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On the initial- and boundary-value problem for 2D micropolar equations with only angular velocity dissipation

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Abstract. This paper focuses on the initial- and boundary-value problem for the two-dimensional micropolar equations with only angular velocity dissipation in a smooth bounded domain. The aim here is to establish the global existence and uniqueness of solutions by imposing natural boundary conditions and minimal regularity assumptions on the initial data. Besides, the global solution is shown to possess higher regularity when the initial datum is more regular. To obtain these results, we overcome two main difficulties: one due to the lack of full dissipation and one due to the boundary conditions. In addition to the global regularity problem, we also examine the large time behavior of solutions and obtain explicit decay rates.

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1. Introduction and main results

This paper studies the global existence and uniqueness and large time behavior of solutions to the two-dimensional (2D) micropolar equations in a bounded domain D with smooth boundary. The 2D micropolar equations are a special case of the 3D micropolar equations. The standard 3D incompressible micropolar equations are given by

$$\begin{cases} u_t - (\nu + \kappa)\Delta u + u \cdot \nabla u + \nabla \pi = 2\kappa \nabla \times w, \\ w_t - \gamma \Delta w + 4\kappa w - \mu \nabla \nabla \cdot w + u \cdot \nabla w = 2\kappa \nabla \times u, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the fluid velocity, $\pi(x,t)$ denotes the scalar pressure, $w(x,t)$ denotes the microrotation field, the parameter ν represents the Newtonian kinematic viscosity, κ represents the microrotation viscosity, γ and μ represent the angular viscosities.

In the special case, when

$$u = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0),\ \pi = \pi(x_1,x_2,t),\ w = (0,0,w_3(x_1,x_2,t)),$$

the 3D micropolar equations reduce to the 2D micropolar equations,

$$\begin{cases} u_t - (\nu + \kappa)\Delta u + u \cdot \nabla u + \nabla \pi = 2\kappa \nabla^{\perp} w, \\ w_t - \gamma \Delta w + 4\kappa w + u \cdot \nabla w = 2\kappa \nabla \times u, \\ \nabla \cdot u = 0. \end{cases} \tag{1.2}$$

Here $u = (u_1, u_2)$ is a 2D vector with the corresponding scalar vorticity Ω given by

$$\Omega \equiv \nabla \times u = \partial_1 u_2 - \partial_2 u_1,$$

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while ω represents $\omega_3(x_1, x_2, t)$ for simplicity, which is a scalar function with

$$\nabla^{\perp} w = (\partial_2 w, -\partial_1 w).$$

The micropolar equations were introduced in 1965 by C.A. Eringen to model micropolar fluids (see, e.g., [7]). Micropolar fluids are fluids with microstructure. Certain anisotropic fluids, e.g., liquid crystals which are made up of dumbbell molecules, are of this type. They belong to a class of non-Newtonian fluids with nonsymmetric stress tensor (called polar fluids) and include, as a special case, the classical fluids modeled by the Navier–Stokes equations. In fact, when microrotation effects are neglected, namely $w = 0$, (1.1) reduces to the incompressible Navier–Stokes equations. The micropolar equations are significant generalizations of the Navier–Stokes equations and cover many more phenomena such as fluids consisting of particles suspended in a viscous medium. The micropolar equations have been extensively applied and studied by many engineers and physicists.

Due to their physical applications and mathematical significance, the well-posedness problem and large time decay issue on the micropolar equations have attracted considerable attention recently from the community of mathematical fluids [2,3,8,15]. Lukaszewicz in his monograph [15] studied the well-posedness problem on the 3D stationary as well as the time-dependent micropolar equations. In [4], Dong and Chen obtained the global existence and uniqueness, and sharp algebraic time decay rates for the 2D micropolar equations (1.2).

More recent efforts are focused on the 2D micropolar equation with partial dissipation, which naturally bridge the inviscid micropolar equation and the micropolar equation with full dissipation. The global regularity problem for the inviscid equation is currently out of reach. In [6], Dong and Zhang examined (1.2) with the microrotation viscosity $\gamma = 0$ and established the global regularity by observing that the combined quantity

$$\Omega - \frac{2\kappa}{\nu + \kappa} w$$

obeys a transport-diffusion equation. Another partial dissipation case, (1.2) with $\nu = 0$, $\gamma > 0$, $\kappa > 0$, and $\kappa \neq \gamma$, was examined by Xue, who was able to obtain the global well-posedness in the frame work of Besov spaces [20]. We remark that the requirement $\kappa \neq \gamma$ in [20] is not crucial and it is not difficult to see that the global well-posedness remains valid even when $\kappa = \gamma$. Recently Dong, Li and Wu took on the case when (1.2) involves only the angular viscosity dissipation [5]. They proved the global (in time) regularity by fully exploiting the structure of system and controlling the vorticity via the evolution of a combined quantity of the vorticity and the microrotation angular velocity. In addition, [5] introduced a diagonalization process to eliminate the linear terms in order to obtain the large time behavior of the solutions.

Most of the results we mentioned above are for the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$. In many real-world applications, the flows are often restricted to bounded domains with suitable constraints imposed on the boundaries and these applications naturally lead to the studies of the initial- and boundary-value problems. In addition, solutions of the initial- and boundary-value problems may exhibit much richer phenomena than those of the whole space counterparts. The case when $\nu > 0$, $\kappa > 0$ and $\gamma > 0$ has been extensively analyzed. [18] proves the existence and uniqueness of a global solution for 2D micropolar fluid equation with periodic boundary conditions. [21] obtains the global existence of strong solutions with small initial data for the three-dimensional case. However, the global regularity problem for the partial viscosity case in (1.3) is not answered and solved in this paper.

This paper is devoted to the initial- and boundary-value problem for the 2D micropolar equations with only angular viscosity dissipation,

$$\begin{cases} u_t + u \cdot \nabla u + \nabla \pi = 2\kappa \nabla^{\perp} w, \\ w_t - \gamma \Delta w + 4\kappa w + u \cdot \nabla w = 2\kappa \nabla \times u, \\ \nabla \cdot u = 0, \end{cases} \tag{1.3}$$

with the natural boundary condition

$$u \cdot \boldsymbol{n}|_{\partial D} = w|_{\partial D} = 0 \tag{1.4}$$

and the initial condition

$$(u, w)(x, 0) = (u_0, w_0)(x), \quad \text{in } D, \tag{1.5}$$

where $D \subset \mathbb{R}^2$ represents a bounded domain with smooth boundary and $\boldsymbol{n}$ is the unit outward normal vector. In addition, for global existence of smooth solutions, we also impose the following compatibility conditions

$$\begin{cases} u_0 \cdot \boldsymbol{n}|_{\partial D} = w_0|_{\partial D} = 0, \\ \nabla \cdot u_0 = 0, \\ 4\kappa w_0 + u_0 \cdot \nabla w_0 = \gamma \Delta w_0 + 2\kappa \nabla \times u_0, \text{ on } \partial D. \end{cases} \tag{1.6}$$

Our first goal here is to establish the global existence and uniqueness of solutions to (1.3)–(1.5) by imposing the least regularity assumptions on the initial data. We assume here that the initial vorticity $\Omega_0 = \nabla \times u_0$ is in the Yudovich class and $w_0 \in H^2(D)$, and obtain the following result.

Theorem 1.1. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume (u_0, w_0) satisfies*

$$u_0 \in H^1(D), \quad \Omega_0 \in L^\infty(D), \quad w_0 \in H^2(D).$$

Then, for any $T > 0$, (1.3)–(1.5) have a unique global strong solution (u, w) satisfying

$$u \in L^\infty(0, T; H^1(D)), \quad \Omega \in L^\infty(0, T; L^\infty(D)), \quad w \in L^\infty(0, T; H^2(D)). \tag{1.7}$$

We further establish higher regularity of the solution stated in Theorem 1.1. By imposing the higher regularity

$$u_0 \in H^3(D), \quad w_0 \in H^4(D),$$

we obtain that the corresponding solution (u, w) remains this class for all time. More precisely, we have the following theorem.

Theorem 1.2. *In addition to the conditions in Theorem 1.1, we further assume*

$$u_0 \in H^3(D), \quad w_0 \in H^4(D),$$

and the compatibility conditions (1.6), then, the solution is smooth and remains the same regularity for all time, namely

$$u \in L^\infty(0, T; H^3(D)), \qquad w \in L^\infty(0, T; H^4(D)),$$

for any $T > 0$.

The proof of Theorem 1.1 is divided into several major steps. The first step is to establish the global H^1-bound for (u, w). We make use of the equation of the vorticity $\Omega = \nabla \times u$,

$$\partial_t \Omega + u \cdot \nabla \Omega = -2\kappa \Delta w \tag{1.8}$$

and the equation of ∇w, and bound the vortex stretching term in the equation of ∇w suitably. The second step combines the global H^1-bound obtained in the first step and Schauder's fixed point theorem to prove the global existence of weak solutions. The third step establishes the global boundedness of the vorticity in L^∞. Due to the presence of the bad term $-\Delta w$, (1.8) itself does not allow us to extract the desired global bound. To overcome this difficulty, we consider the combined quantity

$$Z = \Omega + \frac{2\kappa}{\gamma} w,$$

which satisfies

$$\partial_t Z + u \cdot \nabla Z - \frac{4\kappa^2}{\gamma} Z + \frac{8\kappa^2}{\gamma}\left(1 + \frac{\kappa}{\gamma}\right) w = 0. \tag{1.9}$$

This equation eliminates the difficulty and yields the desired bound. The fourth step is to prove the global H^2-bound for w. Due to the no-slip boundary condition for w, the standard energy estimates would not work. Instead, we obtain a global bound for $\|\partial_t w\|_{L^2(D)}$ first and the global bound for $\|w\|_{H^2(D)}$ follows as a consequence.

To prove the higher regularity bounds specified in Theorem 1.2, we further exploit the equation of Z, namely (1.9). By taking the gradient of (1.9) and making use of a logarithmic Sobolev type inequality, we obtain the global bounds for

$$\|\nabla\Omega\|_{L^\infty(0,T;L^q(D))} \quad \text{and} \quad \|\nabla u\|_{L^\infty(0,T;L^\infty(D))},$$

where $q \in (1,\infty)$. These bounds allow us to further consecutively establish the bounds for

$$\|\Delta w\|_{L^q(0,T;L^q(D))}, \quad \|\nabla\partial_t w\|_{L^\infty(0,T;L^2(D))} \quad \text{and} \quad \|w\|_{L^\infty(0,T;H^4(D))}.$$

We remark that, in contrast to the whole space case, the estimates here are not so straightforward. We use extensively the estimates on the heat equation and general elliptic equations. Due to the bounded domain setup, many of the estimates here are more delicate than those in the whole space case.

