COMMUN. MATH. SCI.
Vol. 18, No. 4, pp. 987–1022

UNIQUE WEAK SOLUTIONS OF THE NON-RESISTIVE MAGNETOHYDRODYNAMIC EQUATIONS WITH FRACTIONAL DISSIPATION*

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Abstract. This paper examines the uniqueness of weak solutions to the d-dimensional magnetohydrodynamic (MHD) equations with the fractional dissipation $(-\Delta)^\alpha u$ and without the magnetic diffusion. Important progress has been made on the standard Laplacian dissipation case $\alpha=1$. This paper discovers that there are new phenomena with the case $\alpha<1$. The approach for $\alpha=1$ can not be directly extended to $\alpha<1$. We establish that, for $\alpha<1$, any initial data (u_0,b_0) in the inhomogeneous Besov space $B^\sigma_{2,\infty}(\mathbb{R}^d)$ with $\sigma>1+\frac{d}{2}-\alpha$ leads to a unique local solution. For the case $\alpha\geq 1$, u_0 in the homogeneous Besov space $\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d)$ and b_0 in $\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}(\mathbb{R}^d)$ guarantees the existence and uniqueness. These regularity requirements appear to be optimal.

Keywords. Besov spaces; magnetohydrodynamic equations; uniqueness; weak solution.

AMS subject classifications. 35A05; 35Q35; 76D03.

1. Introduction

The MHD equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. They consist of a coupled system of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Since their initial derivation by the Nobel Laureate H. Alfvén in 1942 [1], the MHD equations have played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [5,18]). Besides their wide physical applicability, the MHD equations are also of great interest in mathematics. As a coupled system, the MHD equations contain much richer structures than the Navier-Stokes equations. They are not merely a combination of two parallel Navier-Stokes-type equations but an interactive and integrated system. Their distinctive features make analytic studies a great challenge but offer new opportunities.

Attention here is focused on the d-dimensional non-resistive MHD equations with fractional dissipation,

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nu(-\Delta)^\alpha u=-\nabla P+b\cdot\nabla b, \quad x\in\mathbb{R}^d, t>0,\\ \partial_t b+u\cdot\nabla b=b\cdot\nabla u, \quad x\in\mathbb{R}^d, t>0,\\ \nabla\cdot u=\nabla\cdot b=0, \quad x\in\mathbb{R}^d, t>0,\\ u(x,0)=u_0(x), \quad b(x,0)=b_0(x), \quad x\in\mathbb{R}^d,\end{cases} \tag{1.1}$$

where u, P and b represent the velocity, the pressure and the magnetic field, respectively, and $\nu>0$ is the kinematic viscosity and $\alpha>0$ is a parameter. The fractional Laplacian

*Received: June 06, 2019; Accepted (in revised form): January 13, 2020. Communicated by Song Jiang.

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operator $(-\Delta)^\alpha$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi),$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

When $\alpha = 1$, (1.1) reduces to the standard MHD equations without magnetic diffusion, which models electrically conducting fluids that can be treated as perfect conductors such as strongly collisional plasmas. When $\alpha > 0$ is fractional, (1.1) may be applicable in some geophysical and astrophysical circumstances. One example is in the thinning of atmosphere. Flows in the middle atmosphere traveling upward undergo changes due to the changes in atmospheric properties. The effect of kinematic diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian. In addition, (1.1) may be used to model nonlocal and long-range diffusive interactions. Mathematically (1.1) allows us to study a family of equations simultaneously and provides a broad view on how the solutions are related to the sizes of α. Fractional Laplacian diffusion has recently been applied to model many real-world phenomena, ranging from quasi-geostrophic flows [8,17,31], flame propagation [7] to jumping processes in probability and finance [16]. Several books and monographs have been exclusively devoted to nonlocal diffusion (see, e.g., [6]).

One of the most fundamental issues on the MHD equations is the well-posedness problem. Mathematically rigorous foundational work has been laid by G. Duvaut and J. L. Lions in [21] and by M. Sermange and R. Temam in [40]. The MHD equations have recently gained renewed interests and there have been substantial developments on the well-posedness problem, especially when the MHD equations involve only partial or fractional dissipation (see, e.g., [10–12, 15, 19, 20, 22, 29, 30, 45, 49–54]). A summary on some of the recent results can be found in a review paper [46]. Equations (1.1) with $\alpha \geq 1 + \frac{d}{2}$ always has a unique global solution (see [46]). Yamazaki was able to improve this result by weakening the dissipation by a logarithm [52]. It remains an outstanding open problem whether or not (1.1) with $\alpha < 1 + \frac{d}{2}$ can have finite-time singular classical solutions. Even the global existence of Leray-Hopf weak solutions is not known due to the lack of suitable strong convergence in b. In spite of the difficulties due to the lack of magnetic diffusion, significant progress has been made on the global well-posedness of solutions near background magnetic fields and many exciting results have been obtained (see, e.g., [3, 9, 25–27, 33, 36–38, 41, 44, 47, 48, 55]).

Another direction of research on the non-resistive MHD equations has resulted in a steady stream of progress. This direction has been seeking the weakest possible functional setting for which one still has the uniqueness. The results currently available are for (1.1) with $\alpha = 1$. Q. Jiu and D. Niu [28] proved the local well-posedness of (1.1) with $\alpha = 1$ in the Sobolev space H^s with $s \geq 3$. Fefferman, McCormick, Robinson and Rodrigo were able to weaken the regularity assumption to $(u_0, b_0) \in H^s$ with $s > \frac{d}{2}$ in [23] and then to $u_0 \in H^{s-1+\epsilon}$ and $b_0 \in H^s$ with $s > \frac{d}{2}$ in [24]. Chemin, McCormick, Robinson and Rodrigo [14] made further improvement by assuming only $u_0 \in B_{2,1}^{\frac{d}{2}-1}$ and $b_0 \in B_{2,1}^{\frac{d}{2}}$. Here $B_{p,q}^s$ denotes an inhomogeneous Besov space. They obtained the local existence for $d = 2$ and 3, and the uniqueness for $d = 3$. R. Wan [43] obtained the uniqueness for $d = 2$. J. Li, W. Tan and Z. Yin [32] recently made an important progress by reducing the functional setting to homogeneous Besov space $u_0 \in \mathring{B}_{p,1}^{\frac{d}{p}-1}$ and $b_0 \in \mathring{B}_{p,1}^{\frac{d}{p}}$ with $p \in [1, 2d]$. The definitions of the Besov spaces are provided in the Appendix.

The aim of this paper is to establish the local existence and uniqueness of weak solutions with the minimal initial regularity assumption and for the largest possible range of α's. The case when $\alpha>1$ can be handled similarly as the case when $\alpha=1$. We can show that, for $\alpha>1$, any initial data (u_0,b_0) with $u_0\in\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)$ and $b_0\in\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d)$ leads to a unique local solution.

However, when $\alpha<1$, the situation is different and there are new phenomena. The approach for the case $\alpha=1$ can not be directly extended to $\alpha<1$. We tested several seemingly natural classes of initial data:

$$(1)\, u_0\in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d) \quad \text{and} \quad b_0\in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d); \tag{1.2}$$

$$(2)\, u_0\in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d) \quad \text{and} \quad b_0\in B_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d); \tag{1.3}$$

$$(3)\, u_0\in B_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d) \quad \text{and} \quad b_0\in B_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d), \tag{1.4}$$

but it appears impossible to prove the local existence and uniqueness of weak solutions in these functional settings. Our investigation with these initial data leads to several discoveries. First, we realize that, in order to attain the uniqueness, the regularity level of the Besov space for b_0 has to have at least $\frac{d}{2}-\alpha+1$-derivative, which is more than $\frac{d}{2}$ for $\alpha<1$. Second, we discover that if the derivative of the Besov setting for b_0 exceeds $\frac{d}{2}$, then u_0 and b_0 should have the same Besov setting in order to establish the existence of solutions. Furthermore, one needs to combine the term of $b\cdot\nabla b$ in the equation of u and $u\cdot\nabla b$ in the equation of b in order to generate the cancellation. More technical explanations are given in Section 5. As a consequence of these findings, we choose the following Besov spaces for $\alpha<1$,

$$u_0\in B_{2,\infty}^{\sigma}(\mathbb{R}^d), \quad b_0\in B_{2,\infty}^{\sigma}(\mathbb{R}^d), \quad \sigma>\frac{d}{2}+1-\alpha.$$

These functional settings appear to be optimal when $\alpha<1$. More technical evidence is provided in Section 5. Our precise result is stated in the following theorem.

THEOREM 1.1. *Let $d\geq 2$ and consider (1.1) with $0\leq\alpha<1+\frac{d}{4}$. Assume the initial data (u_0,b_0) satisfies $\nabla\cdot u_0=\nabla\cdot b_0=0$, and is in the following Besov spaces*

$$\text{for } \alpha\geq 1, \qquad u_0\in\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d), \quad b_0\in\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d), \tag{1.5}$$

$$\text{for } \alpha<1, \qquad u_0\in B_{2,\infty}^{\sigma}(\mathbb{R}^d), \quad b_0\in B_{2,\infty}^{\sigma}(\mathbb{R}^d), \quad \sigma>\frac{d}{2}+1-\alpha. \tag{1.6}$$

Then there exist $T>0$ and a unique local solution (u,b) of (1.1) satisfying, in the case of $\alpha\geq 1$,

$$u\in C([0,T];\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d))\cap L^1(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1}(\mathbb{R}^d)), b\in C([0,T];\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d))$$

and, in the case of $\alpha<1$,

$$u\in C([0,T];B_{2,\infty}^{\sigma}(\mathbb{R}^d))\cap\widetilde{L}^2(0,T;B_{2,\infty}^{\alpha+\sigma}(\mathbb{R}^d)), b\in C([0,T];B_{2,\infty}^{\sigma}(\mathbb{R}^d)).$$

Theorem 1.1 covers a full range of $\alpha\in[0,1+\frac{d}{4})$ and includes $\alpha=1$ and $\alpha=0$ as two special cases. $\alpha<1+\frac{d}{4}$ is imposed to satisfy a technical requirement in bounding the high frequency interaction terms in the paraproduct decomposition. When α reaches this upper bound, the functional setting for u_0 is $\mathring{B}^{-1}_{2,1}$. When $\alpha \geq 1$, the initial data (u_0, b_0) and the corresponding solution are in the homogeneous Besov spaces. For $\alpha < 1$, the functional setting are the inhomogeneous Besov spaces. We may not be able to reduce the assumption for $\alpha < 1$ to the corresponding homogeneous Besov spaces.

As aforementioned, the regularity assumptions imposed on (u_0, b_0) in Theorem 1.1 may be the minimal requirements we need for the existence and uniqueness. We present a detailed explanation in Section 5. Roughly speaking, when $\alpha \geq 1$, $u_0 \in \mathring{B}^{\frac{d}{2}+1-2\alpha}_{2,1}(\mathbb{R}^d)$ in (1.5) is necessary in order for the solution $u \in L^1(0,T;\mathring{B}^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d))$, which is more or less the regularity level for the uniqueness. The regularity setting for u_0 leads to the corresponding choice for b_0, namely $b_0 \in \mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1}(\mathbb{R}^d)$. In the case when $\alpha < 1$, (1.6) may be optimal due to our findings discovered in working with three other initial Besov settings described above in (1.2), (1.3) and (1.4). Another hint comes from the uniqueness requirement for the ideal MHD equations. When α is zero or $\alpha > 0$ is small, (1.6) is the regularity class that guarantees the uniqueness of solutions to the ideal MHD equations.

The statement of Theorem 1.1 clearly indicates that the case $\alpha \geq 1$ is handled differently from the case $\alpha < 1$. The existence part of Theorem 1.1 is proven through a successive approximation process. Naturally we divide the consideration into two cases: $\alpha \geq 1$ and $\alpha < 1$. In the case when $\alpha \geq 1$, the successive approximation sequence $(u^{(n)}, b^{(n)})$ satisfies

$$
\begin{cases}
u^{(1)} = \mathring{S}_1 u_0, \quad b^{(1)} = \mathring{S}_1 b_0, \\
\partial_t u^{(n+1)} + \nu(-\Delta)^\alpha u^{(n+1)} = \mathbb{P}(-u^{(n)} \cdot \nabla u^{(n+1)} + b^{(n)} \cdot \nabla b^{(n)}), \\
\partial_t b^{(n+1)} = -u^{(n)} \cdot \nabla b^{(n+1)} + b^{(n)} \cdot \nabla u^{(n)}, \\
\nabla \cdot u^{(n+1)} = \nabla \cdot b^{(n+1)} = 0, \\
u^{(n+1)}(x,0) = \mathring{S}_{n+1} u_0, \quad b^{(n+1)}(x,0) = \mathring{S}_{n+1} b_0,
\end{cases}
\tag{1.7}
$$

where $\mathbb{P}$ is the standard Leray projection and $\mathring{S}_j$ is the standard homogeneous low frequency cutoff operator (see the Appendix for its definition). The functional setting for $(u^{(n)}, b^{(n)})$ is given by

$$
\begin{aligned}
M &= 2\left(\|u_0\|_{\mathring{B}^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|b_0\|_{\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1}} \right) \\
Y &\equiv \Big\{ (u,b) \,\Big|\, \|u\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-2\alpha}_{2,1})} \leq M, \quad \|b\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})} \leq M, \\
&\qquad \|u\|_{L^1(0,T;\mathring{B}^{\frac{d}{2}+1}_{2,1})} \leq \delta, \quad \|u\|_{\widetilde{L}^2(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})} \leq \delta \Big\},
\end{aligned}
\tag{1.8}
$$

where $T > 0$ is chosen to be sufficiently small and $0 < \delta < 1$ is specified in Section 2. In the case when $\alpha < 1$, $(u^{(n)}, b^{(n)})$ satisfies

