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# BOUNDARY CONTROL FOR OPTIMAL MIXING VIA NAVIER–STOKES FLOWS*

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**Abstract.** We discuss the problem of optimal mixing of an inhomogeneous distribution of a scalar field $\theta$ via an active control of the flow velocity $v$, governed by the incompressible Navier–Stokes equations, in an open bounded and connected domain $\Omega \subset \mathbb{R}^2$. We consider the velocity field generated by a control input that acts tangentially on the boundary of the domain through the Navier slip boundary conditions. This problem is motivated by mixing the fluids within a cavity or vessel by moving the walls or stirring at the boundaries. Our main objective is to design an optimal Navier slip boundary control that optimizes mixing at a given final time $T > 0$. Nondissipative scalars, both passive and active, governed by the transport equation will be addressed. In the absence of diffusion, transport and mixing occur due to pure advection. This essentially leads to a nonlinear control problem of a semidissipative system. Sobolev norm for the dual space $(H^1(\Omega))'$ of $H^1(\Omega)$ is adopted to quantify mixing due to the property of weak convergence. The challenge arises from the vanishing diffusivity and nonlinear coupling of the system, which results in requiring the velocity field to satisfy $\int_0^T \|\nabla v\|_{L^\infty(\Omega)}\, d\tau < \infty$. We present a rigorous proof to show the existence of an optimal controller for both passive and active scalars and that the compatibility conditions for initial and boundary data are not required for Navier slip boundary control. Finally, we establish the first-order necessary conditions for optimality for both cases by using a variational inequality.

**Key words.** optimal mixing, passive and active scalars, Navier–Stokes flows, Navier slip boundary control, nonlinear control

**AMS subject classifications.** 35B65, 35Q35, 35Q93, 49J20, 49K20, 76B03, 76B75, 76F25

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## 1. Introduction.

Transport and mixing play central roles in many natural phenomena and engineering problems, such as the circulation of the atmosphere and oceans, the spreading of environmental pollutants, the ventilation in buildings, or the mixing of chemical substances in combustion. Effectively enhancing mixing has attracted increasing attention in both academic and industrial communities; see, e.g., [20, 21, 34, 36, 37, 38, 39, 45, 47]. The current work is concerned with the problem of optimal mixing of a scalar field via an active control of the incompressible Navier–Stokes equations. More precisely, we aim at determining an optimal flow velocity with optimal control inputs that optimizes mixing in a two-dimensional open bounded domain. In particular, nondissipative scalars, both *passive* and *active*, governed by the transport equation will be investigated in this paper.

Optimal mixing and stirring of passive scalar fields due to pure advection has been widely discussed; see, e.g., [1, 28, 34, 37, 38, 45, 50]. However, the prescribed constraints were often imposed on the flow fields in order to formulate an optimization problem. A final time optimal control problem was discussed for mixing in stationary Stokes flows by Matthew et al. [38]. The flow was assumed to be induced by a finite

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set of force fields that can be controlled in time. Moreover, a fixed value of the control action was imposed. No dynamics was incorporated for the flow velocity during the mixing process. Recently, Foures, Caufield, and Schmid [15] considered the case where the velocity field was governed by Navier–Stokes equations and derived the optimal initial and boundary conditions to be prescribed for the flow velocity. However, the Navier–Stokes equations were not controlled in real time.

Motivated by the observation that moving walls accelerate mixing (see, e.g., [17, 18, 19, 46]), we consider the problem of Navier boundary control design to steer the advection by applying the controls tangentially along the wall. In fact, it is found that fixed walls with no-slip boundary condition can turn an exponential decay in time into a power decay due to the presence of separatrices on the walls, which slow down the whole mixing region [17, 18]. However, this can be overcome by moving the walls to create closed orbits near the walls, which effectively insulate the central mixing region from the walls [19, 46]. In our present work, we consider that the velocity field is generated by the control inputs acting on the domain boundary through Navier slip boundary conditions, which allow the fluid to slip with resistance on the boundary [9, 40, 42]. Due to vanishing diffusivity, transport and mixing of the scalars are determined solely through advection by the nonlocal velocity field. This essentially leads to a nonlinear control problem of a semidissipative system.

To quantify mixing, a classical measure is the variance of the concentration of the scalar, which can be related to the $L^2$-norm of the scalar field [12]. However, this measurement fails in the case of zero diffusivity since it is unable to quantify pure stirring effects [39]. It can be shown that the scalar field is conserved in terms of any $L^p$-norm for $1 \leq p \leq \infty$. Recently, the mix-norm and negative Sobolev norms are adopted to quantify mixing based on ergodic theory, which are sensitive to both stirring and diffusion [34, 39, 45]. Mathew, Mezić, and Petzold in [39] first showed the equivalence of the mix-norm to the $H^{-1/2}$-norm. In fact, any negative Sobolev norm $H^{-s}$ for $s > 0$ can be used as a mix-norm since the bridge that connects negative Sobolev norms and mixing is the property of weak convergence, as first stated in [34]. To have the negative Sobolev norms well-defined, periodic boundary conditions are often considered. In our present approach, we consider a general domain for the scalar field without imposing any additional boundary conditions other than no-penetration on the velocity field. We replace the negative Sobolev norm by the norm of the dual space $(H^s(\Omega))'$ of $H^s(\Omega)$ with $s > 0$. Very recently, Hu in [23] applied Navier slip boundary control for enhancing mixing in unsteady Stokes flows at a given final time $T > 0$, where $(H^1(\Omega))'$ was adopted for qualifying mixing [23]. We shall continue to use this norm in our current work.

**1.1. Mathematical models.** Consider a scalar field that is advected by an incompressible flow in an open bounded and connected domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\Gamma$. The transport equation is used to describe the mass distribution or scalar concentration, where molecular diffusion is assumed to be negligible. Although active and passive scalars are governed by the same transport equation considered in this paper, their nature is essentially different. In the case of passive scalars, there is a one-way coupling between the scalar and the flow: The transported scalar does not influence the velocity field. The feedback of the scalar field is negligible, and the velocity determines the properties of the scalar. In contrast, in the case of active scalars, which, while transported, act on the velocity through local forces (such as buoyancy), the presence of the feedback couples the transported scalar to the velocity. The complexity of the two-way coupling between the scalar and the flow presents a major challenge in analysis. The understanding of the active transport is far behind that of the passive counterpart [5]. This motivates the use of case studies of mixing of passive and active scalars via flow advection. Especially for active scalars, this work focuses on mixing via the buoyancy-driven flow modeled with the Boussinesq approximation in the absence of diffusivity. We introduce the following two models to address the problems in details.

**Model I.** Consider a *passive* scalar field advected by an incompressible flow. The system is governed by

$$\partial_t\theta + v\cdot\nabla\theta = 0 \tag{1.1}$$
$$\partial_t v - \nu\Delta v + v\cdot\nabla v + \nabla p = 0 \tag{1.2}$$
$$\nabla\cdot v = 0, \tag{1.3}$$

where $\theta$ is the mass distribution or scalar concentration, $v$ is the velocity of the flow, $\nu > 0$ is the viscosity, and $p$ is the pressure. As a result of one-way coupling, investigating the optimal control design for the coupled system (1.1)–(1.3) is tied to understanding the control problems of the Navier–Stokes equations.

