem 1.3 and Theorem 1.4, the local well-posedness of the logarithmically regularized 3D Euler equations and the logarithmically regularized 3D Boussinesq equations in the borderline space. The approach here is to identify the Sobolev norm H^s with the Besov space $B^s_{2,2}$ and apply the Besov space techniques. Direct manipulations with the Sobolev space H^s does not appear to work.

Proof of Theorem 1.3: For notational convenience, we write s for $\frac{3}{2}$ in the entire proof. Since the Sobolev norm H^s is equivalent to the norm in the Besov space $B^s_{2,2}$, the proof takes advantage of the Besov space techniques.

Applying Δ_j to the first equation in (1.8), taking the inner product with $\Delta_j\omega$, multiplying by 2^{2sj} and summing over integers $j \geq -1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{H^s} = J_1 + J_2, \tag{4.1}$$

where

$$J_1 = \sum_{j=-1}^{\infty} 2^{2sj} \int \Delta_j(\omega\cdot\nabla u)\cdot\Delta_j\omega,$$

$$J_2 = -\sum_{j=-1}^{\infty} 2^{2sj} \int \Delta_j\,(u\cdot\nabla\omega)\cdot\Delta_j\omega.$$

To estimate J_1, we decompose $\Delta_j(\omega \cdot \nabla u)$ into paraproducts and write J_1 as

$$J_1 = J_{11} + J_{12} + J_{13},$$

where

$$J_{11} = \sum_{j=-1}^{\infty} 2^{2sj} \int \sum_{|k-j|\leq 2} \Delta_j(S_{k-1}\omega\cdot\nabla\Delta_k u)\cdot\Delta_j\omega,$$

$$J_{12} = \sum_{j=-1}^{\infty} 2^{2sj} \int \sum_{|k-j|\leq 2} \Delta_j(\Delta_k\omega\cdot\nabla S_{k-1} u)\cdot\Delta_j\omega,$$

$$J_{13} = \sum_{j=-1}^{\infty} 2^{2sj} \int \sum_{k\geq j-1} \Delta_j(\Delta_k\omega\cdot\nabla\widetilde{\Delta}_k u)\cdot\Delta_j\omega.$$

By Hölder's inequality, we have

$$|J_{11}| \leq C \sum_{j=-1}^{\infty} 2^{2sj}\|S_{j-1}\omega\cdot\nabla\Delta_j u\|_2\,\|\Delta_j\omega\|_2. \tag{4.2}$$

Noticing that $\nabla u = \nabla\Delta^{-1}\nabla \times P(\Lambda)\omega$ and the boundedness of Riesz transforms on L^2, we have

$$\begin{aligned}\|S_{j-1}\omega\cdot\nabla\Delta_j u\|_2 &\leq \|S_{j-1}\omega\|_\infty\,\|\nabla\Delta_j u\|_2\\ &\leq C\,\|S_{j-1}\omega\|_\infty\,\|\Delta_j P(\Lambda)\omega\|_2.\end{aligned} \tag{4.3}$$

By the definition of S_{j-1} and Hölder's inequality,

$$\begin{aligned}\|S_{j-1}\omega\|_\infty &\leq \int |\widehat{S_{j-1}\omega}(\xi)|\,d\xi \leq \int_{|\xi|\leq 2^j} |\widehat{\omega}(\xi)|\,d\xi\\ &\leq C\,\|\omega\|_{H^s}\,(\log(1+2^j))^{\frac{1}{2}}.\end{aligned} \tag{4.4}$$

Therefore, by Proposition 3.1,

$$\|S_{j-1}\omega\cdot\nabla\Delta_j u\|_2 \le C\,\|\omega\|_{H^s}\,(\log(1+2^j))^{\frac{1}{2}}\,P(2^j)\,\|\Delta_j\omega\|_2. \tag{4.5}$$

Inserting (4.5) in (4.2) and noticing that, due to (1.6), for some $j_0 > 0$,

$$\sup_{j\ge j_0}\left(\log(1+2^j)\right)^{\frac{1}{2}}\;P(2^j)\le C, \tag{4.6}$$

we find

$$\begin{aligned}|J_{11}| &\le C\,\sup_{j\ge j_0}(\log(1+2^j))^{\frac{1}{2}}\,P(2^j)\,\|\omega\|_{H^s}\sum_{j=-1}^{\infty}2^{2sj}\|\Delta_j\omega\|_2^2\\ &\le C\,\|\omega\|_{H^s}^3.\end{aligned} \tag{4.7}$$