This paper also looks into the large time behavior of solutions of (1.3) with an extra velocity damping term and without the term $4\kappa w$ in the w equation, namely

$$\begin{cases} u_t + \kappa u + u\cdot\nabla u + \nabla\pi = 2\kappa\nabla^\perp w, \\ w_t - \gamma\Delta w + u\cdot\nabla w = 2\kappa\nabla\times u, \\ \nabla\cdot u = 0. \end{cases} \tag{1.10}$$

This system is not explicitly derived as a physical model, although it may be relevant for physical circumstances in which the drag force on the system obeys Stokes' law (the drag force is proportional to the velocity field u). It is clear that Theorems 1.1 and 1.2 remain valid for (1.10). The damping term in (1.10) and the elimination of $4\kappa w$ term do not affect the regularity results. Our last focus is the large time behavior, and we show that if

$$\gamma > 4\kappa, \tag{1.11}$$

then the H^1-norm of (u,w) decays exponentially in time,

$$\|(u(t), w(t)\|_{H^1(D)} \le C\, e^{-\widetilde{C}t},$$

where $C > 0$ is a constant depending on the H^1-norm of (u_0, w_0) and $\widetilde{C} > 0$ depends on γ and κ only. More precisely, we have the following theorem.

Theorem 1.3. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume the conditions on (u_0, w_0), as stated in Theorem 1.1. If we further assume*

$$\gamma > 4\kappa, \tag{1.12}$$

then the solution (u,w) of (1.10) satisfies

$$\|u\|_{H^1(D)}^2 + \|w\|_{H^1(D)}^2 \le C\, e^{-\widetilde{C}t}, \tag{1.13}$$

where $C > 0$ is a constant depending on the H^1-norm of (u_0, w_0) and $\widetilde{C} > 0$ depends on γ and κ only.

The condition in (1.11) is necessary and sharp. As pointed out in [5], when (1.11) is violated, the solution may grow in time.

The rest of this paper is divided into five sections. The second section serves as a preparation and presents a list of facts and tools for bounded domains such as embedding inequalities and logarithmic type interpolation inequalities. Section 3 establishes the global existence of H^1-weak solutions, one major step in the proof of Theorem 1.1. Section 4 proves the global L^∞ bound for Ω and the global H^2 bound for w. This step completes the proof of Theorem 1.1. Section 5 proves Theorem 1.2, the higher global regularity bounds. The last section is devoted to the large time behavior result stated in Theorem 1.3.

2. Preliminaries

This section serves as a preparation. We list a few basic tools for bounded domains to be used in the subsequent sections. In particular, we provide the Gagliardo–Nirenberg type inequalities, the logarithmic type interpolation inequalities, and regularization estimates for elliptic and parabolic equations in bounded domains. These estimates will also be handy for future studies on PDEs in bounded domains.

We start with the well-known Gagliardo–Nirenberg inequality for bounded domains (see, e.g., [17]).

Lemma 2.1. *Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $1 \le p, q, r \le \infty$ be real numbers and $j \le m$ be nonnegative integers. If a real number α satisfies*

$$\frac{1}{p} - \frac{j}{n} = \alpha \left(\frac{1}{r} - \frac{m}{n} \right) + (1-\alpha)\frac{1}{q}, \qquad \frac{j}{m} \le \alpha \le 1,$$

then

$$\|\mathrm{D}^j f\|_{L^p(D)} \le C_1 \|\mathrm{D}^m f\|^{\alpha}_{L^r(D)} \|f\|^{1-\alpha}_{L^q(D)} + C_2 \|f\|_{L^s(D)},$$

where $s > 0$, and the constants C_1 and C_2 depend upon D and the indices p, q, r, m, j, s only.

In particular, the following special cases will be used.

Corollary 2.1. *Suppose $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then,*

(1) $\|f\|_{L^4(D)} \le C\,(\|f\|^{\frac{1}{2}}_{L^2(D)} \|\nabla f\|^{\frac{1}{2}}_{L^2(D)} + \|f\|_{L^2(D)}), \ \forall f \in H^1(D)$;

(2) $\|\nabla f\|_{L^4(D)} \le C\,(\|f\|^{\frac{1}{4}}_{L^2(D)} \|\nabla^2 f\|^{\frac{3}{4}}_{L^2(D)} + \|f\|_{L^2(D)}), \ \forall f \in H^2(D)$;

(3) $\|f\|_{L^\infty(D)} \le C\,(\|f\|^{\frac{1}{2}}_{L^2(D)} \|\nabla^2 f\|^{\frac{1}{2}}_{L^2(D)} + \|f\|_{L^2(D)}), \ \forall f \in H^2(D)$;

(4) $\|f\|_{L^\infty(D)} \le C\,(\|f\|^{\frac{2}{3}}_{L^2(D)} \|\nabla^3 f\|^{\frac{1}{3}}_{L^2(D)} + \|f\|_{L^2(D)}), \ \forall f \in H^3(D)$.

The lemma below provides estimates for products and commutators (see, e.g., [16]).

Lemma 2.2. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then, for any multi-indices α and β,*

$$\|D^\alpha (fg)\|_{L^2(D)} \le C(\|f\|_{L^\infty(D)} \|g\|_{H^{|\alpha|}(D)} + \|f\|_{H^{|\alpha|}(D)} \|g\|_{L^\infty(D)}) \tag{2.1}$$

for some constant C depending on D and α, and

$$\|D^\beta (fg) - f D^\beta g\|_{L^2(D)} \le C(\|\nabla f\|_{L^\infty(D)} \|g\|_{H^{|\beta|-1}(D)} + \|f\|_{H^{|\beta|}(D)} \|g\|_{L^\infty(D)}) \tag{2.2}$$

for some constant C depending on D and β.

We will need the Poincaré type inequality (see, e.g., [9]).

Lemma 2.3. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $u \in H^1_0(D)$. Then, there exists a constant C depending on D only such that*

$$\|u\|_{L^2(D)} \le C\,\|\nabla u\|_{L^2(D)}.$$

It is well known that the standard singular integral operators are bounded on $L^q(\mathbb{R}^n)$ for any $q \in (1, \infty)$ (see, e.g., [19]). The lemma below provides the bounded domain version of this fact (see, e.g., [1,10,19]). More precisely, this lemma allows us to control the gradient of a vector field in L^q in terms of its curl.

Lemma 2.4. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and let m be a positive integer. If a vector field $u \in L^q(D)$ with $q \in (1, \infty)$ satisfies*

$$\nabla \times u \in W^{m-1,q}(D), \qquad \nabla \cdot u = 0, \qquad u \cdot \boldsymbol{n}|_{\partial D} = 0,$$

then there is a constant $K = K(D,m) > 0$ (independent of q) such that

$$\|u\|_{W^{m,q}(D)} \le K\,q\,(\|\nabla \times u\|_{W^{m-1,q}(D)} + \|u\|_{L^q(D)}).$$

The lemma next presents a logarithmic interpolation inequality for vector fields defined on bounded domains. It is taken from [13].

Lemma 2.5. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then, for any $0 < \alpha < 1$,*

$$\|\nabla u\|_{L^\infty(D)} \le C\|\nabla \times u\|_{L^\infty(D)}(1 + \log(e + \|\nabla \times u\|_{C^\alpha(D)})).$$

The next two lemmas state the regularization estimates for elliptic and parabolic equations defined on bounded domains (see, e.g., [9,11,14]).

Lemma 2.6. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Consider the elliptic boundary-value problem*

$$\begin{cases} -\Delta f = g & \text{in } D, \\ f = 0 & \text{on } \partial D. \end{cases} \tag{2.3}$$

If, for $p \in (1,\infty)$ and an integer $m \ge -1$, $g \in W^{m,p}(D)$, then (2.3) has a unique solution f satisfying

$$\|f\|_{W^{m+2,p}(D)} \le C\|g\|_{W^{m,p}(D)},$$

where C depending only on D, m and p.

Lemma 2.7. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume, for $q \in [2,\infty)$,*

$$\varphi \in W_0^{1,q}(D), \qquad g \in L^q(0,T;L^q(D)).$$

Assume $f \in L^2(0,T;H_0^1(D))$ with $f_t \in L^2(0,T;H^{-1}(D))$ is a weak solution of the parabolic system

$$\begin{cases} f_t - \Delta f = g, & \text{in } D, \\ f = 0 & \text{on } \partial D, \\ f|_{t=0} = \varphi & \text{in } D. \end{cases}$$

Then, there exists a constant C depending only on q, D, such that

$$\|f_t\|_{L^q(0,T;L^q(D))} + \|\Delta f\|_{L^q(0,T;L^q(D))} \le C(\|g\|_{L^q(0,T;L^q(D))} + \|\varphi\|_{W_0^{1,q}(D)}).$$

We also recall Kato's well-posedness result on the 2D Euler equations (see [12]).

Lemma 2.8. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Consider the initial- and boundary-value problem*

$$\begin{cases} u_t - u \cdot \nabla u + \nabla p = f, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0(x), \ \ u \cdot \mathbf{n}|_{\partial D} = 0. \end{cases}$$

Assume $u_0 \in C^{1+\gamma}(\overline{D})$ with $0 < \gamma < 1$ satisfying $\nabla \cdot u_0 = 0$ and $u_0 \cdot \mathbf{n}|_{\partial D} = 0$. Let $T > 0$ and $f \in C([0,T];C^{1+\gamma}(\overline{D}))$. Then, there exists a unique solution (u,p) such that $(u,p) \in C^{1+\gamma}(\overline{D} \times [0,T])$.

3. Global existence of H^1-weak solutions

This section proves the global existence of H^1-weak solutions of (1.3)–(1.5). This result is an important step in the proof of Theorem 1.1. To be more precise, we first provide the definition of weak solutions of (1.3)–(1.5) and then state the main result of this section as a proposition.

Definition 3.1. Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $(u_0, w_0) \in L^2(D)$. A pair of measurable functions (u, w) is called a weak solution of (1.3)–(1.5) if

(1) $$u \in C(0,T;L^2(D)),\ \ w \in C(0,T;L^2(D)) \cap L^2(0,T;H_0^1(D));$$

(2) $$\int_D u_0 \cdot \varphi_0 \mathrm{d}x + \int_0^T \int_D \left[u \cdot \varphi_t + u \cdot \nabla\varphi \cdot u + 2\kappa w \nabla \times \varphi\right] \mathrm{d}x\mathrm{d}t = 0,$$

$$\int_D w_0 \psi_0 \mathrm{d}x + \int_0^T \int_D \left[w\psi_t - \gamma \nabla w \cdot \nabla \psi - 4\kappa w \psi + u \cdot \nabla\psi \cdot w + 2\kappa u \cdot \nabla^\perp \psi\right] \mathrm{d}x\mathrm{d}t = 0;$$

holds for any test functions $\varphi = (\varphi_1,\ \varphi_2) \in C_0^\infty([0,T) \times D)^2$ satisfying $\nabla \cdot \varphi = 0$ and $\psi \in C_0^\infty([0,T) \times D)$.

The main result of this section is stated in the following proposition.

