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An approximating approach for boundary control of optimal mixing via Navier-Stokes flows

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Abstract

The present work focuses on an approximating control design for optimal mixing of a non-dissipative scalar via Navier-Stokes flows in an open bounded and connected domain $\Omega \subset \mathbb{R}^2$. The objective is to achieve optimal mixing at a given final time $T > 0$, via the active control of the flow velocity through the Navier slip boundary control, where Sobolev norm for the dual space $(H^1(\Omega))'$ of $H^1(\Omega)$ is adopted for quantifying mixing. Both passive and active scalars governed by the transport equation will be investigated. Our current approach will lead to a more transparent optimality system for characterizing the optimal solution compared to our previous work [12]. This is achieved by first introducing a small diffusivity to the transport equation and then establishing a rigorous analysis of convergence of the approximating control problem to the original one as the diffusivity converges to zero. Moreover, uniqueness of the optimal solution is obtained.

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1. Introduction

Consider a scalar field that is advected by an incompressible flow in an open bounded and connected domain $\Omega \subset \mathbb{R}^2$, with a sufficiently smooth boundary Γ. The transport equation is used to describe the mass distribution or scalar concentration, where molecular diffusion is assumed to be negligible. Consider the flow velocity induced by control inputs acting tangentially on the boundary of the domain through the Navier slip boundary conditions as addressed in [12]. Our motivation is based on the observation that moving walls accelerate mixing compared to fixed walls; see, e.g., [5–7,24]. The objective is to design an optimal Navier slip boundary control that optimizes mixing at a given final time $T > 0$. The system of equations reads

$$\partial_t \theta + v \cdot \nabla \theta = 0, \tag{1.1}$$

$$\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla p = \xi \theta e_2, \tag{1.2}$$

$$\nabla \cdot v = 0, \tag{1.3}$$

where θ is the mass distribution or scalar concentration, v is the velocity of the flow, $\nu > 0$ is the viscosity, and p is the pressure. For the convenience of notation, we set $\xi = 0$ for the passive transport and $\xi = 1$ for the active transport, via buoyancy-driven flows modeled by the 2D Boussinesq approximation. The Navier slip boundary conditions are given by [19],

$$v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha v \cdot \tau)|_\Gamma = g \cdot \tau, \tag{1.4}$$

where n and τ denote the outward unit normal and tangential vectors with respect to the domain Ω, and $\mathbb{D}(v) = (1/2)(\nabla v + (\nabla v)^T)$. The friction between the fluid and the wall is proportional to $-v$ with the positive coefficient of proportionality α. The nonhomogeneous boundary term g with $g \cdot n|_\Gamma = 0$, is the control input, which is employed to generate the velocity field for mixing. The initial condition is given by

$$(\theta(0), v(0)) = (\theta_0, v_0). \tag{1.5}$$

With the help of divergence-free condition (1.3) and no-penetration boundary condition $v \cdot n|_\Gamma = 0$ in (1.4), it is easy to verify that; see, e.g., [3,8], any L^p-norm of θ is conserved, i.e.,

$$\|\theta(t)\|_{L^p(\Omega)} = \|\theta_0\|_{L^p(\Omega)}, \quad t \geq 0, \quad p \in [1, \infty]. \tag{1.6}$$

Throughout this paper, we use $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$, without ambiguity, for the L^2-inner products as well as the duality in the interior of the domain Ω and on the boundary Γ, respectively. The symbol C denotes a generic positive constant, which is allowed to depend on the domain as well as on indicated parameters.

1.1. Boundary control for optimal mixing

As discussed in our previous work [12], we study the vortex-enhanced mixing since it is widely recognized that mixing can be enhanced by introducing strong streamwise vortices [4,26,27]. Consider the optimal control problem as follows: For a given $T > 0$, find a control g minimizing the cost functional

$$J(g)=\frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'}+\frac{\gamma}{2}\|g\|^2_{U_{\rm ad}}-\frac{\zeta}{2}\int_0^T\|\nabla\times v\|^2_{L^2}\,dt, \qquad \text{(P)}$$

where $\nabla\times v=\partial_1 v_2-\partial_2 v_1$ stands for the vorticity, $\zeta>0$ is the regularization parameter for vorticity, $U_{\rm ad}$ is the set of admissible controls, and the parameter $\gamma>0$ is chosen to establish the relative weight depending on the first and the third term. However, it is also true that the long-time dynamics may be dominated by strong coherent vortices that can possibly slow down mixing. Therefore, parameter ζ can be used to test the sensitivity of mixing rate with respect to vorticity. Soblev norm $\|\cdot\|_{(H^1(\Omega))'}$ is adopted to quantify the degree of mixedness because of the property of weak convergence [18,23,25], which is defined by

$$\|f\|_{(H^s(\Omega))'}=\sup_{\phi\in H^s(\Omega)}\frac{|(f,\phi)_{((H^s(\Omega))',H^s(\Omega))}|}{\|\phi\|_{H^s}},\quad f\in(H^s(\Omega))'\quad\text{for}\quad s>0, \tag{1.7}$$

where $(f,\phi)_{((H^s(\Omega))',H^s(\Omega))}=\int_\Omega f\bar{\phi}\,dx$. We have the Gelfand triple

$$H^s(\Omega)\subset L^2(\Omega)\subset(H^s(\Omega))',\quad s>0,$$

with the embeddings being continuous and compact. The space $H^s(\Omega)$ may be defined as the domain of an operator Λ^s equipped with the norm $\|\cdot\|_{H^s}$, where Λ is self-adjoint, positive and unbounded in $L^2(\Omega)$. Correspondingly, the space $(H^s(\Omega))'$ can be identified as the domain of Λ^{-s} equipped with the norm $\|\cdot\|_{(H^s(\Omega))'}$, and hence $\Lambda^{2s}\in\mathcal{L}(H^s(\Omega),(H^s(\Omega))')$. In this paper, we continue to adopt $\|\cdot\|_{(H^1(\Omega))'}$ for qualifying mixing as in [9,12]. In particular, we set $\Lambda=\mathcal{A}^{-1/2}$, where $\mathcal{A}$ is given by

$$\mathcal{A}\phi=(-\Delta+I)\phi,\quad \phi\in D(\mathcal{A})=\{\phi\in H^2(\Omega)\colon \frac{\partial\phi}{\partial n}|_\Gamma=0\}. \tag{1.8}$$

Then $D(\Lambda)=D(\mathcal{A}^{1/2})=H^1(\Omega)$ and $D(\Lambda^{-1})=D(\mathcal{A}^{-1/2})=(H^1(\Omega))'$.

To set up the abstract formulation for the velocity field, we use the same notations as defined in [12]

$$V^s_n(\Omega)=\{v\in H^s(\Omega):\operatorname{div} v=0,\ v\cdot n|_\Gamma=0\},\quad s\ge 0,$$
$$V^s_n(\Gamma)=\{g\in H^s(\Gamma):g\cdot n|_\Gamma=0\},\quad s\ge 0.$$

1.2. Preliminary results

To solve problem (P), it is crucial to identify the set of admissible controls U_{ad}, which is usually chosen to establish the well-posedness of problem (P) and the existence of an optimal solution. In fact, derivation of $U_{\rm ad}$ requires a more in-depth analysis in studying differentiability properties of the control-state map. In the current approach, the first-order optimality conditions for solving the minimum of J subject to model (1.1)–(1.4) will be derived by using the variational inequality, that is, if u is an optimal solution of problem (P1), then

$$J'(u)\cdot(g-u)\ge 0,\quad \forall g\in U_{ad}, \tag{1.9}$$

where $J'(u) \cdot g$ stands for the Gâteaux derivative of J with respect to u in every direction $g \in U_{ad}$. As a first step to carry out (1.9), one needs the map $u \mapsto (\theta, v)$ to be Gâteaux differentiable, which gives rise to the following major difficulties [12].

Due to zero diffusivity and the nonlinear coupling $v \cdot \nabla\theta$ in the transport equation, establishing the well-posedness of the Gâteaux derivative of θ with respect to u, i.e., $\theta'(u) \cdot g$, for $g \in U_{ad}$, requires $\sup_{t\in[0,T]} \|\nabla\theta\|_{L^2(\Omega)} < \infty$, which in turn demands the flow velocity to satisfy

$$\int_0^T \|\nabla v\|_{L^\infty(\Omega)}\, d\tau < \infty. \tag{1.10}$$

Therefore, the initial condition and the control input for system (1.1)–(1.5) have to be identified such that (1.10) holds. As shown in our recent work [12], for boundary control the time regularity on the boundary data has to be imposed on the cost functional J. Furthermore, establishing the *a priori* estimate (1.10) needs a sharp estimate on the state space so that the compatibility conditions for the boundary and initial data can be possibly avoided. The *a priori* estimate (1.10) becomes the major obstruction since $\|\nabla v\|_{L^\infty}$ can not be bounded by $\|v\|_{H^2}$ based on Sobolev embedding in 2D domain [8,12,13]. Utilizing spectral decomposition analysis in [12], we were able to establish the existence of an optimal Navier slip boundary control to problem (P) for $(\theta_0, v_0) \in (L^\infty \cap H^1) \times H^1$ and the first-order optimality conditions for solving such a controller in both passive and active cases. This indicates that the compatibility conditions for initial and boundary data are not required for Navier slip boundary control, and hence the control input can act only on a portion of the boundary Γ. In addition, vorticity can be clearly addressed on the boundary using Navier slip boundary conditions. However, due to (1.10) the time derivative of control input was imposed on the cost functional in order to obtain the Gâteaux differentiability of (θ, v) with respect to u. For computational convenience, the first derivative $\partial u/\partial t$ was employed rather than the lower order fractional time derivatives. The set of admissible controls was chosen to be

$$U_{ad} = \left\{ u \in L^2(0, T; V_n^{1/2+\delta}(\Gamma)) : \frac{\partial u}{\partial t} \in L^2(0, T; V_n^0(\Gamma)) \right\}, \quad \forall\delta > 0, \tag{1.11}$$

equipped with the norm

$$\|u\|_{U_{ad}} = \|u\|_{L^2(0,T;V_n^{1/2+\delta}(\Gamma))} + \|\frac{\partial u}{\partial t}\|_{L^2(0,T;V_n^0(\Gamma))}.$$

As a consequence, the second derivative $\frac{\partial^2 u}{\partial t^2}$ appears in the optimality conditions, and therefore it is intractable to characterize the features of the optimal velocity field [12, Theorems 4.6 and 5.5].

1.3. An approximating control approach

In this paper, we propose an approximating control strategy to lower the regularity required on the boundary data, which is applicable for solving more general optimal control problems governed by the semi-dissipative systems. A dynamical system is called semi-dissipative if the system is dissipative in some variables, but not in others [3].

To relax the regularity required on the boundary data, we consider an approximating control approach by adding a small diffusion term $\epsilon\Delta\theta$, for $\epsilon > 0$, to the transport equation, and then establish a rigorous analysis of convergence of the approximating control problem to the original one as the diffusivity approaches to zero. The approximating control problem is formulated as follows: For a given $T > 0$, find a control g_ϵ minimizing the cost functional

$$J_\epsilon(g_\epsilon) = \frac{1}{2}\|\theta_\epsilon(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g_\epsilon\|^2_{U_{\epsilon_{\rm ad}}} - \frac{\zeta}{2}\int_0^T \|\nabla \times v_\epsilon\|^2_{L^2(\Omega)}\,dt, \qquad (P_\epsilon)$$

subject to an approximating system governed by

$$\partial_t\theta_\epsilon + v_\epsilon\cdot\nabla\theta_\epsilon = \epsilon\Delta\theta_\epsilon, \quad \epsilon > 0, \tag{1.12}$$

$$\partial_t v_\epsilon + v_\epsilon\cdot\nabla v_\epsilon + \nabla p = \nu\Delta v_\epsilon + \xi\theta_\epsilon e_2, \quad \xi\in\{0,1\}, \tag{1.13}$$

with the Neumann boundary condition is imposed on the scalar equation

$$\epsilon\frac{\partial\theta_\epsilon}{\partial n}|_\Gamma = 0 \tag{1.14}$$

and the Navier slip boundary conditions

$$v_\epsilon\cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n\cdot\mathbb{D}(v_\epsilon)\cdot\tau + \alpha v_\epsilon\cdot\tau)|_\Gamma = g_\epsilon\cdot\tau. \tag{1.15}$$

The initial condition is given by

$$(\theta_\epsilon(0), v_\epsilon(0)) = (\theta_0, v_0) \in L^\infty(\Omega)\times V_n^1(\Omega). \tag{1.16}$$

With a small diffusivity in (1.12), it is possible to prove the existence of an optimal control to problem (P_ϵ) for

$$g_\epsilon \in U_{\epsilon_{\rm ad}} = L^2(0,T;V_n^0(\Gamma)).$$

Moreover, the Gâteaux derivative $J'_\epsilon(g_\epsilon)\cdot h_\epsilon$, for $h_\epsilon\in U_{\epsilon_{\rm ad}}$, is well-defined, and thus the variational inequality (1.9) can be established without involving the time derivatives of g_ϵ. It is also worth to point out that although the set of the admissible controls $U_{\epsilon_{\rm ad}}$ has low regularity in time and space, the optimal control g^*_ϵ solved from the resulting optimality system usually gains regularity [12]. The key is to investigate the relation between the approximating control problem (P_ϵ) and the original one (P). This approach has been applied to solve the passive mixing via Stokes flows in [10]. Although the parabolic regularization of the transport equation is standard, new and significant challenges are encountered in the treatment of the active scalar case when $\xi = 1$.

The outline of the rest of this paper is as follows. In Section 2 we first address the well-posedness of the nonhomogeneous Navier slip boundary value problem (1.1)–(1.5) with low regularity on the boundary data. Next we analyze the convergence of the approximating system governed by (1.12)–(1.16) to the original one governed by (1.1)–(1.5). In Section 3 and Section 4 we establish the existence of an optimal solution to the approximating control problem

(P_ϵ) and derive the first-order necessary conditions of optimality by using a variational inequality. Then we show that the optimal solution $(g^*_\epsilon, v^*_\epsilon, \theta^*_\epsilon)$ to problem (P_ϵ) strongly converges to some (g^*, v^*, θ^*) as $\epsilon \to 0$, which turns out to be the optimal solution to the original problem (P). In Section 5, we present the sufficient conditions on the parameters ζ, γ, and T, to obtain the uniqueness of the optimal solution (g^*, v^*, θ^*).

2. Well-posedness of the nonhomogeneous Navier slip boundary value problems

We first discuss the global well-posedness of the nonhomogeneous Navier slip boundary value problem (1.1)–(1.5) with low regularity on the boundary data. First recall the Stokes problem with Navier slip boundary conditions

$$-\nu\Delta v + \nabla p = 0, \tag{2.1}$$

$$\nabla \cdot v = 0, \tag{2.2}$$

$$v\cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n\cdot \mathbb{D}(v)\cdot\tau + \alpha v\cdot\tau)|_\Gamma = g\cdot\tau. \tag{2.3}$$

The following results and lemmas are provided in [11,12,14] and the references cited therein. To be self-contained, we present the complete statements.

Lemma 2.1. *Assume that Ω is an open bounded and connected domain with boundary $\Gamma \in C^{1,1}$. Let $g \in H^{-1/2}(\Gamma)$. Then, there exists the pressure unique up to a constant such that*

$$\|v\|^2_{H^1} + \|p\|^2_{L^2} \le c\|g\|^2_{H^{-1/2}(\Gamma)}. \tag{2.4}$$

Moreover, if $\Gamma \in C^{2,1}$ and $g \in V_n^{1/2}(\Gamma)$, then $(v, p) \in V_n^2(\Omega) \times H^1(\Omega)$ and

$$\|v\|^2_{H^2} + \|p\|^2_{H^1} \le c\|g\|^2_{H^{1/2}(\Gamma)}. \tag{2.5}$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded and connected domain with boundary $\Gamma \in C^2$. Let $v, u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying the Navier boundary conditions (1.4). Then*

$$\int_\Omega \Delta v\cdot\psi\, dx = -2\int_\Omega \mathbb{D}(v)\cdot\mathbb{D}(\psi)\, dx + \int_\Gamma (\frac{1}{\nu}g\cdot\tau)(\psi\cdot\tau)\, dx - \int_\Gamma \frac{\alpha}{\nu}(v\cdot\tau)(\psi\cdot\tau)\, dx. \tag{2.6}$$

In particular, when $\psi = v$, we have

$$\int_\Omega \Delta v\cdot v\, dx = -2\int_\Omega |\mathbb{D}(v)|^2\, dx + \int_\Gamma (\frac{1}{\nu}g\cdot\tau)(v\cdot\tau)\, dx - \int_\Gamma \frac{\alpha}{\nu}(v\cdot\tau)^2\, dx. \tag{2.7}$$

In addition, if letting $\omega = \nabla \times v$, then by [12, Lemma 2.2] we get

$$\omega = (2\kappa - \frac{\alpha}{\nu})(v\cdot\tau) + \frac{1}{\nu}g\cdot\tau \quad \text{on} \quad \Gamma, \tag{2.8}$$

where κ denotes the curvature of Γ. If each component of Γ is parameterized by arc length s, then $\frac{\partial n}{\partial \tau} = \frac{dn}{ds} = \kappa\tau$.