$$
\begin{cases}
u^{(1)} = S_1 u_0, \quad b^{(1)} = S_1 b_0, \\
\partial_t u^{(n+1)} + \nu(-\Delta)^\alpha u^{(n+1)} = \mathbb{P}(-u^{(n)} \cdot \nabla u^{(n+1)} + b^{(n)} \cdot \nabla b^{(n+1)}), \\
\partial_t b^{(n+1)} = -u^{(n)} \cdot \nabla b^{(n+1)} + b^{(n)} \cdot \nabla u^{(n+1)}, \\
\nabla \cdot u^{(n+1)} = \nabla \cdot b^{(n+1)} = 0, \\
u^{(n+1)}(x,0) = S_{n+1} u_0, \quad b^{(n+1)}(x,0) = S_{n+1} b_0
\end{cases}
\tag{1.9}
$$

and the corresponding functional setting is

$$\begin{aligned} &M = 2\max\left\{\|(u_0,b_0)\|_{B^{\sigma}_{2,\infty}}, \frac{1}{\sqrt{C_0}}\|(u_0,b_0)\|_{B^{\sigma}_{2,\infty}}\right\}, \\ &Y \equiv \left\{(u,b)\Big|\ \|(u,b)\|_{\widetilde{L}^\infty(0,T;B^{\sigma}_{2,\infty})} \le M,\ \|u\|_{\widetilde{L}^2(0,T;B^{\alpha+\sigma}_{2,\infty})} \le M\right\}, \end{aligned} \tag{1.10}$$

where $C_0 > 0$ is a pure constant as defined in (3.1). The main effort is devoted to showing that $(u^{(n)},b^{(n)})$ actually converges to a weak solution of (1.1). The process of obtaining a subsequence of $(u^{(n)},b^{(n)})$ that converges to a weak solution (u,b) of (1.1) is divided into two main steps. The first step is to assert the uniform boundedness of $(u^{(n)},b^{(n)})$ in Y while the second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma. The strong convergence then ensures that the limit is indeed a weak solution of (1.1). The uniform boundedness is shown via an iterative process. We assume $(u^{(n)},b^{(n)}) \in Y$ and show $(u^{(n+1)},b^{(n+1)}) \in Y$.

The technical approach to proving the uniform boundedness for the case $\alpha \ge 1$ is different from that for the case when $\alpha < 1$. For $\alpha < 1$, we estimate $u^{(n+1)}$ and $b^{(n+1)}$ in $\widetilde{L}^\infty(0,T;B^{\sigma}_{2,\infty})$, and $u^{(n+1)}$ in $\widetilde{L}^2(0,T;B^{\alpha+\sigma}_{2,\infty})$ simultaneously. The purpose is to make use of the cancellation resulting from adding the equations for $\|\Delta_j u^{(n+1)}\|_{L^2}$ and $\|\Delta_j b^{(n+1)}\|_{L^2}$. The cancellation is in the sum

$$\int_{\mathbb{R}^d} \Delta_j(b^{(n)}\cdot\nabla b^{(n+1)})\cdot\Delta_j u^{(n+1)}\,dx + \int_{\mathbb{R}^d} \Delta_j(b^{(n)}\cdot\nabla u^{(n+1)})\cdot\Delta_j b^{(n+1)}\,dx,$$

whose paraproduct decomposition contains

$$\int_{\mathbb{R}^d}\left(S_j b^{(n)}\cdot\nabla\Delta_j b^{(n+1)}\cdot\Delta_j u^{(n+1)} + S_j b^{(n)}\cdot\nabla\Delta_j u^{(n+1)}\cdot\Delta_j b^{(n+1)}\right)dx = 0$$

as $\nabla\cdot b^{(n)} = 0$. This appears to be the only approach that allows us to show the existence of solutions in functional spaces with the order of the derivative exceeding $\frac{d}{2}$. In the case when $\alpha \ge 1$, $b_0 \in \mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1}(\mathbb{R}^d)$ and the order of derivative is $\frac{d}{2}+1-\alpha \le \frac{d}{2}$. The desired norms of $u^{(n+1)}$ and $b^{(n+1)}$ can be suitably estimated without the cancellation. In addition, some upper bounds on products in Besov spaces are valid only for $\alpha \ge 1$ and break down when $\alpha < 1$. When $\alpha \ge 1$,

$$\begin{aligned} \|b^{(n)}\cdot\nabla b^{(n+1)}\|_{B^{\frac{d}{2}-2\alpha+1}_{2,1}(\mathbb{R}^d)} &\le \|b^{(n)}\otimes b^{(n+1)}\|_{B^{\frac{d}{2}-2\alpha+2}_{2,1}(\mathbb{R}^d)} \\ &\le C\|b^{(n)}\|_{B^{\frac{d}{2}-\alpha+1}_{2,1}(\mathbb{R}^d)}\|b^{(n+1)}\|_{B^{\frac{d}{2}-\alpha+1}_{2,1}(\mathbb{R}^d)} \end{aligned}$$

based on the following lemma (see, e.g., [2, p.90] or Lemma 2.6 in [32]).

LEMMA 1.1. *Let* $1 \le p \le \infty$, $s_1,s_2 \le \frac{d}{p}$ *and* $s_1+s_2 > d\max\{0,\frac{2}{p}-1\}$. *Then*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}(\mathbb{R}^d)} \le C\|f\|_{\dot{B}^{s_1}_{p,1}(\mathbb{R}^d)}\|g\|_{\dot{B}^{s_2}_{p,1}(\mathbb{R}^d)}.$$

However, when $\alpha < 1$, Lemma 1.1 breaks down since $s_1 = \frac{d}{2}-\alpha+1$ and $s_2 = \frac{d}{2}-\alpha+1$ no longer satisfy the condition $s_1,s_2 \le \frac{d}{2}$. This difficulty is overcome by performing a detailed analysis on different frequencies of this product when $\alpha < 1$.

The rest of this paper is divided into four sections and an appendix. Section 2 focuses on the proof of the existence part in Theorem 1.1 for the case when $\alpha \ge 1$

while Section 3 is devoted to the case when $\alpha<1$. Section 4 presents the proof for the uniqueness part of Theorem 1.1. We again distinguish between the case when $\alpha\geq 1$ and the case when $\alpha<1$. Section 5 explains in detail why the regularity assumptions on the initial data in Theorem 1.1 may be optimal. In particular, we describe the difficulties when the regularity assumptions are reduced to those in (1.2), (1.3) and (1.4). The Appendix provides the definitions of Besov spaces and other closely related tools.

2. Proof of the existence part in Theorem 1.1 for $\alpha\geq 1$

This section proves the existence part of Theorem 1.1 for the case $\alpha\geq 1$. The approach is to construct a successive approximation sequence and show that the limit of a subsequence actually solves (1.1) in the weak sense.

Proof. **(Proof for the existence part of Theorem 1.1 in the case when $\alpha\geq 1$.)** We consider a successive approximation sequence $\{(u^{(n)},b^{(n)})\}$ satisfying (1.7). We define the functional setting Y as in (1.8) . Our goal is to show that $\{(u^{(n)},b^{(n)})\}$ has a subsequence that converges to a weak solution of (1.1). This process consists of three main steps. The first step is to show that $(u^{(n)},b^{(n)})$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma while the last step is to show that the limit is indeed a weak solution of (1.1). Our main effort is devoted to showing the uniform bound for $(u^{(n)},b^{(n)})$ in Y. This is proven by induction.

We show inductively that $(u^{(n)},b^{(n)})$ is bounded uniformly in Y. Recall that (u_0,b_0) is in the regularity class (1.5). According to (1.7),

$$u^{(1)}=\mathring{S}_1u_0, \qquad b^{(1)}=\mathring{S}_1b_0.$$

Clearly,

$$\|u^{(1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha})}\leq M, \qquad \|b^{(1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha})}\leq M.$$

If $T>0$ is sufficiently small, then

$$\begin{aligned}
&\|u^{(1)}\|_{L^1(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1})}\leq T\|\mathring{S}_2u_0\|_{\mathring{B}_{2,1}^{\frac{d}{2}+1}}\leq TC\|u_0\|_{\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha}}\leq\delta,\\
&\|u^{(1)}\|_{\widetilde{L}^2(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha})}\leq\sqrt{T}C\|u_0\|_{\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha}}\leq\delta.
\end{aligned}$$

Assuming that $(u^{(n)},b^{(n)})$ obeys the bounds defined in Y, namely

$$\begin{aligned}
&\|u^{(n)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha})}\leq M, \qquad \|b^{(n)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha})}\leq M,\\
&\|u^{(n)}\|_{L^1(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1})}\leq\delta, \qquad \|u^{(n)}\|_{\widetilde{L}^2(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha})}\leq\delta,
\end{aligned}$$

we prove that $(u^{(n+1)},b^{(n+1)})$ obeys the same bounds for sufficiently small $T>0$ and suitably selected $\delta>0$. For the sake of clarity, the rest of this section is divided into five subsections.

2.1. The estimate of $u^{(n+1)}$ in $\widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d))$.

Let j be an integer. Applying $\mathring{\Delta}_j$ (we shall just use Δ_j to simplify the notation) to the second equation in (1.7) and then dotting with $\Delta_ju^{(n+1)}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_ju^{(n+1)}\|_{L^2}^2+\nu\|\Lambda^\alpha\Delta_ju^{(n+1)}\|_{L^2}^2=A_1+A_2, \tag{2.1}$$

where

$$A_1 = -\int \Delta_j(u^{(n)}\cdot\nabla u^{(n+1)})\cdot\Delta_j u^{(n+1)}\,dx,$$
$$A_2 = \int \Delta_j(b^{(n)}\cdot\nabla b^{(n)})\cdot\Delta_j u^{(n+1)}\,dx.$$

We remark that the projection operator $\mathbb{P}$ has been eliminated due to the divergence-free condition $\nabla\cdot u^{(n+1)}=0$. The dissipative part admits a lower bound

$$\nu\|\Lambda^\alpha\Delta_j u^{(n+1)}\|_{L^2}^2 \ge C_0\,2^{2\alpha j}\,\|\Delta_j u^{(n+1)}\|_{L^2}^2,$$

where $C_0>0$ is a constant. According to Lemma 6.2, A_1 can be bounded by

$$\begin{aligned}|A_1| &\le C\,\|\Delta_j u^{(n+1)}\|_{L^2}^2 \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\\ &\quad + C\,\|\Delta_j u^{(n+1)}\|_{L^2}\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2}\\ &\quad + C\,\|\Delta_j u^{(n+1)}\|_{L^2}\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\Delta_k u^{(n)}\|_{L^2}\,\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}.\end{aligned}$$

Also by Lemma 6.2, A_2 is bounded by

$$\begin{aligned}|A_2| &\le C\,\|\Delta_j u^{(n+1)}\|_{L^2}\,2^j\,\|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2}\\ &\quad + C\,\|\Delta_j u^{(n+1)}\|_{L^2}\,\|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n)}\|_{L^2}\\ &\quad + C\,\|\Delta_j u^{(n+1)}\|_{L^2}\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\Delta_k b^{(n)}\|_{L^2}\,\|\widetilde{\Delta}_k b^{(n)}\|_{L^2}.\end{aligned}$$

Inserting the estimates above in (2.1) and eliminating $\|\Delta_j u^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^2} + C_0\,2^{2\alpha j}\|\Delta_j u^{(n+1)}\|_{L^2} \le J_1+\cdots+J_6, \tag{2.2}$$

where

$$\begin{aligned}J_1 &= C\,\|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},\\ J_2 &= C\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2}\\ J_3 &= C\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\Delta_k u^{(n)}\|_{L^2}\,\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2},\\ J_4 &= C\,2^j\,\|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2},\\ J_5 &= C\,\|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n)}\|_{L^2},\\ J_6 &= C\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\Delta_k b^{(n)}\|_{L^2}\,\|\widetilde{\Delta}_k b^{(n)}\|_{L^2}.\end{aligned}$$

Integrating (2.2) in time yields

$$\|\Delta_j u^{(n+1)}(t)\|_{L^2} \le e^{-C_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2} + \int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}(J_1+\cdots+J_6)\,d\tau. \tag{2.3}$$

Taking the $L^\infty(0,T)$ of (2.3), then multiplying by $2^{(1+\frac{d}{2}-2\alpha)j}$ and summing over j, we have

$$\begin{aligned}&\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})}\\ &\le \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\left\|\int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}(J_1+\cdots+J_6)\,d\tau\right\|_{L^\infty(0,T)}.\end{aligned}$$

Applying Young's inequality to the convolution in time yields

$$\begin{aligned}&\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})}\\ &\le \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + \sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\|J_1+\cdots+J_6\|_{L^1(0,T)}.\end{aligned} \tag{2.4}$$

The terms on the right-hand side can be estimated as follows. Recalling the definition of J_1 and using the inductive assumption on $u^{(n)}$, we have

$$\begin{aligned}&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^T J_1\,d\tau\\ &= C\int_0^T \sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\|\Delta_j u^{(n+1)}\|_{L^2}\sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\,d\tau\\ &\le C\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})}\|u^{(n)}\|_{L^1(0,T;\mathring{B}^{1+\frac{d}{2}}_{2,1})}\\ &\le C\delta\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})}.\end{aligned}$$