Proposition 3.1. *Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $(u_0, w_0) \in H^1(D)$. Then, (1.3)–(1.5) has a global weak solution.*

The proof of this proposition relies on the following global H^1-bound.

Lemma 3.1. *Under the assumptions of Proposition 3.1, for any $T > 0$, there exists a constant C depending only on T and the initial data such that*

$$\|u\|_{L^\infty(0,T;H^1(D))} + \|w\|_{L^\infty(0,T;H^1(D))} + \|w\|_{L^2(0,T;H^2(D))} \le C.$$

Proof of Lemma 3.1. We start with the global L^2-bound. Taking the inner product of (1.3) with (u, w) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&(\|u\|_{L^2(D)}^2 + \|w\|_{L^2(D)}^2) + \gamma\|\nabla w\|_{L^2(D)}^2 + 4\kappa\|w\|_{L^2(D)}^2\\ &= 2\kappa \int_D \nabla^\perp w \cdot u \,\mathrm{d}x + 2\kappa \int_D \nabla \times u w \mathrm{d}x.\end{aligned}$$

Noticing that $\nabla \times u = \partial_1 u_2 - \partial_2 u_1$ and $\nabla^\perp w = (\partial_2 w, -\partial_1 w)$, we have

$$\nabla \times u w = (\partial_1 u_2 - \partial_2 u_1) w = \partial_1(u_2\, w) - \partial_2(u_1\, w) + \nabla^\perp w \cdot u.$$

Integrating by parts and applying the boundary condition for w, we have

$$\begin{aligned}2\kappa &\int_D \nabla^\perp w \cdot u \,\mathrm{d}x + 2\kappa \int_D \nabla \times u w \,\mathrm{d}x\\ &= 4\kappa \int_D \nabla^\perp w \cdot u \,\mathrm{d}x + 2\kappa \int_{\partial D} u \cdot \boldsymbol{n}^\perp w \mathrm{d}s\\ &= 4\kappa \int_D \nabla^\perp w \cdot u \,\mathrm{d}x\\ &\le \frac{\gamma}{2}\|\nabla w\|_{L^2(D)}^2 + C\|u\|_{L^2(D)}^2,\end{aligned} \tag{3.1}$$

where $\mathbf{n}^\perp = (-n_2, n_1)$. It then follows, after integration in time, that

$$\begin{aligned}
&\|u\|^2_{L^2(D)} + \|w\|^2_{L^2(D)} + \gamma \int_0^t \|\nabla w\|^2_{L^2(D)}\,\mathrm{d}\tau + 8\kappa \int_0^t \|w\|^2_{L^2(D)}\mathrm{d}\tau \\
&\quad \le e^{C\,t}(\|u_0\|^2_{L^2(D)} + \|w_0\|^2_{L^2(D)}) \equiv A_1(t, \|(u_0, w_0)\|_{L^2}),
\end{aligned} \tag{3.2}$$

where $C = C(\gamma, \kappa)$. To obtain the global H^1-bound for (u, w), we invoke the vorticity equation

$$\Omega_t + u\cdot\nabla\Omega = -2\kappa\Delta w, \tag{3.3}$$

Multiplying (3.3) by Ω and the equation of w in (1.3) by Δw, and applying the boundary condition $w|_{\partial D} = 0$, the Cauchy–Schwarz inequality, and Corollary 2.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Omega\|^2_{L^2(D)} + \|\nabla w\|^2_{L^2(D)}) + \gamma\|\Delta w\|^2_{L^2(D)} + 4\kappa\|\nabla w\|^2_{L^2(D)} \\
&\quad = \int_D u\cdot\nabla w\,\Delta w\,\mathrm{d}x - 4\kappa\int_D \Delta w\,\Omega\,\mathrm{d}x \\
&\quad \le \|\Delta w\|_{L^2(D)}\|u\|_{L^4(D)}\|\nabla w\|_{L^4(D)} + 4\kappa\|\Omega\|_{L^2(D)}\|\Delta w\|_{L^2(D)} \\
&\quad \le C\|\Delta w\|^{\frac{3}{2}}_{L^2(D)}\|\Omega\|^{\frac{1}{2}}_{L^2(D)}\|\nabla w\|^{\frac{1}{2}}_{L^2(D)} + C\|\Delta w\|_{L^2(D)}\|\Omega\|^{\frac{1}{2}}_{L^2(D)}\|\nabla w\|_{L^2(D)} \\
&\qquad + 4\kappa\|\Omega\|_{L^2(D)}\|\Delta w\|_{L^2(D)} \\
&\quad \le \frac{\gamma}{2}\|\Delta w\|^2_{L^2(D)} + C\|\nabla w\|^2_{L^2(D)}(1 + \|\Omega\|^2_{L^2(D)}) + C\|\Omega\|^2_{L^2(D)}.
\end{aligned} \tag{3.4}$$

Gronwall's inequality and (3.2) then yield the following global H^1-bound

$$\begin{aligned}
&\|\nabla u(t)\|^2_{L^2(D)} + \|\nabla w(t)\|^2_{L^2(D)} + \gamma\int_0^t \|\Delta w\|^2_{L^2(D)}\,\mathrm{d}\tau + 4\kappa\int_0^t \|\nabla w\|^2_{L^2(D)}\,\mathrm{d}\tau \\
&\quad \le C_1\, e^{C_2\, e^{C_3 t}}\,\|(\nabla u_0, \nabla w_0)\|^2_{L^2(D)} \equiv A_2(t),
\end{aligned} \tag{3.5}$$

where $C_1 = C_1(\gamma, \kappa)$, $C_2 = C_2(\gamma, \kappa, \|(u_0, w_0)\|_{L^2(D)})$ and $C_3 = C_3(\gamma, \kappa)$. This completes the proof of Lemma 3.1. □

We now prove Proposition 3.1.

Proof of Proposition 3.1. The proof is a consequence of Schauder's fixed point theorem. We shall only provide the sketches.

To define the functional setting, we fix $T > 0$ and R_0 to be specified later. For notational convenience, we write

$$X \equiv C(0, T; L^2(D)) \cap L^2(0, T; H^1_0(D))$$

with $\|g\|_X \equiv \|g\|_{C(0,T;L^2(D))} + \|g\|_{L^2(0,T;H^1_0(D))}$, and define

$$B = \{g \in X \mid \|g\|_X \le R_0\}.$$

Clearly, $B \subset X$ is closed and convex.

We fix $\epsilon \in (0, 1)$ and define a continuous map on B. For any $g \in B$, we regularize it and the initial data (u_0, w_0) via the standard mollifying process,

$$g^\epsilon = \rho^\epsilon * g, \quad u_0^\epsilon = \rho^\epsilon * u_0, \quad w_0^\epsilon = \rho^\epsilon * w_0,$$

where ρ^ϵ is the standard mollifier. According to Lemma 2.8, the 2D incompressible Euler equations with smooth external forcing $2\kappa\nabla^\perp g^\epsilon$ and smooth initial data u_0^ϵ

$$\begin{cases} u_t + u\cdot\nabla u + \nabla\pi = 2\kappa\nabla^\perp g^\epsilon, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0^\epsilon(x), \quad u\cdot\boldsymbol{n}|_{\partial D} = 0, \end{cases} \tag{3.6}$$

have a unique solution u^ϵ. We then solve the linear parabolic equation with the smooth initial data w_0^ϵ

$$\begin{cases} w_t - \gamma\Delta w + 4\kappa w + u^\epsilon\cdot\nabla w = 2\kappa\nabla\times u^\epsilon, \\ w(x,0) = w_0^\epsilon(x), \quad w|_{\partial D} = 0, \end{cases} \tag{3.7}$$

and denote the solution by w^ϵ. This process allows us to define the map

$$F^\epsilon(g) = w^\epsilon.$$

We then apply Schauder's fixed point theorem to construct a sequence of approximate solutions to (1.3)–(1.5). It suffices to show that, for any fixed $\epsilon\in(0,1)$, $F^\epsilon: B\to B$ is continuous and compact. More precisely, we need to show

(a) $\|w^\epsilon\|_B \le R_0$;
(b) $\|w^\epsilon\|_{C(0,T;H_0^1(\Omega))} + \|w^\epsilon\|_{L^2(0,T;H^2(\Omega))} \le C$;
(c) $\|F^\epsilon(g_1) - F^\epsilon(g_2)\|_B \le C\|g_1-g_2\|_B$ for C independent of ϵ and any g_1, $g_2\in B$.

We verify (a) first. A simple L^2-estimate on (3.6) leads to

$$\begin{aligned} \|u^\epsilon(t)\|_{L^2(D)} &\le \|u_0^\epsilon\|_{L^2(D)} + 2\kappa\int_0^t \|\nabla g^\epsilon\|_{L^2(D)}\,\mathrm{d}\tau, \\ &\le \|u_0\|_{L^2(D)} + 2\kappa\int_0^t \|\nabla g\|_{L^2(D)}\,\mathrm{d}\tau. \end{aligned}$$

Similar to (3.2), we have

$$\|w^\epsilon\|_{L^2(D)}^2 + \gamma\int_0^t \|\nabla w^\epsilon\|_{L^2(D)}^2\,\mathrm{d}\tau + 4\kappa\int_0^t \|w^\epsilon\|_{L^2(D)}^2\,\mathrm{d}\tau \le \|w_0\|_{L^2(D)}^2 + \frac{2\kappa^2}{\gamma}\int_0^t \|u^\epsilon\|_{L^2(D)}^2\,\mathrm{d}\tau.$$

In order for F^ϵ to map B to B, it suffices for the right-hand side to be bounded by R_0. Invoking the bounds for $\|u^\epsilon\|_{L^2}$ and $\|w^\epsilon\|_{L^2}$, we obtain a condition for T and R_0,

$$\|w_0\|_{L^2(D)}^2 + CT\,(\|u_0\|_{L^2(D)}^2 + TR_0^2) \le R_0^2, \tag{3.8}$$

where the constants C depend only on the parameters κ and γ. It is not difficult to see that, if T is sufficiently small, (3.8) would hold. Similarly, we can show (b) and (c) under the condition that T is sufficiently small. Schauder's fixed point theorem then allows us to conclude that the existence of a solution on a finite time interval T. These uniform estimates would allow us to pass the limit to obtain a weak solution (u,w).

We remark that the local solution obtained by Schauder's fixed point theorem can be easily extended into a global solution via Picard type extension theorem due to the global bounds obtained in (3.2) and (3.5). This allows us to obtain the desired global weak solution. This completes the proof. □

4. Yudovich regularity and proof of Theorem 1.1

The goal of this section is to complete the proof of Theorem 1.1. To do so, we first establish the Yudovich type regularity for the vorticity Ω, namely $\Omega \in L^\infty$ for all time, and then show that $w \in L^\infty(0,T;H^2(D))$. The regularity obtained here for Ω and w allows us to prove the uniqueness of the weak solutions established in the previous section.