**Model II.** In the case of *active* scalars, we focus on the transport and mixing in the buoyancy-driven flow modeled by the Boussinesq approximation with zero diffusivity in the scalar equation. The system is now governed by

$$\partial_t\theta + v\cdot\nabla\theta = 0 \tag{1.4}$$
$$\partial_t v - \nu\Delta v + v\cdot\nabla v + \nabla p = \theta e_2 \tag{1.5}$$
$$\nabla\cdot v = 0, \tag{1.6}$$

where $e_2 = (0,1)^T$ is a unit vector in the direction of gravitational acceleration. Because of the complexity of two-way coupling, whether singularities of $\nabla\theta$ can develop as the molecular diffusivity vanishes has been a challenging problem in fluid mechanics literature [41]. A particular difficulty if the domain is bounded is the creation of vorticity on the domain boundary, which requires a careful mathematical treatment of the nonlinearity and the coupling.

Navier slip boundary conditions are defined as follows [42]:

$$v\cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n\cdot\mathbb{D}(v)\cdot\tau + \alpha v\cdot\tau)|_\Gamma = g\cdot\tau, \tag{1.7}$$

where $n$ and $\tau$ denote the outward unit normal and tangential vectors with respect to the domain $\Omega$ and $\mathbb{D}(v) = (1/2)(\nabla v + (\nabla v)^T)$. The friction between the fluid and the wall is proportional to $-v$ with the positive coefficient of proportionality $\alpha$. The nonhomogeneous boundary term $g$ with $g\cdot n|_\Gamma = 0$ is the control input depending on both space and time, which is employed to generate the velocity field for mixing. The initial condition is given by

$$(\theta(0), v(0)) = (\theta_0, v_0). \tag{1.8}$$

Throughout this paper, we use $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, without ambiguity, for the $L^2$-inner products as well as the duality in the interior of the domain $\Omega$ and on the boundary $\Gamma$, respectively.

**1.2. Formulation of the optimal control problem.** It is widely recognized that mixing can be enhanced by introducing strong streamwise vortices [7, 16, 48, 49].

The fundamental idea is to utilize a streamwise vortical structure with its associated cross-stream circulation to augment the rate of mixing. In fact, when the scalar field is convected in the velocity field of a vortex, the stretching of the interface between two fluids of different properties creates two interrelated effects. First, the interfacial surface area is increased. Second, the magnitude of gradients normal to the interface is increased. Both effects augment mixing [48]. This naturally motives the study of vortex-enhanced mixing.

We formulate the optimal control problem as follows: For a given $T > 0$, find a control $g$ minimizing the cost functional

$$J(g) = \frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'} + \frac{\gamma}{2}\|g\|^2_{U_{\mathrm{ad}}} - \frac{\zeta}{2}\int_0^T \|\nabla \times v\|^2_{L^2}\, dt, \qquad \text{(P)}$$

where $\nabla\times v = \partial_1 v_2 - \partial_2 v_1$ stands for the vorticity, $\zeta > 0$ is the regularization parameter for vorticity, $U_{\mathrm{ad}}$ is the set of admissible controls, and $\gamma > 0$ is the control weight parameter, which is chosen to establish the relative weight depending on the first and the third terms; $\gamma$ and $\zeta$ can be adjusted based on specific physical applications. On the other hand, it is also true that the long-time dynamics may be dominated by strong coherent vortices that can possibly slow down mixing. Thus, $\zeta$ can be used to test the sensitivity of mixing rate with respect to vorticity.

The Soblev norm $\|\cdot\|_{(H^1(\Omega))'}$ is adopted to quantify the degree of mixedness and is defined by

$$\text{(1.9)} \qquad \|f\|_{(H^s(\Omega))'} = \sup_{\phi\in H^s(\Omega)} \frac{|(f,\phi)_{((H^s(\Omega))',H^s(\Omega))}|}{\|\phi\|_{H^s}}, \quad f \in (H^s(\Omega))' \quad \text{for} \quad s > 0,$$

where $(f,\phi)_{((H^s(\Omega))',H^s(\Omega))} = \int_\Omega f\bar{\phi}\, dx$. We have the Gelfand triple

$$H^s(\Omega) \subset L^2(\Omega) \subset (H^s(\Omega))', \quad s > 0,$$

with the embeddings being continuous and compact. The space $H^s(\Omega)$ may be defined as the domain of an operator $\Lambda^s$ equipped with the norm $\|\cdot\|_{H^s}$, where $\Lambda$ is self-adjoint, positive, and unbounded in $L^2(\Omega)$. Correspondingly, the space $(H^s(\Omega))'$ can be identified as the domain of $\Lambda^{-s}$ equipped with the norm $\|\cdot\|_{(H^s(\Omega))'}$. Thus, $\Lambda^{2s} \in \mathcal{L}(H^s(\Omega),(H^s(\Omega))')$. We write $\|\theta(T)\|_{(H^1(\Omega))'} = \|\Lambda^{-1}\theta(T)\|_{L^2(\Omega)}$.

Problem (P) is called well-posed if for any initial condition $(\theta_0, v_0)$, there exist a control $g \in U_{\mathrm{ad}}$ and the corresponding solution $(\theta, v)$ to the governing system such that the cost functional $J$ is finite. The choice of $U_{\mathrm{ad}}$ is influenced by the physical properties as well as the need to establish the existence of an optimal solution. Note that boundary control of the velocity field essentially leads to a nonlinear control problem of the scalar equation. Thus, problem (P) becomes nonconvex. Investigating the optimal control problem of Model I and Model II is tied to understanding the Navier slip boundary control problem of the Navier–Stokes equations. However, the nonlinear coupling creates technical difficulties in studying the existence and uniqueness of an optimal control. A critical challenge arises in deriving the first-order necessary conditions of optimality. To establish the well-posedness of the optimality system, one needs $\sup_{t\in[0,T]}\|\nabla\theta\|_{L^2(\Omega)} < \infty$, which in turn demands a priori control on $\int_0^T \|\nabla v\|_{L^\infty(\Omega)}\, dt$ due to zero diffusivity. As a result, the initial condition and control input should be chosen such that

$$\int_0^T \|\nabla v\|_{L^\infty(\Omega)}\, dt < \infty.$$

This requires a sharp estimate on the state space in order to possibly avoid the compatibility conditions to come into play, which is particularly difficult in the active scalar case to establish the global well-posedness in low-regularity spaces. The main obstacle is because $\|\nabla v\|_{L^\infty}$ cannot be bounded by $\|v\|_{H^2}$ using Sobolev embedding in a two-dimensional domain. In fact, the global well-posedness with zero diffusivity (or zero viscosity) of the Boussinesq equations has been open until recently [4, 6, 11, 22, 24, 25, 26, 29, 31, 32]. In the case of a bounded domain with no-slip boundary condition, Lai, Pan, and Zhan [31] proved the existence of a unique global solution for initial data $(\theta_0, v_0) \in H^3(\Omega) \times H^3(\Omega)$ with some extra compatibility conditions on $v_0$. Hu, Kukavica, and Ziane [25] proved the existence of a unique global solution for $(\theta_0, v_0) \in H^1(\Omega) \times H^2(\Omega)$ with no additional compatibility condition required on $v_0$ beyond $v_0|_\Gamma = 0$. Ju [29] further improved this result and obtained the global regularity for $(\theta_0, v_0) \in H^1(\Omega) \times H^1(\Omega)$ by using spectral decomposition analysis. Very recently, Hu et al. [27] proved the global regularity for $(\theta_0, v_0) \in L^\infty(\Omega) \times H^1(\Omega)$ with Navier sip boundary conditions. The last two results indicate that the compatibility condition for initial and boundary data may not be needed when using the Navier slip boundary control.