The term involving J_2 admits the same bound. In fact,

$$\begin{aligned}&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^T J_2\,d\tau\\ &= C\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1} 2^{2\alpha(m-j)}2^{(1+\frac{d}{2}-2\alpha)m}\|\Delta_m u^{(n+1)}\|_{L^2}\,d\tau\\ &\le C\int_0^T \|u^{(n)}(\tau)\|_{\mathring{B}^{1+\frac{d}{2}}_{2,1}}\|u^{(n+1)}(\tau)\|_{\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1}}\,d\tau\\ &\le C\delta\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})},\end{aligned}$$

where we have used the fact that $2\alpha(m-j)\le 0$. The term with J_3 is bounded by

$$\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^T J_3\,d\tau = \int_0^T \sum_j 2^{(1+\frac{d}{2}-2\alpha)j}2^j\sum_{k\ge j-1} 2^{\frac{d}{2}k}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}\,d\tau$$

$$
\begin{aligned}
&= C\int_0^T \sum_j \sum_{k\ge j-1} 2^{(2+\frac{d}{2}-2\alpha)(j-k)}\, 2^{(1+\frac{d}{2})k}\|\Delta_k u^{(n)}\|_{L^2} \\
&\qquad\qquad \times 2^{(1+\frac{d}{2}-2\alpha)k}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\, d\tau \\
&\le C\int_0^T \|u^{(n)}(\tau)\|_{\mathring{B}^{1+\frac{d}{2}}_{2,1}} \|u^{(n+1)}(\tau)\|_{\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1}}\, d\tau \\
&\le C\delta \|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})},
\end{aligned}
$$

where we have used Young's inequality for series convolution and the fact $(2+\frac{d}{2}-2\alpha)(j-k)<0$. This is the place where we need $\alpha<1+\frac{d}{4}$. We now estimate the terms involving J_4 through J_6. The term with J_4 is bounded by,

$$
\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^T J_4\, d\tau \\
&= \sum_j \int_0^T 2^{(1+\frac{d}{2}-2\alpha)j}\, 2^j\, \|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2}\, d\tau \\
&= \int_0^T \sum_j 2^{(1+\frac{d}{2}-\alpha)j}\, \|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1-\alpha)(j-m)}\, 2^{(1+\frac{d}{2}-\alpha)m}\|\Delta_m b^{(n)}\|_{L^2}\, d\tau \\
&\le CT\|b^{(n)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})} \|b^{(n)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})} \\
&\le CTM^2,
\end{aligned}
$$

where we have used the fact that $\alpha\ge 1$ and $(1-\alpha)(j-m)\le 0$. The terms with J_5 and J_6 are estimated similarly and they obey the same bound. Inserting the bounds above in (2.4), we find

$$
\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})} \le \|u_0^{(n+1)}\|_{\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1}} + C\delta\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1})} + CTM^2. \tag{2.5}
$$

2.2. The estimate of $\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}-\alpha+1}_{2,1})}$. We use the third equation of (1.7). Applying Δ_j to the third equation in (1.7) and then dotting with $\Delta_j b^{(n+1)}$, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\Delta_j b^{(n+1)}\|_{L^2}^2 \le B_1 + B_2, \tag{2.6}
$$

where

$$
\begin{aligned}
B_1 &= -\int \Delta_j(u^{(n)}\cdot\nabla b^{(n+1)})\cdot\Delta_j b^{(n+1)}\, dx, \\
B_2 &= \int \Delta_j(b^{(n)}\cdot\nabla u^{(n)})\cdot\Delta_j b^{(n+1)}\, dx.
\end{aligned}
$$

By Lemma 6.2,

$$
|B_1| \le C\|\Delta_j b^{(n+1)}\|_{L^2}^2 \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}
$$

$$+C\|\Delta_j b^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n+1)}\|_{L^2}$$
$$+C\|\Delta_j b^{(n+1)}\|_{L^2}2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k b^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2}$$

and

$$|B_2|\le C\|\Delta_j b^{(n+1)}\|_{L^2}2^j\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1}2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2}$$
$$+C\|\Delta_j b^{(n+1)}\|_{L^2}\|\Delta_j b^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}$$
$$+C\|\Delta_j b^{(n+1)}\|_{L^2}2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\Delta_k b^{(n)}\|_{L^2}\|\widetilde{\Delta}_k u^{(n)}\|_{L^2}.$$

Inserting the estimates above in (2.6) and eliminating $\|\Delta_j b^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$\frac{d}{dt}\|\Delta_j b^{(n+1)}\|_{L^2}\le K_1+\cdots+K_6, \tag{2.7}$$

where

$$K_1=C\|\Delta_j b^{(n+1)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},$$
$$K_2=C\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n+1)}\|_{L^2}$$
$$K_3=C2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k b^{(n+1)}\|_{L^2}\|\Delta_k u^{(n)}\|_{L^2},$$
$$K_4=C2^j\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1}2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2},$$
$$K_5=C\|\Delta_j b^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},$$
$$K_6=C2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\Delta_k b^{(n)}\|_{L^2}\|\widetilde{\Delta}_k u^{(n)}\|_{L^2}.$$

Integrating (2.7) in time yields,

$$\|\Delta_j b^{(n+1)}(t)\|_{L^2}\le\|\Delta_j b_0^{(n+1)}\|_{L^2}+\int_0^t(K_1+\cdots+K_6)\,d\tau. \tag{2.8}$$

Taking the $L^\infty(0,T)$ of (2.8), multiplying by $2^{(\frac{d}{2}-\alpha+1)j}$ and summing over j, we have

$$\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}-\alpha+1}_{2,1})}$$
$$\le\|b_0^{(n+1)}\|_{\mathring{B}^{\frac{d}{2}-\alpha+1}_{2,1}}+\sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T(K_1+\cdots+K_6)\,d\tau. \tag{2.9}$$

The terms on the right can be bounded similarly as those in the previous subsection. In fact,

$$\sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T K_1\,d\tau\le C\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})}\|u^{(n)}\|_{L^1(0,T;\mathring{B}^{1+\frac{d}{2}}_{2,1})}$$

$$\leq C\delta \|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})}.$$

Similarly,

$$\sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T K_2\, d\tau \leq C\delta \|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})},$$
$$\sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T K_3\, d\tau \leq C\delta \|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})}.$$

The terms with K_4, K_5 and K_6 are bounded as follows.

$$\begin{aligned}
&\sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T K_4\, d\tau\\
&= C\sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T 2^j\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\leq j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2}\, d\tau\\
&= C\int_0^T \sum_j 2^{(\frac{d}{2}+1)j}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\leq j-1} 2^{(1-\alpha)(j-m)}\,2^{(\frac{d}{2}+1-\alpha)m}\|\Delta_m b^{(n)}\|_{L^2}\, d\tau\\
&\leq C\|u^{(n)}\|_{L^1(0,T;\mathring{B}^{1+\frac{d}{2}}_{2,1})}\|b^{(n)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})}\\
&\leq C\delta M,
\end{aligned}$$

where we have used the fact that $\alpha\geq 1$ and $(1-\alpha)(j-m)\leq 0$. Similarly,

$$\sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T K_5\, d\tau \leq C\delta M, \quad \sum_j 2^{(\frac{d}{2}-\alpha+1)j}\int_0^T K_6\, d\tau \leq C\delta M.$$

Inserting the estimates above in (2.9), we find

$$\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}-\alpha+1}_{2,1})} \leq \|b_0^{(n+1)}\|_{\mathring{B}^{\frac{d}{2}-\alpha+1}_{2,1}} + C\delta\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})} + C\delta M. \tag{2.10}$$

2.3. The estimate of $\|u^{(n+1)}\|_{L^1\left(0,T;\mathring{B}^{1+\frac{d}{2}}_{2,1}\right)}$. We multiply (2.3) by $2^{(1+\frac{d}{2})j}$, sum over j and integrate in time to obtain

$$\begin{aligned}
\|u^{(n+1)}\|_{L^1\left(0,T;\mathring{B}^{1+\frac{d}{2}}_{2,1}\right)} \leq& \int_0^T\sum_j 2^{(1+\frac{d}{2})j}e^{-C_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt\\
&+\int_0^T\sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)}(J_1+\cdots+J_6)\, d\tau\, ds.
\end{aligned}$$

We estimate the terms on the right and start with the first term.

$$\begin{aligned}
&\int_0^T\sum_j 2^{(1+\frac{d}{2})j}e^{-C_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt\\
&= C\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\left(1-e^{-C_0 2^{2\alpha j}T}\right)\|\Delta_j u_0^{(n+1)}\|_{L^2}.
\end{aligned}$$

Since $u_0 \in \mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1}$, it follows from the dominated convergence theorem that

$$\lim_{T\to 0}\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\left(1-e^{-C_0\,2^{2\alpha j}T}\right)\|\Delta_j u_0^{(n+1)}\|_{L^2}=0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\, e^{-C_0\,2^{2\alpha j}t}\,\|\Delta_j u_0^{(n+1)}\|_{L^2}\,dt \le \frac{\delta}{4}.$$

Applying Young's inequality for the time convolution, we have

$$\begin{aligned}
&\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0\,2^{2\alpha j}(s-\tau)} J_1\,d\tau\,ds\\
&= C\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0\,2^{2\alpha j}(s-\tau)}\|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\\
&\qquad\qquad \times \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}(\tau)\|_{L^2}\,d\tau\,ds\\
&\le C\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\,\|\Delta_j u^{(n+1)}\|_{L^2}\sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\,d\tau\times\int_0^T e^{-C_0\,2^{2\alpha j}s}ds\\
&\le C\int_0^T \sum_j 2^{(1+\frac{d}{2}-\alpha)j}\,\|\Delta_j u^{(n+1)}(\tau)\|_{L^2}\\
&\qquad\qquad \times \sum_{m\le j-1} 2^{(m-j)\alpha}\,2^{(1+\frac{d}{2}-\alpha)m}\|\Delta_m u^{(n)}(\tau)\|_{L^2}\,d\tau\\
&\le C\,\|u^{(n)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}\\
&\le C\,\delta\,\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}.
\end{aligned}$$

The terms with J_2 and J_3 can be estimated similarly and they share the same upper bound with the term involving J_1,

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0\,2^{2\alpha j}(s-\tau)} J_2\,d\tau\,ds \le C\,\delta\,\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})},$$

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0\,2^{2\alpha j}(s-\tau)} J_3\,d\tau\,ds \le C\,\delta\,\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}.$$

We now examine the terms involving J_4 through J_6. Again by Young's inequality,

$$\begin{aligned}
&\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0\,2^{2\alpha j}(s-\tau)} J_4\,d\tau\,ds\\
&= C\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0\,2^{2\alpha j}(s-\tau)}2^j\,\|\Delta_j b^{(n)}(\tau)\|_{L^2}\times\sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}(\tau)\|_{L^2}\,d\tau\,ds\\
&\le C\int_0^T \sum_j 2^{(\frac{d}{2}+2-2\alpha)j}\,\|\Delta_j b^{(n)}(\tau)\|_{L^2}\sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}(\tau)\|_{L^2}\,d\tau
\end{aligned}$$

$$\le C\int_0^T \sum_j 2^{(\frac{d}{2}+1-\alpha)j}\|\Delta_j b^{(n)}(\tau)\|_{L^2}\sum_{m\le j-1} 2^{(1-\alpha)(j-m)}2^{(\frac{d}{2}+1-\alpha)m}\|\Delta_m b^{(n)}(\tau)\|_{L^2}\,d\tau$$
$$\le C\int_0^T \|b^{(n)}(\tau)\|^2_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\,d\tau$$
$$\le CT\|b^{(n)}\|^2_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}\le CTM^2,$$

where we have used the fact that $\alpha\ge 1$ and $(1-\alpha)(j-m)\le 0$ again. The other two terms involving J_5 and J_6 obey the same bound,

$$\int_0^T\sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)}J_5\,d\tau\,ds\le CTM^2,$$
$$\int_0^T\sum_j 2^{(1+\frac{d}{2})j}\int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)}J_6\,d\tau\,ds\le CTM^2.$$

Here we have used $\alpha<1+\frac{d}{4}$ in the estimate of J_6. Collecting the estimates above leads to

$$\|u^{(n+1)}\|_{L^1\left(0,T;\mathring{B}^{1+\frac{d}{2}}_{2,1}\right)}\le\frac{\delta}{4}+C\delta\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}+CTM^2. \tag{2.11}$$

2.4. The bound for $\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}$. We multiply (2.3) by $2^{(1+\frac{d}{2}-\alpha)j}$, take the $L^2(0,T)$-norm and sum over j to obtain

$$\begin{aligned}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}&\le\sum_j 2^{(1+\frac{d}{2}-\alpha)j}\left\|e^{-C_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\right\|_{L^2(0,T)}\\&+\sum_j 2^{(1+\frac{d}{2}-\alpha)j}\left\|\int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)}(J_1+\cdots+J_6)\,d\tau\right\|_{L^2(0,T)}.\end{aligned}\tag{2.12}$$

The first term on the right is bound by

$$\begin{aligned}&\sum_j 2^{(1+\frac{d}{2}-\alpha)j}\left\|e^{-C_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\right\|_{L^2(0,T)}\\&=C\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\left(1-e^{-2C_0 2^{2\alpha j}T}\right)^{\frac12}\|\Delta_j u_0^{(n+1)}\|_{L^2}.\end{aligned}$$

Since $u_0\in\mathring{B}^{1+\frac{d}{2}-2\alpha}_{2,1}$, it follows from the dominated convergence theorem that

$$\lim_{T\to 0}\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\left(1-e^{-2C_0 2^{2\alpha j}T}\right)^{\frac12}\|\Delta_j u_0^{(n+1)}\|_{L^2}=0.$$

Therefore we can choose $T>0$ sufficiently small such that

$$\sum_j 2^{(1+\frac{d}{2}-\alpha)j}\left\|e^{-C_0 2^{2\alpha j}t}\|\Delta_j u_0^{(n+1)}\|_{L^2}\right\|_{L^2(0,T)}\le\frac{\delta}{4}.$$