Define the bilinear form

$$a_0(v, \psi) = 2(\mathbb{D}(v), \mathbb{D}(\psi)) + \frac{\alpha}{\nu}\langle v, \psi\rangle, \quad v, \psi \in V_n^1(\Omega).$$

By Korn and Poincare's inequalities and trace theorem, it is easy to check that

$$c_1\|v\|_{H^1}^2 \le a_0(v, v) \le c_2\|v\|_{H^1}^2,$$

for some constants $c_1, c_2 > 0$. Thus $a_0(\cdot,\cdot)$ is H^1-coercive. Define the operator $A : V_n^1(\Omega) \to (V_n^1(\Omega))'$ by

$$(Av, \psi) = a_0(v, \psi).$$

The Lax-Milgram Theorem implies that $A \in \mathcal{L}(V_n^1(\Omega), (V_n^1(\Omega))')$. This also allows us to identify A as an operator acting on $V_n^0(\Omega)$ with the domain

$$\mathscr{D}(A) = \{v \in V_n^1(\Omega) \colon \psi \mapsto a_0(v, \psi) \text{ is } L^2\text{-continuous}\}.$$

According to (2.6), $A = -\mathbb{P}\Delta$ is the Stokes operator associated with the Navier slip boundary conditions, where $\mathbb{P}$ is the Leray projector on $L^2(\Omega)$ on the space $V_n^0(\Omega)$ [22, p. 13]. Note that A is self-adjoint, strictly positive, and hence the fractal powers of A are well-defined. It immediately follows that for $v \in V_n^1(\Omega)$

$$c_1\|A^{1/2}v\|_{L^2} \le \|\mathbb{D}(v)\|_{L^2} \le c_2\|A^{1/2}v\|_{L^2}. \tag{2.9}$$

Moreover, it is proven in [12, Proposition 2.7] that the domains of A^σ for $0 \le \sigma \le 1$ can be identified as follows

$$\mathscr{D}(A^\sigma) = V_n^{2\sigma}(\Omega), \quad 0 \le \sigma < \frac{3}{4}, \tag{2.10}$$

$$\text{and} \quad \mathscr{D}(A^\sigma) = \{v \in V_n^{2\sigma}(\Omega) \colon (2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha v \cdot \tau)|_\Gamma = 0\}, \quad \frac{3}{4} < \sigma \le 1.$$

Furthermore, the Poincaré inequality holds

$$\|A^\alpha v\|_{L^2} \le \lambda_1^{\alpha-\beta}\|A^\beta v\|_{L^2}, \quad 0 \le \alpha \le \beta \le 1. \tag{2.11}$$

To address the nonhomogeneous boundary value problem, we define the Navier slip boundary operator $N \colon L^2(\Gamma) \to V_n^0(\Omega)$ by

$$Ng = v \iff a_0(v, \psi) = \langle \frac{1}{\nu} g, \psi\rangle, \quad \psi \in V_n^1(\Omega).$$

Based on (2.6) in Lemma 2.2, $v = Ng$ satisfies the Stokes problem (2.1)–(2.3) and

$$N^*A\psi = \frac{1}{\nu}\psi|_\Gamma, \quad \psi \in \mathscr{D}(A). \tag{2.12}$$

Moreover, by Lemma 2.1 and (2.10)

$$N: L^2(\Gamma) \to V_n^{3/2}(\Omega) \subset V_n^{3/2-\varepsilon}(\Omega) = \mathcal{D}(A^{3/4-\varepsilon/2}), \quad \varepsilon > 0,$$

and thus

$$A^{3/4-\varepsilon/2}N \in \mathcal{L}(L^2(\Gamma), V_n^0(\Omega)). \tag{2.13}$$

By virtue of (2.4) and (2.10) it is also true that

$$A^{1/2}N \in \mathcal{L}(H^{-1/2}(\Gamma), V_n^0(\Omega)). \tag{2.14}$$

Making a change of variable, we can rewrite the solution to the nonhomogeneous boundary value problem (1.2)–(1.5) by using the variation of parameters formula

$$v(t) = e^{-\nu At}v_0 + \int_0^t e^{-\nu A(t-\tau)}\mathbb{P}(v\cdot\nabla v)\,d\tau + \int_0^t e^{-\nu A(t-\tau)}\mathbb{P}(\xi\theta e_2)\,d\tau + (Lg)(t), \tag{2.15}$$

where $e^{-\nu At}$, $t \geq 0$, is an analytic semigroup generated by $-\upsilon A$ on $V_n^0(\Omega)$ and L in (2.15) is given by

$$(Lg)(t) = \int_0^t \nu Ae^{-\nu A(t-\tau)}Ng(\tau)\,d\tau. \tag{2.16}$$

The following properties hold for analytic semigroups (cf. [20, p. 74, Theorem 6.13], [15, Proposition 0.1]),

$$e^{-\nu At} \in \mathcal{L}\left(V_n^0(\Omega), L^2(0,T;\mathcal{D}(A^{1/2}))\right), \tag{2.17}$$

$$\|(\nu A)^\sigma e^{-\nu At}\| \leq M_0 t^{-\sigma}e^{-\omega t}, \quad \sigma \geq 0, \tag{2.18}$$

for $M_0 \geq 1$, $\omega > 0$. Also,

$$\int_0^t e^{-\nu A(t-\tau)}\cdot\,d\tau: \text{continuous } L^2(0,T;V_n^0(\Omega)) \to L^2(0,T;\mathcal{D}(A)) \tag{2.19}$$

$$\text{and} \quad \int_0^t e^{-\nu A(t-\tau)}\cdot\,d\tau: \text{continuous } L^2(0,T;V_n^0(\Omega)) \to C([0,T];\mathcal{D}(A^{1/2})). \tag{2.20}$$

For $v \cdot n|_\Gamma = 0$, the regularity properties of L can be derived by following the similar approaches as in (cf. [1, Theorem 3.1.4, Theorem 3.1.8], [15, Lemmas 3.2.2–3.2.3], and [21, Theorems 2.5–2.6]). For $0 \le s < 1/2$,

$$\begin{aligned} L \in \mathcal{L}\big(&L^2(0,T;V_n^{2s}(\Gamma) \cap H^s(0,T;V_n^0(\Gamma)),\\ &L^2(0,T;V_n^{2s+3/2}(\Omega)) \cap H^{s+3/4}(0,T;V_n^0(\Omega))\big). \end{aligned} \tag{2.21}$$

For $1/2 < s \le 1$, (2.21) holds if $g(0) = 0$. With the help of (2.14), (2.10), and (2.19) we get

$$L \in \mathcal{L}\big(L^2(0,T;V_n^{-1/2}(\Gamma)), L^2(0,T;V_n^1(\Omega))\big). \tag{2.22}$$

Furthermore, the $L^2(0,T;\cdot)$-adjoint operator L^* of L is given by

$$(L^*\psi)(t) = \int_t^T \nu N^* A e^{-\nu A(\tau - t)}\psi(\tau)\,d\tau = (\int_t^T e^{-\nu A(\tau-t)}\psi(\tau)\,d\tau)|_\Gamma. \tag{2.23}$$

Slightly modifying the proof in [1, Theorem 3.1.9] yields

$$L^* \in \mathcal{L}\left(L^2(0,T;V_n^{2s}(\Omega)) \cap H^s(0,T;V_n^0(\Omega)), L^2(0,T;V_n^{2s+3/2}(\Gamma)) \cap H^{s+3/4}(0,T;V_n^0(\Gamma))\right), \tag{2.24}$$

for $0 \le s \le 1$. The results in (2.21) and (2.24) are obtained by using the regularity of N given by (2.13), the properties of an analytic semigroup given by (2.17)–(2.20), and the intermediate derivative theorem, based on the similar procedures as in the proofs of (cf. [15, Lemmas 3.2.2–3.2.3], [1, Theorem 3.1.4 and Theorem 3.1.8], and [21, Theorems 2.5–2.6]).

Now we are in a position to discuss the well-posedness of the nonhomogeneous Navier slip boundary value problem (1.1)–(1.5) with different initial data.

Let

$$\mathcal{S} = L^2(0,T;V_n^{1/2+\delta}(\Gamma)) \cap H^{1/4+2/\delta}(0,T;V_n^0(\Gamma)), \quad \forall \delta > 0, \tag{2.25}$$

equipped with the norm

$$\|g\|_{\mathcal{S}} = \|g\|_{L^2(0,T;V_n^{1/2+\delta}(\Gamma))} + \|g\|_{H^{1/4+2/\delta}(0,T;V_n^0(\Gamma))}. \tag{2.26}$$

The following theorem provides a sharper estimate on the boundary data in order to establish the well-posedness of (1.1)–(1.5) compared to [12, Theorem 4.1 and Theorem 5.2], where the first order time derivative is required on g.

Theorem 2.3. *Consider the nonhomogeneous boundary value problem* (1.1)–(1.5).

(1) *If* $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^1(\Omega)$, $g \in \mathcal{S}$, *then for any* $T > 0$, *there exists a unique global solution to* (1.1)–(1.5) *satisfying*

$$(\theta, v) \in L^\infty(0,T;L^\infty(\Omega)) \times (C([0,T];V_n^1(\Omega)) \cap L^2(0,T;V_n^2(\Omega)), \tag{2.27}$$

and

$$\|\theta\|_{L^\infty(0,T;L^\infty(\Omega))} + \|v\|_{L^\infty(0,T;H^1(\Omega))} + \|v\|_{L^2(0,T;H^2(\Omega))} + \|\frac{\partial v}{\partial t}\|_{L^2(0,T;L^2(\Omega))} \leq C(\|\theta_0\|_{L^\infty}, \|v_0\|_{H^1}, \|g\|_{\mathcal{S}}, T). \tag{2.28}$$

(2) *If* $(\theta_0, v_0) \in H^1(\Omega) \times V_n^1(\Omega)$ *and* $g \in \mathcal{S}$, *then for any* $T > 0$, *there exists a unique global solution to* (1.1)–(1.5) *satisfying*

$$(\theta, v) \in L^\infty(0, T; H^1(\Omega)) \times (C([0, T]; V_n^1(\Omega)) \cap L^2(0, T; V_n^2(\Omega))), \tag{2.29}$$

$$\|\theta\|_{L^\infty(0,T;H^1(\Omega))} + \|v\|_{L^\infty(0,T;H^1(\Omega))} + \|v\|_{L^2(0,T;H^2(\Omega))} + \|\frac{\partial v}{\partial t}\|_{L^2(0,T;L^2(\Omega))} \leq C(\|\theta_0\|_{H^1}, \|v_0\|_{H^1}, \|g\|_{\mathcal{S}}, T), \tag{2.30}$$

and (1.10).

Proof. It suffices to prove these results hold for the active scalar case, i.e., $\xi = 1$.

Part (1). The proof for (2.27) with homogeneous Naiver slip boundary conditions has been established in [11, Theorem 1.1], where to show the existence of a unique solution, one needs

$$\int_0^T \|v\|_{H^2}\, dt < \infty. \tag{2.31}$$

For $g \neq 0$, it suffices to identify the regularity of the boundary datum g such that

$$Lg \in L^\infty(0, T; V_n^1(\Omega)) \cap L^2(0, T; V_n^2(\Omega)). \tag{2.32}$$

Applying the variation of parameters formula (2.15) and the regularity of L given by (2.21), we obtain

$$\|Lg\|_{L^2(0,T;V_n^2(\Omega))\cap H^1(0,T;V_n^0(\Omega))} \leq \|g\|_{\mathcal{S}}, \tag{2.33}$$

where $L^2(0, T; V_n^2(\Omega)) \cap H^1(0, T; V_n^0(\Omega)) \subset C([0, T]; V_n^1(\Omega))$ by [16, Theorem 3.1, p. 19]. Thus (2.32) follows immediately.

Part (2). To complete the proof for (2), we make use of the results in [12, Theorem 5.2] for $(\theta_0, v_0) \in H^1(\Omega) \times V_n^1(\Omega)$. It remains to show that

$$\int_0^T \|\nabla(Lg)\|_{L^\infty}\, dt < \infty,$$

for $g \in \mathcal{S}$. Using Agmon's inequality for $d = 2$ together with (2.21) yields

$$\int_0^T \|\nabla(Lg)\|_{L^\infty}\,dt \le C\sqrt{T}\Big(\int_0^T \|Lg\|^2_{H^{2+\varepsilon}}\,dt\Big)^{1/2} \le C\sqrt{T}\|g\|_{\mathcal{S}}, \tag{2.34}$$

for some $0 < \varepsilon < 1/2$ and $\varepsilon \le \delta$. □

Next theorem establishes the global well-posedess of the approximating system (1.12)–(1.16). The proof can be easily carried out using the variation of parameters formula together with energy estimates for $(v_\epsilon, \theta_\epsilon)$.

Theorem 2.4. *Consider the approximating system* (1.12)–(1.16).

(1) *If* $(\theta_{\epsilon 0}, v_{\epsilon 0}) \in L^\infty(\Omega) \times V_n^1(\Omega)$ *and* $g_\epsilon \in L^2(0,T; V_n^0(\Gamma))$*, there exists a unique weak solution* $(\theta_\epsilon, v_\epsilon)$ *satisfying*

$$\begin{aligned}(\theta_\epsilon, v_\epsilon) \in &\big(C([0,T]; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega))\big)\\ &\times \big(C([0,T]; V_n^{1/2}(\Omega)) \cap L^2(0,T; V_n^{3/2}(\Omega))\big).\end{aligned} \tag{2.35}$$

Moreover, if $(\theta_{\epsilon 0}, v_{\epsilon 0}) \in L^\infty(\Omega) \times V_n^1(\Omega)$ *and* $g_\epsilon \in \mathcal{S}$*, then*

$$(\theta_\epsilon, v_\epsilon) \in \big(C([0,T]; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega))\big) \times \big(C([0,T]; V_n^1(\Omega)) \cap L^2(0,T; V_n^2(\Omega))\big). \tag{2.36}$$

(2) *If* $(\theta_{\epsilon 0}, v_{\epsilon 0}) \in H^1(\Omega) \times V_n^1(\Omega)$ *and* $g_\epsilon \in \mathcal{S}$*, then*

$$(\theta_\epsilon, v_\epsilon) \in \big(C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))\big) \times \big(C([0,T]; V_n^1(\Omega)) \cap L^2(0,T; V_n^2(\Omega))\big) \tag{2.37}$$

and

$$\begin{aligned}&\|\theta_\epsilon\|_{L^\infty(0,T;H^1(\Omega))} + \sqrt{\epsilon}\|\theta_\epsilon\|_{L^2(0,T;H^2(\Omega))} + \Big\|\frac{\partial \theta_\epsilon}{\partial t}\Big\|_{L^2(0,T;L^2(\Omega))}\\ &\quad + \|v_\epsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|v_\epsilon\|_{L^2(0,T;H^2(\Omega))} + \Big\|\frac{\partial v_\epsilon}{\partial t}\Big\|_{L^2(0,T;L^2(\Omega))}\\ &\le C(\|\theta_{\epsilon 0}\|_{H^1}, \|v_{\epsilon 0}\|_{H^1}, \|g_\epsilon\|_{\mathcal{S}}, T).\end{aligned} \tag{2.38}$$

2.1. Convergence of the approximating system

This section addresses the convergence issues of the approximating system (1.12)–(1.16) to the original system (1.1)–(1.5).

Proposition 2.5. *Assume that* $(\theta_{\epsilon 0}, v_{\epsilon 0}) = (\theta_0, v_0) \in L^\infty(\Omega) \times V_n^1(\Omega)$ *and* $g_\epsilon = g \in \mathcal{S}$*. Let* $(\theta_\epsilon, v_\epsilon)$ *and* (θ, v) *be the corresponding solutions to the approximating system* (1.12)–(1.16) *and the original system* (1.1)–(1.5)*, respectively. Then for any* $T > 0$,

$$\|\theta_\epsilon - \theta\|_{(H^1(\Omega))'} \to 0 \quad \textit{uniformly in } t \in [0,T], \quad \textit{as} \quad \epsilon \to 0, \tag{2.39}$$

$$\|v_\epsilon - v\|_{L^2} \to 0 \quad \textit{uniformly in } t \in [0,T], \quad \textit{as} \quad \epsilon \to 0, \tag{2.40}$$

$$\textit{and} \quad \int_0^T \|A^{1/2}v_\epsilon - A^{1/2}v\|_{L^2}^2 \to 0 \quad \textit{as} \quad \epsilon \to 0. \tag{2.41}$$

The proof utilizes the Yudovich techniques and the Osgood inequality as in [11, Theorem 1.2]. We present the necessary details for the convenience of the reader. The following Osgood type inequality will be used, which can be found in [11, Lemma 2.5].