Recall that Ω satisfies

$$\Omega_t + u\cdot\nabla\Omega = -2\kappa\Delta w. \tag{4.1}$$

Due to the bad term $-\Delta w$ on the right-hand side, this equation itself does not allow us to extract a global bound on Ω. To bypass this difficulty, we combine (4.1) with the equation of w,

$$w_t - \gamma\Delta w + 4\kappa w - 2\kappa\nabla\times u + u\cdot\nabla w = 0$$

to eliminate the bad term. More precisely, we consider the combined quantity

$$Z = \Omega + \frac{2\kappa}{\gamma} w$$

and the equation that it satisfies

$$\partial_t Z + u\cdot\nabla Z - \frac{4\kappa^2}{\gamma}Z + \frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)w = 0, \tag{4.2}$$

which leads us to the desired global bound. More precisely, we have the following proposition.

Proposition 4.1. *Assume that (u_0, w_0) satisfies the conditions stated in Theorem 1.1. Let (u,w) be the global weak solution obtained in Proposition 3.1. Then, the corresponding vorticity Ω obeys the global bound, for any $2\le p<\infty$, and any $T>0$ and $0<t\le T$,*

$$\|\Omega\|_{L^\infty(0,T;L^p(D))} \le C,$$

where the constant C depends only on D, T and the initial data.

Proof. We start with the equation of Z, namely (4.2). For any $2\le p<\infty$, multiplying (4.2) with $|Z|^{p-2}Z$ and integrating on D, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|^p_{L^p(D)} \le \frac{4\kappa^2}{\gamma}\|Z\|^p_{L^p(D)} + \frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\|w\|_{L^p(D)}\|Z\|^{p-1}_{L^p(D)},$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z\|_{L^p(D)} \le \frac{4\kappa^2}{\gamma}\|Z\|_{L^p(D)} + \frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\|w\|_{L^p(D)},$$

which, according to Gronwall's inequality, implies

$$\|Z\|_{L^p(D)} \le e^{\frac{4\kappa^2}{\gamma}t}\left(\|Z_0\|_{L^p(D)} + C\int_0^t \|w(\tau)\|_{L^p(D)}\mathrm{d}\tau\right).$$

Noting that C is independent of p, we obtain, by letting $p\to\infty$,

$$\begin{aligned}\|Z\|_{L^\infty(D)} &\le e^{\frac{4\kappa^2}{\gamma}t}\left(\|Z_0\|_{L^\infty(D)} + C\int_0^t \|w(\tau)\|_{L^\infty(D)}\mathrm{d}\tau\right)\\ &\le e^{\frac{4\kappa^2}{\gamma}t}\left(\|Z_0\|_{L^\infty(D)} + C\int_0^t \|w(\tau)\|_{H^2(D)}\mathrm{d}\tau\right).\end{aligned}$$

Thus, by noticing that $\|w\|_{L^2(0,T;H^2(D))} \leq C$ from Lemma 3.1, it is clear that

$$\|Z\|_{L^\infty(0,T;L^p(D))} \leq C, \tag{4.3}$$

for any $2 \leq p \leq \infty$. By the definition of Z, Sobolev embedding, and Lemma 3.1, we have

$$\begin{aligned}\|\Omega\|_{L^\infty(0,T;L^p(D))} &\leq (\|Z\|_{L^\infty(0,T;L^p(D))} + \|w\|_{L^\infty(0,T;L^p(D))})\\ &\leq C(\|Z\|_{L^\infty(0,T;L^p(D))} + \|w\|_{L^\infty(0,T;H^1(D))})\\ &\leq C.\end{aligned}$$

for any $2 \leq p < \infty$. This completes the proof of Proposition 4.1. □

Next we prove the global bound for $\|w\|_{H^2(D)}$. In contrast to the whole space case, we need to estimate the time derivatives of (u, w) in order to obtain the desired bound.

Proposition 4.2. *Assume that (u_0, w_0) satisfies the conditions stated in Theorem 1.1. Let (u, w) be the global weak solution obtained in Proposition 3.1. Then, for any $T > 0$ and $0 < t < T$,*

$$\|u_t\|_{L^\infty(0,T;L^2(D))} + \|w_t\|_{L^\infty(0,T;L^2(D))} + \|w\|_{L^\infty(0,T;H^2(D))} \leq C,$$

where the constant C depends only on D, T, and the initial data.

Proof. We first estimate $\|u_t\|_{L^2(D)}$. Dotting the equation of u in (1.3) with u_t, we have

$$\begin{aligned}\|u_t\|^2_{L^2(D)} &= -\int_D u\cdot\nabla u\cdot u_t\,\mathrm{d}x + 2\kappa\int_D \nabla^\perp w\cdot u_t\,\mathrm{d}x\\ &\leq \frac{1}{2}\|u_t\|^2_{L^2(D)} + C\,\|u\cdot\nabla u\|^2_{L^2(D)} + C\|\nabla^\perp w\|^2_{L^2(D)}\\ &\leq \frac{1}{2}\|u_t\|^2_{L^2(D)} + C\,\|u\|^2_{L^4(D)}\|\nabla u\|^2_{L^4(D)} + C\|\nabla w\|^2_{L^2(D)}\\ &\leq \frac{1}{2}\|u_t\|^2_{L^2(D)} + C\,\|u\|^2_{H^1(D)}(\|u\|^2_{L^2(D)} + \|\Omega\|^2_{L^4(D)}) + C\|\nabla w\|^2_{L^2(D)}.\end{aligned}$$

The global bounds in Lemma 3.1 and Proposition 4.1 then imply

$$\|u_t\|_{L^\infty(0,T;L^2(D))} \leq C. \tag{4.4}$$

To estimate $\|w_t\|_{L^2(D)}$, we take the temporal derivative of the w-equation in (1.3) to get

$$w_{tt} + u\cdot\nabla w_t + u_t\cdot\nabla w + 4\kappa w_t = \gamma\Delta w_t + 2\kappa\Omega_t. \tag{4.5}$$

Multiplying (4.5) with w_t and integrating on D, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|^2_{L^2(D)} + \gamma\|\nabla w_t\|^2_{L^2(D)} + 4\kappa\|w_t\|^2_{L^2(D)}\\ &\quad = -\int_D w_t u_t\cdot\nabla w\,\mathrm{d}x + 2\kappa\int_D \Omega_t w_t\,\mathrm{d}x.\end{aligned} \tag{4.6}$$

By integration by parts, Hölder's inequality, and Young's inequality,

$$\begin{aligned}-\int_D w_t u_t\cdot\nabla w\mathrm{d}x &= \int_D w u_t\cdot\nabla w_t\,\mathrm{d}x\\ &\leq \frac{\gamma}{4}\|\nabla w_t\|^2_{L^2(D)} + \frac{C}{\gamma}\|u_t\|^2_{L^2(D)}\|w\|^2_{L^\infty(D)}\\ &\leq \frac{\gamma}{4}\|\nabla w_t\|^2_{L^2(D)} + \frac{C}{\gamma}\|u_t\|^2_{L^2(D)}\|w\|^2_{H^2(D)}.\end{aligned}$$

To estimate the second term in the right of (4.6), we make use of the vorticity equation (4.1) and integrate by parts on D to get

$$2\kappa\int\limits_D \Omega_t w_t \mathrm{d}x = -2\kappa\int\limits_D u\cdot\nabla\Omega w_t\mathrm{d}x - 4\kappa^2\int\limits_D \Delta w w_t \mathrm{d}x$$
$$= 2\kappa\int\limits_D u\cdot\nabla w_t\Omega\mathrm{d}x + 4\kappa^2\int\limits_D \nabla w\cdot\nabla w_t\mathrm{d}x,$$

which yields

$$2\kappa\int\limits_D \Omega_t w_t\mathrm{d}x \le \frac{\gamma}{4}\|\nabla w_t\|_{L^2(D)}^2 + \frac{8\kappa^2}{\gamma}\|\Omega\|_{L^4(D)}^2\|u\|_{H^1(D)}^2 + \frac{32\kappa^4}{\gamma}\|\nabla w\|_{L^2(D)}^2.$$

Combining the two estimates above with (4.6), applying (4.4), and invoking Lemma 3.1 and Proposition 4.1, we conclude

$$\|w_t\|_{L^\infty(0,T;L^2(D))} \le C.$$

By Lemma 2.6,

$$\|w\|_{H^2(D)} \le C\left(\|w_t\|_{L^2(D)} + \|u\cdot\nabla w\|_{L^2(D)} + \|w\|_{L^2(D)} + \|\Omega\|_{L^2(D)}\right). \tag{4.7}$$

By Hölder's inequality, Sobolev embedding inequality, and Lemma 2.4, we have

$$\begin{aligned}\|u\cdot\nabla w\|_{L^2(D)} &\le \|u\|_{L^\infty(D)}\|\nabla w\|_{L^2(D)}\\ &\le C\|u\|_{W^{1,p}(D)}\|\nabla w\|_{L^2(D)}\\ &\le C(\|\Omega\|_{L^p(D)} + \|u\|_{H^1(D)})\|\nabla w\|_{L^2(D)},\end{aligned}$$

where $2 \le p < \infty$. Together with Lemma 3.1 and Proposition 4.1, we obtain the desired global bound. This completes the proof of Proposition 4.2. □

The global bound for $\|\Omega\|_{L^\infty(D)}$ follows as an easy consequence of Proposition 4.2.

Proposition 4.3. *Assume that (u_0, w_0) satisfies the conditions stated in Theorem 1.1. Let (u, w) be the global weak solution obtained in Proposition 3.1. Then, for any $T > 0$ and $0 < t \le T$,*

$$\|\Omega\|_{L^\infty(0,T;L^\infty(D))} \le C,$$

where the constant C depends only on D, T, and the initial data.

Proof. According to (4.3),

$$\|Z\|_{L^\infty(0,T;L^\infty(D))} \le C,$$

which implies, by the definition of Z, Sobolev's embedding, and Proposition 4.2,

$$\begin{aligned}\|\Omega\|_{L^\infty(0,T;L^\infty(D))} &\le (\|Z\|_{L^\infty(0,T;L^\infty(D))} + \|w\|_{L^\infty(0,T;L^\infty(D))})\\ &\le C(\|Z\|_{L^\infty(0,T;L^\infty(D))} + \|w\|_{L^\infty(0,T;H^2(D))})\\ &\le C.\end{aligned}$$

This completes the proof of Proposition 4.3. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In view of Propositions 3.1, 4.1 and 4.2, it suffices to prove the uniqueness. We employ the method of Yudovich.