J_{12} can be bounded easily. In fact, by Hölder's inequality,

$$\begin{aligned}|J_{12}| &\le C\sum_{j=-1}^{\infty}2^{2sj}\|\Delta_j\omega\cdot\nabla S_{j-1}u\|_2\,\|\Delta_j\omega\|_2\\ &\le C\,\|\nabla u\|_\infty\,\|\omega\|_{H^s}^2.\end{aligned}$$

By a similar calculation as in (2.3), we have

$$\|\nabla u\|_\infty \le C\|\omega\|_{H^s}. \tag{4.8}$$

Therefore,

$$|J_{12}| \le C\,\|\omega\|_{H^s}^3. \tag{4.9}$$

To bound J_{23}, we apply a different Hölder's inequality to obtain

$$\begin{aligned}|J_{13}| &\le C\sum_{j=-1}^{\infty}2^{2sj}\|\Delta_j\omega\|_\infty\sum_{k\ge j-1}\|\Delta_k\omega\cdot\nabla\widetilde{\Delta}_k u\|_1\\ &\le C\sum_{j=-1}^{\infty}2^{2sj}\|\Delta_j\omega\|_\infty\sum_{k\ge j-1}\|\Delta_k\omega\|_2\,\|\nabla\widetilde{\Delta}_k u\|_2\\ &\le C\sum_{k=-1}^{\infty}\|\Delta_k\omega\|_2\,\|\nabla\widetilde{\Delta}_k u\|_2\sum_{j\le k+1}2^{2sj}\|\Delta_j\omega\|_\infty.\end{aligned} \tag{4.10}$$

Similarly as in (4.4), we have

$$\|\Delta_j\omega\|_\infty \le C\,\|\omega\|_{H^s}\,(\log(1+2^j))^{\frac{1}{2}}. \tag{4.11}$$

In addition, by Proposition 3.1 again,

$$\|\nabla\widetilde{\Delta}_k u\|_2 \le C\,P(2^k)\,\|\Delta_k\omega\|_2. \tag{4.12}$$

Inserting (4.11) and (4.12) in (4.10), we find

$$|J_{13}| \le C\,\|\omega\|_{H^s}\sum_{k=-1}^{\infty}2^{2ks}\|\Delta_k\omega\|_2^2\sum_{j\le k+1}2^{2s(j-k)}P(2^k)\,(\log(1+2^j))^{\frac{1}{2}}. \tag{4.13}$$

Thanks to (1.6), we have, for large k,

$$P(2^k)\;\le C(\log(1+2^k))^{-\frac{1}{2}}$$

and thus

$$|J_{13}| \le C\,\|\omega\|_{H^s}\sum_{k=-1}^{\infty}2^{2ks}\|\Delta_k\omega\|_2^2 = C\,\|\omega\|_{H^s}^3. \tag{4.14}$$

Combining (4.7), (4.9), and (4.14), we have

$$|J_1| \le C\, \|\omega\|_{H^s}^3. \tag{4.15}$$

We now turn to J_2. By paraproducts decomposition, we write

$$J_2 = J_{21} + J_{22} + J_{23} + J_{24} + J_{25},$$

where

$$J_{21} = \sum_{j=-1}^{\infty} 2^{2sj} \int \Delta_j\omega \cdot \sum_{|j-k|\le 2} [\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\omega,$$

$$J_{22} = \sum_{j=-1}^{\infty} 2^{2sj} \int \Delta_j\omega \cdot \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k\omega,$$

$$J_{23} = \sum_{j=-1}^{\infty} 2^{2sj} \int \Delta_j\omega \cdot S_j u\cdot\nabla\Delta_j\omega,$$

$$J_{24} = \sum_{j=-1}^{\infty} 2^{2sj} \int \Delta_j\omega \cdot \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}\omega),$$

$$J_{25} = \sum_{j=-1}^{\infty} 2^{2sj} \int \Delta_j\omega \cdot \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\omega).$$

By Höler's inequality and a standard commutator estimate, we have

$$\begin{aligned} |J_{21}| &\le C \sum_{j=-1}^{\infty} 2^{2sj}\|\Delta_j\omega\|_2\, \|\nabla S_{j-1}u\|_\infty \|\Delta_j\omega\|_2 \\ &\le C\,\|\nabla u\|_\infty \|\omega\|_{H^s}^2 \le C\,\|\omega\|_{H^s}^3. \end{aligned} \tag{4.16}$$