Lemma 2.6. *Let $T > 0$ and $I = [0, T)$. Let $f \geq 0$ be a measurable function on I. Let $\mathcal{A} \geq 0$ and $\mathcal{B} \geq 0$, and $\mathcal{A}, \mathcal{B} \in L^1(I)$. Let $M > 0$ be a fixed constant. Assume that f satisfies, for $t \in I$,*

$$\frac{df}{dt} \leq \mathcal{A}f + \mathcal{B}f(\ln M - \ln f).$$

Then, for $t \in I$,

$$f(t) \leq f(0)^{e^{-\int_0^t \mathcal{B}(\tau)d\tau}} M^{1-e^{-\int_0^t \mathcal{B}(\tau)d\tau}} e^{\int_0^t \mathcal{A}(s)e^{\int_s^t \mathcal{B}(\tau)d\tau}ds}. \tag{2.42}$$

Especially, $f(0) = 0$ implies $f(t) = 0$ for $t \in I$.

Proof of Proposition 2.5. Let $\Theta = \theta_\epsilon - \theta$, $P = p_\epsilon - p$ and $V = v_\epsilon - v$. Then

$$\partial_t \Theta + v \cdot \nabla\Theta + V \cdot \nabla\theta_\epsilon = \epsilon\Delta\theta_\epsilon, \tag{2.43}$$

$$\partial_t V + v \cdot \nabla V + V \cdot \nabla v_\epsilon = -\nabla P + \nu\Delta V + \xi\Theta\vec{e}_2, \tag{2.44}$$

with boundary conditions

$$V \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(V) \cdot \tau + \alpha V \cdot \tau)|_\Gamma = 0. \tag{2.45}$$

The initial condition is given by

$$(\Theta(0), V(0)) = (0, 0).$$

Define η and η_ϵ by

$$\Delta\eta = \theta \quad \text{in } \Omega, \quad \frac{\partial\eta}{\partial n} + \eta = 0 \quad \text{on } \Gamma, \tag{2.46}$$

$$\Delta\eta_\epsilon = \theta_\epsilon \quad \text{in } \Omega, \quad \frac{\partial\eta_\epsilon}{\partial n} + \eta_\epsilon = 0. \quad \text{on } \Gamma. \tag{2.47}$$

There exist unique solutions to equations (2.46)–(2.47). Let $H = \eta_\epsilon - \eta$. Then

$$\Delta H = \Theta \quad \text{in } \Omega, \quad \frac{\partial H}{\partial n} + H = 0 \quad \text{on } \Gamma. \tag{2.48}$$

Taking the inner product of the velocity equation with V yields

$$\frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2 = \nu\int_\Omega \Delta V\cdot V\,dx + \int_\Omega \xi\Theta\vec{e}_2\cdot V\,dx - \int_\Omega V\cdot\nabla v_\epsilon\cdot V\,dx. \tag{2.49}$$

By Lemma 2.2,

$$\int_\Omega \Delta V\cdot V\,dx = -2\int_\Omega |\mathbb{D}(V)|^2\,dx - \int_\Gamma \alpha(V\cdot\tau)^2\,dx$$

and

$$\begin{aligned}|\int_\Omega V\cdot\nabla v_\epsilon\cdot V\,dx| &\le \|\nabla v_\epsilon\|_{L^2}\|V\|_{L^4}^2 \le C\|\nabla v_\epsilon\|_{L^2}\|V\|_{L^2}\|\nabla V\|_{L^2}\\ &\le \frac{\nu}{4}(2\int_\Omega |\mathbb{D}(V)|^2\,dx + \int_\Gamma \alpha(V\cdot\tau)^2\,dx) + C\|\nabla v_\epsilon\|_{L^2}^2\|V\|_{L^2}^2.\end{aligned}$$

Moreover,

$$|\int \xi\Theta\vec{e}_2\cdot V\,dx| \le \xi\|\nabla V\|_{L^2}\|\nabla H\|_{L^2} \le \xi\frac{\nu}{4}(2\int_\Omega |\mathbb{D}(V)|^2 + \int_\Omega \alpha(V\cdot\tau)^2\,dx) + C\xi\|\nabla H\|_{L^2}^2.$$

Thus,

$$\frac{d}{dt}\|V\|_{L^2}^2 + \nu(2\int_\Omega |\mathbb{D}(V)|^2 + \int_\Gamma \alpha(V\cdot\tau)^2\,dx) \le C\|\nabla v_\epsilon\|_{L^2}^2\|V\|_{L^2}^2 + C\xi\|\nabla H\|_{L^2}^2. \tag{2.50}$$

Taking the inner product of the scalar equation (2.43) with H yields

$$\frac{d}{dt}\|\nabla H\|_{L^2}^2 = \int_\Omega v\cdot\nabla(\Delta H)H\,dx + \int_\Omega V\cdot\nabla\theta_\epsilon H\,dx - \int_\Omega \epsilon(\Delta\theta_\epsilon)H\,dx. \tag{2.51}$$

Using (2.48), we have the first term on the right hand side of (2.51) satisfy

$$\begin{aligned}|\int_\Omega v\cdot\nabla(\Delta H)H\,dx| &= |\int_\Omega v\cdot\nabla((\Delta H)H)\,dx - \int_\Omega (\Delta H)v\cdot\nabla H\,dx|\\ &= |-\int_\Omega \partial_k\partial_k H(v\cdot\nabla H)\,dx|\\ &= |\int_\Omega \partial_k H(\partial_k v\cdot\nabla H + v\cdot\nabla\partial_k H)\,dx - \int_\Gamma \frac{\partial H}{\partial n}(v\cdot\nabla H)\,dx|\end{aligned}$$

$$= |\int_\Omega \partial_k H \partial_k v \cdot \nabla H \, dx + (H, v\cdot\nabla H)_{(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))}|$$

$$\leq |\int_\Omega \nabla H \cdot \nabla v \cdot \nabla H \, dx| + \|\nabla H\|_{L^2} \|v \cdot \nabla H\|_{L^2}, \tag{2.52}$$

where

$$\|\nabla H\|_{L^2} \|v \cdot \nabla H\|_{L^2} \leq C\|v\|_{L^\infty} \|\nabla H\|_{L^2}^2 \leq C\|v\|_{H^2} \|\nabla H\|_{L^2}^2. \tag{2.53}$$

To estimate the first term on the right hand side of (2.52), we employ Yudovich's method. Recall that by Sobolev's embedding inequality, for any $2 \leq q < \infty$,

$$\|\nabla v\|_{L^q(\Omega)} \leq C(\Omega)\, q\, \|\nabla v\|_{L^2(\Omega)} + C(\Omega)\, q\, \|\nabla\nabla v\|_{L^2}$$
$$\leq C(\Omega)\, q\, \|\nabla v\|_{L^2} + C(\Omega)\, q\, \|v\|_{H^2},$$

which gives

$$\sup_{q \geq 2} \frac{\|\nabla v\|_{L^q}}{q} \leq C\, \|v\|_{H^2}. \tag{2.54}$$

For any $2 < q < \infty$, by Hölder's inequality,

$$|\int_\Omega \nabla H \cdot \nabla v \cdot \nabla H \, dx| \leq \|\nabla H\|_{L^2}\, \|\nabla v\|_{L^q}\, \|\nabla H\|_{L^{\frac{2q}{q-2}}}$$
$$\leq \|\nabla H\|_{L^2}\, \|\nabla v\|_{L^q}\, \|\nabla H\|_{L^2}^{1-\frac{2}{q}}\, \|\nabla H\|_{L^\infty}^{\frac{2}{q}}.$$

Let $M \equiv \|\nabla H\|_{L^\infty}$. Then for any $2 < r < \infty$,

$$M \equiv \|\nabla H\|_{L^\infty} \leq C\, \|\nabla\nabla H\|_{L^r} \leq C\|\theta\|_{L^r} < \infty.$$

Thus by (2.54), for any $2 < q < \infty$,

$$|\int_\Omega \nabla H \cdot \nabla v \cdot \nabla H \, dx| \leq C\, M^{\frac{2}{q}}\, \|\nabla H\|_{L^2}^{2(1-\frac{1}{q})}\, \|\nabla v\|_{L^q}$$
$$\leq C\, q\|v\|_{H^2}\, M^{\frac{2}{q}}\, \|\nabla H\|_{L^2}^{2-\frac{2}{q}}$$
$$= C\, \|v\|_{H^2}\, \|\nabla H\|_{L^2}^2 \left(q M^{\frac{2}{q}}\, \|\nabla H\|_{L^2}^{-\frac{2}{q}}\right).$$

By taking $q = 2\ln(M/\|\nabla H\|_{L^2})$, we obtain the minimizer of $qM^{\frac{2}{q}}\|\nabla H\|_{L^2}^{-\frac{2}{q}}$, that is,

$$\min_{2\le q<\infty} qM^{\frac{2}{q}}\|\nabla H\|_{L^2}^{-\frac{2}{q}} = 2e\left(\ln M - \ln\|\nabla H\|_{L^2}\right).$$

Consequently,

$$|\int_\Omega \nabla H\cdot\nabla v\cdot\nabla H\,dx| \le C\|v\|_{H^2}\|\nabla H\|_{L^2}^2(\ln M - \ln\|\nabla H\|_{L^2}). \tag{2.55}$$

For the second term of the right hand of (2.51), we have

$$\begin{aligned}|\int_\Omega V\cdot\nabla\theta_\epsilon H\,dx| &= |\int_\Omega V\cdot\nabla(\theta_\epsilon H)\,dx - \int_\Omega \theta_\epsilon V\cdot\nabla H\,dx|\\ &\le \|\theta_\epsilon\|_{L^\infty}(\|V\|_{L^2}^2 + \|\nabla H\|_{L^2}^2).\end{aligned} \tag{2.56}$$

The third term of the right hand of (2.51) satisfies

$$\begin{aligned}-\epsilon\int(\Delta\theta_\epsilon)H\,dx &= -\epsilon\int(\Delta\Theta)H\,dx - \epsilon\int(\Delta\theta)H\,dx\\ &= -\epsilon\int\Theta^2\,dx - \epsilon\int\theta\Delta H\,dx,\end{aligned} \tag{2.57}$$

where the second term on the right hand is bounded by

$$\epsilon\int\theta\Theta\,dx \le \frac{\epsilon}{2}\int\Theta^2\,dx + \frac{\epsilon}{2}\int\theta^2\,dx. \tag{2.58}$$

Finally, combining (2.51) with (2.52)–(2.58) gives

$$\begin{aligned}\frac{d}{dt}\|\nabla H\|_{L^2}^2 + \frac{\epsilon}{2}\|\Theta\|_{L^2}^2 &\le C\|v\|_{H^2}\|\nabla H\|_{L^2}^2(\ln\tilde{M} - \ln\|\nabla H\|_{L^2})\\ &+ \|\theta_\epsilon\|_{L^\infty}(\|V\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \frac{\epsilon}{2}\|\theta\|_{L^2}^2,\end{aligned} \tag{2.59}$$

which, together with (2.50), yields

$$\begin{aligned}\frac{d}{dt}(\|\nabla H\|_{L^2}^2 + \|V\|_{L^2}^2) &\le C\|\nabla v_\epsilon\|_{L^2}^2\|V\|_{L^2}^2 + C\xi\|\nabla H\|_{L^2}^2\\ &+ C\|v\|_{H^2}\|\nabla H\|_{L^2}^2(\ln\tilde{M} - \ln\|\nabla H\|_{L^2})\\ &+ \|\theta_\epsilon\|_{L^\infty}(\|V\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \frac{\epsilon}{2}\|\theta\|_{L^2}^2.\end{aligned} \tag{2.60}$$

Let $Y(t) = \delta + \|V\|_{L^2}^2 + \|\nabla H\|_{L^2}^2$ for any small $\delta > 0$. Then $Y(t)$ satisfies

$$\frac{d}{dt}Y \le C(\xi + \|\nabla v_\epsilon\|_{L^2}^2 + \|\theta_\epsilon\|_{L^\infty})Y + C\|v\|_{H^2}Y(\ln\tilde{M} - \ln Y) + \frac{\epsilon}{2}\|\theta_0\|_{L^2}^2, \tag{2.61}$$

for any $\epsilon > 0$. By Osgood inequality (2.42) and (2.28), letting $\epsilon, \delta \to 0$ yields $\|V\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \to 0$ uniformly in $t \in [0, T]$, and hence (2.39)–(2.40) hold. (2.41) follows immediately from (2.50). □

To establish the convergence of the adjoint systems later on, we shall need a stronger convergence of θ_ϵ to θ. To this end, we let $(\theta_0, v_0) \in H^1(\Omega) \times V_n^1(\Omega)$ in the following result.

Proposition 2.7. *Assume that $(\theta_{\epsilon 0}, v_{\epsilon 0}) = (\theta_0, v_0) \in H^1(\Omega) \times V_n^1(\Omega)$ and $g_\epsilon = g \in \mathcal{S}$. Let $(\theta_\epsilon, v_\epsilon)$ and (θ, v) be the corresponding solutions to the approximating system (1.12)–(1.16) and the original system (1.1)–(1.5), respectively. Then for any $T > 0$,*

$$\|\theta_\epsilon - \theta\|_{L^2} \to 0, \quad \text{uniformly in } t \in [0, T], \quad \text{as} \quad \epsilon \to 0, \tag{2.62}$$

$$\|v_\epsilon - v\|_{H^1} \to 0, \quad \text{uniformly in } t \in [0, T], \quad \text{as} \quad \epsilon \to 0, \tag{2.63}$$

$$\int_0^T \|Av_\epsilon - Av\|_{L^2}^2 \, dt \to 0, \quad \text{as} \quad \epsilon \to 0. \tag{2.64}$$

Proof. The convergence results (2.62)–(2.64) can be established by applying L^2-estimate for Θ and H^1-estimate for V together with the regularity result (2.29). Taking the inner product of equation (2.43) with Θ gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Theta\|_{L^2}^2 &\le \|V\|_{L^\infty}\|\nabla\theta_\epsilon\|_{L^2}\|\Theta\|_{L^2} + \epsilon\|\Delta\theta_\epsilon\|_{L^2}\|\Theta\|_{L^2} \\
&\le C\|AV\|_{L^2}\|\nabla\theta_\epsilon\|_{L^2}\|\Theta\|_{L^2} + \epsilon\|\Delta\theta_\epsilon\|_{L^2}\|\Theta\|_{L^2} \\
&\le \frac{\nu}{8}\|AV\|_{L^2}^2 + C(\|\nabla\theta_\epsilon\|_{L^2}^2 + 1)\|\Theta\|_{L^2}^2 + \frac{\epsilon^2}{2}\|\Delta\theta_\epsilon\|_{L^2}^2,
\end{aligned} \tag{2.65}$$

which implies

$$\|\Theta\|_{L^2}^2 \le \frac{\nu}{4}\int_0^T \|AV\|_{L^2}^2 \, dt + C\int_0^T (\|\nabla\theta_\epsilon\|_{L^2}^2 + 1)\|\Theta\|_{L^2}^2 \, dt + \epsilon^2 \int_0^T \|\Delta\theta_\epsilon\|_{L^2}^2 \, dt. \tag{2.66}$$