Assume (u, w, π) and $(\widetilde{u}, \widetilde{w}, \widetilde{\pi})$ are two solutions of (1.3)–(1.5) with the regularity specified in (1.7). Consider their difference

$$U = u - \widetilde{u},\ W = w - \widetilde{w},\ \Pi = \pi - \widetilde{\pi},$$

which solves the following initial- and boundary-value problem

$$\begin{cases} U_t + u\cdot\nabla U + U\cdot\nabla\widetilde{u} + \nabla\Pi = 2\kappa\nabla^{\perp}W, \\ W_t + u\cdot\nabla W + U\cdot\nabla\widetilde{w} - \gamma\Delta W + 4\kappa W = 2\kappa\nabla\times U, \\ \nabla\cdot U = 0, \\ U\cdot\boldsymbol{n}|_{\partial\Omega} = W|_{\partial\Omega} = 0, \\ (U,W)(x,0) = 0. \end{cases} \tag{4.8}$$

Dotting the first two equations with (U,W) yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|U\|_{L^2(D)}^2 + \|W\|_{L^2(D)}^2) + \gamma\|\nabla W\|_{L^2(D)}^2 + 4\kappa\|W\|_{L^2(D)}^2 \\ &\quad = -\int_D U\cdot\nabla\widetilde{u}\cdot U\mathrm{d}x - \int_D U\cdot\nabla\widetilde{w}\cdot W\mathrm{d}x \\ &\qquad + 2\kappa\int_D \nabla^{\perp}W\cdot U\mathrm{d}x + 2\kappa\int_D \nabla\times U\cdot W\mathrm{d}x. \end{aligned} \tag{4.9}$$

By the divergence theorem and the boundary condition $W|_{\partial\Omega} = 0$,

$$\begin{aligned} &2\kappa\int_D \nabla^{\perp}W\cdot U\mathrm{d}x + 2\kappa\int_D \nabla\times U\cdot W\mathrm{d}x \\ &\quad = 4\kappa\int_D \nabla^{\perp}W\cdot U\,\mathrm{d}x \le \frac{\gamma}{2}\,\|\nabla W\|_{L^2(D)}^2 + C\,\|U\|_{L^2(D)}^2. \end{aligned}$$

Since $\nabla\widetilde{u}$ is not known to be bounded in L^∞ while the corresponding vorticity $\widetilde{\Omega}$ is, we apply the Yudovich approach to bound the first term on the right of (4.9). Since $\widetilde{\Omega}\in L^\infty(0,T;L^\infty(D))$, we have, by Lemma 2.4,

$$\|\nabla\widetilde{u}\|_{L^q(D)} \le C\,q\,\left(\|\widetilde{\Omega}\|_{L^q(D)} + \|\widetilde{u}\|_{L^q(D)}\right) \le C\,q\,\left(\|\widetilde{\Omega}\|_{L^\infty(D)} + \|\widetilde{\Omega}\|_{L^2(D)} + \|\widetilde{u}\|_{L^2(D)}\right).$$

Therefore,

$$L \equiv \sup_{q\ge 2}\frac{\|\nabla\widetilde{u}\|_{L^q(D)}}{q} < \infty.$$

In addition, by Lemma 2.1,

$$\begin{aligned} M \equiv \|U\|_{L^\infty(D)} &\le C\,\left(\|\nabla U\|_{L^4(D)}^{\frac{2}{3}}\,\|U\|_{L^2(D)}^{\frac{1}{3}} + \|U\|_{L^2(D)}\right) \\ &\le C\,\left(\|\nabla u\|_{L^4(D)} + \|\nabla\widetilde{u}\|_{L^4(D)} + \|u\|_{L^2(D)} + \|\widetilde{u}\|_{L^2(D)}\right) < \infty. \end{aligned}$$

Therefore, by Hölder's inequality, for any $2\le q<\infty$,

$$\left|\int_D U\cdot\nabla\widetilde{u}\cdot U\,\mathrm{d}x\right| \le C\,q\,L\,M^{\frac{2}{q}}\,\left(\|U\|_{L^2(D)}^2 + \delta\right)^{1-\frac{1}{q}},$$

where $\delta>0$ is inserted here to justify some of the later steps. By optimizing the bound above in terms of q, we obtain

$$\left|\int_D U\cdot\nabla\widetilde{u}\cdot U\,\mathrm{d}x\right| \le C\,e\,L\,(\|U\|_{L^2(D)}^2 + \delta)\,\ln\frac{M^2}{\|U\|_{L^2(D)}^2 + \delta}.$$

To bound the second term on the right of (4.9), we integrate by parts and invoke the boundary condition $W|_{\partial\Omega} = 0$ to obtain

$$\begin{aligned} -\int_D U\cdot\nabla\widetilde{w}\cdot W\,\mathrm{d}x &= \int_D \widetilde{w}\,U\cdot\nabla W\,\mathrm{d}x \\ &\le \|\widetilde{w}\|_{L^\infty(D)}\,\|U\|_{L^2(D)}\,\|\nabla W\|_{L^2(D)} \\ &\le \frac{\gamma}{2}\,\|\nabla W\|_{L^2(D)}^2 + C\,\|\widetilde{w}\|_{H^2(D)}^2\,\|U\|_{L^2(D)}^2. \end{aligned}$$

Inserting the estimates above in (4.9) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big((\|U\|_{L^2(D)}^2+\delta)+\|W\|_{L^2(D)}^2\Big) &\le C\,(1+\|\widetilde{w}\|_{H^2(D)}^2)\,(\|U\|_{L^2(D)}^2+\delta) \\ &\quad + C\,L\,(\|U\|_{L^2}^2+\delta)\,\ln\frac{M^2}{\|U\|_{L^2(D)}^2+\delta}. \end{aligned}$$

By Osgood's inequality, we obtain

$$\begin{aligned} &(\|U(t)\|_{L^2(D)}^2+\delta)+\|W(t)\|_{L^2(D)}^2 \\ &\le e^{C\int_0^t(1+\|\widetilde{w}\|_{H^2(D)}^2)\,\mathrm{d}\tau}\Big((\|U_0\|_{L^2(D)}^2+\delta)+\|W_0\|_{L^2(D)}^2\Big)^{e^{-C\,L\,t}} \\ &\quad\times e^{\int_0^t C L\,e^{-C\,L\,(t-\tau)}\,\ln\left(M^2\exp(C\int_0^\tau(1+\|\widetilde{w}\|_{H^2(D)}^2)\,ds\right)\mathrm{d}\tau} \end{aligned}$$

for any $t\in(0,T)$. Letting $\delta\to 0$ and noting that $U_0 = W_0 = 0$, we obtain the desired uniqueness $U = W \equiv 0$. This finishes the proof of Theorem 1.1. □

5. Higher regularity and proof of Theorem 1.2

The section proves Theorem 1.2, the higher regularity of (u, w). These higher regularity bounds are achieved through several steps. The first step is to prove the global bound

$$\nabla u \in L^\infty(0,T;L^\infty(D)) \quad\text{and}\quad \nabla\Omega\in L^\infty(0,T;L^p(D)) \quad\text{with } 2\le p<\infty$$

via the equation of the combined quantity Z. The second step is to show the global bound for

$$w\in L^\infty(0,T;W^{2,p}(D)) \quad\text{with } p\in[2,\infty) \quad\text{and}\quad w\in L^\infty(0,T;H^3(D)).$$

Finally, we prove $w\in L^\infty(0,T;H^4(D))$. To do so, we estimate $\|w_{tt}\|_{L^2(D)}$ and $\Omega\in L^\infty(0,T;H^2(D))$ and invoke the regularization estimates for elliptic equations (Lemma 2.6).

We start with the first step.

Proposition 5.1. *Assume that (u_0, w_0) satisfies the conditions in Theorem 1.2. Let (u, w) be the corresponding solution of (1.3)–(1.5) guaranteed by Proposition 3.1. Then, for any $T > 0$ and $0 < t \le T$, and for any $2 \le p < \infty$,*

$$\|\nabla u\|_{L^\infty(0,T;L^\infty(D))} + \|\nabla\Omega\|_{L^\infty(0,T;L^p(D))} \le C.$$

where the constant C depends only on D, T, and the initial data.

Proof. Taking the first-order partial ∂_i of (4.2) yields

$$\partial_t\partial_i Z + u\cdot\nabla\partial_i Z + \partial_i u\cdot\nabla Z - \frac{4\kappa^2}{\gamma}\partial_i Z + \frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\partial_i w = 0. \tag{5.1}$$

Multiplying (5.1) by $|\partial_i Z|^{p-2}\partial_i Z$, summing over i, and integrating on D, we have

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Z\|^p_{L^p(D)} \le \|\nabla u\|_{L^\infty(D)}\|\nabla Z\|^p_{L^p(D)} + \frac{4\kappa^2}{\gamma}\|\nabla Z\|^p_{L^p(D)} \\ + \frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\|\nabla w\|_{L^p(D)}\|\nabla Z\|^{p-1}_{L^p(D)},\end{aligned}$$

which also implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Z\|_{L^p(D)} \le C\left(\frac{\kappa^2}{\gamma} + \|\nabla u\|_{L^\infty(D)}\right)\|\nabla Z\|_{L^p(D)} + \frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\|\nabla w\|_{L^p(D)}. \tag{5.2}$$

By Lemma 2.5 and the Sobolev embedding $W^{1,p}(D) \hookrightarrow C^\alpha(D)$ for $p > 2$,

$$\|\nabla u\|_{L^\infty(D)} \le C\,\|\Omega\|_{L^\infty(D)}\log(e + \|\Omega\|_{W^{1,p}(D)}),$$

which, together with the definition of Z, yields

$$\|\nabla u\|_{L^\infty(D)} \le C\|\Omega\|_{L^\infty(D)}\log\left(e + \|\nabla Z\|_{L^p(D)} + \|Z\|_{L^p(D)} + \frac{2\kappa}{\gamma}\|w\|_{W^{1,p}(D)}\right). \tag{5.3}$$

Inserting (5.3) in (5.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla Z\|_{L^p(D)} \le C(1 + \log(e + \|\nabla Z\|_{L^p(D)}))\,\|\nabla Z\|_{L^p(D)} + C,$$

where we have invoked the regularity bounds for (u, w) from the previous sections. We obtain, via Gronwall's inequality, the global bound

$$\|\nabla Z\|_{L^\infty(0,T;L^p(D))} \le C. \tag{5.4}$$

By the definition of Z, for any $2 < p < \infty$,

$$\|\nabla \Omega\|_{L^\infty(0,T;L^p(D))} \le \|\nabla Z\|_{L^\infty(0,T;L^p(D))} + \frac{2\kappa}{\gamma}\|\nabla w\|_{L^\infty(0,T;L^p(D))} \le C, \tag{5.5}$$

which, together with (5.3), implies

$$\|\nabla u\|_{L^\infty(0,T;L^\infty(D))} \le C.$$

By the way, the bound of $\|\nabla\Omega\|_{L^\infty(0,T;L^2(D))}$ can be inferred by (5.5), Hölder's inequality, and the boundedness of domain D directly. This completes the proof of Proposition 5.1. $\square$

Our next goal is to show the global bound for $\|w\|_{W^{2,p}(D)}$ and $\|w\|_{H^3(D)}$. To avoid the boundary effects, we make use of the estimates of time derivatives and the regularization bounds in Lemma 2.6.