To estimate J_{22}, we first notice that $S_{k-1}u - S_j u$ contains only a finite number of terms $\Delta_l u$ for l between $k-1$ and j. By Hölder's and Bernstein's inequalities,

$$|J_{22}| \le C \sum_{j=-1}^{\infty} 2^{2sj}\|\Delta_j\omega\|_2\, \|\Delta_j u\|_\infty\, 2^j\, \|\Delta_j\omega\|_2.$$

In order to apply the lower bound part of Bernstein's inequality, we split the summation into the low and high modes. That is,

$$|J_{22}| \le C \left(\sum_{j=-1}^{2} + \sum_{j=3}^{\infty}\right) 2^{2sj}\|\Delta_j\omega\|_2\, \|\Delta_j u\|_\infty\, 2^j\, \|\Delta_j\omega\|_2. \tag{4.17}$$

For the high mode part $j\ge 3$, the lower bound part of Bernstein's inequality and (4.8) imply that

$$\|\Delta_j u\|_\infty\, 2^j \le C\|\nabla\Delta_j u\|_\infty \le C\,\|\nabla u\|_\infty \le C\,\|\omega\|_{H^s}.$$

For the low mode part $j\le 2$, by the Hardy-Littlewood-Sobolev inequality

$$\|\Delta_j u\|_\infty\, 2^j \le C\|\Delta_j u\|_6 = C\,\|\nabla\times\Delta^{-1}P(\Lambda)\Delta_j\omega\|_6 \le C\|\Delta_j\omega\|_2.$$

Inserting these estimates in (4.17), we find

$$|J_{22}| \le C\,\|\omega\|_{H^s}^3. \tag{4.18}$$

By $\nabla \cdot S_j u = 0$, we have $J_{23} = 0$. To bound J_{24}, we first apply Hölder's inequality and Bernstein's inequality to obtain

$$|J_{24}| \leq C \sum_{j=-1}^{\infty} 2^{2sj} \|\Delta_j \omega\|_2 \, \|\Delta_j u\|_2 \, 2^j \, \|S_{j-1}\omega\|_\infty.$$

As in (4.17), we split the summation into two parts. Since the low mode part can be easily handled, we shall only present the details for the high mode part. Then

$$|J_{24}| \leq C \sum_{j=-1}^{\infty} 2^{2sj} \|\Delta_j \omega\|_2 \, \|\nabla \Delta_j u\|_2 \, \|S_{j-1}\omega\|_\infty. \tag{4.19}$$

Bounding $\|\nabla \Delta_j u\|_2 \, \|S_{j-1}\omega\|_\infty$ as in (4.3), we can bound J_{24} in the same way as for J_{11},

$$|J_{24}| \leq C \, \|\omega\|_{H^s}^3.$$

Finally we bound J_{25}. The idea is to first shift the derivative from ω to u as we just did in estimating J_{24} and then to bound it as in J_{13}. The bound is still the same,

$$|J_{25}| \leq C \, \|\omega\|_{H^s}^3.$$

Collecting all the estimates, we obtain that

$$\frac{d}{dt} \|\omega\|_{H^s}^2 \leq C \, \|\omega\|_{H^s}^3,$$

which yields a local bound for $\|\omega\|_{H^s}$. This completes the proof of Theorem 1.3. □

We now prove Theorem 1.4.

Proof of Theorem 1.4: The proof of this theorem is parallel to the previous proof. Since the Boussinesq vorticity equation only differs from the Euler vorticity equation by the term $\nabla \times (\theta \mathbf{e}_3)$, a similar procedure as in the proof of Theorem 1.3 yields

$$\frac{d}{dt} \|\omega\|_{H^{\frac{3}{2}}}^2 \leq C \, \|\omega\|_{H^{\frac{3}{2}}}^3 + \|\theta\|_{H^{\frac{5}{2}}} \, \|\omega\|_{H^{\frac{3}{2}}}. \tag{4.20}$$

We now estimate the evolution of $\|\theta\|_{H^{\frac{5}{2}}}$. Applying Δ_j to the second equation in (1.9), taking the inner product with $\Delta_j \theta$, multiplying by 2^{5j} and summing over integers $j \geq -1$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_{H^{\frac{5}{2}}}^2 = -\sum_{j=-1}^{\infty} 2^{5j} \int \Delta_j (u \cdot \nabla \theta) \cdot \Delta_j \theta. \tag{4.21}$$