The other six terms on the right of (2.12) are estimated as follows. Applying Young's inequality for the time convolution, we have

$$
\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_1 \, d\tau \right\|_{L^2(0,T)} \\
&= C \sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \right. \\
&\qquad\qquad \left. \times \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \right\|_{L^2(0,T)} \\
&\le C \sum_j 2^{(1+\frac{d}{2}-\alpha)j} \|e^{-C_0 2^{2\alpha j} s}\|_{L^2(0,T)} \\
&\qquad\qquad \times \left\| \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \right\|_{L^1(0,T)} \\
&\le C \int_0^T \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}\|_{L^2} \, d\tau \\
&\le C \int_0^T \sum_j 2^{(1+\frac{d}{2}-\alpha)j} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \\
&\qquad\qquad \times \sum_{m\le j-1} 2^{(m-j)\alpha} 2^{(1+\frac{d}{2}-\alpha)m} \|\Delta_m u^{(n)}(\tau)\|_{L^2} \, d\tau \\
&\le C \|u^{(n)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})} \|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})} \\
&\le C\delta \|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}.
\end{aligned}
$$

The terms with J_2 and J_3 share the same upper bound,

$$
\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_2 \, d\tau \right\|_{L^2(0,T)} \le C\delta \|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}, \\
&\sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_3 \, d\tau \right\|_{L^2(0,T)} \le C\delta \|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})}.
\end{aligned}
$$

The estimate of the term with J_4 is similar. Again by Young's inequality,

$$
\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_4 \, d\tau \right\|_{L^2(0,T)} \\
&= C \sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} 2^j \|\Delta_j b^{(n)}(\tau)\|_{L^2} \right. \\
&\qquad\qquad \left. \times \sum_{m\le j-1} 2^{\frac{d}{2}m} \|\Delta_m b^{(n)}(\tau)\|_{L^2} \, d\tau \right\|_{L^2(0,T)} \\
&\le C \sum_j 2^{(\frac{d}{2}+2-2\alpha)j} \left\| \|\Delta_j b^{(n)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m} \|\Delta_m b^{(n)}(\tau)\|_{L^2} \right\|_{L^1(0,T)} \\
&\le C \int_0^T \sum_j 2^{(\frac{d}{2}+1-\alpha)j} \|\Delta_j b^{(n)}(\tau)\|_{L^2}
\end{aligned}
$$

$$\times \sum_{m\le j-1} 2^{(1-\alpha)(j-m)} 2^{(\frac{d}{2}+1-\alpha)m} \|\Delta_m b^{(n)}(\tau)\|_{L^2}\, d\tau$$

$$\le C\int_0^T \|b^{(n)}(\tau)\|^2_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\, d\tau$$

$$\le CT\|b^{(n)}\|^2_{\widetilde{L}^\infty(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})} \le CTM^2.$$

The other two terms involving J_5 and J_6 obey the same bound,

$$\sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_5\, d\tau \right\|_{L^2(0,T)} \le CTM^2,$$

$$\sum_j 2^{(1+\frac{d}{2}-\alpha)j} \left\| \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} J_6\, d\tau \right\|_{L^2(0,T)} \le CTM^2.$$

Collecting the estimates above leads to

$$\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})} \le \frac{\delta}{4} + C\delta\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1})} + CTM^2. \tag{2.13}$$

2.5. Completion of the proof for the existence part in the case when $\alpha\ge 1$. The bounds in (2.5), (2.10), (2.11) and (2.13) allow us to conclude that, if we choose $T>0$ sufficiently small and $\delta>0$ suitably, then

$$\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-2\alpha}_{2,1})} \le M, \qquad \|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})} \le M,$$

$$\|u^{(n+1)}\|_{L^1(0,T;\mathring{B}^{\frac{d}{2}+1}_{2,1})} \le \delta, \qquad \|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})} \le \delta.$$

In fact, if T and δ in (2.5) satisfy

$$C\delta \le \frac{1}{4}, \qquad CTM\le \frac{1}{4},$$

then (2.5) implies

$$\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-2\alpha}_{2,1})} \le \frac{1}{2}M + \frac{1}{4}\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-2\alpha}_{2,1})} + \frac{1}{4}M$$

or

$$\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-2\alpha}_{2,1})} \le M.$$

Similarly, if $C\delta\le\frac{1}{4}$ in (2.10), then (2.10) states

$$\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})} \le M.$$

According to (2.13), if we choose $C\delta\le\frac{1}{4}$ and $CTM^2\le\frac{1}{2}\delta$, then

$$\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;\mathring{B}^{\frac{d}{2}+1-\alpha}_{2,1})} \le \delta$$

and consequently, if $C\delta\le\frac{1}{4}$ and $CTM^2\le\frac{1}{2}\delta$ in (2.11), then

$$\|u^{(n+1)}\|_{\widetilde{L}^1(0,T;\mathring{B}^{\frac{d}{2}+1}_{2,1})} \le \delta.$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is, there is $(u,b)\in Y$ such that a subsequence of $(u^{(n)},b^{(n)})$ (still denoted by $(u^{(n)},b^{(n)})$) satisfies

$$u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{in} \quad \widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha}),$$
$$b^{(n)} \overset{*}{\rightharpoonup} b \quad \text{in} \quad \widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha}).$$

In order to show that (u,b) is a weak solution of (1.1), we need to further extract a subsequence which converges strongly to (u,b). This is done via the Aubin-Lions Lemma. We can show by making use of the equations in (1.7) that $(\partial_t u^{(n)},\partial_t b^{(n)})$ is uniformly bounded in

$$\partial_t u^{(n)} \in L^1(0,T;\mathring{B}_{2,1}^{\frac{d}{2}-2\alpha+1})\cap\widetilde{L}^2(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-3\alpha}),$$
$$\partial_t b^{(n)} \in L^2(0,T;\mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha})$$

Since we are in the case of the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions Lemma to show that a subsequence of the weakly convergent subsequence, still denoted by $(u^{(n)},b^{(n)})$, has the following strongly convergent property,

$$(u^{(n)},b^{(n)})\to(u,b) \qquad \text{in} \quad L^2(0,T;\mathring{B}_{2,1}^{\gamma}(Q)),$$

where $\frac{d}{2}+1-2\alpha\le\gamma<\frac{d}{2}+1-\alpha$ and $Q\subset\mathbb{R}^d$ is any compact subset. This strong convergence property would allow us to show that (u,b) is indeed a weak solution of (1.1). This process is routine and we omit the details. This completes the proof for the existence part of Theorem 1.1 in the case when $\alpha\ge 1$. □

3. Proof of the existence part in Theorem 1.1 with $\alpha<1$

This section proves the existence part of Theorem 1.1 for the case when $\alpha<1$. The idea is still to construct a successive approximation sequence and show that the limit of a subsequence actually solves (1.1) in the weak sense. Some of the technical approaches here are different from those for $\alpha\ge 1$.

Proof. **(Proof for the existence part of Theorem 1.1 in the case when $\alpha<1$.)** We consider a successive approximation sequence $\{(u^{(n)},b^{(n)})\}$ satisfying (1.9). We define the functional setting Y as in (1.10). Our goal is to show that $\{(u^{(n)},b^{(n)})\}$ has a subsequence that converges to the weak solution of (1.1). This process consists of three main steps. The first step is to show that $(u^{(n)},b^{(n)})$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma while the last step is to show that the limit is indeed a weak solution of (1.1). Our main effort is devoted to showing the uniform bound for $(u^{(n)},b^{(n)})$ in Y. This is proven by induction. We start with the basis step. Recall that (u_0,b_0) is in the regularity class (1.6). According to (1.9),

$$u^{(1)}=S_2u_0, \qquad b^{(1)}=S_2b_0.$$

Clearly,

$$\|u^{(1)}\|_{\widetilde{L}^\infty(0,T;B_{2,\infty}^\sigma)}\le M, \qquad \|b^{(1)}\|_{\widetilde{L}^\infty(0,T;B_{2,\infty}^\sigma)}\le M.$$

If $T>0$ is sufficiently small, then

$$\|u^{(1)}\|_{\widetilde{L}^2(0,T;B_{2,\infty}^{\alpha+\sigma})}\le\sqrt{T}\|S_2u_0\|_{B_{2,\infty}^{\alpha+\sigma}}\le\sqrt{T}C\|u_0\|_{B_{2,\infty}^\sigma}\le M.$$

Assuming that $(u^{(n)}, b^{(n)})$ obeys the bounds defined in Y, namely

$$\|u^{(n)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})} \le M, \quad \|b^{(n)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})} \le M, \quad \|u^{(n)}\|_{\widetilde{L}^2(0,T;B^{\alpha+\sigma}_{2,\infty})} \le M,$$

we prove that $(u^{(n+1)}, b^{(n+1)})$ obeys the same bound for sufficiently small $T > 0$.

The proof involves inhomogeneous dyadic block operator Δ_j and the inhomogeneous Besov spaces. Let $j \ge 0$ be an integer. Applying Δ_j to the second and third equation in (1.9) and then dotting by $(\Delta_j u^{(n+1)}, \Delta_j b^{(n+1)})$, we have

$$\frac{d}{dt}\left(\|\Delta_j u^{(n+1)}\|_{L^2}^2 + \|\Delta_j b^{(n+1)}\|_{L^2}^2\right) + C_0\, 2^{2\alpha j}\, \|\Delta_j u^{(n+1)}\|_{L^2}^2 \le E_1 + E_2 + E_3, \tag{3.1}$$

where $C_0 > 0$ is constant and

$$\begin{aligned}
E_1 &= -2\int \Delta_j(u^{(n)} \cdot \nabla u^{(n+1)}) \cdot \Delta_j u^{(n+1)}\, dx,\\
E_2 &= -2\int \Delta_j(u^{(n)} \cdot \nabla b^{(n+1)}) \cdot \Delta_j b^{(n+1)}\, dx,\\
E_3 &= 2\int \Delta_j(b^{(n)} \cdot \nabla b^{(n+1)}) \cdot \Delta_j u^{(n+1)}\, dx + \int \Delta_j(b^{(n)} \cdot \nabla u^{(n+1)}) \cdot \Delta_j b^{(n+1)}\, dx.
\end{aligned}$$

According to Lemma 6.2, E_1 is bounded by

$$\begin{aligned}
|E_1| \le{}& C\|\Delta_j u^{(n+1)}\|_{L^2}^2 \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\\
&+ C\|\Delta_j u^{(n+1)}\|_{L^2}\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2}\\
&+ C\|\Delta_j u^{(n+1)}\|_{L^2}\, 2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\, \|\Delta_k u^{(n)}\|_{L^2}\, \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\\
:={}& L_1 + L_2 + L_3.
\end{aligned}$$

E_2 is bounded by

$$\begin{aligned}
|E_2| \le{}& C\|\Delta_j b^{(n+1)}\|_{L^2}^2 \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\\
&+ C\|\Delta_j b^{(n+1)}\|_{L^2}\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n+1)}\|_{L^2}\\
&+ C\|\Delta_j b^{(n+1)}\|_{L^2}\, 2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\, \|\widetilde{\Delta}_k b^{(n+1)}\|_{L^2}\, \|\Delta_k u^{(n)}\|_{L^2}\\
:={}& L_4 + L_5 + L_6.
\end{aligned}$$

E_3 is bounded by

$$\begin{aligned}
|E_3| \le{}& C\|\Delta_j u^{(n+1)}\|_{L^2}\,\|\Delta_j b^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n)}\|_{L^2}\\
&+ C\|\Delta_j u^{(n+1)}\|_{L^2}\,\|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n+1)}\|_{L^2}\\
&+ C\|\Delta_j u^{(n+1)}\|_{L^2}\, 2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\, \|\Delta_k b^{(n)}\|_{L^2}\, \|\widetilde{\Delta}_k b^{(n+1)}\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+C\|\Delta_j b^{(n+1)}\|_{L^2}\|\Delta_j b^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2}\\
&+C\|\Delta_j b^{(n+1)}\|_{L^2}2^j\sum_{k\ge j-1}2^{\frac{d}{2}k}\|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\|\Delta_k b^{(n)}\|_{L^2}\\
:=&\, L_7+L_8+L_9+L_{10}+L_{11}.
\end{aligned}
$$

Inserting these bounds in (3.1) and then integrating in time yield

$$
\begin{aligned}
&\|\Delta_j u^{(n+1)}\|_{L^2}^2+\|\Delta_j b^{(n+1)}\|_{L^2}^2+C_0 2^{2\alpha j}\int_0^t\|\Delta_j u^{(n+1)}\|_{L^2}^2\,d\tau\\
\le&\|\Delta_j u_0^{(n+1)}\|_{L^2}^2+\|\Delta_j b_0^{(n+1)}\|_{L^2}^2+\int_0^t(L_1+\cdots+L_{11})\,d\tau. \qquad (3.2)
\end{aligned}
$$

When $j=-1$, all the nonlinear terms can be estimated similarly and the only difference is on the dissipative term. When $j=-1$, the dissipative term no longer admits a lower bound because the support of the Fourier transform includes the origin. This does not cause a problem. It is easy to see from (1.9) that the L^2-norm of $(u^{(n+1)},b^{(n+1)})$ is bounded uniformly,

$$
\|u^{(n+1)}(t)\|_{L^2}^2+\|b^{(n+1)}(t)\|_{L^2}^2+2\nu\|\Lambda^\alpha u^{(n+1)}(t)\|_{L_t^2L^2}^2\le\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2.
$$

Since $\|\Delta_{-1}u^{(n+1)}(t)\|_{L^2}\le\|u^{(n+1)}(t)\|_{L^2}$ is uniformly bounded and the time interval on which we are seeking the solution is finite, (3.2) remains valid for $j=-1$ if we add a constant term on the right. Taking $L^\infty(0,T)$ of (3.2), then multiplying by $2^{2\sigma j}$ and taking the sup in j yield

$$
\begin{aligned}
&\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}^2+\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}^2+C_0\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}^2\\
\le&\|u_0\|_{B^\sigma_{2,\infty}}^2+\|b_0\|_{B^\sigma_{2,\infty}}^2+\sup_j 2^{2\sigma j}\int_0^T(L_1+\cdots+L_{11})\,d\tau. \qquad (3.3)
\end{aligned}
$$