The rest of this paper is organized as follows. To apply the Navier slip boundary control for generating the incompressible Navier–Stokes flows, we first address the Stokes problem with nonhomogeneous Navier slip boundary conditions in section 2 and then provide the explicit formulation for the vorticity on the boundary. In section 3, we identify the conditions that the initial and boundary data of the governing system have to satisfy in order to establish the well-posedness of the optimal control problem and define the set of admissible controls. In sections 4 and 5, we discuss the existence of an optimal solution for Model I and Model II, respectively, and then derive the first-order necessary optimality conditions by employing a variational inequality. Moreover, since it is not practical to create arbitrarily distributed force fields for stirring, we consider that the control inputs are finite dimensional. This will lead to a more transparent optimality system. Furthermore, the compatibility conditions will not be required; thus, the controls can act on only a portion of the boundary. However, we do not have any uniqueness results, mainly due to the nonconvexity of $J$.

In what follows, the symbol $c$ denotes a generic positive constant, which is allowed to depend on the domain as well as on indicated parameters.

**2. Preliminary: Stokes problem with nonhomogeous Navier slip boundary conditions.** Consider the Stokes problem with Navier slip boundary conditions

$$-\nu\Delta v + \nabla p = 0 \tag{2.1}$$

$$\nabla \cdot v = 0 \tag{2.2}$$

$$v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha v \cdot \tau)|_\Gamma = g \cdot \tau. \tag{2.3}$$

To set up the abstract formulation for the velocity field, we define

$$V_n^s(\Omega) = \{v \in H^s(\Omega) : \text{div } v = 0,\ v \cdot n|_\Gamma = 0\}, \quad s \geq 0,$$
$$V_n^s(\Gamma) = \{g \in H^s(\Gamma) : g \cdot n|_\Gamma = 0\}, \quad s \geq 0.$$

The regularity for the Stokes problem with nonhomogeneous slip-type boundary conditions has been well addressed in [2]. To be more specific, the following results are stated in [2].

LEMMA 2.1. *Assume that* $\Omega$ *is an open bounded and connected domain with boundary* $\Gamma \in C^{1,1}$. *Let* $g \in H^{-1/2}(\Gamma)$. *Then there exists the pressure unique up to a constant such that*

$$\|v\|_{H^1}^2 + \|p\|_{L^2}^2 \leq c\|g\|_{H^{-1/2}(\Gamma)}^2. \tag{2.4}$$

*Moreover, if* $\Gamma \in C^{2,1}$ *and* $g \in V_n^{1/2}(\Gamma)$, *then* $(v,p) \in V_n^2(\Omega) \times H^1(\Omega)$ *and*

$$\|v\|_{H^2}^2 + \|p\|_{H^1}^2 \leq c\|g\|_{H^{1/2}(\Gamma)}^2. \tag{2.5}$$

To understand the vorticity on the boundary, we introduce the following lemmas and provide the complete proofs for the convenience of the reader. Some components can be found in [8, 27, 30].

LEMMA 2.2. *Let* $\Omega \subset \mathbb{R}^2$ *be an open bounded and connected domain with boundary* $\Gamma \in C^2$.

(1) *Assume* $v \in C^1(\overline{\Omega})$ *with* $v \cdot n = 0$ *on* $\partial\Omega$. *Writing* $\tau \cdot \nabla v \cdot n = \tau_k \partial_k v_j\, n_j$ *with Einstein's summation convention, we have*

$$\tau \cdot \nabla v \cdot n + \kappa\, v \cdot \tau = 0 \quad \text{on} \quad \Gamma, \tag{2.6}$$

*where* $\kappa$ *denotes the curvature of* $\Gamma$. *If each component of* $\Gamma$ *is parameterized by arc length* $s$, *then* $\frac{\partial n}{\partial \tau} = \frac{dn}{ds} = \kappa\tau$.

(2) *Assume that* $v \in C^1(\overline{\Omega})$ *satisfies the Navier boundary conditions*

$$v \cdot n = 0 \quad \text{and} \quad 2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha\, v \cdot \tau = g \cdot \tau \quad \text{on} \quad \Gamma. \tag{2.7}$$

*Then*

$$2n \cdot \mathbb{D}(v) \cdot \tau + 2\kappa(v \cdot \tau) = \omega \quad \text{on} \quad \Gamma \tag{2.8}$$

*and*

$$\omega = \left(2\kappa - \frac{\alpha}{\nu}\right)(v \cdot \tau) + \frac{1}{\nu} g \cdot \tau \quad \text{on} \quad \Gamma. \tag{2.9}$$

*Especially,* $\omega = g \cdot \tau$ *on* $\Gamma$ *if and only if* $\kappa = \frac{\alpha}{2}$.

*Proof of Lemma* 2.2. (1) Since $v \cdot n = 0$ on $\Gamma$, the directional derivative of $v \cdot n$ along $\Gamma$ is zero, that is,

$$0 = \frac{\partial(v \cdot n)}{\partial \tau} = \frac{\partial v}{\partial \tau} \cdot n + v \cdot \frac{\partial n}{\partial \tau} = 0 \quad \text{or} \quad \tau \cdot \nabla v \cdot n + v \cdot (\tau \cdot \nabla n) = 0 \quad \text{on} \quad \Gamma.$$

Moreover, $v = (v \cdot n)n + (v \cdot \tau)\tau = (v \cdot \tau)\tau$ on $\Gamma$. Therefore, due to $\kappa = \tau \cdot \nabla n \cdot \tau$, we get

$$\tau \cdot \nabla v \cdot n + (\tau \cdot \nabla n \cdot \tau)(v \cdot \tau) = 0 \quad \text{or} \quad \tau \cdot \nabla v \cdot n + \kappa\, v \cdot \tau = 0 \quad \text{on} \quad \Gamma.$$

(2) To prove (2.8), we recall $2\mathbb{D}(v) = \nabla v + (\nabla v)^T$ and use (2.6) to get

$$\begin{aligned} n \cdot \nabla v \cdot \tau &= 2n \cdot \mathbb{D}(v) \cdot \tau - n \cdot (\nabla v)^T \cdot \tau \\ &= 2n \cdot \mathbb{D}(v) \cdot \tau - \tau \cdot \nabla v \cdot n \\ &= 2n \cdot \mathbb{D}(v) \cdot \tau + \kappa(v \cdot \tau). \end{aligned} \tag{2.10}$$

On the other hand,

$$\nabla v = \mathbb{D}(v) + \frac{1}{2}\left(\nabla u - (\nabla v)^T\right) = \mathbb{D}(v) + \frac{1}{2}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Thus,

$$
\begin{aligned}
n \cdot \nabla v \cdot \tau &= n \cdot \mathbb{D}(v) \cdot \tau + n \cdot \frac{1}{2} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \cdot \tau \\
&= n \cdot \mathbb{D}(v) \cdot \tau + \frac{\omega}{2}(-\tau_1 n_2 + n_1 \tau_2) \\
&= n \cdot \mathbb{D}(v) \cdot \tau + \frac{\omega}{2}, \qquad (2.11)
\end{aligned}
$$

where we used $-\tau_1 n_2 + n_1\tau_2 = \tau_1^2 + \tau_2^2 = 1$. Combining (2.10) with (2.11) gives (2.8). As a result, (2.9) holds immediately from (2.7)–(2.8). This completes the proof. ∎

As we shall see in the following sections, $n \cdot \nabla v \cdot \tau$ plays a key role in dealing with the dissipation, and the identities stated here will be very helpful. Notice that $\tau \cdot \nabla v \cdot n$ is different from $n \cdot \nabla v \cdot \tau$ in general.