The term on the right-hand side can be further decomposed into the sum of

$$K_1 = -\sum_{j=-1}^{\infty} 2^{5j} \int \Delta_j \theta \cdot \sum_{|j-k| \leq 2} [\Delta_j, \, S_{k-1} u \cdot \nabla] \Delta_k \theta,$$

$$K_2 = -\sum_{j=-1}^{\infty} 2^{5j} \int \Delta_j \theta \cdot \sum_{|j-k| \leq 2} (S_{k-1} u - S_j u) \cdot \nabla \Delta_j \Delta_k \theta,$$

$$K_3 = -\sum_{j=-1}^{\infty} 2^{5j} \int \Delta_j \theta \cdot S_j u \cdot \nabla \Delta_j \theta,$$

$$K_4 = -\sum_{j=-1}^{\infty} 2^{5j} \int \Delta_j \theta \cdot \sum_{|j-k| \leq 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta),$$

$$K_5 = -\sum_{j=-1}^{\infty} 2^{5j} \int \Delta_j \theta \cdot \sum_{k \geq j-1} \Delta_j (\Delta_k u \cdot \nabla \widetilde{\Delta}_k \theta).$$

As in the estimates of J_{21} and J_{22}, namely (4.16) and (4.18), we have

$$|K_1| \le C\|\nabla u\|_\infty \|\theta\|^2_{H^{\frac{5}{2}}} \le C\,\|\omega\|_{H^{\frac{3}{2}}}\,\|\theta\|^2_{H^{\frac{5}{2}}}, \quad |K_2| \le C\,\|\omega\|_{H^{\frac{3}{2}}}\,\|\theta\|^2_{H^{\frac{5}{2}}}.$$

Thanks to $\nabla \cdot S_j u = 0$, $K_3 = 0$. By Hölder's inequality,

$$|K_4| \le C \sum_{j=-1}^{\infty} 2^{5j}\,\|\Delta_j\theta\|_2\,\|\Delta_j u\|_2\,\|S_{j-1}\nabla\theta\|_\infty.$$

As in (3.8), for $j \ge 3$,

$$2^j\,\|\Delta_j u\|_2 \le C\|\nabla\Delta_j u\|_2 \le C\,P(2^j)\,\|\Delta_j\omega\|_2. \tag{4.22}$$

Following a similar calculation as in (4.4), we have

$$\|S_{j-1}\nabla\theta\|_\infty \le C\,\|\theta\|_{H^{\frac{5}{2}}}\,(\log(1+2^j))^{\frac{1}{2}}. \tag{4.23}$$

Due to (1.6) and thus (4.6), we have

$$|K_4| \le C \sum_{j=-1}^{\infty} 2^{4j}\,\|\Delta_j\theta\|_2\,\|\Delta_j\omega\|_2\,\|\theta\|_{H^{\frac{5}{2}}} \le C\,\|\omega\|_{H^{\frac{3}{2}}}\,\|\theta\|^2_{H^{\frac{5}{2}}}.$$

To bound K_5, we employ the idea used in dealing with J_{13}. By Hölder's inequality,

$$\begin{aligned}|K_5| &\le C \sum_{j=-1}^{\infty} 2^{5j}\,\|\Delta_j\theta\|_\infty \sum_{k\ge j-1} \|\Delta_k u\|_2\,\|\widetilde{\Delta}_k\nabla\theta\|_2\\ &= C \sum_{k=-1}^{\infty} \|\Delta_k u\|_2\,\|\widetilde{\Delta}_k\nabla\theta\|_2 \sum_{j\le k+1} 2^{5j}\,\|\Delta_j\theta\|_\infty.\end{aligned}$$

Similarly as in (4.22) and (4.23), we have

$$2^k\,\|\Delta_k u\|_2 \le C\,P(2^k)\,\|\Delta_k\omega\|_2, \qquad 2^j\,\|\Delta_j\theta\|_\infty \le \|\theta\|_{H^{\frac{5}{2}}}\,(\log(1+2^j))^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}|K_5| \le C\,\|\theta\|_{H^{\frac{5}{2}}} &\sum_{k=-1}^{\infty} 2^{\frac{3}{2}k}\,\|\Delta_k\omega\|_2\,2^{\frac{3}{2}k}\,\|\Delta_k\nabla\theta\|_2\\ &\times \sum_{j\le k+1} 2^{-3(k-j)}\,P(2^k)\,(\log(1+2^j))^{\frac{1}{2}}.\end{aligned}$$