To estimate $\int_0^T \|AV\|_{L^2} \, dt$, we employ the following variation of parameters formulation for V

$$V(t) = e^{-\nu At}V(0) - \int_0^t e^{-\nu A(t-\tau)}\mathbb{P}(v \cdot \nabla V + V \cdot \nabla v_\epsilon) \, d\tau + \int_0^t e^{-\nu A(t-\tau)}\mathbb{P}(\xi\Theta e_2) \, d\tau, \tag{2.67}$$

where $V(0) = 0$. According to (2.19)–(2.20) we get

$$\|A^{1/2}V\|_{L^2}^2 + \nu \int_0^T \|AV\|_{L^2}^2\,dt \le C\int_0^T \|\mathbb{P}(v\cdot\nabla V + V\cdot\nabla v_\epsilon)\|_{L^2}^2\,dt + C\xi^2 \int_0^T \|\Theta\|_{L^2}^2\,dt$$

$$\le C\int_0^T (\|v\|_{L^4}^2\|\nabla V\|_{L^4}^2 + \|V\|_{L^\infty}^2\|\nabla v_\epsilon\|_{L^2}^2)\,dt + C\xi^2\int_0^T \|\Theta\|_{L^2}^2\,dt$$

$$\le C\int_0^T \|v\|_{L^2}\|A^{1/2}v\|_{L^2}\|A^{1/2}V\|_{L^2}\|AV\|_{L^2}\,dt$$

$$+ C\int_0^T \|A^{1/2}V\|_{L^2}^{3/2}\|AV\|_{L^2}^{1/2}\|A^{1/2}v_\epsilon\|_{L^2}^2\,dt + C\xi^2\int_0^T \|\Theta\|_{L^2}^2\,dt$$

$$\le C(\int_0^T \|v\|_{L^2}^2\|A^{1/2}v\|_{L^2}^2\|A^{1/2}V\|_{L^2}^2\,dt)^{1/2}(\int_0^T \|AV\|_{L^2}^2\,dt)^{1/2}$$

$$+ C(\int_0^T \|A^{1/2}V\|_{L^2}^2\|A^{1/2}v_\epsilon\|_{L^2}^{8/3})^{3/4}(\int_0^T \|AV\|_{L^2}^2\,dt)^{1/4} + C\xi^2\int_0^T \|\Theta\|_{L^2}^2\,dt$$

$$\le C\int_0^T \|v\|_{L^2}^2\|A^{1/2}v\|_{L^2}^2\|A^{1/2}V\|_{L^2}^2\,dt + \frac{\nu}{4}\int_0^T \|AV\|_{L^2}^2\,dt$$

$$+ C\int_0^T \|A^{1/2}V\|_{L^2}^2\|A^{1/2}v_\epsilon\|_{L^2}^{8/3}\,dt + \frac{\nu}{4}\int_0^T \|AV\|_{L^2}^2\,dt + C\xi^2\int_0^T \|\Theta\|_{L^2}^2\,dt. \quad (2.68)$$

Note from (2.38) that $\epsilon\int_0^T \|\Delta\theta_\epsilon\|_{L^2}^2\,dt < \infty$. Thus combining (2.66) with (2.68) and letting $\epsilon \to 0$ yield

$$\|\Theta\|_{L^2}^2 + \|A^{1/2}V\|_{L^2}^2 + \frac{\nu}{4}\int_0^T \|AV\|_{L^2}^2\,dt \le C\int_0^T \|v\|_{L^2}^2\|A^{1/2}v\|_{L^2}^2\|A^{1/2}V\|_{L^2}^2\,dt$$

$$+ C\int_0^T \|A^{1/2}V\|_{L^2}^2\|A^{1/2}v_\epsilon\|_{L^2}^{8/3}\,dt + C\int_0^T (\|\nabla\theta_\epsilon\|_{L^2}^2 + 1 + \xi^2)\|\Theta\|_{L^2}^2\,dt$$

$$\le \int_0^T C(\|v\|_{L^2}^2\|A^{1/2}v\|_{L^2}^2 + \|A^{1/2}v_\epsilon\|_{L^2}^{8/3} + \|\nabla\theta_\epsilon\|_{L^2}^2 + 1 + \xi^2)(\|\Theta\|_{L^2}^2 + \|A^{1/2}V\|_{L^2})\,dt. \quad (2.69)$$

By Poincaré inequality (2.11), $\|v\|_{L^2} \le \lambda_1^{-1/2}\|A^{1/2}v\|_{L^2}$, where $\lambda_1 > 0$ is the lowest eigenvalue of A. According to (2.28) and (2.38), $\|A^{1/2}v\|_{L^2}$, $\|A^{1/2}v_\epsilon\|_{L^2}$ and $\|\nabla\theta_\epsilon\|_{L^2}$ are bounded. Applying Gronwall inequality to (2.69), we obtain $\|\Theta\|_{L^2} = \|A^{1/2}V\|_{L^2} = 0$ and $\int_0^T \|AV\|_{L^2}^2\,dt = 0$. This completes the proof. □

3. Existence of an optimal solution in control space with low regularity

In this section we establish the existence of an optimal solution to problems (P) and (P_ϵ) for $g \in U_{\mathrm{ad}}$, where

$$U_{ad} = L^2(0,T; V_n^0(\Gamma)). \tag{3.1}$$

Recall the definition of a weak solution to (1.1)–(1.5).

Definition 3.1. For $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^1(\Omega)$ and $g \in L^2(0,T; V_n^0(\Gamma))$, $(\theta, v) \in C([0,T], (H^1(\Omega))') \times C([0,T]; V_n^0(\Omega)) \cap L^2(0,T; V_n^1(\Omega))$ is said to be a weak solution of equation (1.1)–(1.5), if (θ, v) satisfies

$$(\frac{\partial\theta}{\partial t}, \phi) - (v\theta, \nabla\phi) = 0, \quad \forall\phi \in H^1(\Omega), \tag{3.2}$$

$$(\frac{\partial v}{\partial t}, \psi) + 2\nu(\mathbb{D}(v), \mathbb{D}(\psi)) + \alpha\langle v, \psi\rangle + (v\cdot\nabla v, \psi) = \langle g, \psi\rangle + (\xi\mathbb{P}\theta e_2, \psi), \quad \forall\psi \in V_n^1(\Omega). \tag{3.3}$$

The existence of a weak solution to (3.2)–(3.3) is proven in [11, Propositions 3.1-3.2], which satisfies

$$(\theta, v) \in L^\infty(0,T; L^\infty(\Omega)) \times \big(C([0,T]; V_n^{1/2}(\Omega)) \cap L^2(0,T; V_n^{3/2}(\Omega))\big)$$

and

$$\begin{aligned}&\|\theta\|_{L^\infty(0,T;L^\infty(\Omega))} + \|v\|_{L^\infty(0,T;H^{1/2}(\Omega))} + \|v\|_{L^2(0,T;H^{3/2}(\Omega))} + \|\frac{\partial v}{\partial t}\|_{L^2(0,T;L^2(\Omega))}\\ &\quad \le C(\|\theta_0\|_{L^\infty}, \|v_0\|_{H^1}, \|g\|_{U_{ad}}, T).\end{aligned} \tag{3.4}$$

However, in the active scalar case the uniqueness can not be obtained for the boundary data with such low regularity, as the *a priori* estimate (2.31) on velocity does not hold.

Theorem 3.2. *Assume that* $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^1(\Omega)$. *There exists an optimal solution* $g^* \in U_{ad}$ *to problem (P).*

Proof. The proof follows the similar procedures as in [12, Theorm 5.3]. We provide a complete one for the convenience of the reader. Since J is bounded from below, we can choose a minimizing sequence $\{g_m\} \subset U_{\mathrm{ad}}$ such that

$$\lim_{m\to\infty} J(g_m) = \inf_{g\in U_{\mathrm{ad}}} J(g). \tag{3.5}$$

This also indicates that $\{g_m\}$ is uniformly bounded in U_{ad}, and hence there exists a weakly convergent subsequence, still denoted by $\{g_m\}$, such that

$$g_m \to g^* \quad \text{weakly in} \quad L^2(0,T;V_n^0(\Gamma)). \tag{3.6}$$

Correspondingly, by (3.4) we can extract subsequences $\{v_m\}$ and $\{\theta_m\}$ such that

$$v_m \to v^* \quad \text{weakly in} \quad L^2(0,T;H^{3/2}(\Omega)), \tag{3.7}$$

$$\frac{\partial v_m}{\partial t} \to \frac{\partial v^*}{\partial t} \quad \text{weakly in} \quad L^2(0,T;H^{-1/2}(\Omega)), \tag{3.8}$$

and

$$\theta_m \to \theta^* \quad \text{weakly}^* \text{ in} \quad L^\infty(0,T;L^\infty(\Omega)), \tag{3.9}$$

where by the compactness theorem [22, Theorem 2.2, p. 186], it follows that

$$v_m \to v^* \quad \text{strongly in} \quad L^2(0,T;V_n^0(\Omega)). \tag{3.10}$$

Next we verify that (θ^*, v^*) is the weak solution based on Definition 3.1. Note that g_m and (θ_m, v_m) satisfy

$$(\frac{\partial \theta_m}{\partial t}, \phi) - (v_m\theta_m, \nabla\phi) = 0, \quad \phi \in H^1(\Omega), \tag{3.11}$$

$$\begin{aligned}(\frac{\partial v_m}{\partial t}, \psi) + 2\nu(\mathbb{D}(v_m), \mathbb{D}(\psi)) + \alpha\langle v_m, \psi\rangle + (v_m\cdot\nabla v_m, \psi)&\\ = \langle g, \psi\rangle + (\xi\theta_m e_2, \psi), \quad \psi \in V_n^1(\Omega),&\end{aligned} \tag{3.12}$$

with $(\theta_m, v_m) = (\theta_0, v_0)$. Let (φ, Ψ) be a vector of continuously differentiable function on $[0,T]$ with $(\varphi(T), \Psi(T)) = (0,0)$. For each $(\phi,\psi) \in H^1(\Omega)\times V_n^1(\Omega)$, we multiply (3.11) by φ and (3.12) by Ψ, respectively, and then integrate by parts. After integrating the first term by parts for each equation, we get

$$-\int_0^T (\theta_m, \phi\dot{\varphi})\,dt - \int_0^T (v_m\theta_m, \nabla\phi\varphi)\,dt = (\theta_0, \phi\varphi(0)), \tag{3.13}$$

$$\begin{aligned}-\int_0^T (v_m, \psi\dot{\Psi})\,dt + 2\nu(\mathbb{D}(v_m), \mathbb{D}(\psi)) + \alpha\langle v_m, \psi\rangle + (v_m\cdot\nabla v_m, \psi\dot{\Psi})&\\ = \langle g_m, \psi\rangle + (\theta_m e_2, \psi) + (v_0, \psi\Phi(0)).&\end{aligned} \tag{3.14}$$

Since $\phi\dot{\varphi} \in L^1(0,T;L^1(\Omega))$, it is straightforward to pass to the limit in the first term of the left hand side of (3.13) with the help of (3.9). To estimate the second term of the left hand side of (3.13), we use the convergence results (3.9)–(3.10) and get

$$
\begin{aligned}
&|\int_0^T\int_\Omega (v_m\theta_m)\cdot\nabla(\phi\varphi)\,dx\,dt - \int_0^T\int_\Omega (v^*\theta^*)\cdot\nabla(\phi\varphi)\,dx\,dt| \\
&\quad\le |\int_0^T\int_\Omega (v_m\theta_m)\cdot\nabla(\phi\varphi) - (v^*\theta_m)\cdot\nabla(\phi\varphi)\,dx\,dt| \\
&\qquad + |\int_0^T\int_\Omega (v^*\theta_m)\cdot\nabla(\phi\varphi) - (v^*\theta^*)\cdot\nabla(\phi\varphi)\,dx\,dt| \\
&\quad\le \int_0^T \|v_m - v^*\|_{L^2}\|\theta_m\|_{L^\infty}\|\nabla\phi\|_{L^2}|\varphi|\,dt + |\int_0^T\int_\Omega (\theta_m-\theta^*)v^*\cdot\nabla(\phi\varphi)\,dx\,dt|,
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^T \|v_m - v^*\|_{L^2}\|\theta_m\|_{L^\infty}\|\nabla\phi\|_{L^2}|\varphi|\,dt \\
&\qquad\le \|v_m - v^*\|_{L^2(0,T;V_n^0(\Omega))}\|\theta_0\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\varphi\|_{L^2(0,T)} \to 0. \qquad (3.15)
\end{aligned}
$$

Further note that $v^*\cdot\nabla(\phi\varphi)\in L^1(0,T;L^1(\Omega))$, and hence by (3.9)

$$
|\int_0^T\int_\Omega (\theta_m-\theta^*)v^*\cdot\nabla(\phi\varphi)\,dx\,dt| \to 0. \qquad (3.16)
$$

Passing to the limit in (3.11) yields

$$
-\int_0^T (\theta^*,\phi\dot\varphi)\,dt - \int_0^T (v^*\theta^*,\nabla\phi\varphi) = (\theta_0,\phi\varphi(0)), \quad \phi\in H^1(\Omega). \qquad (3.17)
$$

To show

$$
(v_m\cdot\nabla v_m,\psi\dot\Psi)\to(v^*\cdot\nabla v^*,\psi\dot\Psi), \quad \psi\in V_n^1(\Omega),
$$

we write

$$
\begin{aligned}
(v_m\cdot\nabla v_m,\psi\Psi) &= \int_\Omega v_{im}\partial_i(v_{jm}\psi_j\Psi)\,dx - \int_\Omega v_{im}v_{jm}\partial_i(\psi_j\Psi)\,dx \\
&= -\int_\Omega v_{im}v_{jm}\partial_i(\psi_j\Psi)\,dx.
\end{aligned}
$$

Since $\{g_m\}$ is uniformly bounded in $L^2(0, T; V_n^0(\Omega))$, by (3.4) $\{v_m\}$ is uniformly bounded in $L^2(0, T; H^{3/2}(\Omega))$. Further utilizing (3.10) we have

$$
\begin{aligned}
&|\int_0^T\int_\Omega v_{im}v_{jm}\partial_i(\psi_j\Psi)\,dx\,dt - \int_0^T\int_\Omega v_i^*v_j^*\partial_i(\psi_j\Psi)\,dx\,dt| \\
&\quad\le |\int_0^T\int_\Omega (v_{im}-v_i^*)v_{jm}\partial_i(\psi_j\Psi)\,dx\,dt| + |\int_0^T\int_\Omega v_i^*(v_{jm}-v_j^*)\partial_i(\psi_j\Psi)\,dx\,dt| \\
&\quad\le \int_0^T \|v_m - v^*\|_{L^2}\|v_m\|_{L^\infty}\|\nabla\psi\|_{L^2}|\Psi|\,dt + |\int_0^T \|v_i^*\|_{L^\infty}\|v_{jm}-v_j^*\|_{L^2}\|\partial_i\psi_j\|_{L^2}|\Psi|\,dt| \\
&\quad\le \|v_m - v^*\|_{L^2(0,T;V_n^0(\Omega))}\|v_m\|_{L^2(0,T;H^{1+\varepsilon}(\Omega))}\|\nabla\psi\|_{L^2}\|\Psi\|_{L^\infty(0,T)} \\
&\qquad + \|v_i^*\|_{L^2(0,T;H^{1+\varepsilon}(\Omega))}\|v_m - v^*\|_{L^2(0,T;V_n^0(\Omega))}\|\nabla\psi\|_{L^2}\|\Psi\|_{L^\infty(0,T)} \to 0, \quad 0<\varepsilon\le 1/2,
\end{aligned}
$$

Moreover, it is straightforward to verify that $(\theta^*(0) - \theta_0, \phi\varphi(0)) = 0$ for any $\phi \in H^1(\Omega)$ and $(v^*(0) - v_0, \psi\Phi(0)) = 0$ for any $\psi \in V_n^1(\Omega)$. Thus $(\theta^*(0), v^*(0)) = (\theta_0, v_0)$. Finally, using the weakly lower semicontinuity property of norms defined in J yields

$$J(g^*) \le \lim_{m\to\infty} \inf J(g_m).$$

This completes the proof. □

Since the existence of an optimal controller to problem (P) is independent of ϵ, the existence of an optimal controller to problem (P_ϵ) can be obtained in a similar fashion.

Theorem 3.3. *Assume that* $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^1(\Omega)$. *There exists an optimal solution* $g_\epsilon^* \in U_{ad}$ *to problem* (P_ϵ).

4. Optimality system of problem (P_ϵ) and its convergence

In this section we derive the first-order necessary optimality conditions for problem (P_ϵ) by using a variational inequality (cf. [16]), that is, if g_ϵ is an optimal solution of problem (P_ϵ), then

$$J_\epsilon'(g_\epsilon)\cdot(f_\epsilon - g_\epsilon) \ge 0, \quad f_\epsilon \in U_{ad}. \tag{4.1}$$

We first present the following two lemmas to address the linearized problem of (1.12)–(1.16) and its adjoint system.