We now estimate the eleven terms on the right. By Hölder's inequality,

$$
\begin{aligned}
&\sup_j 2^{2\sigma j}\int_0^T L_1\,d\tau\\
=&\,C\sup_j 2^{2\sigma j}\int_0^T\|\Delta_j u^{(n+1)}\|_{L^2}^2\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\,d\tau\\
\le&\,C\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}^2\sup_j\sum_{m\le j-1}2^{(1+\frac{d}{2}-(\alpha+\sigma))m}\int_0^T 2^{(\alpha+\sigma)m}\|\Delta_m u^{(n)}\|_{L^2}\,d\tau\\
\le&\,C\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}^2\sum_{m\le j-1}2^{(1+\frac{d}{2}-(\alpha+\sigma))m}\sqrt{T}\,\|2^{(\alpha+\sigma)m}\|\Delta_m u^{(n)}\|_{L^2}\|_{L^2(0,T)}\\
\le&\,C\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}^2\sqrt{T}\sup_m\|2^{(\alpha+\sigma)m}\|\Delta_m u^{(n)}\|_{L^2}\|_{L^2(0,T)}\\
=&\,C\sqrt{T}\|u^{(n)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}^2\\
\le&\,C\sqrt{T}M\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}^2,
\end{aligned}
$$

where we have used the fact that $\alpha+\sigma>1+\frac{d}{2}$ and we are working with inhomogeneous dyadic blocks. The terms with L_2, L_3 and L_4 can be bounded very similarly and the bounds for them are

$$\sup_j 2^{2\sigma j}\int_0^T L_2\,d\tau \le C\sqrt{T}M\,\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})},$$

$$\sup_j 2^{2\sigma j}\int_0^T L_3\,d\tau \le C\sqrt{T}M\,\|u^{(n+1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})},$$

$$\sup_j 2^{2\sigma j}\int_0^T L_4\,d\tau \le C\sqrt{T}M\,\|b^{(n+1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}.$$

The term with L_5 is estimated slightly differently.

$$\begin{aligned}
&\sup_j 2^{2\sigma j}\int_0^T L_5\,d\tau\\
&= C\sup_j 2^{2\sigma j}\int_0^T \|\Delta_j b^{(n+1)}\|_{L^2}\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n+1)}\|_{L^2}\,d\tau\\
&= C\sup_j\int_0^T 2^{\sigma j}\|\Delta_j b^{(n+1)}\|_{L^2}\,2^{(\alpha+\sigma)j}\|\Delta_j u^{(n)}\|_{L^2}\\
&\qquad\qquad \times\sum_{m\le j-1}2^{\alpha(m-j)}\,2^{(1+\frac{d}{2}-(\alpha+\sigma))m}\,2^{\sigma m}\,\|\Delta_m b^{(n+1)}\|_{L^2}\,d\tau\\
&\le C\|b^{(n+1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\sup_j\int_0^T 2^{(\alpha+\sigma)j}\|\Delta_j u^{(n)}\|_{L^2}\,d\tau\\
&\le C\|b^{(n+1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\sqrt{T}\sup_j\|2^{(\alpha+\sigma)j}\|\Delta_j u^{(n)}\|_{L^2}\|_{L^2(0,T)}\\
&= C\sqrt{T}\|u^{(n)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}\|b^{(n+1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\\
&\le C\sqrt{T}M\|b^{(n+1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})},
\end{aligned}$$

where we have used the fact that $m-j<0$ and $1+\frac{d}{2}-\alpha-\sigma<0$. The estimates of the other terms are similar,

$$\sup_j 2^{2\sigma j}\int_0^T L_6\,d\tau \le C\sqrt{T}M\,\|b^{(n+1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})},$$

$$\sup_j 2^{2\sigma j}\int_0^T L_7\,d\tau \le C\sqrt{T}M\,\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})},$$

$$\sup_j 2^{2\sigma j}\int_0^T L_8\,d\tau \le C\sqrt{T}M\,\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})},$$

$$\sup_j 2^{2\sigma j}\int_0^T L_9\,d\tau \le C\sqrt{T}M\,\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})},$$

$$\sup_j 2^{2\sigma j}\int_0^T L_{10}\,d\tau \le C\sqrt{T}M\,\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})},$$

$$\sup_j 2^{2\sigma j}\int_0^T L_{11}\,d\tau \le C\sqrt{T}M\,\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}.$$

Inserting the bounds above in (3.3) yields

$$\|(u^{(n+1)},b^{(n+1)})\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}+C_0\|u^{(n+1)}\|^2_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}$$

$$\begin{aligned}&\le \|(u_0,b_0)\|^2_{B^\sigma_{2,\infty}} + C\sqrt{T}M\|(u^{(n+1)},b^{(n+1)})\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\\&\quad + C\sqrt{T}M\|(u^{(n+1)},b^{(n+1)})\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}.\end{aligned} \tag{3.4}$$

We choose $T>0$ to be sufficiently small such that

$$\begin{aligned}&C\sqrt{T}M\|(u^{(n+1)},b^{(n+1)})\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\\&\qquad + C\sqrt{T}M\|(u^{(n+1)},b^{(n+1)})\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}\\&\le \frac{3}{4}\|(u^{(n+1)},b^{(n+1)})\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})} + \frac{3}{4}C_0\|u^{(n+1)}\|^2_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})}.\end{aligned}$$

Then (3.4) implies

$$\begin{aligned}&\|(u^{(n+1)},b^{(n+1)})\|_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})} \le 2\|(u_0,b_0)\|_{B^\sigma_{2,\infty}} = M,\\&\|u^{(n+1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})} \le \frac{2}{\sqrt{C_0}}\|(u_0,b_0)\|_{B^\sigma_{2,\infty}} \le M.\end{aligned}$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is, there is $(u,b)\in Y$ such that a subsequence of $(u^{(n)},b^{(n)})$ (still denoted by $(u^{(n)},b^{(n)})$) satisfies

$$\begin{aligned}&u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{in} \quad \widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})\cap\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty}),\\&b^{(n)} \overset{*}{\rightharpoonup} b \quad \text{in} \quad \widetilde{L}^\infty(0,T;B^\sigma_{2,\infty}).\end{aligned}$$

In order to show that (u,b) is a weak solution of (1.1), we need to further extract a subsequence which converges strongly to (u,b). This is done via the Aubin-Lions Lemma. We can show by making use of the equations in (1.9) that $(\partial_t u^{(n)},\partial_t b^{(n)})$ is uniformly bounded in

$$\partial_t u^{(n)} \in \widetilde{L}^2(0,T;B^{\sigma-\alpha}_{2,\infty}), \qquad \partial_t b^{(n)} \in \widetilde{L}^2(0,T;B^{\frac{d}{2}}_{2,\infty}).$$

Since the domain here is the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions Lemma to show that a subsequence of the weakly convergent subsequence, still denoted by $(u^{(n)},b^{(n)})$, has the following strongly convergent property,

$$\begin{aligned}&u^{(n)} \to u \quad \text{in} \quad L^2(0,T;B^{\gamma_1}_{2,\infty}(Q)) \quad \text{for} \quad \gamma_1\in(\sigma-\alpha,\sigma+\alpha)\\&b^{(n)} \to b \quad \text{in} \quad L^2(0,T;B^{\gamma_2}_{2,\infty}(Q)) \quad \text{for} \quad \gamma_2\in(d/2,\sigma),\end{aligned}$$

where $Q\subset\mathbb{R}^d$ is any compact subset. This strong convergence property would allow us to show that (u,b) is indeed a weak solution of (1.1). This process is routine and we omit the details. This completes the proof for the existence part of Theorem 1.1 in the case when $\alpha<1$. □

4. Proof for the uniqueness part of Theorem 1.1

This section proves the uniqueness part of Theorem 1.1. The case $\alpha\ge 1$ is treated differently from the case $\alpha<1$. When $\alpha\ge 1$, estimating the L^2-difference of two solutions does not lead to the desired uniqueness due to the difficulty in suitably bounding one of the terms. Instead we need to estimate the velocity difference $\widetilde{u}$ in a different setting from that for the difference for the magnetic field $\widetilde{b}$. More precisely, we combine the estimate of $\|\widetilde{u}\|_{\widetilde{L}^1_t \mathring{B}^{\frac{d}{2}}_{2,\infty}(\mathbb{R}^d)}$ with $\|\widetilde{b}\|_{L^\infty_t \mathring{B}^{\frac{d}{2}-\alpha}_{2,\infty}(\mathbb{R}^d)}$. After invoking a logarithmic Besov-type inequality, we are able to obtain the desired uniqueness. The effectiveness of this approach for the uniqueness was discovered in [43] for the 2D MHD equations with the standard Laplacian dissipation and used in [32] for the 2D and 3D MHD equations again with the standard Laplacian dissipation. The case $\alpha<1$ is simpler. We can directly work with the L^2-difference to obtain the uniqueness.

Naturally this section is divided into two subsections. The first subsection deals with the simpler case $\alpha<1$ while the second subsection is devoted to the case $\alpha\geq 1$.

4.1. The case $\alpha<1$.

Proof. Assume that $(u^{(1)},b^{(1)})$ and $(u^{(2)},b^{(2)})$ are two solutions. Their difference $(\widetilde{u},\widetilde{b})$ with

$$\widetilde{u}=u^{(2)}-u^{(1)}, \qquad \widetilde{b}=b^{(2)}-b^{(1)}$$

satisfies

$$\begin{cases}\partial_t\widetilde{u}+\nu(-\Delta)^\alpha\widetilde{u}=-\mathbb{P}(u^{(2)}\cdot\nabla\widetilde{u}+\widetilde{u}\cdot\nabla u^{(1)})+\mathbb{P}(b^{(2)}\cdot\nabla\widetilde{b}+\widetilde{b}\cdot\nabla b^{(1)}),\\ \partial_t\widetilde{b}=-u^{(2)}\cdot\nabla\widetilde{b}-\widetilde{u}\cdot\nabla b^{(1)}+b^{(2)}\cdot\nabla\widetilde{u}+\widetilde{b}\cdot\nabla u^{(1)},\\ \nabla\cdot\widetilde{u}=\nabla\cdot\widetilde{b}=0,\\ \widetilde{u}(x,0)=0, \quad \widetilde{b}(x,0)=0.\end{cases} \tag{4.1}$$

We estimate the difference $(\widetilde{u},\widetilde{b})$ in $L^2(\mathbb{R}^d)$. Dotting (4.1) by $(\widetilde{u},\widetilde{b})$ and applying the divergence-free condition, we find

$$\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|^2_{L^2}+\|\widetilde{b}\|^2_{L^2})+\nu\|\Lambda^\alpha\widetilde{u}\|^2_{L^2}=Q_1+Q_2+Q_3+Q_4+Q_5, \tag{4.2}$$

where

$$\begin{aligned} Q_1&=-\int\widetilde{u}\cdot\nabla u^{(1)}\cdot\widetilde{u}\,dx,\\ Q_2&=\int b^{(2)}\cdot\nabla\widetilde{b}\cdot\widetilde{u}\,dx+\int b^{(2)}\cdot\nabla\widetilde{u}\cdot\widetilde{b}\,dx,\\ Q_3&=\int\widetilde{b}\cdot\nabla b^{(1)}\cdot\widetilde{u}\,dx,\\ Q_4&=-\int\widetilde{u}\cdot\nabla b^{(1)}\cdot\widetilde{b}\,dx,\\ Q_5&=\int\widetilde{b}\cdot\nabla u^{(1)}\cdot\widetilde{b}\,dx.\end{aligned}$$

As $\nabla\cdot b^{(2)}=0$, we find $Q_2=0$ after integration by parts. We remark that Q_3+Q_4 is not necessarily zero. The operator Δ_j in this subsection denotes the inhomogeneous dyadic block operators and the Besov spaces are inhomogeneous. By Hölder's inequality,

$$|Q_1|\leq\|\nabla u^{(1)}\|_{L^\infty}\|\widetilde{u}\|^2_{L^2}, \qquad |Q_5|\leq\|\nabla u^{(1)}\|_{L^\infty}\|\widetilde{b}\|^2_{L^2}.$$

To bound Q_3, we set

$$\frac{1}{p}=\frac{1}{2}-\frac{\alpha}{d}, \qquad \frac{1}{q}+\frac{1}{p}=\frac{1}{2}$$

and apply Hölder's inequality and Sobolev's inequality to obtain

$$\begin{aligned}|Q_3| &\leq \|\widetilde{b}\|_{L^2} \|\nabla b^{(1)}\|_{L^q} \|\widetilde{u}\|_{L^p} \\ &\leq C\|\widetilde{b}\|_{L^2} \|\nabla b^{(1)}\|_{L^q} \|\Lambda^\alpha \widetilde{u}\|_{L^2} \\ &\leq \frac{\nu}{4}\|\Lambda^\alpha \widetilde{u}\|_{L^2}^2 + C\|\nabla b^{(1)}\|_{L^q}^2 \|\widetilde{b}\|_{L^2}^2.\end{aligned}$$

Q_4 obeys exactly the same bound. Inserting these bounds in (4.2), we find

$$\begin{aligned}&\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2) + \nu\|\Lambda^\alpha \widetilde{u}\|_{L^2}^2 \\ &\leq C\|\nabla u^{(1)}\|_{L^\infty}(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2) + C\|\nabla b^{(1)}\|_{L^q}^2 \|\widetilde{b}\|_{L^2}^2.\end{aligned} \tag{4.3}$$