LEMMA 2.3. *Assume that $\Omega$ obeys the same conditions as in Lemma 2.2. Let $v, \psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying the Navier boundary conditions (2.7). Then*

$$
\int_\Omega \Delta v \cdot \psi \, dx = -2 \int_\Omega \mathbb{D}(v) \cdot \mathbb{D}(\psi) \, dx + \int_\Gamma \left( \frac{1}{\nu} g \cdot \tau \right) (\psi \cdot \tau) \, dx - \int_\Gamma \frac{\alpha}{\nu} (v \cdot \tau)(\psi \cdot \tau) \, dx. \qquad (2.12)
$$

*In particular, when $\psi = v$, we have*

$$
\int_\Omega \Delta v \cdot v \, dx = -2 \int_\Omega |\mathbb{D}(v)|^2 \, dx + \int_\Gamma \left( \frac{1}{\nu} g \cdot \tau \right) (v \cdot \tau) \, dx - \int_\Gamma \frac{\alpha}{\nu} (v \cdot \tau)^2 \, dx.
$$

*Proof of Lemma* 2.3. First, we know that

$$
\int_\Omega \Delta v \cdot \psi \, dx = - \int_\Omega \nabla v \cdot \nabla \psi \, dx + \int_\Gamma n \cdot \nabla v \cdot \psi \, dx.
$$

Since $v \cdot n = 0$ on $\Gamma$, we write $\psi = (\psi \cdot \tau)\tau$. By Lemma 2.2, we have

$$
\begin{aligned}
\int_\Gamma n \cdot \nabla v \cdot \psi \, dx &= \int_\Gamma n \cdot \nabla v \cdot \tau (\psi \cdot \tau) \, dx \\
&= \int_\Gamma \left( \frac{1}{\nu} g \cdot \tau \right) (\psi \cdot \tau) \, dx + \int_\Gamma \left( \kappa - \frac{\alpha}{\nu} \right) (v \cdot \tau)(\psi \cdot \tau) \, dx.
\end{aligned}
$$

Therefore,

$$
\int_\Omega \Delta v \cdot \psi \, dx = - \int_\Omega \nabla v \cdot \nabla \psi \, dx + \int_\Gamma \left( \frac{1}{\nu} g \cdot \tau \right) (\psi \cdot \tau) \, dx + \int_\Gamma \left( \kappa - \frac{\alpha}{\nu} \right) (v \cdot \tau)(\psi \cdot \tau) \, dx. \qquad (2.13)
$$

Next, we write out the terms in $\mathbb{D}(v) \cdot \mathbb{D}(\psi)$ using Einstein's summation convention:

$$
\begin{aligned}
2 \int_\Omega \mathbb{D}(v) \cdot \mathbb{D}(\psi) \, dx &= \int_\Omega \left( \nabla v \cdot \nabla \psi + \nabla v \cdot (\nabla \psi)^T \right) dx \\
&= \int_\Omega \nabla v \cdot \nabla \psi \, dx + \int_\Omega \partial_j v_k \partial_k \psi_j \, dx \\
&= \int_\Omega \nabla v \cdot \nabla \psi \, dx + \int_\Omega [\partial_k (\partial_j v_k \psi_j) - \partial_j \partial_k v_k \psi_j] \, dx \\
&= \int_\Omega \nabla v \cdot \nabla \psi \, dx + \int_\Gamma n_k \partial_j v_k \psi_j \, dx = \int_\Omega \nabla v \cdot \nabla \psi \, dx + \int_\Gamma \psi \cdot \nabla v \cdot n \, dx,
\end{aligned}
$$

where $\partial_j \partial_k v_k \psi_j = 0$ due to the divergence-free condition. Writing $\psi = (\psi \cdot \tau)\tau$ and applying Lemma 2.2, we have

$$2 \int_\Omega \mathbb{D}(v) \cdot \mathbb{D}(\psi)\, dx = \int_\Omega \nabla v \cdot \nabla \psi \, dx - \kappa \int_\Gamma (v \cdot \tau)(\psi \cdot \tau)\, dx. \tag{2.14}$$

Combining (2.13) and (2.14) yields (2.12). This completes the proof of Lemma 2.3. ◻

We now introduce the Stokes operator associated with Navier slip boundary conditions and identify the domains of its fractional powers. With the help of Lemmas 2.2–2.3, we define the bilinear form

$$a_0(v,\psi) = 2(\mathbb{D}(v), \mathbb{D}(\psi)) + \frac{\alpha}{\nu}\langle v, \psi\rangle, \quad v, \psi \in V_n^1(\Omega).$$

By Korn and Poincare's inequalities and the trace theorem, it is easy to check that

$$c_1 \|v\|_{H^1}^2 \le a_0(v,v) \le c_2 \|v\|_{H^1}^2$$

for some constants $c_1, c_2 > 0$. Thus, $a_0(\cdot,\cdot)$ is $H^1$-coercive. Define the operator $A : V_n^1(\Omega) \to (V_n^1(\Omega))'$ by

$$(Av, \psi) = a_0(v, \psi).$$

The Lax–Milgram theorem implies that $A \in \mathcal{L}(V_n^1(\Omega), (V_n^1(\Omega))')$. This also allows us to identify $A$ as an operator acting on $V_n^0(\Omega)$ with the domain

$$\mathscr{D}(A) = \{v \in V_n^1(\Omega) \colon \psi \mapsto a_0(v,\psi) \text{ is } L^2\text{-continuous}\}.$$

In fact, by (2.12), $A = -\mathbb{P}\Delta$ is the Stokes operator associated with the Navier slip boundary conditions, where $\mathbb{P}$ is the Leray projector on $L^2(\Omega)$ on the space $V_n^0(\Omega)$ [44, p. 13]. Note that $A$ is self-adjoint and strictly positive, and hence the fractal powers of $A$ are well-defined. Moreover, as proven in Lemma 2.2 of [8], operator $A$ has a countable set of positive eigenvalues $\{\lambda_j\}$ with $\lambda_j \to +\infty$ as $j \to +\infty$. The corresponding eigenfunctions $\{\psi^{(j)}\} \subset H^3(\Omega)$ form an orthonormal basis for $V_n^0(\Omega)$ and satisfy

$$\psi^{(j)} \cdot n = 0 \quad \text{and} \quad 2\nu n \cdot \mathbb{D}(\psi^{(j)}) \cdot \tau + \alpha \psi^{(j)} \cdot \tau = 0.$$

Therefore, we have, for $\alpha \ge 0$,

$$A^\sigma v = \sum_{j=1}^\infty \lambda_j^\sigma (v, \psi^{(j)})\psi^{(j)}, \quad v \in D(A^\sigma),$$

with domain

$$\mathscr{D}(A^\sigma) = \left\{ v \in V_n^0(\Omega) \colon \sum_{j=1}^\infty \lambda_j^{2\sigma} |(v, \psi^{(j)})|^2 < \infty \right\}. \tag{2.15}$$

Furthermore, the Poincaré inequality holds:

$$\|A^\alpha v\|_{L^2} \le \lambda_1^{\alpha-\beta} \|A^\beta v\|_{L^2}, \quad 0 \le \alpha \le \beta \le 1. \tag{2.16}$$

The domain of $A^\sigma$ with $0 \le \sigma \le 1$ defined in (2.15) can be made more explicit and identified with Sobolev spaces. We note that the domain of $A^\sigma$ with $0 \le \sigma < 3/4$ is different from that of $A^\sigma$ with $3/4 < \sigma \le 1$. The details are given by the following proposition.