Lemma 4.1. *Assume* $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^1(\Omega)$ *and* $g \in U_{ad}$. *Let* $y_\epsilon = (v_\epsilon'(g_\epsilon)\cdot h_\epsilon)$, $z_\epsilon = \theta_\epsilon'(g_\epsilon)\cdot h_\epsilon$, *and* $q_\epsilon = p_\epsilon'(g)\cdot h_\epsilon$ *be the Gâteaux derivatives of* v_ϵ, θ_ϵ, *and* p_ϵ *with respect to* g_ϵ *in every direction* h_ϵ *in* $U_{\epsilon_{ad}}$, *respectively. Then* (y_ϵ, z_ϵ) *is the solution of the linearized problem*

$$\frac{\partial z_\epsilon}{\partial t} - \epsilon\Delta z_\epsilon + y_\epsilon \cdot \nabla\theta_\epsilon + v_\epsilon \cdot \nabla z_\epsilon = 0 \quad in \quad \Omega, \tag{4.2}$$

$$\frac{\partial y_\epsilon}{\partial t} - \nu\Delta y_\epsilon + y_\epsilon \cdot \nabla v_\epsilon + v_\epsilon \cdot \nabla y_\epsilon + \nabla q_\epsilon = \xi z_\epsilon e_2 \quad in \quad \Omega, \tag{4.3}$$

$$\nabla \cdot y_\epsilon = 0 \quad in \quad \Omega, \tag{4.4}$$

with the Neumann boundary condition

$$\epsilon\frac{\partial z_\epsilon}{\partial n}|_\Gamma = 0 \tag{4.5}$$

and the Navier slip boundary conditions

$$y_\epsilon \cdot n|_\Gamma = 0 \quad and \quad (2\nu n \cdot \mathbb{D}(y_\epsilon) \cdot \tau + \alpha y_\epsilon \cdot \tau)|_\Gamma = h_\epsilon \cdot \tau. \tag{4.6}$$

The initial condition is given by $(z_\epsilon(0), y_\epsilon(0)) = (0, 0)$. *Moreover,*

$$(z_\epsilon, y_\epsilon) \in \Big(L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))\Big) \times \Big(L^\infty(0,T; V_n^{1/2}(\Omega)) \cap L^2(0,T; V_n^{3/2}(\Omega))\Big). \tag{4.7}$$

The regularity result (4.7) can be easily derived by using the variation of parameters formula, with the help of the regularity of $h_\epsilon \in L^2(0,T; V_n^0(\Gamma))$ and the regularity of θ_ϵ and v_ϵ given by (2.35).

Lemma 4.2. *The adjoint state* $(\rho_\epsilon, \bar{y}_\epsilon, \bar{q}_\epsilon)$ *associated with the cost functional* J_ϵ *in* (P_ϵ) *satisfies*

$$-\frac{\partial \rho_\epsilon}{\partial t} - \epsilon\Delta\rho_\epsilon - v_\epsilon \cdot \nabla\rho_\epsilon - \xi\bar{y}_\epsilon \cdot e_2 = 0 \quad in \quad \Omega, \tag{4.8}$$

$$-\frac{\partial \bar{y}_\epsilon}{\partial t} - \nu\Delta\bar{y}_\epsilon + (\nabla v_\epsilon)^T\bar{y}_\epsilon - v_\epsilon \cdot \nabla\bar{y}_\epsilon + \nabla\bar{q}_\epsilon = \theta_\epsilon\nabla\rho_\epsilon + \zeta\nabla^\perp(\nabla \times v_\epsilon) \quad in \quad \Omega, \tag{4.9}$$

$$\nabla \cdot \bar{y}_\epsilon = 0 \quad in \quad \Omega, \tag{4.10}$$

with the Neumann boundary condition

$$\epsilon\frac{\partial \rho_\epsilon}{\partial n}|_\Gamma = 0 \tag{4.11}$$

and the Navier slip boundary conditions

$$\bar{y}_\epsilon \cdot n|_\Gamma = 0 \quad and \quad (2\nu n \cdot \mathbb{D}(\bar{y}_\epsilon) \cdot \tau + \alpha\bar{y}_\epsilon \cdot \tau)|_\Gamma = -\zeta\nabla \times v_\epsilon. \tag{4.12}$$

The final time condition is given by

$$(\rho_\epsilon(T), \bar{y}_\epsilon(T)) = (\Lambda^{-2}\theta_\epsilon(T), 0). \tag{4.13}$$

Moreover, for $(\theta_\epsilon, v_\epsilon)$ *satisfying* (2.35), *we have*

$$(\rho_\epsilon, \bar{y}_\epsilon) \in (L^\infty(0,T; L^\infty(\Omega) \cap H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))) \times \left(L^\infty(0,T; V_n^{1/2}(\Omega)) \cap L^2(0,T; V_n^{3/2}(\Omega))\right). \tag{4.14}$$

Since $\rho_\epsilon(T) = \Lambda^{-2}\theta_\epsilon(T) \in H^2(\Omega)$, the compatibility condition for final and boundary data is required to hold, i.e., $\epsilon \frac{\partial \rho_\epsilon(T)}{\partial n}|_\Gamma = 0$. This is indeed true because of (1.8) that $\Lambda^{-2}\theta_\epsilon(T) = \mathcal{A}^{-1}\theta_\epsilon(T)$ satisfies the Neumann boundary condition (4.11). The regularity property (4.14) holds due to $\theta_\epsilon \nabla \rho_\epsilon \in L^2(0,T; L^2(\Omega))$, $\nabla^\perp(\nabla \times v_\epsilon) \in L^2(0,T; H^{-1/2}(\Omega))$, and (2.8) that $\nabla \times v_\epsilon|_\Gamma = (2\kappa - \frac{\alpha}{\nu})(v_\epsilon \cdot \tau) + \frac{1}{\nu} g_\epsilon \cdot \tau \in L^2(0,T; V_n^0(\Gamma))$. As a result of the trace theorem [17, Theorem 2.1, p. 9],

$$\bar{y}_\epsilon|_\Gamma \in H^{1/2}(0,T; V_n^1(\Gamma)). \tag{4.15}$$

We now turn to derive the optimality system of problem (P_ϵ) by employing the variational inequity (4.1), and then prove its convergence as $\epsilon \to 0$. First, rewrite the cost functional J_ϵ as

$$J_\epsilon(g_\epsilon) = \frac{1}{2}(\Lambda^{-2}\theta_\epsilon(T), \theta_\epsilon(T)) + \frac{\gamma}{2}\int_0^T \langle g_\epsilon, g_\epsilon\rangle\, dt - \frac{\zeta}{2}\int_0^T (\nabla \times v_\epsilon, \nabla \times v_\epsilon)\, dt. \quad (P'_\epsilon)$$

Then the variational inequality (4.1) becomes

$$J'_\epsilon(g_\epsilon)\cdot h_\epsilon = (\Lambda^{-2}\theta_\epsilon(T), z_\epsilon(T)) + \gamma\int_0^T \langle g_\epsilon, h_\epsilon\rangle\, dt - \frac{\zeta}{2}\int_0^T (\nabla \times v_\epsilon, \nabla \times w_\epsilon)\, dt \ge 0, \quad h_\epsilon \in U_{\epsilon_{ad}}. \tag{4.16}$$

Theorem 4.3. *Let $(\theta_0, v_0) \in L^\infty(\Omega) \times V_n^1(\Omega)$. Assume that g_ϵ^* is an optimal controller of problem (P_ϵ). If $(v_\epsilon, \theta_\epsilon)$ is the corresponding solution of (1.12)–(1.16) and $(\rho_\epsilon, y_\epsilon)$ is the solution of the adjoint equations (4.8)–(4.13) associated with $(v_\epsilon, \theta_\epsilon)$, then*

$$g_\epsilon^* = -\frac{1}{\gamma}\bar{y}_\epsilon \in H^{3/4}(0,T; V_n^{3/2}(\Gamma)). \tag{4.17}$$

Proof. Multiplying (4.2) by ρ_ϵ and integrating the first term with respect to t give

$$-\int_0^T \left(\frac{\partial}{\partial t}\rho_\epsilon, z_\epsilon\right) dt + (\rho_\epsilon(T), z_\epsilon(T) - \int_0^T (\epsilon\Delta\rho_\epsilon, z_\epsilon) + \int_0^T (y_\epsilon \cdot \nabla\theta_\epsilon, \rho_\epsilon)\, dt - \int_0^T (v_\epsilon \cdot \nabla\rho_\epsilon, z_\epsilon)\, dt = 0.$$

Based on the adjoint equation (4.8) and the final condition (4.13), we have

$$
\begin{aligned}
(\Lambda^{-2}\theta_\epsilon(T), z_\epsilon(T)) = (\rho_\epsilon(T), z_\epsilon(T)) &= -\int_0^T (\xi \bar{y}_\epsilon \cdot e_2, z_\epsilon)\, dt - \int_0^T (y_\epsilon \cdot \nabla\theta_\epsilon, \rho_\epsilon)\, dt \\
&= -\int_0^T (\xi z_\epsilon e_2, \bar{y}_\epsilon)\, dt + \int_0^T (y_\epsilon, \theta_\epsilon \nabla\rho_\epsilon)\, dt.
\end{aligned} \tag{4.18}
$$

Thus (4.16) becomes

$$
\begin{aligned}
J'(g_\epsilon)\cdot h_\epsilon = &-\int_0^T (\xi z_\epsilon e_2, \bar{y})\, dt + \int_0^T (y, \theta_\epsilon \nabla\rho_\epsilon + \zeta\nabla^\perp(\nabla\times v_\epsilon))\, dt + \gamma\int_0^T \langle g_\epsilon, h_\epsilon\rangle\, dt \\
&- \zeta(2\kappa-\alpha)\int_0^T \langle v_\epsilon\cdot\tau, y_\epsilon\cdot\tau\rangle\, dt - \zeta\int_0^T \langle g_\epsilon\cdot\tau, y_\epsilon\cdot\tau\rangle\, dt,
\end{aligned} \tag{4.19}
$$

where by (4.9) we get

$$
\begin{aligned}
&\int_0^T (y_\epsilon, \theta_\epsilon\nabla\rho_\epsilon + \zeta\nabla^\perp(\nabla\times v_\epsilon))\, dt \\
&= \int_0^T (\frac{dy_\epsilon}{dt}, \bar{y}_\epsilon)\, dt + \int_0^T [(-\nu\Delta y_\epsilon, \bar{y}_\epsilon) + \langle h_\epsilon, \bar{y}\rangle + \langle y_\epsilon\cdot\tau, \zeta((2\kappa-\alpha)v_\epsilon\cdot\tau + h_\epsilon\cdot\tau)\rangle \\
&\quad + ((y_\epsilon\cdot\nabla)v_\epsilon + (v_\epsilon\cdot\nabla)y_\epsilon, \bar{y}_\epsilon) + (\nabla q_\epsilon, \bar{y}_\epsilon)]\, dt \\
&= \int_0^T (\xi z_\epsilon e_2, \bar{y}_\epsilon)\, dt + \int_0^T \langle h_\epsilon, \bar{y}_\epsilon\rangle\, dt + \int_0^T \langle y_\epsilon\cdot\tau, \zeta((2\kappa-\alpha)v_\epsilon\cdot\tau + h_\epsilon\cdot\tau\rangle\, dt.
\end{aligned} \tag{4.20}
$$

Therefore, combining (4.19) with (4.20) follows

$$
J'(g_\epsilon^*)\cdot h_\epsilon = \int_0^T \langle h_\epsilon, \bar{y}_\epsilon\rangle\, dt + \gamma\int_0^T \langle h_\epsilon, g_\epsilon^*\rangle\, dt \geq 0, \quad \forall h \in U_{ad},
$$

which implies

$$
g_\epsilon^* = -\frac{1}{\gamma}\bar{y}_\epsilon|_\Gamma. \tag{4.21}
$$

According to (4.15), we have $g_\epsilon^* \in H^{1/2}(0,T;V_n^1(\Gamma))$. Let $(\theta_\epsilon^*, v_\epsilon^*)$ be the optimal solution to (1.13)–(1.16) associated with g_ϵ^*. Then by a bootstrap argument and (2.36) in Theorem 2.4 (2), we have $v_\epsilon^* \in L^2(0,T;V_n^2(\Omega))$, thus

$$\nabla^{\perp}(\nabla \times v_{\epsilon}^{*}) \in L^{2}(0,T;L^{2}(\Omega)) \tag{4.22}$$

and $\nabla \times v_{\epsilon}^{*}|_{\Gamma} = (2\kappa - \frac{\alpha}{\nu})(v_{\epsilon}^{*} \cdot \tau) + \frac{1}{\nu} g_{\epsilon}^{*} \cdot \tau \in H^{1/2}(0,T;V_{n}^{1}(\Gamma))$. Moreover, $\theta_{\epsilon}^{*}(T) \in L^{\infty}(\Omega)$. With the help of the final condition (4.13), we get

$$\begin{aligned}(\rho_{\epsilon}^{*}, \bar{y}_{\epsilon}^{*}) \in &(L^{\infty}(0,T;L^{\infty}(\Omega) \cap H^{1}(\Omega)) \cap L^{2}(0,T;H^{2}(\Omega))) \\ &\times \left(L^{\infty}(0,T;V_{n}^{1}(\Omega)) \cap L^{2}(0,T;V_{n}^{2}(\Omega))\right)\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}&\|\rho_{\epsilon}^{*}\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} + \|\rho_{\epsilon}^{*}\|_{L^{\infty}(0,T;H^{1}(\Omega))} + \sqrt{\epsilon}\|\rho_{\epsilon}^{*}\|_{L^{2}(0,T;H^{2}(\Omega))} + \|\frac{\partial \rho_{\epsilon}^{*}}{\partial t}\|_{L^{2}(0,T;L^{2}(\Omega))} \\ &\quad + \|\bar{y}_{\epsilon}^{*}\|_{L^{\infty}(0,T;H^{1}(\Omega))} + \sqrt{\nu}\|\bar{y}_{\epsilon}^{*}\|_{L^{2}(0,T;H^{2}(\Omega))} + \|\frac{\partial \bar{y}_{\epsilon}^{*}}{\partial t}\|_{L^{2}(0,T;L^{2}(\Omega))} \\ &\leq C(\|\Lambda^{-2}\theta_{\epsilon}^{*}(T)\|_{L^{2}}, \|g_{\epsilon}^{*}\|_{\mathcal{S}}, T).\end{aligned} \tag{4.24}$$

As a result of (4.21) and (4.23)–(4.24),

$$g_{\epsilon}^{*} \in H^{3/4}(0,T;V_{n}^{3/2}(\Gamma)),$$

which completes the proof. □

4.1. Convergence of the adjoint system

This section is concerned with the convergence of the optimality system of the approximating problem (P_{ϵ}) to that of problem (P). Recall that the convergence of the approximating system (1.12)–(1.16) to the original system (1.1)–(1.5) has been established in Proposition 2.7. To establish the convergence of the adjoint system (4.8)–(4.13), we need $\theta_0 \in L^{\infty}(\Omega) \cap H^{1}(\Omega)$.

Lemma 4.4. *For $(\theta_{\epsilon 0}, v_{\epsilon 0}) = (\theta_0, v_0) \in (L^{\infty}(\Omega) \cap H^{1}(\Omega)) \times V_{n}^{1}(\Omega)$ and $g \in \mathcal{S}$, the solution to the adjoint problem (4.8)–(4.13) of the approximating system (1.12)–(1.16), associated with cost functional J_{ϵ}, weakly converges to the following problem*

$$-\frac{\partial \rho}{\partial t} - v \cdot \nabla \rho - \xi \bar{y} \cdot e_2 = 0 \quad in \quad \Omega, \tag{4.25}$$

$$-\frac{\partial \bar{y}}{\partial t} - \nu \Delta \bar{y} + (\nabla v)^{T} \bar{y} - (v \cdot \nabla)\bar{y} + \nabla \bar{q} = \theta \nabla \rho + \zeta \nabla^{\perp}(\nabla \times v) \quad in \quad \Omega, \tag{4.26}$$

$$\nabla \cdot \bar{y} = 0 \quad in \quad \Omega, \tag{4.27}$$

with the Navier slip boundary conditions

$$\bar{y} \cdot n|_{\Gamma} = 0 \quad and \quad (2\nu n \cdot \mathbb{D}(\bar{y}) \cdot \tau + \alpha \bar{y} \cdot \tau)|_{\Gamma} = -\zeta \nabla \times v \tag{4.28}$$

and the final time condition

$$(\rho(T), \bar{y}(T)) = (\Lambda^{-2}\theta(T), 0), \tag{4.29}$$

where (θ, v) *is the solution to* (1.1)–(1.5) *corresponding to* (θ_0, v_0) *and* g.