Using the fact that $P(2^k) \le C(\log(1+2^k))^{-\frac{1}{2}}$, we obtain

$$|K_5| \le C\,\|\theta\|^2_{H^{\frac{5}{2}}}\,\|\omega\|_{H^{\frac{3}{2}}}.$$

Combining the estimates and inserting them in (4.21), we obtain

$$\frac{d}{dt}\|\theta\|^2_{H^{\frac{5}{2}}} \le C\,\|\theta\|^2_{H^{\frac{5}{2}}}\,\|\omega\|_{H^{\frac{3}{2}}}. \tag{4.24}$$

(4.20) and (4.24) together then yield the desired local bound. This completes the proof of Theorem 1.4. □

V. GLOBAL SOLUTIONS OF LOGARITHMICALLY REGULARIZED 2D EULER

This last section presents a global regularity result for the logarithmically regularized 2D Euler equation (1.5). The global bound is obtained in a slightly different functional setting from the borderline space.

Theorem 5.1: *Let $p > 2$ and $\omega_0 \in W^{1,p}(\mathbb{R}^2)$. Then the vorticity ω of solution to (1.5) with P satisfying (1.6) obeys*

$$\omega \in C(0, \infty; W^{1,p}(\mathbb{R}^2)).$$

Proof: Applying ∇ on (1.5), and then taking $L^2(\mathbb{R}^2)$ inner product with $\nabla\omega|\nabla\omega|^{p-2}$, we obtain after integration by part,

$$\frac{1}{p}\frac{d}{dt}\|\nabla\omega\|_{L^p}^p \le \|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^p}^p,$$

and thus

$$\frac{d}{dt}\|\nabla\omega\|_{L^p} \le \|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^p}. \tag{5.1}$$

Since $\frac{d}{dt}\|\omega\|_{L^p} = 0$, we have from (5.1) that

$$\frac{d}{dt}\|\omega\|_{W^{1,p}} \le C\|\nabla u\|_{L^\infty}\|\omega\|_{W^{1,p}}. \tag{5.2}$$

We recall the following inequality proved in Ref. 9,

$$\|f\|_{L^\infty} \le C\left\{1 + \|f\|_{BMO}\ln(e + \|f\|_{W^{1,p}})\right\}, \quad p > d,$$

where d is the dimension of space. Then, we have

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\le C\left\{1 + \|\nabla u\|_{BMO}\ln(e + \|\nabla u\|_{W^{1,p}})\right\}\\
&\le C\left\{1 + \|\omega\|_{BMO}\ln(e + \|\omega\|_{W^{1,p}})\right\}\\
&\le C\left\{1 + \|\omega\|_{L^\infty}\ln(e + \|\omega\|_{W^{1,p}})\right\}\\
&= C\left\{1 + \|\omega_0\|_{L^\infty}\ln(e + \|\omega\|_{W^{1,p}})\right\},
\end{aligned} \tag{5.3}$$

where we used the fact

$$\widehat{\nabla u}(\xi) = Q(\xi)\widehat{\omega}(\xi), \quad Q(\xi) := \xi\xi^\perp|\xi|^{-2}P(|\xi|),$$

and the operator defined by the multiplier Q maps BMO into itself. Substituting (5.3) into (5.2), one obtains

$$\frac{d}{dt}\|\omega\|_{W^{1,p}} \le C\|\omega\|_{W^{1,p}}\left\{1 + \|\omega_0\|_{L^\infty}\log(e + \|\omega\|_{W^{1,p}})\right\}. \tag{5.4}$$

By Gronwall's inequality this provides us with the desired global bound. □

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APPENDIX: BESOV SPACES AND RELATED FACTS

This appendix provides the definitions of Δ_j, S_j, and inhomogeneous Besov spaces. Related useful facts such as the Bernstein inequality are also provided here. Materials presented in this appendix here can be found in several books and papers (see, e.g., Refs. 1 and 4 or 12).