PROPOSITION 2.4. *Consider the Stokes problem with Navier slip boundary conditions*

$$\nu A v = 0 \tag{2.17}$$

$$\nabla \cdot v = 0 \tag{2.18}$$

$$v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha v \cdot \tau)|_\Gamma = g \cdot \tau. \tag{2.19}$$

*The domains of $A^\sigma$ for $0 \leq \sigma \leq 1$ can be identified as follows:*

$$\mathscr{D}(A^\sigma) = V_n^{2\sigma}(\Omega), \quad 0 \leq \sigma < \frac{3}{4}, \quad \text{and}$$

$$\mathscr{D}(A^\sigma) = \{v \in V_n^{2\sigma}(\Omega) \colon (2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha v \cdot \tau)|_\Gamma = 0\}, \quad \frac{3}{4} < \sigma \leq 1.$$

*Proof of Proposition* 2.4. Following the approach in Lemma 2.2 of [8], we introduce $\omega^{(j)} = \nabla \times \psi^{(j)}$, where $\psi^{(j)}$ is the eigenfunction of $A$ associated with eigenvalue $\lambda_j$. Then with the help of (2.9) and setting $g = 0$, we know that $\omega^{(j)}$ is a solution to the following Dirichlet problem for the Laplacian $\tilde{A} = -\Delta$:

$$\tilde{A}\omega^{(j)} = \lambda_j \omega^{(j)} \tag{2.20}$$

$$\omega^{(j)}|_\Gamma = \left(2\kappa - \frac{\alpha}{\nu}\right)(\psi^{(j)} \cdot \tau). \tag{2.21}$$

Moreover, $\{\omega^{(j)}\}$ is a basis for $L^2(\Omega)$ and an orthonormal basis for $H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$. In other words, $\{\tilde{A}^{-1/2}\omega^{(j)}\}$ forms an orthonormal basis for $L^2(\Omega)$. By [33, 35], we can identify $D(\tilde{A}^\sigma)$ for $0 \leq \sigma \leq 1$, the domains of the fractional powers of the Laplacian with Dirichlet boundary conditions, as follows:

$$\mathscr{D}(\tilde{A}^\sigma) = H^{2\sigma}(\Omega), \quad 0 \leq \sigma < \frac{1}{4}, \quad \text{and} \tag{2.22}$$

$$\mathscr{D}(\tilde{A}^\sigma) = \left\{\omega \in H^{2\sigma}(\Omega) \colon \left(\omega - \left(2\kappa - \frac{\alpha}{\nu}\right)(v \cdot \tau)\right)|_\Gamma = 0 \text{ with } \omega = \nabla \times v\right\}, \quad \frac{1}{4} < \sigma \leq 1. \tag{2.23}$$

For the eigenfunctions $\psi^{(i)}$ and $\psi^{(j)}$ of $A$, we have

$$\begin{aligned} |(\psi^{(i)}, \psi^{(j)})| &\leq c|(\tilde{A}^{-1/2}(\nabla \times \psi^{(i)}), \tilde{A}^{-1/2}(\nabla \times \psi^{(j)}))| \\ &= c\lambda_i^{-1/2}|(\omega^{(i)}, \tilde{A}^{-1/2}\omega^{(j)})|. \end{aligned}$$

Thus, for every $v \in D(A^\sigma)$, we have

$$\sum_{j=1}^\infty \lambda_j^{2\sigma}|(v, \psi^{(j)})|^2 \leq c\sum_{j=1}^\infty \lambda_j^{2\sigma}\lambda_i^{-1}|(\omega, \tilde{A}^{-1/2}\omega^{(j)})|^2,$$

where $\omega = \nabla \times v$. Therefore, according to (2.22), we derive

$$\mathscr{D}(A^\sigma) = V_n^{2\sigma}(\Omega) \quad \text{for} \quad 2\sigma - 1 < \frac{1}{2} \quad \text{or} \quad \sigma < \frac{3}{4}.$$

Moreover, based on (2.23) and (2.8), we rewrite $(\omega - (2\kappa - \frac{\alpha}{\nu})(v \cdot \tau))|_\Gamma = 0$ as $(2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha v \cdot \tau)|_\Gamma = 0$; then we obtain

$$\mathscr{D}(A^\sigma) = \{v \in V_n^{2\sigma}(\Omega) \colon (2\nu n \cdot \mathbb{D}(v) \cdot \tau + \alpha v \cdot \tau)|_\Gamma = 0\} \quad \text{for} \quad \frac{3}{4} < \sigma \leq 1.$$

This completes the proof. ∎

It immediately follows for $v \in V_n^1(\Omega)$ that

$$c_1\|A^{1/2}v\|_{L^2} \le \|\mathbb{D}(v)\|_{L^2} \le c_2\|A^{1/2}v\|_{L^2}. \tag{2.24}$$

To handle the nonhomogeneous boundary conditions, we define the Navier slip boundary operator $N\colon L^2(\Gamma) \to V_n^0(\Omega)$ by

$$Ng = v \iff a_0(v,\psi) = \left\langle \frac{1}{\nu} g, \psi \right\rangle, \quad \psi \in V_n^1(\Omega).$$

By (2.12) in Lemma 2.3, we know that $v = Ng$ satisfies the Stokes problem (2.17)–(2.19). Furthermore, by Lemma 2.1 and Proposition 2.4, we have

$$N\colon L^2(\Gamma) \to V_n^{3/2}(\Omega) \subset V_n^{3/2-\epsilon}(\Omega) = \mathcal{D}(A^{3/4-\epsilon/2}), \quad \epsilon > 0.$$

This implies that

$$A^{3/4-\epsilon/2}N \in \mathcal{L}(L^2(\Gamma), V_n^0(\Omega)). \tag{2.25}$$

**3. Well-posedness of the optimal control problem and identification of the set of admissible controls.** To establish the well-posedness of problem (P), we first derive the conditions such that $\theta \in C([0,T];(H^1(\Omega))')$. For this purpose, we recall some basic properties of the scalar equation. Taking the inner product of (1.1) with $\theta$ and using divergence-free condition and boundary condition (1.7) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 &= -\int_\Omega (v\cdot\nabla\theta)\theta\,dx = -\frac{1}{2}\int_\Omega v\cdot\nabla(\theta^2)\,dx\\ &= \frac{1}{2}\int_\Omega(\nabla\cdot v)\theta^2\,dx - \int_\Gamma (v\cdot n)\theta\,dx = 0,\end{aligned}$$

from where

$$\|\theta(t)\|_{L^2} = \|\theta_0\|_{L^2}, \quad t \ge 0. \tag{3.1}$$

In fact, it can be verified that any $L^p$-norm of $\theta$ is conserved, i.e.,

$$\|\theta(t)\|_{L^p(\Omega)} = \|\theta_0\|_{L^p(\Omega)}, \quad t \ge 0, \quad p \in [1,\infty], \tag{3.2}$$

if no-penetration boundary condition $v\cdot n|_\Gamma = 0$ is imposed on the velocity field [3, 25]. Here $n$ denotes the outward unit normal vector with respect to the domain $\Omega$. To have $\theta \in C([0,T];(H^1(\Omega))')$, it suffices to have $v\cdot\nabla\theta \in L^1(0,T;(H^1(\Omega))')$ in terms of (1.1). By using (1.9) and (3.2), we get

$$\begin{aligned}\|v\cdot\nabla\theta\|_{(H^1(\Omega))'} &= \sup_{\phi\in H^1(\Omega)} \frac{|\int_\Omega v\cdot\nabla\theta\phi\,dx|}{\|\phi\|_{H^1}}\\ &\le \sup_{\phi\in H^1(\Omega)} \frac{|\int_\Omega v\cdot\nabla(\theta\phi)\,dx - \int_\Omega v\theta\cdot\nabla\phi\,dx|}{\|\phi\|_{H^1}}\\ &\le c\sup_{\phi\in H^1(\Omega)} \frac{\|v\|_{L^2}\|\theta\|_{L^\infty}\|\phi\|_{H^1}}{\|\phi\|_{H^1}} = c\|v\|_{L^2}\|\theta\|_{L^\infty}\\ &= c\|v\|_{L^2}\|\theta_0\|_{L^\infty},\end{aligned}$$

which yields

$$\int_0^T \|v\cdot\nabla\theta\|_{(H^1(\Omega))'}\,dt \le c\int_0^T \|v\|_{L^2}\,dt\,\|\theta_0\|_{L^\infty}\,dt.$$