By Bernstein's inequality,

$$\begin{aligned}\int_0^T \|\nabla u^{(1)}\|_{L^\infty}\, dt &\leq \sum_{j\geq -1}\int_0^T \|\Delta_j \nabla u^{(1)}\|_{L^\infty}\, dt \\ &\leq \sum_{j\geq -1}\int_0^T 2^{(1+\frac{d}{2})j}\|\Delta_j u^{(1)}\|_{L^2} dt \\ &\leq \sum_{j\geq -1} 2^{(1+\frac{d}{2}-\alpha-\sigma)j}\int_0^T 2^{(\alpha+\sigma)j}\|\Delta_j u^{(1)}\|_{L^2} dt \\ &\leq \sum_{j\geq -1} 2^{(1+\frac{d}{2}-\alpha-\sigma)j}\sqrt{T}\|2^{(\alpha+\sigma)j}\|\Delta_j u^{(1)}\|_{L^2}\|_{L^2(0,T)} \\ &\leq \leq C\sqrt{T}\|u^{(1)}\|_{\widetilde{L}^2(0,T;B^{\sigma+\alpha}_{2,\infty})},\end{aligned} \tag{4.4}$$

where we have used the fact that $\sigma > 1+\frac{d}{2}-\alpha$. In addition,

$$\begin{aligned}\|\nabla b^{(1)}\|_{L^q} &\leq \sum_{j\geq -1}\|\Delta_j \nabla b^{(1)}\|_{L^q} \\ &\leq C\sum_{j\geq -1} 2^{j+dj(\frac{1}{2}-\frac{1}{q})}\|\Delta_j b^{(1)}\|_{L^2} \\ &\leq C\sum_{j\geq -1} 2^{j+dj(\frac{1}{2}-\frac{\alpha}{d})}\|\Delta_j b^{(1)}\|_{L^2} \\ &= C\sum_{j\geq -1} 2^{(1+\frac{d}{2}-\alpha-\sigma)j}2^{\sigma j}\|\Delta_j b^{(1)}\|_{L^2} \\ &\leq C\|b^{(1)}\|_{B^\sigma_{2,\infty}},\end{aligned}$$

where again we have used the fact that $\sigma > 1+\frac{d}{2}-\alpha$. Therefore,

$$\int_0^T \|\nabla b^{(1)}\|_{L^q}^2\, dt \leq CT\|b^{(1)}\|^2_{\widetilde{L}^\infty(0,T;B^\sigma_{2,\infty})}. \tag{4.5}$$

Applying Grönwall's inequality to (4.3) and invoking (4.4) and (4.5), we obtain

$$\|\widetilde{u}\|_{L^2} = \|\widetilde{b}\|_{L^2} = 0,$$

which leads to the desired uniqueness. This completes the proof of the uniqueness part of Theorem 1.1 for the case when $\alpha < 1$. ∎

4.2. The case $\alpha \geq 1$. As we have mentioned at the beginning of this section, the uniqueness for this case can no longer be established by estimating the difference in L^2. Before we give the proof, we explain why one of the terms can not be bounded suitably in the L^2-setting. Q_1 and Q_5 can still be bounded as before. Q_3 has to be bounded differently. By integration by parts,

$$Q_3 = -\int \widetilde{b}\cdot\nabla\widetilde{u}\cdot b^{(1)}\,dx.$$

For p and q defined by

$$\frac{1}{p} = \frac{1}{2} + \frac{1-\alpha}{d}, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{2},$$

we have, by Hölder's inequality,

$$\begin{aligned}
|Q_3| &\leq \|\widetilde{b}\|_{L^2}\,\|\nabla\widetilde{u}\|_{L^p}\,\|b^{(1)}\|_{L^q}\\
&\leq C\,\|\widetilde{b}\|_{L^2}\,\|\Lambda^{\alpha-1}\nabla\widetilde{u}\|_{L^2}\,\|b^{(1)}\|_{L^q}\\
&\leq \frac{\nu}{4}\|\Lambda^{\alpha}\widetilde{u}\|_{L^2}^2 + C\,\|b^{(1)}\|_{L^q}^2\,\|\widetilde{b}\|_{L^2}^2.
\end{aligned}$$

Q_4 can not be bounded similarly as Q_3. In fact, Q_4 is the main trouble that prevents us from deriving the uniqueness. In order to prove the uniqueness, we have to abandon the L^2-setting and combine the estimate of $\|\widetilde{u}\|_{\widetilde{L}_t^1\mathring{B}_{2,\infty}^{\frac{d}{2}}(\mathbb{R}^d)}$ with $\|\widetilde{b}\|_{L_t^\infty\mathring{B}_{2,\infty}^{\frac{d}{2}-\alpha}(\mathbb{R}^d)}$. The precise proof is given as follows.

Proof. Throughout the following proof, Δ_j denotes the homogeneous dyadic block operators for the simplicity of notation. As in (2.2), we have from (4.1) that

$$\begin{aligned}
&\frac{d}{dt}\|\Delta_j\widetilde{u}\|_{L^2} + C_0\,2^{2\alpha j}\|\Delta_j\widetilde{u}\|_{L^2}\\
&\leq \|[\Delta_j, u^{(2)}\cdot\nabla]\widetilde{u}\|_{L^2} + \|\Delta_j(\widetilde{u}\cdot\nabla u^{(1)})\|_{L^2} + \|\Delta_j(b^{(2)}\cdot\nabla\widetilde{b})\|_{L^2} + \|\Delta_j(\widetilde{b}\cdot\nabla b^{(1)})\|_{L^2}.
\end{aligned}$$

Integrating in time and taking into account the zero initial condition, we find

$$\begin{aligned}
\|\Delta_j\widetilde{u}\|_{L^2} \leq \int_0^t e^{-C_0 2^{2\alpha j}(t-\tau)}\Big(&\|[\Delta_j, u^{(2)}\cdot\nabla]\widetilde{u}\|_{L^2} + \|\Delta_j(\widetilde{u}\cdot\nabla u^{(1)})\|_{L^2}\\
&+\|\Delta_j(b^{(2)}\cdot\nabla\widetilde{b})\|_{L^2} + \|\Delta_j(\widetilde{b}\cdot\nabla b^{(1)})\|_{L^2}\Big)\,d\tau.
\end{aligned}$$

For $1\leq q\leq\infty$, we take the L^q-norm in time and apply Young's inequality for the time convolution to obtain

$$\begin{aligned}
\|\Delta_j\widetilde{u}\|_{L_t^qL^2} \leq \|e^{-C_0 2^{2\alpha j}t}\|_{L^q}\Big(&\|[\Delta_j, u^{(2)}\cdot\nabla]\widetilde{u}\|_{L_t^1L^2} + \|\Delta_j(\widetilde{u}\cdot\nabla u^{(1)})\|_{L_t^1L^2}\\
&+\|\Delta_j(b^{(2)}\cdot\nabla\widetilde{b})\|_{L_t^1L^2} + \|\Delta_j(\widetilde{b}\cdot\nabla b^{(1)})\|_{L_t^1L^2}\Big).
\end{aligned}$$

Multiplying each side by $2^{\frac{2\alpha}{q}j+(\frac{d}{2}-2\alpha)j}$ and taking the supremum with respect to j, we have

$$\begin{aligned}
\|\widetilde{u}\|_{\widetilde{L}_t^q\mathring{B}_{2,\infty}^{\frac{d}{2}-2\alpha+\frac{2\alpha}{q}}} \leq C\sup_{j\in\mathbb{Z}} 2^{(\frac{d}{2}-2\alpha)j}\Big(&\|[\Delta_j, u^{(2)}\cdot\nabla]\widetilde{u}\|_{L_t^1L^2} + \|\Delta_j(\widetilde{u}\cdot\nabla u^{(1)})\|_{L_t^1L^2}\\
&+\|\Delta_j(b^{(2)}\cdot\nabla\widetilde{b})\|_{L_t^1L^2} + \|\Delta_j(\widetilde{b}\cdot\nabla b^{(1)})\|_{L_t^1L^2}\Big).
\end{aligned}$$

The four terms on the right of the inequality above can be bounded as in the proof of the existence part. More precisely, the first two terms are bounded by

$$\sup_{j\in\mathbb{Z}} 2^{(\frac{d}{2}-2\alpha)j}\|[\Delta_j, u^{(2)}\cdot\nabla]\widetilde{u}\|_{L^1_tL^2} \le C\|u^{(2)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}}\|\widetilde{u}\|_{L^\infty_t\mathring{B}^{\frac{d}{2}-2\alpha}_{2,\infty}},$$
$$\sup_{j\in\mathbb{Z}} 2^{(\frac{d}{2}-2\alpha)j}\|\Delta_j(\widetilde{u}\cdot\nabla u^{(1)})\|_{L^1_tL^2} \le C\|u^{(1)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}}\|\widetilde{u}\|_{L^\infty_t\mathring{B}^{\frac{d}{2}-2\alpha}_{2,\infty}}.$$

The second two terms are bounded slightly differently,

$$\begin{aligned}\sup_{j\in\mathbb{Z}} 2^{(\frac{d}{2}-2\alpha)j}\|\Delta_j(b^{(2)}\cdot\nabla\widetilde{b})\|_{L^1_tL^2} &\le \int_0^t \sup_{j\in\mathbb{Z}} 2^{(\frac{d}{2}-2\alpha)j}\|\Delta_j(b^{(2)}\cdot\nabla\widetilde{b})\|_{L^2}\,d\tau\\ &\le C\int_0^t \|b^{(2)}\|_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\|\widetilde{b}\|_{\mathring{B}^{\frac{d}{2}-\alpha}_{2,\infty}}\,d\tau,\\ \sup_{j\in\mathbb{Z}} 2^{(\frac{d}{2}-2\alpha)j}\|\Delta_j(\widetilde{b}\cdot\nabla b^{(1)})\|_{L^1_tL^2} &\le C\int_0^t \|b^{(1)}\|_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\|\widetilde{b}\|_{\mathring{B}^{\frac{d}{2}-\alpha}_{2,\infty}}\,d\tau.\end{aligned}$$

Using these bounds and taking $q=\infty$ and $q=1$, we find

$$\begin{aligned}\|\widetilde{u}\|_{L^\infty_t\mathring{B}^{\frac{d}{2}-2\alpha}_{2,\infty}} + \|\widetilde{u}\|_{\widetilde{L}^1_t\mathring{B}^{\frac{d}{2}}_{2,\infty}} &\le C\left(\|u^{(1)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}} + \|u^{(2)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}}\right)\|\widetilde{u}\|_{L^\infty_t\mathring{B}^{\frac{d}{2}-2\alpha}_{2,\infty}}\\ &\quad + C\int_0^t\left(\|b^{(1)}\|_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}} + \|b^{(2)}\|_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\right)\|\widetilde{b}\|_{\mathring{B}^{\frac{d}{2}-\alpha}_{2,\infty}}\,d\tau. \end{aligned}\tag{4.6}$$

Working with the equation of $\widetilde{b}$ in (4.1), we have

$$\begin{aligned}\|\widetilde{b}\|_{L^\infty_tB^{\frac{d}{2}-\alpha}_{2,\infty}} \le \sup_{j\in\mathbb{Z}} 2^{(\frac{d}{2}-\alpha)j}\int_0^t&\Big(\|[\Delta_j, u^{(2)}\cdot\nabla]\widetilde{b}\|_{L^2} + \|\Delta_j(\widetilde{u}\cdot\nabla b^{(1)})\|_{L^2}\\ &+\|\Delta_j(b^{(2)}\cdot\nabla\widetilde{u})\|_{L^2} + \|\Delta_j(\widetilde{b}\cdot\nabla u^{(1)})\|_{L^2}\Big)\,d\tau.\end{aligned}$$

Estimating the four terms on the right in a similar fashion as in the existence proof, we find that

$$\begin{aligned}\|\widetilde{b}\|_{L^\infty_tB^{\frac{d}{2}-\alpha}_{2,\infty}} &\le C\left(\|u^{(1)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}} + \|u^{(2)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}}\right)\|\widetilde{b}\|_{L^\infty_t\mathring{B}^{\frac{d}{2}-\alpha}_{2,\infty}}\\ &\quad + C\left(\|b^{(1)}\|_{L^\infty_t\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}} + \|b^{(2)}\|_{L^\infty_t\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\right)\|\widetilde{u}\|_{L^1_t\mathring{B}^{\frac{d}{2}}_{2,1}}. \end{aligned}\tag{4.7}$$

Using a basic fact from real analysis (stated as a lemma below), we can choose T_0 (smaller than the maximal existence time T) such that, for any $0<t\le T_0$,

$$C\left(\|u^{(1)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}} + \|u^{(2)}\|_{L^1_t\mathring{B}^{1+\frac{d}{2}}_{2,1}}\right) \le \frac{1}{2}.$$

It then follows from (4.6) and (4.7) that

$$\frac{1}{2}\|\widetilde{u}\|_{L^\infty_t\mathring{B}^{\frac{d}{2}-2\alpha}_{2,\infty}} + \|\widetilde{u}\|_{\widetilde{L}^1_t\mathring{B}^{\frac{d}{2}}_{2,\infty}} \le C\int_0^t\left(\|b^{(1)}\|_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}} + \|b^{(2)}\|_{\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\right)\|\widetilde{b}\|_{\mathring{B}^{\frac{d}{2}-\alpha}_{2,\infty}}\,d\tau,$$
$$\frac{1}{2}\|\widetilde{b}\|_{L^\infty_tB^{\frac{d}{2}-\alpha}_{2,\infty}} \le C\left(\|b^{(1)}\|_{L^\infty_t\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}} + \|b^{(2)}\|_{L^\infty_t\mathring{B}^{1+\frac{d}{2}-\alpha}_{2,1}}\right)\|\widetilde{u}\|_{L^1_t\mathring{B}^{\frac{d}{2}}_{2,1}}.$$

Inserting the second inequality into the first one and invoking the fact that, for any $0<t\le T_0$,

$$\|b^{(1)}\|_{L_t^\infty \mathring{B}_{2,1}^{1+\frac{d}{2}-\alpha}}+\|b^{(2)}\|_{L_t^\infty \mathring{B}_{2,1}^{1+\frac{d}{2}-\alpha}}\le C(T),$$

we obtain

$$\|\widetilde{u}\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}\le C\int_0^t \|\widetilde{u}\|_{L_\tau^1 \mathring{B}_{2,1}^{\frac{d}{2}}}\,d\tau.$$

Invoking the logarithmic Besov-type inequality (see Lemma 4.2 below)

$$\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{\frac{d}{2}}}\le C\|\widetilde{u}\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}\log\left(e+\frac{\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}}+\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}}}{\|\widetilde{u}\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}}\right),$$

we obtain

$$\|\widetilde{u}\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}\le C\int_0^t \|\widetilde{u}\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}\log\left(e+\frac{\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}}+\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}}}{\|\widetilde{u}\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}}\right)d\tau.$$

Due to the boundedness

$$\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}}+\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}}<\infty,$$

Osgood's inequality then implies that, for any $t\le T_0$,

$$\|\widetilde{u}\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}=0\quad\text{and}\quad\|\widetilde{u}\|_{L_t^1 \mathring{B}_{2,1}^{\frac{d}{2}}}=0.$$

Therefore,

$$\|\widetilde{b}\|_{L_t^\infty B_{2,\infty}^{\frac{d}{2}-\alpha}}=0.$$

In particular, $\widetilde{u}=\widetilde{b}=0$ almost everywhere for $t\le T_0$. Repeating this process yields the uniqueness on the whole time interval $[0,T]$. This completes the proof of the uniqueness part of Theorem 1.1 for the case when $\alpha\ge 1$. $\square$

In the proof above, we have used two facts stated in the following lemmas. The first lemma can be found in [35].