Proposition 5.2. *Assume that (u_0, w_0) satisfies the conditions in Theorem 1.2. Let (u, w) be the corresponding solution of (1.3)–(1.5) guaranteed by Proposition 3.1. Then, for any $T > 0$ and $0 < t \le T$, and for any $2 \le p < \infty$,*

$$\|w\|_{L^\infty(0,T;\,W^{2,p}(D))} + \|w\|_{L^\infty(0,T;\,H^3(D))} \le C, \tag{5.6}$$

where the constant C depends only on D, T, p and the initial data.

Proof. To prove this proposition, we first show that

$$\|\nabla w_t\|_{L^\infty(0,T;\,L^2(D))} \le C.$$

By Lemma 2.7, for any $2 \le p < \infty$,

$$\begin{aligned}
&\int_0^T \|\Delta w\|_{L^p(D)}^p \mathrm{d}t \\
&\quad \le C \int_0^T \Big[\|w_0\|_{H^2(D)}^p + \|w\|_{L^p(D)}^p + \|\Omega\|_{L^p(D)}^p + \|u \cdot \nabla w\|_{L^p(D)}^p \Big] \,\mathrm{d}t \\
&\quad \le C \int_0^T \Big[\|w_0\|_{H^2(D)}^p + \|w\|_{H^1(D)}^p + \|\Omega\|_{L^p(D)}^p + \|u\|_{L^{2p}(D)}^p \|\nabla w\|_{L^{2p}(D)}^p \Big] \,\mathrm{d}t.
\end{aligned}$$

Due to the embedding

$$\|\nabla w\|_{L^{2p}(D)} \le C \, \|w\|_{H^2(D)}$$

and the global bound on $\|w\|_{L^\infty(0,T;\, H^2(D))}$ in Proposition 4.2, we obtain

$$\int_0^T \|w\|_{W^{2,p}(D)}^p \,\mathrm{d}t \le C. \tag{5.7}$$

Multiplying (4.5) by $-\Delta w_t$ and integrating on D, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla w_t\|_{L^2(D)}^2 + \gamma \|\Delta w_t\|_{L^2(D)}^2 + 4\kappa \|\nabla w_t\|_{L^2(D)}^2 \\
&\quad = \int_D u \cdot \nabla w_t \Delta w_t \mathrm{d}x - \int_D u_t \cdot \nabla w \Delta w_t \mathrm{d}x - 2\kappa \int_D \Omega_t \Delta w_t \mathrm{d}x.
\end{aligned} \tag{5.8}$$

By Hölder's inequality, Sobolev's embedding, and Lemma 2.4,

$$\begin{aligned}
\int_D u \cdot \nabla w_t \Delta w_t \mathrm{d}x &\le \|u\|_{L^\infty(D)} \|\nabla w_t\|_{L^2(D)} \|\Delta w_t\|_{L^2(D)} \\
&\le \frac{\gamma}{6} \|\Delta w_t\|_{L^2(D)}^2 + \frac{C}{\gamma} \|u\|_{H^2(D)}^2 \|\nabla w_t\|_{L^2(D)}^2 \\
&\le \frac{\gamma}{6} \|\Delta w_t\|_{L^2(D)}^2 + \frac{C}{\gamma} [\|u\|_{L^2(D)}^2 + \|\nabla \Omega\|_{L^2(D)}^2] \|\nabla w_t\|_{L^2(D)}^2
\end{aligned}$$

and

$$\begin{aligned}
-\int_D u_t \cdot \nabla w \Delta w_t \mathrm{d}x &\le \|u_t\|_{L^2(D)} \|\nabla w\|_{L^\infty(D)} \|\Delta w_t\|_{L^2(D)} \\
&\le \frac{\gamma}{6} \|\Delta w_t\|_{L^2(D)}^2 + \frac{C}{\gamma} \|u_t\|_{L^2(D)}^2 \|w\|_{W^{2,p}(D)}^2.
\end{aligned}$$

By the vorticity Eq. (3.3) and Lemma 2.4,

$$\begin{aligned}-2\kappa\int_D \Omega_t\Delta w_t \mathrm{d}x &= 2\kappa\int_D u\cdot\nabla\Omega\Delta w_t\mathrm{d}x + 4\kappa^2\int_D \Delta w\Delta w_t\mathrm{d}x\\
&\le \frac{\gamma}{6}\|\Delta w_t\|^2_{L^2(D)} + \frac{C\kappa}{\gamma}\|u\|^2_{H^2(D)}\|\nabla\Omega\|^2_{L^2(D)} + \frac{C\kappa^2}{\gamma}\|\Delta w\|^2_{L^2(D)}\\
&\le \frac{\gamma}{6}\|\Delta w_t\|^2_{L^2(D)} + \frac{C\kappa}{\gamma}\left[\|u\|^2_{L^2(D)} + \|\nabla\Omega\|^2_{L^2(D)}\right]\|\nabla\Omega\|^2_{L^2(D)}\\
&\quad + \frac{C\kappa^2}{\gamma}\|w\|^2_{H^2(D)}.\end{aligned}$$

Inserting the estimates above in (5.8) and invoking the regularity bounds obtained before, we have

$$\|\nabla w_t\|_{L^2(D)} \le C. \tag{5.9}$$

By Lemma 2.6,

$$\begin{aligned}\|w\|_{W^{2,p}(D)} &\le C\big[\|w_t\|_{L^p(D)} + \|u\cdot\nabla w\|_{L^p(D)} + \|w\|_{L^p(D)} + \|\Omega\|_{L^p(D)}\big]\\
&\le C\big[\|w_t\|_{H^1(D)} + \|u\|_{L^{2p}(D)}\|\nabla w\|_{L^{2p}(D)} + \|w\|_{H^1(D)} + \|\Omega\|_{L^p(D)}\big]\\
&\le C\big[\|w_t\|_{H^1(D)} + \|u\|_{H^1(D)}\|\nabla w\|_{H^1(D)} + \|w\|_{H^1(D)} + \|\Omega\|_{L^p(D)}\big].\end{aligned}$$

According to (5.9) and Proposition 4.1,

$$\|w\|_{L^\infty(0,T;\,W^{2,p}(D))} \le C. \tag{5.10}$$

As a consequence, by Lemma 2.6,

$$\begin{aligned}\|w\|_{H^3(D)} &\le C\big[\|w_t\|_{H^1(D)} + \|u\cdot\nabla w\|_{H^1(D)} + \|w\|_{H^1(D)} + \|\Omega\|_{H^1(D)}\big]\\
&\le C\big[\|w_t\|_{H^1(D)} + \|u\|_{L^\infty(D)}\|\nabla w\|_{H^1(D)} + \|\nabla w\|_{L^\infty(D)}\|u\|_{H^1(D)}\\
&\quad + \|w\|_{H^1(D)} + \|\Omega\|_{H^1(D)}\big]\\
&\le C\big[\|w_t\|_{H^1(D)} + \|u\|_{H^2(D)}\|w\|_{H^2(D)} + \|w\|_{W^{2,p}(D)}\|u\|_{H^1(D)}\\
&\quad + \|w\|_{H^1(D)} + \|\Omega\|_{H^1(D)}\big]\\
&\le C.\end{aligned} \tag{5.11}$$

This completes the proof of Proposition 5.2. □

Finally, we prove the global H^4-bound for w by making full use of the structure of the micropolar equations and classical elliptic regularization theory.

Proposition 5.3. *Assume that (u_0, w_0) satisfies the conditions in Theorem 1.2. Let (u, w) be the corresponding solution of (1.3)–(1.5) guaranteed by Proposition 3.1. Then, for any $T > 0$ and $0 < t \le T$,*

$$\|w\|_{L^\infty(0,T;H^4(D))} \le C,$$

where the constant C depends only on D, T, and the initial data.

Proof. By Lemma 2.6,

$$\|w\|_{H^4(D)} \le C\big[\|w_t\|_{H^2(D)} + \|u\cdot\nabla w\|_{H^2(D)} + \|w\|_{H^2(D)} + \|\Omega\|_{H^2(D)}\big].$$

$\|w\|_{H^2(D)}$ is globally bounded due to Proposition 4.2. To bound $\|u\cdot\nabla w\|_{H^2(D)}$, we apply Lemmas 2.2, 2.4 and the Sobolev embedding $H^2(D) \hookrightarrow L^\infty(D)$ to get

$$\begin{aligned}\|u\cdot\nabla w\|_{H^2(D)} &\le C\big[\|u\|_{L^\infty(D)}\|\nabla w\|_{H^2(D)} + \|\nabla w\|_{L^\infty(D)}\|u\|_{H^2(D)}\big]\\
&\le C\|u\|_{H^2(D)}\|\nabla w\|_{H^2(D)}\\
&\le C\big[\|\Omega\|_{H^1(D)} + \|u\|_{L^2(D)}\big]\|w\|_{H^3(D)} \le C.\end{aligned}$$

according to Propositions 5.1–5.2. Our main efforts are devoted to bounding

$$\|w_t\|_{H^2(D)} \quad \text{and} \quad \|\Omega\|_{H^2(D)}.$$

The following two lemmas establish these desired global bounds. With the help of these lemmas, we obtain

$$\|w\|_{L^\infty(0,T;\, H^4(D))} \leq C.$$

The two lemmas and their proofs are given below. This completes the proof of Proposition 5.3. □

We have used the fact that $\|w_t\|_{H^2(D)}$ is globally bounded. The following lemma states this fact and then prove this fact.

Lemma 5.1. *Assume that (u_0, w_0) satisfies the conditions in Theorem 1.2. Let (u, w) be the corresponding solution of (1.3)–(1.5) guaranteed by Proposition 3.1. Then, for any $T > 0$ and $0 < t \leq T$,*

$$\|\Omega_t\|_{L^\infty(0,T;L^p(D))} \leq C \quad \text{and} \quad \|w_t\|_{L^\infty(0,T;\, H^2(D)} \leq C,$$

where the constant C depends only on D, T, and the initial data.

Proof. Applying Lemma 2.6 to (4.5) yields

$$\begin{aligned}\|w_t\|_{H^2(D)} &\leq C\big[\|w_{tt}\|_{L^2(D)} + \|u\cdot\nabla w_t\|_{L^2(D)} + \|u_t\cdot\nabla w\|_{L^2(D)} \\ &\qquad + \|w_t\|_{L^2(D)} + \|\Omega_t\|_{L^2(D)}\big] \\ &\leq C\big[\|w_{tt}\|_{L^2(D)} + \|u\|_{H^2(D)}\|\nabla w_t\|_{L^2(D)} + \|u_t\|_{L^2(D)}\|\nabla w\|_{H^2(D)} \\ &\qquad + \|w_t\|_{L^2(D)} + \|\Omega_t\|_{L^2(D)}\big].\end{aligned} \tag{5.12}$$

Clearly, the terms $\|u\|_{H^2(D)}\|\nabla w_t\|_{L^2(D)} + \|u_t\|_{L^2(D)}\|\nabla w\|_{H^2(D)} + \|w_t\|_{L^2(D)}$ are all bounded. Therefore, it suffices to bound $\|\Omega_t\|_{L^2(D)}$ and $\|w_{tt}\|_{L^2(D)}$.