Next, we identify the initial and boundary conditions of the velocity field such that $\int_0^T \|\nabla \times v\|_{L^2}^2 \, dt < \infty$. We first consider the passive scalar case described by Model I. The active scalar case will be handled similarly in section 5. With the help of Lemma 2.2, we know that vorticity $\omega = \nabla \times v$ satisfies

$$\partial_t \omega - \nu \Delta \omega + v \cdot \nabla \omega = 0 \tag{3.3}$$

$$\omega|_\Gamma = \left(2\kappa - \frac{\alpha}{\nu}\right)(v \cdot \tau) + \frac{1}{\nu} g \cdot \tau. \tag{3.4}$$

However, instead of estimating $\int_0^T \|\nabla \times v\|_{L^2}^2 \, dt$ based on (3.3)–(3.4), we use the fact that $\int_0^T \|\nabla \times v\|_{L^2}^2 \, dt \leq \int_0^T \|\nabla v\|_{L^2}^2 \, dt$ and establish an a priori estimate on $v$. Applying the $L^2$-estimate to the velocity equation (1.2) and then making use of Poincaré inequality (2.16) and Lemma 2.3 yields

$$\begin{aligned} \frac{1}{2} \frac{d\|v\|_{L^2}^2}{dt} + \nu \|\mathbb{D} v\|_{L^2}^2 + \alpha \|v\|_{L^2(\Gamma)}^2 &= \langle g, v \rangle \leq \|g\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \\ &\leq c \|g\|_{L^2(\Gamma)} \|v\|_{H^{1/2}} \leq c \lambda_1^{-1/4} \|g\|_{L^2(\Gamma)} \|A^{1/2} v\|_{L^2}. \end{aligned}$$

Using (2.24), we get

$$\frac{1}{2} \frac{d\|v\|_{L^2}^2}{dt} + c\nu \|A^{1/2} v\|_{L^2}^2 + \alpha \|v\|_{L^2(\Gamma)}^2 \leq c \lambda_1^{-1/2} \|g\|_{L^2(\Gamma)}^2 + \frac{c\nu}{2} \|A^{1/2} v\|_{L^2}^2.$$

Thus,

$$\frac{d\|v\|_{L^2}^2}{dt} + c\nu \|A^{1/2} v\|_{L^2}^2 + 2\alpha \|v\|_{L^2(\Gamma)}^2 \leq c \lambda_1^{-1/2} \|g\|_{L^2(\Gamma)}^2, \tag{3.5}$$

which implies

$$\frac{d\|v\|_{L^2}^2}{dt} + c\nu \lambda_1 \|v\|_{L^2}^2 + 2\alpha \|v\|_{L^2(\Gamma)}^2 \leq c \lambda_1^{-1/2} \|g\|_{L^2(\Gamma)}^2.$$

Assume for every $T > 0$ that

$$\sup_{t \in [0,T]} \|g\|_{L^2(\Gamma)} \leq M_1 \tag{3.6}$$

for some $M_1 > 0$. Further, we may assume without loss of generality that

$$\|v_0\|_{L^2} \leq M_0 = c \left( \frac{\lambda_1^{-3/2}}{\nu} \right)^{1/2} M_1;$$

then

$$\sup_{t \in [0,T]} \|v\|_{L^2(\Omega)} \leq M_0 \tag{3.7}$$

and

$$\int_0^T \|v\|_{L^2}^2 \, dt \leq c \left( \int_0^T \|g\|_{L^2(\Gamma)}^2 \, dt + M_0 \right), \tag{3.8}$$

while (3.5) gives

$$\int_t^{t+1} \|A^{1/2} v\|_{L^2}^2 \, d\tau \leq C(M_0, M_1).$$

This also indicates that

$$\int_0^T \|\nabla \times v\|_{L^2}^2\, dt \le c \int_0^T \|A^{1/2} v\|_{L^2}^2\, dt \le c(\|g\|^2_{L^2(0,T;L^2(\Gamma))} + \|v_0\|^2_{L^2}). \tag{3.9}$$

Thus, $J$ is bounded for $g \in L^2(0,T;V_n^0(\Gamma))$ and $v_0 \in V_n^0(\Omega)$.

We now determine the initial and boundary conditions such that the Gâteaux derivative of $J$ with respect to the control input $g$ is well-defined. To this end, we first establish the well-posedness of the Gâteaux derivative of $\theta$ with respect to $g$. This lemma was presented in [23]. To be self-contained, we provide a complete proof.

LEMMA 3.1. *For given $\theta_0 \in H^1(\Omega)$, $v$ and $y$ satisfying*

$$\int_0^T \|\nabla v\|_{L^\infty}\, dt < \infty \quad \textit{and} \quad \int_0^T \|y\|_{L^\infty}\, dt < \infty, \tag{3.10}$$

*there exists a unique solution to the linear transport problem*

$$\frac{\partial z}{\partial t} + y \cdot \nabla \theta + v \cdot \nabla z = 0 \tag{3.11}$$

$$z(0) = 0, \tag{3.12}$$

*and $z \in L^\infty(0,T;L^2(\Omega))$, where $\theta$ satisfies* (1.1) *and $\theta(0) = \theta_0$.*

*Proof of Lemma* 3.1. To derive the well-posedness of (3.11)–(3.12), it suffices to show that $y \cdot \nabla\theta \in L^1(0,T;L^2(\Omega))$ [13]. Note that for $\theta_0 \in H^1(\Omega)$ and $v$ satisfying (3.10), there exists a unique solution $\theta \in L^\infty(0,T;H^1(\Omega))$ satisfying (1.1) and $\theta(0) = \theta_0$. In fact, as shown in [25], applying $\nabla$ to the density equation (1.1) and taking the inner product with $\nabla\theta$ yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 &= -\int_\Omega \partial_j(v_i\partial_i\theta)\partial_j\theta\, dx = -\int_\Omega \partial_j v_i \partial_i\theta\partial_j\theta\, dx - \frac{1}{2}\int_\Omega v_i\partial_i(\partial_j\theta\partial_j\theta)\, dx \\
&= -\int_\Omega \partial_j v_i\partial_i\theta\partial_j\theta\, dx + \frac{1}{2}\int_\Omega \partial_i v_i \partial_j\theta\partial_j\theta\, dx - \frac{1}{2}\int_\Gamma v_i n_i \partial_j\theta\partial_j\theta\, dx \\
&= -\int_\Omega \partial_j v_i\partial_i\theta\partial_j\theta\, dx \le \|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^2}^2.
\end{aligned}$$