Proof. First of all, in light of Theorem 2.4, for $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$ and $g \in \mathcal{S}$ we have

$$(\theta, v) \in L^\infty(0, T; (L^\infty(\Omega) \cap H^1(\Omega)) \times (L^\infty(0, T; V_n^1(\Omega)) \cap L^2(0, T; V_n^2(\Omega)), \tag{4.30}$$

thus $\nabla^\perp(\nabla \times v) \in L^2(0, T; V^0(\Omega))$ and $\nabla \times v|_\Gamma = (2\kappa - \frac{\alpha}{\nu})(v \cdot \tau) + \frac{1}{\nu} g \cdot \tau \in \mathcal{S}$. Combining these with the final condition (4.29), we get

$$(\rho, \bar{y}) \in L^\infty(0, T; L^\infty(\Omega) \cap H^1(\Omega)) \times \Big(L^\infty(0, T; V_n^1(\Omega)) \cap L^2(0, T; V_n^2(\Omega))\Big). \tag{4.31}$$

Based on (4.30)–(4.31) it is easy to check that

$$\begin{aligned}\int_0^T \|v \cdot \nabla\rho\|_{L^2}^2 \, dt &\le \int_0^T \|v\|_{L^\infty}^2 \|\nabla\rho\|_{L^2}^2 \, dt \le \int_0^T \|Av\|_{L^2}^2 \|\nabla\rho\|_{L^2}^2 \, dt \\ &\le \sup_{t\in[0,T]} \|\nabla\rho\|_{L^2}^2 \int_0^T \|Av\|_{L^2}^2 \, dt < \infty,\end{aligned}$$

and hence, by (4.25)

$$\frac{\partial \rho}{\partial t} \in L^2(0, T; L^2(\Omega)). \tag{4.32}$$

Moreover,

$$\int_0^T \|\theta\nabla\rho\|_{L^2}^2 \, dt \le \int_0^T \|\theta\|_\infty^2 \|\nabla\rho\|_{L^2}^2 \, dt \le \sup_{t\in[0,T]} \|\theta_0\|_{L^\infty}^2 \sup_{t\in[0,T]} \|\nabla\rho\|_{L^2}^2 T < \infty.$$

Next we show that the solution $(\rho_\epsilon, \bar{y}_\epsilon, \bar{q}_\epsilon)$ to the adjoint problem (4.8)–(4.13) of the approximating system converges to some $(\tilde{\rho}, \tilde{y}, \tilde{q})$ as $\epsilon \to 0$. According to the estimate (4.24) there exists a subsequence $\{(\rho_\epsilon, \bar{y}_\epsilon)\}$ in terms of ϵ, such that

$$\nabla\rho_\epsilon \to \nabla\tilde{\rho} \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \text{ as } \epsilon \to 0, \tag{4.33}$$

$$\frac{\partial \rho_\epsilon}{\partial t} \to \frac{\partial \tilde{\rho}}{\partial t} \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \text{ as } \epsilon \to 0, \tag{4.34}$$

and

$$\Delta \bar{y}_\epsilon \to \Delta \tilde{y} \quad \text{weakly in } L^2(0,T;V_n^0(\Omega)), \text{ as } \epsilon \to 0, \tag{4.35}$$

$$\nabla \bar{y}_\epsilon \to \nabla \tilde{y} \quad \text{weakly in } L^2(0,T;V_n^1(\Omega)), \text{ as } \epsilon \to 0, \tag{4.36}$$

$$\frac{\partial \bar{y}_\epsilon}{\partial t} \to \frac{\partial \tilde{y}}{\partial t} \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \text{ as } \epsilon \to 0, \tag{4.37}$$

$$\bar{y}_\epsilon \to \tilde{y} \quad \text{strongly in } H^{1-\delta/2}(0,T;V_n^{2-\delta}(\Omega)),\ 0<\delta<2, \text{ as } \epsilon \to 0, \tag{4.38}$$

$$\nabla \bar{q}_\epsilon \to \nabla \tilde{q} \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \text{ as } \epsilon \to 0.$$

It follows immediately from (4.22) and (4.38) that

$$\nabla^\perp(\nabla\times v_\epsilon) \to \nabla^\perp(\nabla \times v) \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$
$$\nabla \times v_\epsilon|_\Gamma \to \nabla\times v|_\Gamma \quad \text{strongly in } H^{1/4-\delta/2}(0,T;V_n^{1/2-\delta}(\Gamma)),\ 0<\delta<1/2.$$

It remains to prove the weak convergence of the product terms $v_\epsilon \cdot \nabla\rho_\epsilon$, $(\nabla v_\epsilon)^T \bar{y}_\epsilon$, $v_\epsilon \cdot \nabla \bar{y}_\epsilon$ and $\theta_\epsilon \nabla \rho_\epsilon$, respectively. For $\phi \in L^2(0,T;H^1(\Omega))$, we have

$$|\int_0^T\int_\Omega (v_\epsilon\cdot\nabla\rho_\epsilon - v\cdot\nabla\tilde{\rho})\phi\,dx\,dt| \le |\int_0^T\int_\Omega (v_\epsilon - v)\cdot\nabla\rho_\epsilon\phi\,dxdt|$$
$$+|\int_0^T\int_\Omega |v\cdot\nabla(\rho_\epsilon-\tilde{\rho})\phi\,dx\,dt|, \tag{4.39}$$

where by Proposition 2.7 and (4.24),

$$|\int_0^T\int_\Omega (v_\epsilon - v)\cdot\nabla\rho_\epsilon\phi\,dxdt| = |\int_0^T\int_\Omega (v_\epsilon - v)\cdot\nabla(\rho_\epsilon\phi) - (v_\epsilon - v)\rho_\epsilon\cdot\nabla\phi\,dxdt|$$
$$= |\int_0^T\int_\Omega (v_\epsilon - v)\rho_\epsilon\cdot\nabla\phi\,dxdt|$$
$$\le \sup_{t\in[0,T]}\|v_\epsilon - v\|_{L^2}\int_0^T \|\rho_\epsilon\|_{L^\infty}\|\nabla\phi\|_{L^2}\,dt$$
$$\le \sup_{t\in[0,T]}\|v_\epsilon - v\|_{L^2}\sup_{t\in[0,T]}\|\rho_\epsilon\|_{L^\infty}\|\nabla\phi\|_{L^2(0,T;L^2(\Omega))}\sqrt{T} \to 0, \text{ as } \epsilon\to 0.$$

Moreover, from (4.30),

$$\int_0^T \|v\phi\|_{L^2}\,dt \le \int_0^T \|v\|_{L^4}\|\phi\|_{L^4}\,dt \le \int_0^T \|A^{1/2}v\|_{L^2}\|\nabla\phi\|_{L^2}\,dt$$
$$\le (\sup_{t\in[0,T]}\|A^{1/2}v\|_{L^2})(\int_0^T \|\nabla\phi\|_{L^2}^2\,dt)^{1/2}\sqrt{T} < \infty. \tag{4.40}$$

With the help of (4.33), we get

$$|\int_0^T\int_\Omega v\cdot\nabla(\rho_\epsilon-\tilde{\rho})\phi\,dx\,dt|\to 0,\ \text{as }\epsilon\to 0.$$

Therefore,

$$(v_\epsilon\cdot\nabla)\rho_\epsilon\to(v\cdot\nabla)\rho\quad\text{weakly in }L^2(0,T;L^2(\Omega)),\ \text{as }\epsilon\to 0. \tag{4.41}$$

For $\psi\in L^2(0,T;H^1(\Omega))$, using the similar approaches as in the estimate of (4.39) together with (4.35)–(4.36) yields

$$(v_\epsilon\cdot\nabla)\bar{y}_\epsilon\to(v\cdot\nabla)\tilde{y}\quad\text{weakly in }L^2(0,T;V_n^0(\Omega)).$$

Furthermore, based on (2.38) in Theorem 2.4 (2) and Proposition 2.7, it is easy to verity that $\|\nabla v_\epsilon\cdot\psi\|_{L^2(0,T;L^1(\Omega))}<\infty$, indenpendent of ϵ, $\tilde{y}\cdot\psi\in L^1(0,T;L^2(\Omega))$, and

$$\begin{aligned}
&|\int_0^T\int_\Omega((\nabla v_\epsilon)^T\bar{y}_\epsilon-(\nabla v)^T\tilde{y})\cdot\psi\,dx\,dt|=|\int_0^T\int_\Omega((\bar{y}_\epsilon-\tilde{y})\cdot\nabla v_\epsilon\cdot\psi+\tilde{y}\cdot(\nabla v_\epsilon-\nabla v)\cdot\psi\,dx\,dt|\\
&\quad\le\|\bar{y}_\epsilon-\tilde{y}\|_{L^2(0,T;L^\infty(\Omega))}\sup_{t\in[0,T]}\|\nabla v_\epsilon\|_{L^2}\|\psi\|_{L^2(0,T;L^2(\Omega))}\\
&\qquad+\sup_{t\in[0,T]}\|A^{1/2}v_\epsilon-A^{1/2}v\|_{L^2}\|\tilde{y}\|_{L^2(0,T;H^1(\Omega))}\|\psi\|_{L^2(0,T;H^1(\Omega))}\to 0,\ \text{as }\epsilon\to 0,
\end{aligned} \tag{4.42}$$

where we used (4.38) to obtain $\|\bar{y}_\epsilon-\tilde{y}\|_{L^2(0,T;L^\infty(\Omega))}\le\|\bar{y}_\epsilon-\tilde{y}\|_{L^2(0,T;H^{2-\delta}(\Omega))}\to 0$ for $0<\delta<1$. Thus (4.42) yields

$$(\nabla v_\epsilon)^T\bar{y}_\epsilon\to(\nabla v)^T\tilde{y}\quad\text{weakly in }L^2(0,T;V_n^0(\Omega)),\ \text{as }\epsilon\to 0.$$

To show the weak convergence of $\theta_\epsilon\nabla\rho_\epsilon$, we have

$$|\int_0^T\int_\Omega(\theta_\epsilon\nabla\rho_\epsilon-\theta\nabla\tilde{\rho})\psi\,dx\,dt|\le|\int_0^T\int_\Omega(\theta_\epsilon-\theta)\nabla\rho_\epsilon\cdot\psi\,dxdt|+|\int_0^T\int_\Omega\theta(\nabla\rho_\epsilon-\nabla\tilde{\rho})\cdot\psi\,dx\,dt|, \tag{4.43}$$

where

$$|\int_0^T\int_\Omega(\theta_\epsilon-\theta)\nabla\rho_\epsilon\cdot\psi\,dxdt|=|\int_0^T\int_\Omega(\theta_\epsilon-\theta)\nabla\cdot(\rho_\epsilon\psi)-(\theta_\epsilon-\theta)\rho_\epsilon\nabla\cdot\psi\,dxdt|$$
$$=|\int_0^T\Big(\int_\Gamma(\theta_\epsilon-\theta)(\rho_\epsilon\psi)\cdot n\,dx-\int_\Omega\nabla(\theta_\epsilon-\theta)\cdot(\rho_\epsilon\psi)\,dx\Big)dt|$$
$$=|\int_0^T\int_\Omega\nabla(\theta_\epsilon-\theta)\cdot(\rho_\epsilon\psi)\,dx\,dt|. \tag{4.44}$$

By (2.38) and (4.30) there exists a subsequence $\{\theta_\epsilon\}$ associated with $\{v_\epsilon\}$ satisfying

$$\theta_\epsilon\to\theta\quad\text{weakly* in } L^\infty(0,T;H^1(\Omega)).$$

Employing the similar idea in the estimates of (4.40) gives $\rho_\epsilon\psi\in L^1(0,T;L^2(\Omega))$. Thus (4.44) converges to 0 as $\epsilon\to 0$. The second term on the right hand side of (4.43) converges to 0 due to (4.33) and $\theta\psi\in L^1(0,T;L^2(\Omega))$. Therefore,

$$\theta_\epsilon\nabla\rho_\epsilon\to\theta\nabla\tilde\rho\quad\text{weakly in } L^2(0,T;L^2(\Omega)),\ \text{as } \epsilon\to 0. \tag{4.45}$$

Lastly, based on (2.29) in Theorem 2.3 (2) and (2.37) in Theorem 2.4 (2), it is clear that $\frac{\partial\theta}{\partial t},\frac{\partial\theta_\epsilon}{\partial t}\in L^2(0,T;L^2(\Omega))$. By virtue of Aubin-Lions Lemma, we get $\theta,\theta_\epsilon\in C([0,T],L^2(\Omega))$. Further utilizing Proposition 2.7 gives

$$\theta_\epsilon(T)\to\theta(T)\quad\text{strongly in } L^2(\Omega),\ \text{as } \epsilon\to 0, \tag{4.46}$$

and hence the final condition

$$\rho_\epsilon(T)=\Lambda^{-2}\theta_\epsilon(T)\to\tilde\rho(T)=\Lambda^{-2}\theta(T)\quad\text{strongly in } L^2(\Omega),\ \text{as } \epsilon\to 0. \tag{4.47}$$

As a result, $(\tilde\rho,\tilde y,\tilde q)$ is the weak solution to the linear system (4.25)–(4.29) corresponding to $(\theta,v,\bar q)$ and $\theta(T)$. Due to the uniqueness of the solution, we obtain $(\tilde\rho,\tilde y,\tilde q)=(\rho,\bar y,\bar q)$ and this completes the proof. □

4.2. Convergence of the optimality system

Let $(g_\epsilon^*,v_\epsilon^*,\theta_\epsilon^*)$ be an optimal solution for (P_ϵ), which solves the optimality system consisted of (1.1)–(1.5), (4.25)–(4.29), and the optimality condition (4.53). Then the following results hold.

Theorem 4.5. *Assume $(\theta_0,v_0)\in(L^\infty(\Omega)\cap H^1(\Omega))\times V_n^1(\Omega)$. Let $(g_\epsilon^*,v_\epsilon^*,\theta_\epsilon^*)$ be an optimal solution for (P_ϵ). Then there exists an optimal solution (g^*,v^*,θ^*) such that*

$$g_\epsilon^*\to g^*\ \text{strongly in}\ H^{3/4-\delta/2}(0,T;V_n^{3/2-\delta}(\Gamma)),\ \forall\delta>0,\ \text{as } \epsilon\to 0,$$
$$\|v_\epsilon^*-v^*\|_{H^1}\to 0\quad\text{uniformly in } t\in[0,T],\ \text{as } \epsilon\to 0, \tag{4.48}$$

$$\int_0^T \|Av_\epsilon^* - Av^*\|_{L^2}^2\,dt \to 0, \quad as \quad \epsilon \to 0, \tag{4.49}$$

and

$$\|\theta_\epsilon^* - \theta^*\|_{L^2} \to 0 \quad \textit{uniformly in } t \in [0,T], \textit{as } \epsilon \to 0. \tag{4.50}$$

Moreover, (g^*, v^*, θ^*) *is an optimal solution to problem* (P).

Proof. Step 1: We first show the convergence of the optimal pair to problem (P_ϵ) as $\epsilon \to 0$. According to (4.17), there exists a subsequence $\{g_\epsilon^*\}$ in terms of ϵ such that

$$g_\epsilon^* \to g^* \quad \text{weakly in} \quad L^2(0,T;V_n^{3/2}(\Gamma)),$$
$$D_t^{3/4} g_\epsilon^* \to D_t^{3/4} g^* \quad \text{weakly in} \quad L^2(0,T;V_n^0(\Gamma)),$$

which give

$$g^* \in H^{3/4}(0,T;V_n^{3/2}(\Gamma)) \tag{4.51}$$

and

$$g_\epsilon^* \to g^* \quad \text{strongly in} \quad H^{3/4-\delta/2}(0,T;V_n^{3/2-\delta}(\Gamma)), \quad \forall \delta > 0. \tag{4.52}$$

Note that the regularity of the optimal controller guarantees a unique weak solution to (3.2)–(3.3) since the estimate (2.31) for velocity holds. Slightly modifying the proof of Proposition 2.7 by adding the nonhomogeneous term $g_\epsilon - g$ to the right hand side of (2.45), we can show that there exists a subsequence $\{(\theta_\epsilon, v_\epsilon)\}$ corresponding to $\{g_\epsilon\}$ satisfying (4.48)–(4.50) where (θ^*, v^*) is the solution corresponding to g^*.

Step 2: We claim that (g^*, v^*, θ^*) is an optimal pair to problem (P). Since $(g_\epsilon^*, v_\epsilon^*, \theta_\epsilon^*)$ is an optimal pair to problem (P_ϵ), we have

$$\|\Lambda^{-1}\theta_\epsilon^*(T)\|_{L^2}^2 + \int_0^T \|g_\epsilon^*\|_{L^2}^2\,dt \le \|\Lambda^{-1}\theta_\epsilon(T)\|_{L^2}^2 + \int_0^T \|g\|_{L^2}^2\,dt,$$

for any $g \in U_{ad}$, where θ_ϵ is the solution of (1.12) associated with (g, v_ϵ). Letting $\epsilon \to 0$, based on (4.48)–(4.50) we get

$$\|\Lambda^{-1}\theta^*(T)\|_{L^2}^2 + \int_0^T \|g^*\|_{L^2}^2\,dt \le \|\Lambda^{-1}\theta(T)\|_{L^2}^2 + \int_0^T \|g\|_{L^2}^2\,dt,$$

for any $g \in U_{ad}$. Thus, (g^*, v^*, θ^*) is an optimal pair to problem (P). In particular, the inf J can be reached by setting $g = g^*$. □

We proceed to prove that any optimal solution to problem (P) is given by (1.1)–(1.5), (4.25)–(4.29), and the optimality condition (4.53).