LEMMA 4.1. *Let $(X,\mathcal{B},\mu)$ be a complete measure space. Let f be an integrable function with respect to the measure μ. Then, given any $\varepsilon>0$, there is $\delta>0$ such that, if $A\in\mathcal{B}$ and $\mu(A)<\delta$, then*

$$\left|\int_A f\,d\mu\right|<\varepsilon.$$

The second lemma states a logarithmic Besov-type inequality, which generalizes Lemma 3.1 of [43]. The proof is parallel, but we provide it for the convenience of readers.

LEMMA 4.2. *Let $\alpha \geq 1$. If F satisfies, for $t>0$,*

$$\|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)} + \|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}(\mathbb{R}^d)} < \infty,$$

then

$$\|F\|_{L_t^1 \mathring{B}_{2,1}^{\frac{d}{2}}} \leq C \|F\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}} \log\left(e + \frac{\|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}} + \|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}}}{\|F\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}} \right).$$

Proof. The proof relies on the definition of Besov spaces in terms of the Littlewood-Paley decomposition. For an integer $N>0$ to be specified later,

$$\begin{aligned} \|F\|_{L_t^1 \mathring{B}_{2,1}^{\frac{d}{2}}} &= \sum_{j\in\mathbb{Z}} 2^{\frac{d}{2}j} \|\Delta_j F\|_{L_t^1 L^2} \\ &= \left(\sum_{j\leq -N} + \sum_{-N\leq j<N} + \sum_{j\geq N} \right) 2^{\frac{d}{2}j} \|\Delta_j F\|_{L_t^1 L^2}. \end{aligned}$$

For $\alpha \geq 1$, the low frequency part can be controlled by

$$\begin{aligned} \sum_{j\leq -N} 2^{\frac{d}{2}j} \|\Delta_j F\|_{L^2} &= \sum_{j\leq -N} 2^{(2\alpha-1)j} 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j F\|_{L_t^1 L^2} \\ &\leq 2^{-(2\alpha-1)N} \|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}} \leq 2^{-N} \|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}}. \end{aligned}$$

The high frequency part can be bounded by

$$\sum_{j\geq N} 2^{\frac{d}{2}j} \|\Delta_j F\|_{L_t^1 L^2} = \sum_{j>N} 2^{-j} 2^{(1+\frac{d}{2})j} \|\Delta_j F\|_{L_t^1 L^2} \leq 2^{-N} \|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}}.$$

Therefore,

$$\|F\|_{L_t^1 \mathring{B}_{2,1}^{\frac{d}{2}}} \leq 2^{-N} \left(\|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}} + \|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}} \right) + 2N \|F\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}.$$

If we take N to be the integer part of

$$\log\left(e + \frac{\|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}} + \|F\|_{L_t^1 \mathring{B}_{2,1}^{1+\frac{d}{2}}}}{\|F\|_{\widetilde{L}_t^1 \mathring{B}_{2,\infty}^{\frac{d}{2}}}} \right),$$

then we obtain the desired inequality. This completes the proof of Lemma 4.2. ∎

5. Conclusion and discussions

We have established that, for $\alpha \geq 1$, any initial data (u_0, b_0) with

$$u_0 \in \mathring{B}_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d), \quad b_0 \in \mathring{B}_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d),$$

and, for $\alpha < 1$, any initial data (u_0, b_0) with

$$u_0 \in B_{2,\infty}^{\sigma}(\mathbb{R}^d), \quad b_0 \in B_{2,\infty}^{\sigma},(\mathbb{R}^d), \qquad \sigma > \frac{d}{2} + 1 - \alpha \tag{5.1}$$

leads to a unique local weak solution of (1.1). The purpose of this section is to explain in some detail why these regularity assumptions may be optimal. The optimality for the case $\alpha \geq 1$ can be easily explained. The index $\frac{d}{2}+1-2\alpha$ is minimal for the velocity in order to achieve the uniqueness. As we know, the velocity u should obey $\int_0^T \|\nabla u\|_{L^\infty}\,dt < \infty$ or a slightly weaker version in order to guarantee the uniqueness. In the Besov setting here, we need

$$\|u\|_{L^1(0,T;\mathring{B}_{2,1}^{1+\frac{d}{2}}(\mathbb{R}^d))} < \infty,$$

which, in turn, requires that

$$u \in \widetilde{L}^\infty(0,T;\mathring{B}_{2,1}^{1+\frac{d}{2}-2\alpha}(\mathbb{R}^d)).$$

This is how the index $\frac{d}{2}+1-2\alpha$ arises. Once the Besov space for u_0 is set, the functional setting for b_0 is determined correspondingly.

We now explain why the initial setup for the case $\alpha < 1$ may be optimal. We have attempted to replace (5.1) by several weaker assumptions, but we failed to establish the desired existence and uniqueness. We now describe the difficulties associated with those weaker initial data.

5.1. Can we replace (5.1) by $u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)$ and $b_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)$? We would have difficulty proving the uniform boundedness of the successive approximation sequence in the existence proof part. If we assume that

$$u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d) \quad \text{and} \quad b_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d),$$

then the corresponding functional space for (u,b) would be

$$\begin{aligned} Y \equiv \Big\{(u,b) \,\Big|\, &\|u\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-2\alpha})} \leq M, \quad \|b\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}})} \leq M, \\ &\|u\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+1})} \leq \delta, \quad \|u\|_{\widetilde{L}^2(0,T;B_{2,1}^{\frac{d}{2}+1-\alpha})} \leq \delta \Big\}. \end{aligned}$$

Suppose we construct the successive approximation sequence by (1.7). We can obtain suitable bounds for

$$\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-2\alpha})}, \quad \|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}})}.$$

We would have difficulty controlling $\|u^{(n+1)}\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+1})}$ due to the term $b^{(n)}\cdot\nabla b^{(n)}$ in (1.7). A quick way to see the difficultly is to count the derivatives needed and the derivatives allowed,

$$\left(\frac{d}{2}+1\right)+\left(\frac{d}{2}+1\right)-2\alpha = 2\left(\frac{d}{2}+1-\alpha\right) > 2\cdot\frac{d}{2}.$$

We explain the meaning of this inequality. The left-hand side $2\left(\frac{d}{2}+1-\alpha\right)$ represents the derivative imposed and the right-hand side $2\cdot\frac{d}{2}$ denotes the derivatives allowed on the two $b^{(n)}$'s. The first $\frac{d}{2}+1$ comes from $B_{2,1}^{\frac{d}{2}+1}$, the second $\frac{d}{2}+1$ represents the derivative when we estimate $\|b^{(n)}\cdot\nabla b^{(n)}\|_{L^2}$ and -2α is due to the dissipation. When $\alpha<1$, the derivatives imposed are more than the derivatives allowed and we can not close the estimates in Y.

5.2. Can we replace (5.1) by $u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)$ and $b_0 \in B_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d)$? Even though we increased the regularity of b_0 to the level that would allow us to overcome one difficulty mentioned in the previous subsection, we would still have trouble proving the uniform boundedness of the successive approximation sequence in the existence proof part. If we assume that

$$u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d) \quad \text{and} \quad b_0 \in B_{2,1}^{\frac{d}{2}+1-\alpha}(\mathbb{R}^d),$$

then the corresponding functional space for (u,b) would be

$$Y \equiv \Big\{(u,b)\Big|\|u\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-2\alpha})} \le M, \quad \|b\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-\alpha})} \le M,$$
$$\|u\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+1})} \le \delta, \quad \|u\|_{\widetilde{L}^2(0,T;B_{2,1}^{\frac{d}{2}+1-\alpha})} \le \delta\Big\}$$

Suppose we construct the successive approximation sequence by (1.7). We can obtain suitable bounds for

$$\|u^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-2\alpha})}, \quad \|u^{(n+1)}\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+1})}.$$

But this new setup would make it impossible to control

$$\|b^{(n+1)}\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-\alpha})}.$$

The difficulty comes from bounding the term $b^{(n)}\cdot\nabla u^{(n)}$ in the equation of $b^{(n+1)}$ in (1.7). In order to bound $\|\Delta_j(b^{(n)}\cdot\nabla u^{(n)})\|_{L^2}$, one naturally decomposes it by paraproducts as in (2.7),

$$\begin{aligned}\|\Delta_j(b^{(n)}\cdot\nabla u^{(n)})\|_{L^2} \le{}& C2^j\,\|\Delta_j u^{(n)}\|_{L^2}\sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2}\\ &+C\,\|\Delta_j b^{(n)}\|_{L^2}\sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}\\ &+C\,2^j\sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\Delta_k b^{(n)}\|_{L^2}\,\|\widetilde{\Delta}_k u^{(n)}\|_{L^2}.\end{aligned}$$

The trouble arises in the first term on the right-hand side. For $\alpha<1$, we can no longer bound

$$2^{(1-\alpha)j}\sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2}$$

by

$$\sum_{m\le j-1} 2^{(\frac{d}{2}+1-\alpha)m}\|\Delta_m b^{(n)}\|_{L^2}$$

and, as a consequence, we are not able to control $b^{(n)}\cdot\nabla u^{(n)}$ by the desired bound $\|u^{(n)}\|_{L^1(0,T;B_{2,1}^{\frac{d}{2}+1})}\|b^{(n)}\|_{\widetilde{L}^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-\alpha})}$. This problem arises when u and b are in different functional settings. We can no longer estimate u and b simultaneously and the good structure of combining the terms $b\cdot\nabla b$ and $b\cdot\nabla u$ can no longer be taken advantage of.

5.3. Can we replace (5.1) by $u_0\in B^{\frac{d}{2}+1-\alpha}_{2,1}(\mathbb{R}^d)$ and $b_0\in B^{\frac{d}{2}+1-\alpha}_{2,1}(\mathbb{R}^d)$? Even u_0 and b_0 are now in the same functional setting, but we are still not able to prove the uniform boundedness of the successive approximation sequence in the existence proof part. We now explain the difficulty. Naturally the corresponding functional setting for (u,b) is

$$Y\equiv\Big\{(u,b)\,\Big|\,\|(u,b)\|_{\widetilde{L}^\infty(0,T;B^{\frac{d}{2}+1-\alpha}_{2,1})}\le M,\qquad \|u\|_{L^1(0,T;B^{\frac{d}{2}+1}_{2,1})}\le\delta,\quad \|u\|_{\widetilde{L}^2(0,T;B^{\frac{d}{2}+1-\alpha}_{2,1})}\le\delta\Big\}$$

Suppose we construct the successive approximation sequence by (1.9). In order to make use of the cancellation in the combination of $b\cdot\nabla b$ and $b\cdot\nabla u$, we have to add the estimates at the L^2-level as in (3.1). However, if we add them at the L^2-level, it is then impossible to control the norm of (u,b) in $B^{\frac{d}{2}+1-\alpha}_{2,1}$. This is exactly why we have selected the functional setting $B^\sigma_{2,\infty}$ with $\sigma>\frac{d}{2}+1-\alpha$ when $\alpha<1$, as in the proof in Section 3.

In conclusion, the regularity assumptions on the initial data in Theorem 1.1 may be optimal.

Acknowledgements. Q. Jiu was partially supported by the National Natural Science Foundation of China (NNSFC) (No. 11671273, No. 11931010), key research project of the Academy for Multidisciplinary Studies of CNU, and Beijing Natural Science Foundation (BNSF) (No. 1192001). J. Wu was partially supported by the National Science Foundation of the United States under grant number DMS-1614246 and by the AT&T Foundation at the Oklahoma State University. H. Yu was partially supported by the National Natural Science Foundation of China (NNSFC) (No. 11901040), Beijing Natural Science Foundation (BNSF) (No. 1204030) and the grant from Beijing Municipal Commission of Education.

Appendix. Besov spaces and related tools. This appendix provides the definition of the Besov spaces and related facts that have been used in the previous sections. Some of the materials are taken from [2]. More details can be found in several books and many papers (see, e.g., [2, 4, 34, 39, 42]). In addition, we also prove several bounds on triple products involving Fourier localized functions. These bounds have been used in the previous sections.