For $\theta_0 \in H^1(\Omega)$, using the Gronwall inequality and (3.10) gives

$$\sup_{t\in[0,T]} \|\nabla\theta\|_{L^2} \le c\|\nabla\theta_0\|_{L^2} e^{\int_0^T \|\nabla v\|_{L^\infty}\, dt} < \infty. \tag{3.13}$$

Thus, by (3.10) and (3.13), we obtain

$$\begin{aligned}
\int_0^T \|y\cdot\nabla\theta\|_{L^2}\, dt &\le \int_0^T \|y\|_{L^\infty}\|\nabla\theta\|_{L^2}\, dt \\
&\le \|y\|_{L^1(0,T;L^\infty(\Omega))}\|\nabla\theta\|_{L^\infty(0,T;L^2(\Omega))} < \infty.
\end{aligned} \tag{3.14}$$

To show that $z \in L^\infty(0,T;L^2(\Omega))$, we take inner product of (3.11) with $z$ and obtain

$$\frac{1}{2}\frac{d\|z\|_{L^2}^2}{dt} = -\int_\Omega (y\cdot\nabla\theta)z\, dx - \int_\Omega (v\cdot\nabla z)z\, dx \le \|y\|_{L^\infty}\|\nabla\theta\|_{L^2}\|z\|_{L^2}.$$

By the Gronwall inequality and (3.14), we get

$$\sup_{t\in[0,T]}\|z\|_{L^2} \le c\int_0^T \|y\|_{L^\infty}\|\nabla\theta\|_{L^2}\, dt < \infty.$$

This completes the proof. ∎

As we see in Lemma 3.1, to derive the Gâteaux derivative of $\theta$ with respect $g$, it is crucial to identify the initial and boundary conditions of the velocity field such that

$$\int_0^T \|\nabla v\|_{L^\infty}\,dt < \infty. \tag{3.15}$$

This indicates that to establish an a priori estimate (3.15), we need the time regularity of $g$. For computational convenience, we consider the first derivative $\partial g/\partial t$ rather than the lower-order fractional time derivative in the cost functional.

**4. Case I: Optimal mixing of a passive scalar via Navier–Stokes flow.** In this section, we focus on the passive scalar case governed by Model I. We first introduce the set of admissible controls and then prove the existence of an optimal control. Using a variational inequality, we derive the first-order necessary conditions of optimality and establish its well-posedness.

**4.1. Well-posedness of the Navier–Stokes equations with nonhomogeneous Navier slip boundary conditions.** Due to one-way coupling, it is critical to understand the problem of Navier slip boundary control for Navier–Stokes equations. To have (3.15) satisfied without going to the state spaces of high regularity, we provide a sharp estimate based on the special decomposition analysis used in [29].

Define the bilinear operator $B\colon V_n^1(\Omega) \to (V_n^1(\Omega))'$ by

$$B(v,w) = \mathbb{P}(v\cdot\nabla w) \quad \forall v,w \in V_n^1(\Omega).$$

We set

$$U_{ad} = \left\{ g \in L^2(0,T;V_n^{1/2+\epsilon}(\Gamma)) : \frac{\partial g}{\partial t} \in L^2(0,T;V_n^0(\Gamma)) \right\}, \quad \epsilon > 0, \tag{4.1}$$

equipped with the norm

$$\|g\|_{U_{ad}} = \|g\|_{L^2(0,T;V_n^{1/2+\epsilon}(\Gamma))} + \left\|\frac{\partial g}{\partial t}\right\|_{L^2(0,T;V_n^0(\Gamma))}. \tag{4.2}$$

THEOREM 4.1. *Assume that $v_0 \in V_n^1(\Omega)$ and $g \in U_{ad}$ with $\|v_0\|_{H^1} \le M_0$ and $\|g\|_{U_{ad}} \le M_1$, where $M_0, M_1 > 0$ are arbitrary. Then there exists a unique global solution $v$ such that $v \in L^\infty(0,\infty); V_n^1(\Omega)) \cap L^2_{loc}(0,\infty); V_n^2(\Omega))$. Moreover, for each $T > 0$, we have*

$$\int_0^T \|\nabla v\|_{L^\infty}\,dt \le C(M_0, M_1, T).$$

To deal with the nonhomogeneous boundary conditions, we first make a change of variable. Let $w = v - Ng$. Then

$$\begin{aligned} \frac{\partial w}{\partial t} &= \nu\Delta w - w\cdot\nabla(Ng) - (Ng)\cdot\nabla w - w\cdot\nabla w - \nabla p \\ &\quad - (Ng)\cdot\nabla(Ng) - \frac{\partial(Ng)}{\partial t} \end{aligned} \tag{4.3}$$

$$\nabla\cdot w = 0 \tag{4.4}$$

with the Navier slip boundary conditions

$$w\cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n\cdot\mathbb{D}(v)\cdot\tau + \alpha v\cdot\tau)|_\Gamma = 0 \tag{4.5}$$

and initial condition

$$w(0) = v_0 - Ng(0). \tag{4.6}$$

Employing the similar idea in [29], we define the spectral projection operators $\{P_m\}_{m=1}^{\infty}$ as

$$P_1 v = \sum_{\lambda_j<2} (v,\psi^{(j)})\psi^{(j)}, \quad P_m v = \sum_{2^{m-1}\le\lambda_j<2^m} (v,\psi^{(j)})\psi^{(j)}, \quad m\ge 2.$$

Thus,

$$2^{(m-1)s}\|P_m v\|_2 \le \|P_m A^s v\|_2 = \|A^s P_m v\| \le 2^{ms}\|P_m v\|_2, \quad m\ge 1, \quad s\ge 0. \tag{4.7}$$

We first introduce the following lemma to address the regularity issue of the translated system (4.3)–(4.6).

Lemma 4.2. *Assume that* $w_0\in V_n^1(\Omega)$ *and* $g\in H^1(0,T;V_n^{1/2}(\Gamma))$ *with* $\|w_0\|_{H^1}\le \|v_0\|_{H^1}+\|Ng(0)\|_{H^1}$ *and*

$$\|g\|_{H^1(0,T;V_n^{1/2}(\Gamma))} = \|g\|_{L^2(0,T;V_n^{1/2}(\Gamma))} + \left\|\frac{\partial g}{\partial t}\right\|_{L^2(0,T;V_n^0(\Gamma))} \le M_1, \tag{4.8}$$

*where* $M_1>0$ *is arbitrary. Then there exists a unique global solution* $w$ *such that*

$$w\in L^\infty(0,\infty;V_n^1(\Omega))\cap L^2_{loc}(0,\infty;V_n^2(\Omega)).$$

*Moreover, for each* $T>0$, *we have*

$$\int_0^T \|\nabla w(t)\|_{L^\infty}\,dt \le C(M_0,M_1,T).$$

*Proof of Lemma* 4.2. By (2.25), we have $N\in\mathcal{L}(L^2(\Gamma),V_n^0(\Gamma))$. Further, due to (3.6)–(3.8), it is clear that

$$\sup_{t\in[0,T]}\|w\|_{L^2} \le \sup_{t\in[0,T]}\|v\|_{L^2} + \sup_{t\in[0,T]}\|Ng\|_{L^2} \le C(M_0,M_1) \tag{4.9}$$

$$\int_0^T \|w\|_{L^2}^2\,dt \le 2\int_0^T \|v\|_{L^2}^2\,dt + 2\int_0^T \|Ng\|_{L^2}^2\,dt \le C(M_0,M_1), \tag{4.10}$$

and

$$\begin{aligned}\int_t^{t+1}\|A^{1/2}w\|_{L^2}\,d\tau &\le \int_t^{t+1}\|A^{1/2}v\|_{L^2}\,d\tau + \int_t^{t+1}\|A^{1/2}Ng\|_{L^2}\,d\tau\\ &\le C(M_0,M_1).\end{aligned} \tag{4.11}$$