Multiplying (3.3) by $|\Omega_t|^{p-2}\Omega_t$ and integrating on D, we have

$$\|\Omega_t\|^p_{L^p(D)} = -\int_D u\cdot\nabla\Omega|\Omega_t|^{p-2}\Omega_t \mathrm{d}x - 2\kappa\int_D \Delta w|\Omega_t|^{p-2}\Omega_t\mathrm{d}x. \tag{5.13}$$

By Hölder's inequality and the embeddings $H^2(D) \hookrightarrow L^\infty(D)$ and $H^1(D) \hookrightarrow L^p(D)$,

$$\begin{aligned}-\int_D u\cdot\nabla\Omega|\Omega_t|^{p-2}\Omega_t\mathrm{d}x &\leq \|u\|_{L^\infty(D)}\|\nabla\Omega\|_{L^p(D)}\|\Omega_t\|^{p-1}_{L^p(D)} \\ &\leq \|u\|_{H^2(D)}\|\nabla\Omega\|_{L^p(D)}\|\Omega_t\|^{p-1}_{L^p(D)}\end{aligned}$$

and

$$\begin{aligned}-2\kappa\int_D \Delta w|\Omega_t|^{p-2}\Omega_t\mathrm{d}x &\leq C\kappa\|\Delta w\|_{L^p(D)}\|\Omega_t\|^{p-1}_{L^p(D)} \\ &\leq C\kappa\|w\|_{H^3(D)}\|\Omega_t\|^{p-1}_{L^p(D)}.\end{aligned}$$

Inserting the estimates above in (5.13) yields

$$\begin{aligned}\|\Omega_t\|_{L^p(D)} &\leq C\big[\|u\|_{H^2(D)}\|\nabla\Omega\|_{L^p(D)} + \|w\|_{H^3(D)}\big] \\ &\leq C\big[(\|\Omega\|_{H^1(D)} + \|u\|_{L^2(D)})\|\nabla\Omega\|_{L^p(D)} + \|w\|_{H^3(D)}\big] \leq C.\end{aligned}$$

Next we prove the global bound

$$\|u_{tt}\|_{L^\infty(0,T;L^2(D))} + \|w_{tt}\|_{L^\infty(0,T;L^2(D))} \leq C.$$

Taking the temporal derivative of the velocity equation in (1.3) yields

$$u_{tt} + u\cdot\nabla u_t + u_t\cdot\nabla u + \nabla p_t = -2\kappa\nabla^\perp w_t. \tag{5.14}$$

Dotting (5.14) with u_{tt} yields

$$\begin{aligned}
\|u_{tt}\|^2_{L^2(D)} &= -\int_D u\cdot\nabla u_t\cdot u_{tt}\mathrm{d}x - \int_D u_t\cdot\nabla u\cdot u_{tt}\mathrm{d}x - 2\kappa\int_D u_{tt}\cdot\nabla^\perp w_t\mathrm{d}x \\
&\le \frac{1}{2}\|u_{tt}\|^2_{L^2(D)} + C\left[\|u\cdot\nabla u_t\|^2_{L^2(D)} + \|u_t\cdot\nabla u\|^2_{L^2(D)} + \|\nabla^\perp w_t\|^2_{L^2(D)}\right] \\
&\le \frac{1}{2}\|u_{tt}\|^2_{L^2(D)} + C\left[\|u\|^2_{L^\infty(D)}\|\nabla u_t\|^2_{L^2(D)} + \|u_t\|^2_{L^2(D)}\|\nabla u\|^2_{L^\infty(D)}\right. \\
&\quad \left. + \|\nabla w_t\|^2_{L^2(D)}\right] \\
&\le \frac{1}{2}\|u_{tt}\|^2_{L^2(D)} + C\left[\|u\|^2_{H^2(D)}(\|u_t\|^2_{L^2(D)} + \|\Omega_t\|^2_{L^2(D)}) + \|u_t\|^2_{L^2(D)}\|u\|^2_{H^2(D)}\right. \\
&\quad \left. + \|\nabla w_t\|^2_{L^2(D)}\right] \\
&\le \frac{1}{2}\|u_{tt}\|^2_{L^2(D)} + C\left[(\|u\|^2_{L^2(D)} + \|\nabla\Omega\|^2_{L^2(D)})(\|u_t\|^2_{L^2(D)} + \|\Omega_t\|^2_{L^2(D)})\right. \\
&\quad \left. + \|\nabla w_t\|^2_{L^2(D)}\right],
\end{aligned}$$

which implies

$$\|u_{tt}\|_{L^2(D)} \le C. \tag{5.15}$$

To estimate $\|w_{tt}\|_{L^2(D)}$, we take the second-order temporal derivative of (4.5)

$$w_{ttt} + u\cdot\nabla w_{tt} + 2u_t\cdot\nabla w_t + u_{tt}\cdot\nabla w + 4\kappa w_{tt} = \gamma\Delta w_{tt} + 2\kappa\Omega_{tt}. \tag{5.16}$$

Multiplying (5.16) by w_{tt} and integrating on D, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{tt}\|^2_{L^2(D)} + \gamma\|\nabla w_{tt}\|^2_{L^2(D)} + 4\kappa\|w_{tt}\|^2_{L^2(D)} \\
&\quad = -2\int_D u_t\cdot\nabla w_t w_{tt}\mathrm{d}x - \int_D u_{tt}\cdot\nabla w w_{tt}\mathrm{d}x + 2\kappa\int_D \Omega_{tt}w_{tt}\mathrm{d}x.
\end{aligned} \tag{5.17}$$

By integration by parts, Hölder's inequality, Young's inequality, Lemma 2.4, and the embedding $H^1(D)\hookrightarrow L^4(D)$,

$$\begin{aligned}
-2\int_D u_t\cdot\nabla w_t w_{tt}\mathrm{d}x &= 2\int_D u_t\cdot\nabla w_{tt}w_t\mathrm{d}x \\
&\le 2\|u_t\|_{L^4(D)}\|w_t\|_{L^4(D)}\|\nabla w_{tt}\|_{L^2(D)} \\
&\le \frac{\gamma}{6}\|\nabla w_{tt}\|^2_{L^2(D)} + \frac{C}{\gamma}\|u_t\|^2_{H^1(D)}\|w_t\|^2_{H^1(D)} \\
&\le \frac{\gamma}{6}\|\nabla w_{tt}\|^2_{L^2(D)} + \frac{C}{\gamma}[\|\Omega_t\|^2_{L^2(D)} + \|u_t\|^2_{L^2(D)}]\|w_t\|^2_{H^1(D)},
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
-\int_D u_{tt}\cdot\nabla w w_{tt}\mathrm{d}x &= \int_D u_{tt}\cdot\nabla w_{tt}w\mathrm{d}x \\
&\le \|u_{tt}\|_{L^2(D)}\|w\|_{L^\infty(D)}\|\nabla w_{tt}\|_{L^2(D)} \\
&\le \frac{\gamma}{6}\|\nabla w_{tt}\|^2_{L^2(D)} + \frac{C}{\gamma}\|u_{tt}\|^2_{L^2(D)}\|w\|^2_{H^2(D)}
\end{aligned} \tag{5.19}$$

and

$$
\begin{aligned}
2\kappa\int\limits_D \Omega_{tt}w_{tt}\mathrm{d}x &= -2\kappa\int\limits_D u_{tt}\cdot\nabla^{\perp}w_{tt}\mathrm{d}x \\
&\le \frac{\gamma}{6}\|\nabla w_{tt}\|_{L^2(D)}^2+\frac{C\kappa^2}{\gamma}\|u_{tt}\|_{L^2(D)}^2.
\end{aligned} \tag{5.20}
$$

Inserting (5.18)–(5.20) in (5.17) yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|w_{tt}\|_{L^2(D)}^2 &\le C(\|\Omega_t\|_{L^2(D)}^2+\|u_t\|_{L^2(D)}^2)\|w_t\|_{H^1(D)}^2 \\
&\quad + C(1+\|u_{tt}\|_{L^2(D)}^2)\|w\|_{H^2(D)}^2.
\end{aligned} \tag{5.21}
$$

Integrating in time yields the desired estimates. This proves Lemma 5.1. □

We now move on to the second lemma asserting the global bound for $\|\Omega\|_{H^2(D)}$.

Lemma 5.2. *Assume that (u_0, w_0) satisfies the conditions in Theorem 1.2. Let (u, w) be the corresponding solution of (1.3)–(1.5) guaranteed by Proposition 3.1. Then, for any $T > 0$ and $0 < t \le T$,*

$$
\|\Omega\|_{L^\infty(0,T;H^2(D))}\le C.
$$

where the constant C depends only on D, T, and the initial data.

Proof. Taking ∂_i^2 of (4.2) yields

$$
\partial_t\partial_i^2 Z+u\cdot\nabla\partial_i^2 Z-\frac{4\kappa^2}{\gamma}\partial_i^2 Z+\frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\partial_i^2 w=u\cdot\nabla\partial_i^2 Z-\partial_i^2(u\cdot\nabla Z). \tag{5.22}
$$

Multiplying (5.22) by $\partial_i^2 Z$ and integrating on D, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i^2 Z\|_{L^2(D)}^2 &= \frac{4\kappa^2}{\gamma}\|\partial_i^2 Z\|_{L^2(D)}^2-\frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\int\limits_D \partial_i^2 w\,\partial_i^2 Z\mathrm{d}x \\
&\quad +\int\limits_D \big(u\cdot\nabla\partial_i^2 Z-\partial_i^2(u\cdot\nabla Z)\big)\partial_i^2 Z\mathrm{d}x.
\end{aligned} \tag{5.23}
$$

By $\nabla\cdot u = 0$ and the commutator estimate (2.2),

$$
\begin{aligned}
&\int\limits_D \big(u\cdot\nabla\partial_i^2 Z-\partial_i^2(u\cdot\nabla Z)\big)\partial_i^2 Z\mathrm{d}x \\
&\quad = \int\limits_D \left(u\cdot\partial_i^2\nabla Z-\partial_i^2\nabla\cdot(uZ)\right)\partial_i^2 Z\mathrm{d}x \\
&\quad \le C\left(\|\nabla u\|_{L^\infty(D)}\|Z\|_{H^2(D)}\|\partial_i^2 Z\|_{L^2(D)}+\|u\|_{H^3(D)}\|Z\|_{L^\infty(D)}\|\partial_i^2 Z\|_{L^2(D)}\right).
\end{aligned} \tag{5.24}
$$