Theorem 4.6. *Assume* $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$. *If* (g^*, v^*, θ^*) *is an optimal pair to problem* (P)*, where* (g^*, θ^*, v^*) *satisfies the system* (1.1)–(1.5) *and* $(\rho^*, \bar{y}^*)$ *is the corresponding solution to the adjoint system* (4.25)–(4.29)*, then* g^* *is given by*

$$g^* = -\frac{1}{\gamma}\bar{y}^*|_\Gamma. \tag{4.53}$$

Proof. Let (g^*, v^*, θ^*) be any optimal solution to the problem (P). Applying the construction shown in [2, Theorem 5] and [10, Theorem 5.5], we first impose a penalization on problem (P_ϵ) in oder to establish a relation between g^* and the optimal solution to the new defined cost functional. For a given $\epsilon > 0$, consider the following minimization problem

$$\min\{J_\epsilon(g) + \frac{1}{2}\int_0^T \|g - g^*\|^2_{L^2(\Gamma)}\, dt\}. \quad (\hat{P}_\epsilon)$$

If let $(\hat{g}_\epsilon, \hat{v}_\epsilon, \hat{\theta}_\epsilon)$ be the optimal solution to problem $(\hat{P}_\epsilon)$, then

$$J_\epsilon(\hat{g}_\epsilon) + \frac{1}{2}\int_0^T \|\hat{g}_\epsilon - g^*\|^2_{L^2}\, dt \le J_\epsilon(g) + \frac{1}{2}\int_0^T \|g - g^*\|^2_{L^2}\, dt, \tag{4.54}$$

for any $g \in L^2(0, T; V_n^0(\Gamma))$. As proved in Theorem 4.5, there exists a subsequence $\{(\hat{g}_\epsilon, \hat{v}_\epsilon, \hat{\theta}_\epsilon)\}$ in terms of ϵ, satisfying

$$\begin{aligned} &\hat{g}_\epsilon \to \hat{g}^* \text{ strongly in } L^2(0, T; V_n^0(\Gamma)), \quad \text{as } \epsilon \to 0, \\ &\hat{\theta}_\epsilon(T) \to \hat{\theta}^*(T) \text{ strongly in } (H^1(\Omega))', \text{ as } \epsilon \to 0. \end{aligned} \tag{4.55}$$

Passing to the limit in (4.54) gives

$$J(\hat{g}^*) + \frac{1}{2}\int_0^T \|\hat{g}^* - g^*\|^2_{L^2}\, dt \le J(g) + \frac{1}{2}\int_0^T \|g - g^*\|^2_{L^2(\Gamma)}\, dt, \tag{4.56}$$

for all $g \in L^2(0, T; V_n^0(\Gamma))$. In particular, setting $g = g^*$ yields

$$J(\hat{g}^*) + \frac{1}{2}\int_0^T \|\hat{g}^* - g^*\|^2_{L^2(\Gamma)}\, dt \le J(g^*), \tag{4.57}$$

which indicates that

$$\int_0^T \|\hat{g}^* - g^*\|_{L^2}^2 \, dt = 0.$$

Therefore, $\hat{g}^* = g^*$, and hence $(\hat{\theta}^*, \hat{v}^*) = (\theta^*, v^*)$. According to (4.55) we have

$$\hat{g}_\epsilon \to g^* \quad \text{strongly in } L^2(0, T; V_n^0(\Gamma)), \text{ as } \epsilon \to 0. \tag{4.58}$$

Following the procedures as in the proof of Theorem 4.3, we derive the optimality condition for problem $(\hat{P}_\epsilon)$ given by

$$\bar{y}_\epsilon^*|_\Gamma + \gamma \hat{g}_\epsilon^* + \hat{g}_\epsilon^* - g^* = 0. \tag{4.59}$$

Letting $\epsilon \to 0$ and using (4.58) yield

$$\lim_{\epsilon \to 0} \bar{y}_\epsilon^*|_\Gamma + \gamma g^* + g^* - g^* = 0. \tag{4.60}$$

Finally, with the help of strong convergence of $\bar{y}_\epsilon^*$ in (4.38), we obtain

$$g^* = -\frac{1}{\gamma} \lim_{\epsilon \to 0} \bar{y}_\epsilon^*|_\Gamma = -\frac{1}{\gamma} \bar{y}^*|_\Gamma,$$

which completes the proof. □

5. Uniqueness of the optimal solution to problem (P)

In the last section we address the uniqueness of the optimal controller to problem (P). The main result is given by the following theorem.

Theorem 5.1. *Let $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$. For a given $\zeta > 0$, if $T > 0$ is sufficiently small and $\gamma > 0$ is sufficiently large, then there exists at most one optimal controller $g \in U_{ad}$ to problem (P). In the passive scalar case, i.e., $\xi = 0$, the small condition on T is not needed.*

Proof. Assume that there are two pairs of optimal solutions to problem (P), denoted by (g_i, v_i, θ_i), $i = 1, 2$. Then $G = g_1 - g_2$, $\vartheta = \theta_1 - \theta_2$, $W = v_1 - v_2$, and $P = p_1 - p_2$ satisfy

$$\frac{\partial \vartheta}{\partial t} + v_1 \cdot \nabla \vartheta + W \cdot \nabla \theta_2 = 0, \tag{5.1}$$

$$\frac{\partial W}{\partial t} - \nu \Delta W + v_1 \cdot \nabla W + W \cdot \nabla v_2 + \nabla P = \xi \vartheta e_2, \tag{5.2}$$

with the Navier slip boundary conditions

$$W \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(W) \cdot \tau + \alpha W \cdot \tau)|_\Gamma = G \tag{5.3}$$

and initial condition $(\vartheta(0), W(0)) = (0, 0)$. Moreover, the corresponding solutions to the adjoint problem (4.25)–(4.29) are denoted by $(\rho_i, \bar{y}_i)$, $i = 1, 2$. Let $Y = \bar{y}_1 - \bar{y}_2$, $\varrho = \rho_1 - \rho_2$ and $Q = \bar{q}_1 - \bar{q}_2$ satisfy

$$-\frac{\partial \varrho}{\partial t} - v_1 \cdot \nabla \varrho - W \nabla \rho_2 - \xi Y \cdot e_2 = 0, \tag{5.4}$$

$$\begin{aligned} -\frac{\partial Y}{\partial t} - \nu \Delta Y + (\nabla v_1)^T Y + (\nabla W)^T \bar{y}_2 - (v_1 \cdot \nabla) Y - W \nabla \bar{y}_2 + \nabla Q \\ = \theta_1 \nabla \varrho + \vartheta \nabla \rho_2 + \zeta \nabla^{\perp} (\nabla \times W), \end{aligned} \tag{5.5}$$

with the Navier slip boundary conditions

$$Y \cdot n|_{\Gamma} = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(Y) \cdot \tau + \alpha Y \cdot \tau)|_{\Gamma} = -\zeta \nabla \times W \tag{5.6}$$

and the final time condition $(\varrho(T), Y(T)) = (\Lambda^{-1} \vartheta(T), 0)$. Then

$$G = \frac{1}{\gamma} Y|_{\Gamma}. \tag{5.7}$$

The goal is to show that $G = 0$. To this end, we first apply *a priori* estimates for ϑ, Y, ϱ, and W, respectively. Estimates (2.28) and (2.30) established in Theorem 2.3 will be often used in the following proof. For convenience, we set $C = C(\|\theta_0\|_{L^\infty \cap H^1}, \|v_0\|_{H^1}, \|g\|_{\mathcal{S}}, T)$. Without loss of generality, we set $\nu = 1$ in the rest of this paper.

Step 1. We first establish the estimates on ϑ and W. Taking the inner product of (5.1) with ϑ yields

$$\|\vartheta\|_{L^2} \le \int_0^T \|W\|_{L^\infty} \|\nabla \theta_2\|_{L^2} \, dt \le \sup_{t \in [0,T]} \|\nabla \theta_2\|_{L^2} \int_0^T \|W\|_{H^{1+\delta}} \, dt, \tag{5.8}$$

where we set $0 < \delta < 1/2$. To estimate $\int_0^T \|W\|_{H^{1+\delta}} \, dt$, we use the variation of parameters formula for W and get

$$W(t) = \int_0^T e^{-A(t-\tau)} \mathbb{P}[-(v_1 \cdot \nabla W + W \cdot \nabla v_2) + \xi \vartheta e_2] \, d\tau + (LG)(t). \tag{5.9}$$

Let $I_0 = -(v_1 \cdot \nabla W + W \cdot \nabla v_2) + \xi \vartheta e_2$. In light of (2.18) and Young's inequality for convolution, we get

$$\begin{aligned} \int_0^T \|A^{1/2+\delta/2} W\|_{L^2} \, dt &\le \int_0^T \| \int_0^t A^{1/2+\delta/2} e^{-A(t-\tau)} \mathbb{P} I_0 \, d\tau \|_{L^2} \, dt \\ &+ \int_0^T \| \int_0^t A^{1/2+\delta/2} A e^{-A(t-\tau)} \mathbb{P} N G(\tau) \, d\tau \|_{L^2} \, dt \end{aligned}$$

$$\leq M_0 \int_0^T t^{-(1/2+\delta/2)} e^{-\omega t}\, dt \int_0^T \|I_0\|_{L^2}\, dt + M_0 \int_0^T t^{-(3/4+\delta)} e^{-\omega t}\, dt \int_0^T \|A^{3/4-\delta/2} N G\|_{L^2}\, dt$$

$$\leq C\Big(\int_0^T \|I_0\|_{L^2}\, dt + \int_0^T \|G\|_{L^2(\Gamma)}\, dt\Big), \tag{5.10}$$

where

$$\int_0^T \|I_0\|_{L^2}\, dt \leq \int_0^T \|v_1 \cdot \nabla W\|_{L^2}\, dt + \int_0^T \|W \cdot \nabla v_2\|_{L^2}\, dt + \xi \int_0^T \|\vartheta\|_{L^2}\, dt$$

$$\leq C \int_0^T \|v_1\|_{L^\infty} \|A^{1/2} W\|_{L^2}\, dt + C \int_0^T \|A^{1/4} W\|_{L^2} \|A^{3/4} v_2\|_{L^2}\, dt + \xi \int_0^T \|\vartheta\|_{L^2}\, dt$$

$$\leq C \left[\Big(\int_0^T \|A v_1\|_{L^2}^2\, dt\Big)^{1/2} + \Big(\int_0^T \|A v_2\|_{L^2}^2)\, dt\Big)^{1/2} \right] \Big(\int_0^T \|A^{1/2} W\|_{L^2}^2\, dt\Big)^{1/2} + \xi \int_0^T \|\vartheta\|_{L^2}\, dt. \tag{5.11}$$

Combining (5.9) with (5.10)–(5.11) yields

$$\sup_{t\in[0,T]} \|\vartheta\|_{L^2} \leq C \int_0^T \|W\|_{L^\infty} \leq C\Big[\Big(\int_0^T \|A^{1/2} W\|_{L^2}^2\, dt\Big)^{1/2} + \xi \int_0^T \|\vartheta\|_{L^2}\, dt + \int_0^T \|G\|_{L^2(\Gamma)}\, dt\Big]$$

$$\leq C\Big[\Big(\int_0^T \|A^{1/2} W\|_{L^2}^2\, dt\Big)^{1/2} + \xi T \sup_{t\in[0,T]} \|\vartheta\|_{L^2} + \sqrt{T}\Big(\int_0^T \|G\|_{L^2(\Gamma)}^2\, dt\Big)^{1/2}\Big].$$

For $\xi = 1$, set T sufficiently small such that $C\xi T < 1$. Then

$$\sup_{t\in[0,T]} \|\vartheta\|_{L^2} \leq C\Big[\Big(\int_0^T \|A^{1/2} W\|_{L^2}^2\, dt\Big)^{1/2} + \sqrt{T}\Big(\int_0^T \|G\|_{L^2(\Gamma)}^2\, dt\Big)^{1/2}\Big], \tag{5.12}$$

and hence

$$\int_0^T \|\vartheta\|_{L^2}^2 \leq C\Big(T \int_0^T \|A^{1/2} W\|_{L^2}^2\, dt + T^2 \int_0^T \|G\|_{L^2(\Gamma)}^2\, dt\Big). \tag{5.13}$$

Now applying L^2-estimate for W and combining (5.13) give

$$
\begin{aligned}
\frac{1}{2}\frac{d\|W\|_{L^2}^2}{dt} + \|A^{1/2}W\|_{L^2}^2 + \alpha\|W\|_{L^2(\Gamma)}^2 &\le \|W\|_{L^4}\|v_2\|_{L^4}\|A^{1/2}W\|_{L^2} \\
&\quad + \xi\|\vartheta\|_{L^2}\|W\|_{L^2} + \|G\|_{L^2(\Gamma)}\|W\|_{L^2(\Gamma)} \\
&\le C\|W\|_{L^2}^{1/2}\|A^{1/2}W\|_{L^2}^{3/2}\|v_2\|_{L^2}^{1/2}\|A^{1/2}v_2\|_{L^2}^{1/2} \\
&\quad + \xi\|\vartheta\|_{L^2}\|W\|_{L^2} + \|G\|_{L^2(\Gamma)}\|W\|_{L^2(\Gamma)} \\
&\le C\|W\|_{L^2}^2\|v_2\|_{L^2}^2\|A^{1/2}v_2\|_{L^2}^2 + \frac{3}{4}\|A^{1/2}W\|_{L^2}^2 \\
&\quad + C\xi\|\vartheta\|_{L^2}^2 + \frac{1}{16}\|A^{1/2}W\|_{L^2}^2 + C\|G\|_{L^2(\Gamma)}^2 + \frac{1}{16}\|A^{1/2}W\|_{L^2}^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sup_{t\in[0,T]}\|W\|_{L^2}^2 + \int_0^T \|A^{1/2}W\|_{L^2}^2\,dt &\le C(\|\theta_0\|_{H^1}, \|v_0\|_{H^1}, \|g\|_{\mathcal{S}})T\sup_{t\in[0,T]}\|W\|_{L^2}^2 \\
&\quad + C\xi\int_0^T \|\vartheta\|_{L^2}^2 + C\int_0^T \|G\|_{L^2(\Gamma)}^2.
\end{aligned}
$$

Again for $\xi = 1$, with the help of (5.13) we set T sufficiently small and obtain

$$
\sup_{t\in[0,T]}\|W\|_{L^2}^2 + \int_0^T \|A^{1/2}W\|_{L^2}^2\,dt \le C(\xi T^2 + 1)\int_0^T \|G\|_{L^2(\Gamma)}^2\,dt). \tag{5.14}
$$

Further applying the optimality condition (5.7), the trace theorem, and (2.10) gives

$$
\int_0^T \|G\|_{L^2(\Gamma)}^2\,dt \le C\frac{1}{\gamma^2}\int_0^T \|Y\|_{H^{1/2}}^2\,dt = C\frac{1}{\gamma^2}\int_0^T \|A^{1/4}Y\|_{L^2}^2\,dt. \tag{5.15}
$$

Therefore, (5.14) becomes

$$
\sup_{t\in[0,T]}\|W\|_{L^2}^2 + \int_0^T \|A^{1/2}W\|_{L^2}^2\,dt \le C(\xi T^2 + 1)\frac{1}{\gamma^2}\int_0^T \|A^{1/4}Y\|_{L^2}^2\,dt. \tag{5.16}
$$

Step 2. We proceed to estimate $\int_0^T \|A^{1/4}Y\|_{L^2}^2\,dt$. Again utilizing the variation of parameters formula for solving Y backward in time yields

$$
\begin{aligned}
Y = \int_T^t & e^{A(t-\tau)}\mathbb{P}[(\nabla v_1)^T Y + (\nabla W)^T \bar{y}_2 - (v_1 \cdot \nabla)Y - W\nabla \bar{y}_2 \\
& - \theta_1 \nabla \varrho - \vartheta \nabla \rho_2 - \zeta \nabla^{\perp}(\nabla \times W)]\, d\tau \\
& + \int_T^t (-A)e^{A(t-\tau)} N(-\zeta \nabla \times W)|_\Gamma \, d\tau.
\end{aligned}
$$

Replacing t by $T - t$ follows

$$
\begin{aligned}
Y(T-t,x) = -\int_0^t e^{-A(t-\tau)}\mathbb{P}[(\nabla v_1)^T Y + (\nabla W)^T \bar{y}_2 - (v_1 \cdot \nabla)Y - W\nabla \bar{y}_2 - \theta_1 \nabla \varrho - \vartheta \nabla \rho_2 \\
- \zeta \nabla^{\perp}(\nabla \times W)]\, d\tau + \int_0^t A e^{-A(t-\tau)} N(-\zeta \nabla \times W)|_\Gamma \, d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T \|A^{1/4} Y\|_{L^2}^2 \, dt \le & \int_0^T \| \int_0^t A^{1/4} e^{-A(t-\tau)}\mathbb{P}[(\nabla v_1)^T Y - (v_1 \cdot \nabla)Y - \zeta \nabla^{\perp}(\nabla \times W)]\, d\tau \|_{L^2}^2 \, dt \\
& + \int_0^T \| \int_0^t A^{1/4} e^{-A(t-\tau)}\mathbb{P}[(\nabla W)^T \bar{y}_2 - W\nabla \bar{y}_2]\, d\tau \|_{L^2}^2 \, dt \\
& + \int_0^T \| \int_0^t A^{1/4} e^{-A(t-\tau)}\mathbb{P}(\theta_1 \nabla \varrho)\, d\tau \|_{L^2}^2 \, dt \\
& + \int_0^T \| \int_0^t A^{1/4} e^{-A(t-\tau)}\mathbb{P}(\vartheta \nabla \rho_2)\, d\tau \|_{L^2}^2 \, dt \\
& + \int_0^T \| \int_0^t A^{1/4} A e^{-A(t-\tau)} N(\zeta \nabla \times W|_\Gamma)(\tau)\, d\tau \|_{L^2}^2 \, dt \\
= & \, I_1 + I_2 + I_3 + I_4 + I_5. \qquad (5.17)
\end{aligned}
$$