We start with the partition of unit. Let $B(0,r)$ and $\mathcal{C}(0,r_1,r_2)$ denote the standard ball and the annulus, respectively,

$$B(0,r)=\left\{\xi\in\mathbb{R}^d:|\xi|\le r\right\},\qquad \mathcal{C}(0,r_1,r_2)=\left\{\xi\in\mathbb{R}^d:r_1\le|\xi|\le r_2\right\}.$$

There are two compactly supported smooth radial functions ϕ and ψ satisfying

$$\begin{aligned}&\operatorname{supp}\phi\subset B(0,4/3),\quad \operatorname{supp}\psi\subset\mathcal{C}(0,3/4,8/3),\\ &\phi(\xi)+\sum_{j\ge0}\psi(2^{-j}\xi)=1\qquad\text{for all }\xi\in\mathbb{R}^d,\\ &\sum_{j\in\mathbb{Z}}\psi(2^{-j}\xi)=1\qquad\text{for }\xi\in\mathbb{R}^d\setminus\{0\}\ .\end{aligned}\tag{6.1}$$

We use $\widetilde{h}$ and h to denote the inverse Fourier transforms of ϕ and ψ respectively,

$$\widetilde{h}=\mathcal{F}^{-1}\phi,\quad h=\mathcal{F}^{-1}\psi.$$

In addition, for notational convenience, we write $\psi_j(\xi)=\psi(2^{-j}\xi)$. By a simple property of the Fourier transform,

$$h_j(x):=\mathcal{F}^{-1}(\psi_j)(x)=2^{dj}\,h(2^j x).$$

The inhomogeneous dyadic block operators Δ_j are defined as follows

$$\begin{aligned}
&\Delta_j f=0 \qquad \text{for } j\le -2,\\
&\Delta_{-1} f=\widetilde{h}*f=\int_{\mathbb{R}^d} f(x-y)\,\widetilde{h}(y)\,dy,\\
&\Delta_j f=h_j*f=2^{dj}\int_{\mathbb{R}^d} f(x-y)\,h(2^j y)\,dy \qquad \text{for } j\ge 0.
\end{aligned}$$

The corresponding inhomogeneous low frequency cut-off operator S_j is defined by

$$S_j f=\sum_{k\le j-1}\Delta_k f.$$

For any function f in the usual Schwarz class $\mathcal{S}$, (6.1) implies

$$\widehat{f}(\xi)=\phi(\xi)\,\widehat{f}(\xi)+\sum_{j\ge 0}\psi(2^{-j}\xi)\,\widehat{f}(\xi) \tag{6.2}$$

or, in terms of the inhomogeneous dyadic block operators,

$$f=\sum_{j\ge -1}\Delta_j f \quad \text{or} \quad \mathrm{Id}=\sum_{j\ge -1}\Delta_j,$$

where Id denotes the identity operator. More generally, for any F in the space of tempered distributions, denoted $\mathcal{S}'$, (6.2) still holds but in the distributional sense. That is, for $F\in\mathcal{S}'$,

$$F=\sum_{j\ge -1}\Delta_j F \quad \text{or} \quad \mathrm{Id}=\sum_{j\ge -1}\Delta_j \qquad \text{in} \quad \mathcal{S}'. \tag{6.3}$$

In fact, one can verify that

$$S_j F:=\sum_{k\le j-1}\Delta_k F \quad \to \quad F \qquad \text{in} \quad \mathcal{S}'.$$

Equation (6.3) is referred to as the Littlewood-Paley decomposition for tempered distributions.

In terms of the inhomogeneous dyadic block operators, we can write the standard product in terms of the paraproducts, namely the Bony decomposition,

$$FG=\sum_{|j-k|\le 2} S_{k-1}F\,\Delta_k G+\sum_{|j-k|\le 2}\Delta_k F\,S_{k-1}G+\sum_{k\ge j-1}\Delta_k F\,\widetilde{\Delta}_k G,$$

where $\widetilde{\Delta}_k=\Delta_{k-1}+\Delta_k+\Delta_{k+1}$.

The inhomogeneous Besov space can be defined in terms of Δ_j specified above.

Definition 6.1. *The inhomogeneous Besov space* $B^s_{p,q}$ *with* $1\le p,q\le\infty$ *and* $s\in\mathbb{R}$ *consists of* $f\in\mathcal{S}'$ *satisfying*

$$\|f\|_{B^s_{p,q}}\equiv\|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q}<\infty.$$

The concepts defined above have their homogeneous version. The homogeneous dyadic block and the homogeneous low frequency cutoff operators are defined by, for any $j\in\mathbb{Z}$,

$$\mathring{\Delta}_j f=h_j*f=2^{dj}\int_{\mathbb{R}^d}f(x-y)h(2^jy)\,dy,$$
$$\mathring{S}_j f=\sum_{k\le j-1}\mathring{\Delta}_k f=2^{jd}\int_{\mathbb{R}^d}\widetilde{h}(2^jy)f(x-y)\,dy.$$

For any function f in the usual Schwarz class $\mathcal{S}$, (6.1) implies

$$\widehat{f}(\xi)=\sum_{j\in\mathbb{Z}}\psi(2^{-j}\xi)\widehat{f}(\xi)\qquad\text{for all }\xi\in\mathbb{R}^d$$

when f satisfies, for any x^β in the set of all polynomials $\mathcal{P}$,

$$\int_{\mathbb{R}^d}x^\beta f(x)\,dx=0.$$

In order to write the Littlewood-Paley decomposition for $F\in\mathcal{S}'$, we need to restrict to the subspace $\mathcal{S}'_h$ consisting of $f\in\mathcal{S}'$ satisfying

$$\lim_{j\to-\infty}\mathring{S}_j f=0\quad\text{in}\quad\mathcal{S}'.$$

Any $f\in\mathcal{S}'$ that has a locally integrable Fourier transform is in $\mathcal{S}'_h$.

The homogeneous Besov space can be defined in terms of $\mathring{\Delta}_j$ specified above.

Definition 6.2. *The homogeneous Besov space* $\mathring{B}^s_{p,q}$ *with* $1\le p,q\le\infty$ *and* $s\in\mathbb{R}$ *consists of* $f\in\mathcal{S}'_h$ *satisfying*

$$\|f\|_{\mathring{B}^s_{p,q}}\equiv\|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q}<\infty.$$

In terms of the homogeneous dyadic blocks, we can also write the standard products in terms of the paraproducts.

The following space-time spaces introduced in [13] have been used in the previous sections.

Definition 6.3. *Let* $s\in\mathbb{R}$ *and* $1\le p,q,r\le\infty$. *Let* $T\in(0,\infty]$. *The space-time space* $\widetilde{L}^r(0,T;B^s_{p,q})$ *consists of tempered distributions satisfying*

$$\|f\|_{\widetilde{L}^r(0,T;B^s_{p,q})}\equiv\|2^{js}\|\|\Delta_j f\|_{L^p}\|_{L^r(0,T)}\|_{l^q}<\infty.$$

$\widetilde{L}^r(0,T;\mathring{B}^s_{p,q})$ *is similarly defined.*

By Minkowski's inequality, the standard space-time space $L^r(0,T;B^s_{p,q})$ is related to $\widetilde{L}^r(0,T;B^s_{p,q})$ as follows

$$L^r(0,T;B^s_{p,q})\subsetneq\widetilde{L}^r(0,T;B^s_{p,q})\qquad\text{if } r<q,$$

$$\widetilde{L}^r(0,T;B^s_{p,q}) \subsetneq L^r(0,T;B^s_{p,q}) \qquad \text{if } r>q,$$
$$L^r(0,T;B^s_{p,q}) = \widetilde{L}^r(0,T;B^s_{p,q}) \qquad \text{if } r=q.$$

Bernstein's inequality is a useful tool on Fourier localized functions and these inequalities trade derivatives for integrability. The following proposition provides Bernstein-type inequalities for fractional derivatives.

Lemma 6.1. *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

1) If f satisfies

$$\mathrm{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq K 2^j\},$$

for some integer j and a constant $K>0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$\mathrm{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0<K_1 \leq K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

We now state and prove bounds for the triple products involving Fourier localized functions. These bounds have been used in the previous sections in the proof of Theorem 1.1.

Lemma 6.2. *Let $j \in \mathbb{Z}$ be an integer. Let Δ_j be a dyadic block operator (either inhomogeneous or homogeneous).*

(1) Let F be a divergence-free vector field. Then there exists a constant C independent of j such that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \Delta_j (F \cdot \nabla G) \cdot \Delta_j H \, dx \right| \\
&\leq C \|\Delta_j H\|_{L^2} \Big(2^j \sum_{m \leq j-1} 2^{\frac{d}{2}m} \|\Delta_m F\|_{L^2} \sum_{|j-k|\leq 2} \|\Delta_k G\|_{L^2} \\
&\qquad + \sum_{|j-k|\leq 2} \|\Delta_k F\|_{L^2} \sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m G\|_{L^2} \\
&\qquad + \sum_{k \geq j-1} 2^j 2^{\frac{d}{2}k} \|\Delta_k F\|_{L^2} \|\widetilde{\Delta}_k G\|_{L^2} \Big). \qquad (6.4)
\end{aligned}$$

(2) Let F be a divergence-free vector field. Then there exists a constant C independent of j such that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \Delta_j (F \cdot \nabla G) \cdot \Delta_j G \, dx \right| \\
&\leq C \|\Delta_j G\|_{L^2} \Big(\sum_{m \leq j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m F\|_{L^2} \sum_{|j-k|\leq 2} \|\Delta_k G\|_{L^2}
\end{aligned}$$

$$+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m G\|_{L^2}$$
$$+\sum_{k\ge j-1}2^j\,2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^2}\Big) \tag{6.5}$$

(3) Let F be a divergence-free vector field. Then there exists a constant C independent of j such that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla H)\cdot\Delta_j G\,dx+\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j H\,dx\right|\\
&\le C\|\Delta_j G\|_{L^2}\Big(\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m F\|_{L^2}\sum_{|j-k|\le 2}\|\Delta_k H\|_{L^2}\\
&\quad+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\sum_{m\le k-1}2^{(1+\frac{d}{2})m}\|\Delta_m H\|_{L^2}\\
&\quad+\sum_{k\ge j-1}2^j\,2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k H\|_{L^2}\Big)\\
&\quad+C\|\Delta_j H\|_{L^2}\Big(\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m F\|_{L^2}\sum_{|j-k|\le 2}\|\Delta_k G\|_{L^2}\\
&\quad+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\sum_{m\le k-1}2^{(1+\frac{d}{2})m}\|\Delta_m G\|_{L^2}\\
&\quad+\sum_{k\ge j-1}2^j\,2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^2}\Big).
\end{aligned}\tag{6.6}$$

Proof. The proof of these inequalities essentially follow from the paraproduct decomposition. By the paraproduct decomposition,

$$\begin{aligned}
\Delta_j(F\cdot\nabla G)=&\sum_{|j-k|\le 2}\Delta_j(S_{k-1}F\cdot\Delta_k\nabla G)+\sum_{|j-k|\le 2}\Delta_j(\Delta_k F\cdot S_{k-1}\nabla G)\\
&+\sum_{k\ge j-1}\Delta_j(\Delta_k F\cdot\nabla\widetilde{\Delta}_k G).
\end{aligned}$$

By Hölder's inequality and Bernstein's inequality in Lemma 6.1,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j H\,dx\right|\\
&\le\|\Delta_j H\|_{L^2}\Big(\sum_{|j-k|\le 2}2^k\|S_{k-1}F\|_{L^\infty}\|\Delta_k G\|_{L^2}+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\|S_{k-1}\nabla G\|_{L^\infty}\\
&\quad+\sum_{k\ge j-1}2^j\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^\infty}\Big),
\end{aligned}$$

where we have used $\nabla\cdot F=0$ in the last part. Equation (6.4) then follows if we invoke the inequalities of the form

$$\|S_{k-1}F\|_{L^\infty}\le\sum_{m\le k-2}2^{\frac{d}{2}m}\|\Delta_m F\|_{L^2}. \tag{6.7}$$

To prove (6.5), we further write the first term as the sum of a commutator and two correction terms,

$$
\begin{aligned}
\Delta_j(F\cdot\nabla G) = & \sum_{|j-k|\le 2}[\Delta_j, S_{k-1}F\cdot\nabla]\Delta_k G \\
& + \sum_{|j-k|\le 2}(S_{k-1}F - S_jF)\cdot\Delta_j\Delta_k\nabla G \\
& + S_jF\cdot\nabla\Delta_j G + \sum_{|j-k|\le 2}\Delta_j(\Delta_k F\cdot S_{k-1}\nabla G) \\
& + \sum_{k\ge j-1}\Delta_j(\Delta_k F\cdot\nabla\widetilde{\Delta}_k G).
\end{aligned}
$$

As $\nabla\cdot F = 0$,

$$
\int_{\mathbb{R}^d} S_jF\cdot\nabla\Delta_j G\cdot\Delta_j G\,dx = 0.
$$

By Hölder's inequality, Bernstein's inequality and a commutator estimate,

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j G\,dx\right| \le \|\Delta_j G\|_{L^2}\Big(& \sum_{|j-k|\le 2}\|\nabla S_{k-1}F\|_{L^\infty}\|\Delta_k G\|_{L^2} \\
& + C2^{(1+\frac{d}{2})j}\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\|\Delta_j G\|_{L^2} \\
& + \sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\|S_{k-1}\nabla G\|_{L^\infty} \\
& + \sum_{k\ge j-1}2^j 2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^2}\Big).
\end{aligned}
$$

(6.5) then follows when we invoke similar inequalities as (6.7). The proof of (6.6) is very similar. This completes the proof of Lemma 6.2. ∎

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