Let $\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}'(\mathbf{R}^d)$ denote the Schwartz class and tempered distributions, respectively. The partition of unity states that there exist two nonnegative radial functions $\psi, \phi \in \mathcal{S}$ such that

$$\begin{aligned}
&\operatorname{supp}\psi \subset \; B\left(0, \frac{11}{12}\right), \quad \operatorname{supp}\phi \subset \; A\left(0, \frac{3}{4}, \frac{11}{6}\right),\\
&\psi(\xi) + \sum_{j\geq 0}\phi_j(\xi) = 1 \quad \text{for} \quad \xi \in \mathbf{R}^d, \qquad \phi_j(\xi) = \phi(2^{-j}\xi),\\
&\operatorname{supp}\psi \cap \operatorname{supp}\phi_j = \emptyset \quad \text{if } j \geq 1,\\
&\operatorname{supp}\phi_j \cap \operatorname{supp}\phi_k = \emptyset \quad \text{if } |j-k| \geq 2,
\end{aligned}$$

where $B(0, r)$ denotes the ball centered at the origin with radius r and $A(0, r_1, r_2)$ is the annulus centered at the origin with the inner radius r_1 and the outer radius r_2.

For any $f \in \mathcal{S}'$, set

$$\begin{aligned}
&\Delta_{-1} f = \mathcal{F}^{-1}\left(\psi(\xi)\mathcal{F}(f)\right) = \Psi * f,\\
&\Delta_j f = \mathcal{F}^{-1}\left(\phi_j(\xi)\mathcal{F}(f)\right) = \Phi_j * f, \quad j = 0, 1, 2, \ldots,\\
&\Delta_j f = 0 \quad \text{for} \quad j \leq -2,\\
&S_j = \sum_{k=-1}^{j-1} \Delta_k \text{ when } \quad j \geq 0,
\end{aligned}$$

where we have used $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote the Fourier and inverse Fourier transforms, respectively. Clearly,

$$\Psi = \mathcal{F}^{-1}(\psi), \quad \Phi_0 = \Phi = \mathcal{F}^{-1}(\phi), \quad \Phi_j(x) = \mathcal{F}^{-1}(\phi_j)(x) = 2^{jd}\,\Phi(2^j x).$$

In addition, we can write

$$\mathcal{F}(S_j f) = \psi\left(\frac{\xi}{2^j}\right)\mathcal{F}(f). \tag{A1}$$

With these notation at our disposal, we now provide the definition of the inhomogeneous Besov space.

Definition A.1: For $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \left\{ f \in \mathcal{S}' : \; \|f\|_{B^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,q}} \equiv \begin{cases} \left(\displaystyle\sum_{j=-1}^{\infty} \left(2^{js}\,\|\Delta_j f\|_{L^p}\right)^q \right)^{1/q}, & \text{if } q < \infty,\\ \displaystyle\sup_{-1\leq j<\infty} 2^{js}\,\|\Delta_j f\|_{L^p}, & \text{if } q = \infty. \end{cases} \tag{A2}$$

The following Bernstein type inequalities are very useful and have been used in Secs. III–IV.

Proposition A.2 (Bernstein inequalities): Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.

(1) *If f satisfies*

$$\operatorname{supp}\widehat{f} \subset \{\xi \in \mathbf{R}^d : \; |\xi| \leq K2^j\},$$

for some integer j and a constant $K > 0$, then

$$\max_{|\beta|=k} \|D^\beta f\|_{L^q(\mathbf{R}^d)} \leq C\, 2^{kj+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbf{R}^d)},$$

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbf{R}^d)} \leq C\, 2^{2\alpha j+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbf{R}^d)}$$

for some constant C depending on K, p, and q only.

(2) *If f satisfies*

$$\mathrm{supp}\, \widehat{f} \subset \{\xi \in \mathbf{R}^d : \ K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and constants $0 < K_1 \le K_2$, *then*

$$C\, 2^{kj} \| f \|_{L^q(\mathbf{R}^d)} \le \max_{|\beta|=k} \| D^\beta f \|_{L^q(\mathbf{R}^d)} \le C\, 2^{kj + jd(\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p(\mathbf{R}^d)},$$

$$C\, 2^{2\alpha j} \| f \|_{L^q(\mathbf{R}^d)} \le \| (-\Delta)^\alpha f \|_{L^q(\mathbf{R}^d)} \le C\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p(\mathbf{R}^d)},$$

where the constants C depend on K_1, K_2, p, *and* q *only.*

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