It is straightforward to verify that

$$I_1 = \int_0^T \| \int_0^t A^{1/4} e^{-A(t-\tau)} \mathbb{P}[(\nabla v_1)^T Y - (v_1 \cdot \nabla) Y - \zeta \nabla^\perp (\nabla \times W)] \, d\tau \|_{L^2}^2 \, dt$$

$$= \int_0^T \| \int_0^t A^{3/4} e^{-A(t-\tau)} A^{-1/2} \mathbb{P}[(\nabla v_1)^T Y - (v_1 \cdot \nabla) Y - \zeta \nabla^\perp (\nabla \times W)] \, d\tau \|_{L^2}^2 \, dt \quad (5.18)$$

$$\le C (\int_0^T (t^{-3/4} e^{-\omega t})^p \, dt)^{2/p} (\int_0^T \| A^{-1/2} \mathbb{P}[(\nabla v_1)^T Y - (v_1 \cdot \nabla) Y - \zeta \nabla^\perp (\nabla \times W)] \|_{L^2}^q \, dt)^{2/q} \quad (5.19)$$

$$\le C (\int_0^T \| A^{-1/2} \mathbb{P}[(\nabla v_1)^T Y - (v_1 \cdot \nabla) Y - \zeta \nabla^\perp (\nabla \times W)] \|_{L^2}^q \, dt)^{2/q}, \quad (5.20)$$

where from (5.18) to (5.19) we used Young's inequality for evolution with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{2}$, $1 < p < 4/3$, and $4/3 < q < 2$. Moreover,

$$\int_0^T \| A^{-1/2} (\nabla v_1)^T Y \|_{L^2}^q \, dt \le \int_0^T (\sup_{\psi \in V_n^1(\Omega)} \frac{| \int_\Omega Y \cdot \nabla v_1 \cdot \psi \, dx |}{\| \psi \|_{H^1}})^q \, dt$$

$$= \int_0^T (\sup_{\psi \in V_n^1(\Omega)} \frac{| \int_\Omega Y \cdot \nabla (v_1 \cdot \psi) - v_1 \cdot (Y \cdot \nabla) \psi \, dx |}{\| \psi \|_{H^1}})^q \, dt$$

$$\le \int_0^T (\sup_{\psi \in V_n^1(\Omega)} \frac{\| A^{1/4} Y \|_{L^2} \| v_1 \|_{L^2}^{1/2} \| A^{1/2} v_1 \|_{L^2}^{1/2} \| \nabla \psi \|_{L^2}}{\| \psi \|_{H^1}})^q \, dt$$

$$\le C \sup_{t \in [0,T]} \| v_1 \|_{L^2}^{q/2} \sup_{t \in [0,T]} \| A^{1/2} v_1 \|_{L^2}^{q/2} \int_0^T \| A^{1/4} Y \|_{L^2}^q \, dt$$

$$\le C T^{1-q/2} \sup_{t \in [0,T]} \| v_1 \|_{L^2}^{q/2} \sup_{t \in [0,T]} \| A^{1/2} v_1 \|_{L^2}^{q/2} (\int_0^T \| A^{1/4} Y \|_{L^2}^2 \, dt)^{q/2}.$$

Similarly,

$$\int_0^T \| A^{-1/2} (v_1 \cdot \nabla) Y \|_{L^2}^q \, dt \le \int_0^T (\sup_{\psi \in V_n^1(\Omega)} \frac{| \int_\Omega (v_1 \cdot \nabla) Y \cdot \psi \, dx |}{\| \psi \|_{H^1}})^q \, dt$$

$$= \int_0^T (\sup_{\psi \in V_n^1(\Omega)} \frac{| \int_\Omega v_1 \cdot \nabla (Y \cdot \psi) - Y \cdot (v_1 \cdot \nabla) \psi \, dx |}{\| \psi \|_{H^1}})^q \, dt$$

$$\leq \int_0^T \Big(\sup_{\psi \in V_n^1(\Omega)} \frac{\|A^{1/4}Y\|_{L^2}\|v_1\|_{L^2}^{1/2}\|A^{1/2}v_1\|_{L^2}^{1/2}\|\nabla\psi\|_{L^2}}{\|\psi\|_{H^1}}\Big)^q\,dt$$

$$\leq CT^{1-q/2}\sup_{t\in[0,T]}\|v_1\|_{L^2}^{q/2}\sup_{t\in[0,T]}\|A^{1/2}v_1\|_{L^2}^{q/2}\Big(\int_0^T\|A^{1/4}Y\|_{L^2}^2\,dt\Big)^{q/2},$$

and

$$\int_0^T\|A^{-1/2}\mathbb{P}[\zeta\nabla^\perp(\nabla\times W)]\|_{L^2}^q\,dt \leq CT^{1-q/2}\zeta^q\Big(\int_0^T\|A^{1/2}W\|_{L^2}^2\,dt\Big)^{q/2}.$$

Therefore,

$$I_1 \leq CT^{2/q-1}\|v_1\|_{L^2}\sup_{t\in[0,T]}\|A^{1/2}v_1\|_{L^2}\int_0^T\|A^{1/4}Y\|_{L^2}^2\,dt + CT^{2/q-1}\zeta^2\int_0^T\|A^{1/2}W\|_{L^2}^2\,dt. \tag{5.21}$$

Next invoking (5.16) we get

$$I_2 = \int_0^T\Big\|\int_0^t A^{1/4}e^{-A(t-\tau)}\mathbb{P}[(\nabla W)^T\bar{y}_2 - W\nabla\bar{y}_2]\,d\tau\Big\|_{L^2}^2\,dt$$

$$\leq C\Big(\int_0^T t^{-1/4}e^{-\omega t}\,dt\Big)^2\int_0^T\|\mathbb{P}[(\nabla W)^T\bar{y}_2 - W\nabla\bar{y}_2]\|_{L^2}^2\,dt$$

$$\leq C\sup_{t\in[0,T]}\|\bar{y}_2\|_{L^\infty}^2\int_0^T\|A^{1/2}W\|_{L^2}^2\,dt + C\int_0^T\|W\|_{L^2}\|A^{1/2}W\|_{L^2}\|\nabla\bar{y}_2\|_{L^2}\|A\bar{y}_2\|_{L^2}\,dt$$

$$\leq C\sup_{t\in[0,T]}\|\bar{y}_2\|_{L^\infty}^2\int_0^T\|A^{1/2}W\|_{L^2}^2\,dt$$

$$+\sup_{t\in[0,T]}\|W\|_{L^2}\sup_{t\in[0,T]}\|\nabla\bar{y}_2\|_{L^2}\Big(\int_0^T\|A^{1/2}W\|_{L^2}^2\Big)^{1/2}\Big(\int_0^T\|A\bar{y}_2\|_{L^2}^2\,dt\Big)^{1/2}$$

$$\leq C(\xi T^2+1)\frac{1}{\gamma^2}\int_0^T\|A^{1/4}Y\|_{L^2}^2\,dt. \tag{5.22}$$

To estimate I_3, we have for $0<\varepsilon<1/2$,

$$
\begin{aligned}
I_3 &\le C \int_0^T \| \int_0^t A^{3/4+\varepsilon/2} e^{-A(t-\tau)} A^{-1/2-\varepsilon/2} \mathbb{P}(\theta_1 \nabla \varrho)\, d\tau \|_{L^2}^2\, dt \\
&\le C (\int_0^T t^{-(3/4+\varepsilon/2)} e^{-\omega t}\, dt)^2 \int_0^T \| A^{-1/2-\varepsilon/2} \mathbb{P}(\theta_1 \nabla \varrho)\|_{L^2}^2\, dt \\
&\le C \int_0^T (\sup_{\psi \in V_n^{1+\varepsilon}(\Omega)} \frac{|\int_\Omega \theta_1 \nabla \varrho \cdot \psi\, dx|}{\|\psi\|_{H^{1+\varepsilon}}})^2\, dt \\
&= C \int_0^T (\sup_{\psi \in V_n^{1+\varepsilon}(\Omega)} \frac{|\int_\Omega \theta_1 \nabla \cdot (\varrho \psi) - \theta_1 \varrho \nabla \cdot \psi\, dx|}{\|\psi\|_{H^{1+\varepsilon}}})^2\, dt \\
&= C \int_0^T (\sup_{\psi \in V_n^{1+\varepsilon}(\Omega)} \frac{|\int_\Omega \nabla \theta_1 \cdot (\varrho \psi) - \theta_1 (\varrho \psi) \cdot n\, dx|}{\|\psi\|_{H^{1+\varepsilon}}})^2\, dt \\
&\le C \int_0^T (\sup_{\psi \in V_n^{1+\varepsilon}(\Omega)} \frac{\|\nabla \theta_1\|_{L^2} \|\varrho\|_{L^2} \|\psi\|_{L^\infty}}{\|\psi\|_{H^{1+\varepsilon}}})^2\, dt \\
&\le C \int_0^T \|\nabla \theta_1\|_{L^2}^2 \|\varrho\|_{L^2}^2\, dt \le C \sup_{t \in [0,T]} \|\nabla \theta_1\|_{L^2}^2 \int_0^T \|\varrho\|_{L^2}^2\, dt. \qquad (5.23)
\end{aligned}
$$

Following the similar idea, we obtain

$$
\begin{aligned}
I_4 &\le C (\int_0^T t^{-(3/4+\varepsilon/2)} e^{-\omega t}\, dt)^2 \int_0^T \| A^{-1/2-\varepsilon/2} \mathbb{P}(\vartheta \nabla \rho_2)\|_{L^2}^2\, dt \\
&\le C \int_0^T (\sup_{\psi \in V_n^{1+\varepsilon}(\Omega)} \frac{|\int_\Omega \vartheta \nabla \rho_2 \cdot \psi\, dx|}{\|\psi\|_{H^{1+\varepsilon}}})^2\, dt \le C \int_0^T (\sup_{\psi \in V_n^{1+\varepsilon}(\Omega)} \frac{\|\vartheta\|_{L^2} \|\nabla \rho_2\|_{L^2} \|\psi\|_{L^\infty}}{\|\psi\|_{H^{1+\varepsilon}}})^2\, dt \\
&\le C \sup_{t \in [0,T]} \|\nabla \rho_2\|_{L^2}^2 \int_0^T \|\vartheta\|_{L^2}^2\, dt. \qquad (5.24)
\end{aligned}
$$

Lastly,

$$
I_5 \le (\int_0^T t^{-3/4} e^{-\omega t}\, dt)^2 \int_0^T \|A^{1/2} N(\zeta \nabla \times W|_\Gamma)\|_{L^2}^2\, dt \qquad (5.25)
$$

$$\leq C\int_0^T \|\zeta\nabla\times W\|^2_{H^{-1/2}(\Gamma)}\,dt \tag{5.26}$$

$$\leq C\zeta\int_0^T \|A^{1/2}W\|^2_{L^2}\,dt. \tag{5.27}$$

From (5.25) to (5.26) we used (2.14). Let T be small enough and γ be large enough. Then combining (5.17) with (5.21)–(5.24) and (5.27) yields

$$\int_0^T \|A^{1/4}Y\|^2_{L^2}\,dt \leq C(\int_0^T \|\varrho\|^2_{L^2}\,dt + \int_0^T \|\vartheta\|^2_{L^2}\,dt + \zeta\int_0^T \|A^{1/2}W\|^2_{L^2}\,dt). \tag{5.28}$$

Now we estimate $\|\varrho\|_{L^2}$ by taking the inner product of (5.4) with ϱ, which yields

$$\begin{aligned}-\frac{1}{2}\frac{d\|\varrho\|^2_{L^2}}{dt} &\leq \|W\nabla\rho_2\varrho\|_{L^2} + \xi\|Y\|_{L^2}\|\varrho\|_{L^2}\\ &\leq C\|W\|_{L^\infty}\|\nabla\rho_2\|_{L^2}\|\varrho\|_{L^2} + \xi\|Y\|_{L^2}\|\varrho\|_{L^2},\end{aligned}$$

or

$$-\frac{d\|\varrho\|_{L^2}}{dt} \leq C\|W\|_{L^\infty}\|\nabla\rho_2\|_{L^2} + \xi\|Y\|_{L^2}. \tag{5.29}$$

Taking the integral of (5.29) from t to T and making using of

$$\|\varrho(T)\|_{L^2} = \|\Lambda^{-2}\vartheta(0)\|_{L^2} \leq C\sup_{t\in[0,T]}\|\vartheta\|_{L^2}$$

together with (5.12) and (5.15) follow that

$$\begin{aligned}\|\varrho\|_{L^2} &\leq C\sup_{t\in[0,T]}\|\nabla\rho_2\|_{L^2}\int_0^T \|W\|_{L^\infty}\,dt + \xi\int_0^T \|Y\|_{L^2}\,dt + C\sup_{t\in[0,T]}\|\vartheta\|_{L^2}\\ &\leq C\sup_{t\in[0,T]}\|\nabla\rho_2\|_{L^2}(\int_0^T \|A^{1/2}W\|^2_{L^2}\,dt)^{1/2}\\ &\quad + C\xi\sqrt{T}(\int_0^T \|A^{1/4}Y\|^2_{L^2}\,dt)^{1/2} + C\sqrt{T}\frac{1}{\gamma}(\int_0^T \|A^{1/4}Y\|^2_{L^2}\,dt)^{1/2}\\ &\leq C\sup_{t\in[0,T]}\|\nabla\rho_2\|_{L^2}(\int_0^T \|A^{1/2}W\|^2_{L^2}\,dt)^{1/2}\end{aligned}$$

$$+ C(\xi + \frac{1}{\gamma})\sqrt{T}(\int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt)^{1/2}. \tag{5.30}$$

Combining (5.28) with (5.13), (5.15), and (5.30) follows

$$\begin{aligned}
\int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt &\le C(\int_0^T \|\varrho\|_{L^2}^2\, dt + \int_0^T \|\vartheta\|_{L^2}^2\, dt + \zeta \int_0^T \|A^{1/2}W\|_{L^2}^2\, dt) \\
&\le C\big(T\int_0^T \|A^{1/2}W\|_{L^2}^2\, dt + (\xi + \frac{1}{\gamma})^2 T^2 \int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt\big) \\
&\quad + C\big(T\int_0^T \|A^{1/2}W\|_{L^2}^2\, dt + \frac{T^2}{\gamma^2}\int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt\big) + C\zeta \int_0^T \|A^{1/2}W\|_{L^2}^2\, dt \\
&= C(2T + \zeta)\int_0^T \|A^{1/2}W\|_{L^2}^2\, dt + C\big((\xi + \frac{1}{\gamma})^2 + \frac{1}{\gamma^2}\big)T^2 \int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt.
\end{aligned} \tag{5.31}$$

By virtue of (5.16) we get

$$\begin{aligned}
\int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt \le& C(2T + \zeta)(\xi T^2 + 1)\frac{1}{\gamma^2}\int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt \\
&+ C\big((\xi + \frac{1}{\gamma})^2 + \frac{1}{\gamma^2}\big)T^2 \int_0^T \|A^{1/4}Y\|_{L^2}^2\, dt.
\end{aligned} \tag{5.32}$$

Finally, for given $\zeta > 0$, if γ is sufficiently large and T is sufficiently small such that

$$C\left((2T + \zeta)(\xi T^2 + 1)\frac{1}{\gamma^2} + \big((\xi + \frac{1}{\gamma})^2 + \frac{1}{\gamma^2}\big)T^2\right) < 1, \tag{5.33}$$

then $\int_0^T \|A^{1/4}Y\|_{L^2}^2 = 0$, and hence $Y = 0$ and $G = 0$. Moreover, (5.16) indicates that $W = 0$. As a consequence of (5.12) and (5.30), we have $\vartheta = 0$ and $\varrho = 0$. The uniqueness of the optimal solution to problem (P) is obtained. In addition, from (5.33) it is easy to see that if $\xi = 0$, then the small condition on T is not required. □

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