Then, by Lemma 2.4 and definition of Z, one has

$$
\|u\|_{H^3(D)}\le C\big(\|\Omega\|_{H^2(D)}+\|u\|_{L^2(D)}\big)\le C\big(\|Z\|_{H^2(D)}+\|w\|_{H^2(D)}+\|u\|_{L^2(D)}\big),
$$

which together with (4.3) and (5.24) yields

$$
\begin{aligned}
&\int\limits_D \big(u\cdot\nabla\partial_i^2 Z-\partial_i^2(u\cdot\nabla Z)\big)\partial_i^2 Z\mathrm{d}x \\
&\quad \le \|\nabla u\|_{L^\infty(D)}\|Z\|_{H^2(D)}\|\partial_i^2 Z\|_{L^2(D)}+\|Z\|_{H^2(D)}\|\partial_i^2 Z\|_{L^2(D)} \\
&\qquad + (\|w\|_{H^2(D)}+\|u\|_{L^2(D)})\|\partial_i^2 Z\|_{L^2(D)}.
\end{aligned}
$$

By Hölder's inequality,

$$-\frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\int_D \partial_i^2 w \partial_i^2 Z \mathrm{d}x \le \frac{8\kappa^2}{\gamma}\left(1+\frac{\kappa}{\gamma}\right)\|\partial_i^2 w\|_{L^2(D)}\|\partial_i^2 Z\|_{L^2(D)}.$$

Inserting the estimates in (5.23) and summing up by i, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 Z\|_{L^2(D)}^2 \le & C(1+\|\nabla u\|_{L^\infty(D)}+\|w\|_{H^2(D)}^2+\|u\|_{L^2(D)}^2)\\ & \times(\|\nabla^2 Z\|_{L^2(D)}^2+\|\nabla Z\|_{L^2(D)}^2+\|Z\|_{L^2(D)}^2).\end{aligned}$$

Gronwall's inequality implies

$$\|\nabla^2 Z\|_{L^2(D)} \le C.$$

By the definition of Z and Proposition 4.2, we have

$$\|\Omega\|_{H^2(D)} \le \|Z\|_{H^2} + \|w\|_{H^2(D)} \le C.$$

This completes the proof of Lemma 5.2. □

6. Large time behavior

This section proves Theorem 1.3, namely the large time estimate stated in (1.13) under the condition that $\gamma > 4\kappa$.

Proof of Theorem 1.3. To begin with, we take the inner product of (1.10) with (u, w) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2(D)}^2+\|w\|_{L^2(D)}^2\right)+\kappa\|u\|_{L^2(D)}^2+\gamma\|\nabla w\|_{L^2(D)}^2\\ &\quad= 2\kappa\int_D u\cdot\nabla^\perp w\,\mathrm{d}x+2\kappa\int_D w\nabla\times u\,\mathrm{d}x=4\kappa\int_D u\cdot\nabla^\perp w\,\mathrm{d}x\end{aligned} \tag{6.1}$$

where we have used (3.1) to combine the two terms on the right. By Hölder's inequality,

$$4\kappa\int_D u\cdot\nabla^\perp w\,\mathrm{d}x \le 4\kappa\|u\|_{L^2(D)}\|\nabla w\|_{L^2(D)} \le \frac{8\kappa^2}{\gamma+4\kappa}\|u(t)\|_{L^2(D)}^2+\frac{\gamma+4\kappa}{2}\|\nabla w(t)\|_{L^2(D)}^2.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2(D)}^2+\|w(t)\|_{L^2(D)}^2\right)+\frac{2\kappa(\gamma-4\kappa)}{\gamma+4\kappa}\|u(t)\|_{L^2(D)}^2+(\gamma-4\kappa)\|\nabla w(t)\|_{L^2(D)}^2 \le 0. \tag{6.2}$$

Noticing that $w|_{\partial D}=0$, we have, by Lemma 2.3,

$$\|w\|_{L^2(D)}^2 \le C\|\nabla w\|_{L^2(D)}^2,$$

which, together with (6.2), yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2(D)}^2+\|w\|_{L^2(D)}^2\right)+\frac{2\kappa(\gamma-4\kappa)}{\gamma+4\kappa}\|u\|_{L^2(D)}^2+\frac{(\gamma-4\kappa)}{C}\|w\|_{L^2(D)}^2 \le 0.$$

Since $\gamma > 4\kappa$, it is then clear that

$$\|u\|_{L^2(D)}^2+\|w\|_{L^2(D)}^2 \le e^{-C_0 t}\left(\|u_0\|_{L^2(D)}^2+\|w_0\|_{L^2(D)}^2\right), \tag{6.3}$$

where C_0 is given by

$$C_0=\min\left\{\frac{2\kappa(\gamma-4\kappa)}{\gamma+4\kappa},\ \frac{(\gamma-4\kappa)}{C}\right\}.$$

In addition, multiplying (6.2) by $e^{\frac{C_0}{2}t}$ and integrating in time yields

$$\begin{aligned}\frac{2\kappa(\gamma-4\kappa)}{\gamma+4\kappa}\int_0^t e^{\frac{C_0}{2}\tau}\,\|u(\tau)\|_{L^2(D)}^2\,\mathrm{d}\tau+(\gamma-4\kappa)\int_0^t e^{\frac{C_0}{2}\tau}\,\|\nabla w(\tau)\|_{L^2(D)}^2\,\mathrm{d}\tau\\ \leq\|u_0\|_{L^2(D)}^2+\|w_0\|_{L^2(D)}^2.\end{aligned}\tag{6.4}$$

We now turn to the decay of the gradient of (u,w). Multiplying (3.3) by Ω and the w-equation in (1.10) by $-\Delta w$ and then integrating on D, one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Omega\|_{L^2(D)}^2+\|\nabla w\|_{L^2(D)}^2\right)+\kappa\|\Omega\|_{L^2(D)}^2+\gamma\|\Delta w\|_{L^2(D)}^2\\ &\quad=-4\kappa\int_D\Delta w\Omega\,\mathrm{d}x+\int_D u\cdot\nabla w\Delta w\,\mathrm{d}x.\end{aligned}\tag{6.5}$$

By Young's inequality,

$$4\kappa\left|\int_D\Delta w\,\Omega\mathrm{d}x\right|\leq\frac{16\kappa^2}{\gamma+12\kappa}\|\Omega\|_{L^2(D)}^2+\frac{\gamma+12\kappa}{4}\|\Delta w\|_{L^2(D)}^2.$$

By Hölder's inequality, Corollary 2.1, Lemma 2.4 and Young's inequality,

$$\begin{aligned}\int_D u\cdot\nabla w\Delta w\mathrm{d}x&\leq C\|u\|_{L^4(D)}\|\nabla w\|_{L^4(D)}\|\Delta w\|_{L^2(D)}\\ &\leq C\|u\|_{L^2(D)}^{\frac12}\|\nabla u\|_{L^2(D)}^{\frac12}(\|\nabla w\|_{L^2(D)}^{\frac12}\|\nabla^2 w\|_{L^2(D)}^{\frac12}\\ &\quad+\|\nabla w\|_{L^2(D)})\|\Delta w\|_{L^2(D)}\\ &\leq C\|u\|_{L^2(D)}^{\frac12}(\|\Omega\|_{L^2(D)}^{\frac12}+\|u\|_{L^2(D)}^{\frac12})(\|\nabla w\|_{L^2(D)}^{\frac12}\|\Delta w\|_{L^2(D)}^{\frac12}\\ &\quad+\|\nabla w\|_{L^2(D)})\|\Delta w\|_{L^2(D)}\\ &\leq\frac{\gamma-4\kappa}{4}\,\|\Delta w\|_{L^2(D)}^2\\ &\quad+C(1+\|u\|_{L^2(D)}^2)(\|u\|_{L^2(D)}^2+\|\Omega\|_{L^2(D)}^2)\|\nabla w\|_{L^2(D)}^2.\end{aligned}$$

Inserting the estimates above in (6.5), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Omega\|_{L^2(D)}^2+\|\nabla w\|_{L^2(D)}^2\right)+\frac{4\kappa(\gamma-4\kappa)}{\gamma+12\kappa}\,\|\Omega\|_{L^2(D)}^2+2(\gamma-4\kappa)\,\|\Delta w\|_{L^2(D)}^2\\ &\quad\leq\ C(1+\|u\|_{L^2(D)}^2)(\|u\|_{L^2(D)}^2+\|\Omega\|_{L^2(D)}^2)\|\nabla w\|_{L^2(D)}^2.\end{aligned}\tag{6.6}$$

We note that $\gamma>4\kappa$. Applying Gronwall's inequality and using (6.4), we have

$$\begin{aligned}&\|\Omega\|_{L^2(D)}^2+\|\nabla w\|_{L^2(D)}^2\leq e^{C(1+\|u\|_{L^2(D)}^2)\int_0^t\|\nabla w(\tau)\|_{L^2(D)}^2\,\mathrm{d}\tau}\left(\|\Omega_0\|_{L^2(D)}^2+\|\nabla w_0\|_{L^2(D)}^2\right)\\ &\quad+e^{C(1+\|u\|_{L^2(D)}^2)\int_0^t\|\nabla w(\tau)\|_{L^2(D)}^2\,\mathrm{d}\tau}\left((\|u\|_{L^2(D)}^2+\|u\|_{L^4(D)}^2)\int_0^t\|\nabla w(\tau)\|_{L^2(D)}^2\mathrm{d}\tau\right)\\ &\leq C(u_0,w_0),\end{aligned}\tag{6.7}$$

where C is a constant depending only on H^1-norm of (u_0, w_0). This global bound, together with (6.3), allows us to obtain the global exponential bound for the gradient of (u, w). In fact, if we set

$$C_1 = \min\left\{\frac{4\kappa(\gamma - 4\kappa)}{\gamma + 12\kappa}, \frac{C_0}{2}\right\}$$

and multiply (6.6) by $e^{C_1 t}$ and integrate in time, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{C_1 t}\|\Omega\|^2_{L^2(D)} + e^{C_1 t}\|\nabla w\|^2_{L^2(D)}\right) \le C_1 e^{C_1 t}\,\|\nabla w\|^2_{L^2(D)}$$
$$+ C\,(1 + \|u\|^2_{L^2(D)})(\|u\|^2_{L^2(D)} + \|\Omega\|^2_{L^2(D)})\, e^{C_1 t}\,\|\nabla w\|^2_{L^2(D)}.$$

Integrating in time and recalling the global bounds in (6.4) and (6.7) yield that

$$e^{C_1 t}\|\Omega(t)\|^2_{L^2(D)} + e^{C_1 t}\|\nabla w(t)\|^2_{L^2(D)} \le C(u_0, w_0),$$

where $C(u_0, w_0)$ depends on $\|u_0\|_{H^1(D)}$ and $\|w_0\|_{H^1(D)}$ only. This completes the proof of Theorem 1.3. □

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