To estimate $\|\nabla w(t)\|_{L^\infty}$, we recall Agmon's inequality that for $w\in D(A)$,

$$\begin{aligned}\|\nabla P_m w\|_\infty &\le c\|\nabla P_m w\|_{L^2}^{1/2}\|A\nabla P_m w\|_{L^2}^{1/2} \le c2^{m/2}\|\nabla P_m w\|_{L^2}\\ &= c2^{m/2}\|A^{1/2}P_m w\|_{L^2} \le c2^m\|P_m w\|_{L^2},\end{aligned} \tag{4.12}$$

where

$$(P_m Aw, P_m w) = \sum_{2^{m-1}\le\lambda_j<2^m} \lambda_j|(w,\psi^{(j)})|^2 \ge 2^{m-1}\|P_m w\|_{L^2}^2 \quad \forall m\ge 1. \tag{4.13}$$

Applying $P_m$ to (4.3) and then taking the inner product with $P_m v$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|P_m w\|_{L^2}^2 + \nu(P_m Aw, P_m w)\\ &\quad= (P_m\mathbb{P}(-w\cdot\nabla(Ng) - (Ng)\cdot\nabla w - w\cdot\nabla w - (Ng)\cdot\nabla(Ng)), P_m w)\\ &\qquad - \left(P_m\mathbb{P}\frac{\partial(Ng)}{\partial t}, P_m w\right).\end{aligned}$$

According to (4.13),

$$\frac{d}{dt}\|P_m w\|_{L^2} + 2^{m-1}\nu\|P_m w\|_{L^2}$$
$$\le \|P_m\mathbb{P}(-w\cdot\nabla(Ng) - (Ng)\cdot\nabla w - w\cdot\nabla w - (Ng)\cdot\nabla(Ng))\|_{L^2}$$
$$+ \|P_m\mathbb{P}\frac{\partial(Ng)}{\partial t}\|_2. \tag{4.14}$$

Let

$$I = -w\cdot\nabla(Ng) - (Ng)\cdot\nabla w - w\cdot\nabla w - (Ng)\cdot\nabla(Ng).$$

Integrating with respect to $t$ on both sides of (4.14) gives

$$2\|P_m w\|_2 + \int_0^T 2^m\nu\|P_m w\|_{L^2}\,dt \le 2\|P_m w_0\|_2 + 2\int_0^T \|P_m\mathbb{P}I\|_{L^2}\,dt$$
$$+ 2\int_0^T \|P_m N\dot{g}\|_{L^2}\,dt. \tag{4.15}$$

Multiplying (4.15) by $2^{\frac{m}{2}}$ and using (4.7), we get

$$\int_0^T 2^{m+\frac{m}{2}}\nu\|P_m w\|_{L^2}\,dt \le c\|A^{1/2}P_m w_0\|_2 + c\int_0^T \|A^{1/2}P_m\mathbb{P}I\|_{L^2}\,dt$$
$$+ c\int_0^T \|A^{1/2}P_m N\dot{g}\|_{L^2}\,dt. \tag{4.16}$$

Squaring both sides of (4.16) and summing up with respect to $m$ yields

$$\sum_{m=1}^{\infty}\left(\int_0^T 2^{m+\frac{m}{2}}\nu\|P_m w\|_{L^2}\,dt\right)^2 \le c\sum_{m=1}^{\infty}\|A^{1/2}P_m w_0\|_2^2$$
$$+ c\sum_{m=1}^{\infty}\left(\int_0^T \|A^{1/2}P_m\mathbb{P}I\|_2\,dt\right)^2 + c\sum_{m=1}^{\infty}\left(\int_0^T \|A^{1/2}P_m N\dot{g}\|_{L^2}\,dt\right)^2$$
$$= c\|A^{1/2}w_0\|_2^2 + c\sum_{m=1}^{\infty}\left(\int_0^T \|A^{1/2}P_m\mathbb{P}I\|_2\,dt\right)^2$$
$$+ c\sum_{m=1}^{\infty}\left(\int_0^T \|A^{1/2}P_m N\dot{g}\|_{L^2}\,dt\right)^2.$$

Next, taking the square root of the above inequality follows:

$$\left[\sum_{m=1}^{\infty}\left(\int_0^T 2^{m+\frac{m}{2}}\nu\|P_m w\|_{L^2}\,dt\right)^2\right]^{1/2} \le c\|A^{1/2}w_0\|_{L^2}$$
$$+ c\left[\sum_{m=1}^{\infty}\left(\int_0^T \|A^{1/2}P_m\mathbb{P}I\|_{L^2}\,dt\right)^2\right]^{1/2} + c\left[\sum_{m=1}^{\infty}\left(\int_0^T \|A^{1/2}P_m N\dot{g}\|_{L^2}\,dt\right)^2\right]^{1/2}$$
$$\le c\|A^{1/2}w_0\|_{L^2} + c\int_0^T\left(\sum_{m=1}^{\infty}\|A^{1/2}P_m\mathbb{P}I\|_2^2\right)^{1/2}dt + c\int_0^T\left(\sum_{m=1}^{\infty}\|A^{1/2}P_m N\dot{g}\|_{L^2}^2\right)^{1/2}dt$$
$$\le c\|A^{1/2}w_0\|_{L^2} + c\int_0^T \|A^{1/2}\mathbb{P}I\|_{L^2}\,dt + c\int_0^T \|A^{1/2}N\dot{g}\|_{L^2}\,dt. \tag{4.17}$$

Therefore, by (4.12) and (4.17), we have

$$
\begin{aligned}
\nu \int_0^T \|\nabla w\|_{L^\infty}\, dt &\le \nu \int_0^T \sum_{m=1}^\infty \|\nabla P_m w\|_\infty\, dt = \sum_{m=1}^\infty 2^{-\frac{m}{2}} \int_0^T 2^{m+\frac{m}{2}} \nu \|P_m w\|_{L^2}\, dt \\
&\le \left[ \sum_{m=1}^\infty \left( \int_0^T 2^{m+\frac{m}{2}} \nu \|P_m w\|_{L^2}\, dt \right)^2 \right]^{1/2} \\
&\le c(\|A^{1/2} w_0\|_{L^2} + \int_0^T \|A^{1/2}\mathbb{P} I\|_{L^2}\, dt + \int_0^T \|A^{1/2} N\dot{g}\|_{L^2}\, dt),
\end{aligned} \tag{4.18}
$$

where, by (2.25),

$$
\int_0^T \|A^{1/2} N\dot{g}\|_{L^2}\, dt \le c \int_0^T \|\dot{g}\|_{L^2}\, dt.
$$

In addition,

$$
\begin{aligned}
\int_0^T \|A^{1/2}\mathbb{P} I\|_2\, dt = &\int_0^T \|A^{1/2}(w\cdot\nabla(Ng))\|_{L^2}\, dt + \int_0^T \|A^{1/2}(Ng\cdot\nabla w)\|_{L^2}\, dt \\
&+ \int_0^T \|A^{1/2}(w\cdot\nabla w)\|_{L^2}\, dt + \int_0^T \|A^{1/2}(Ng\cdot\nabla(Ng))\|_{L^2}\, dt.
\end{aligned} \tag{4.19}
$$

To analyze (4.19), we employ the following inequality [44]:

$$
\|\nabla B(v,w)\|_{L^2} \le c\|v\|_{L^2}^{1/4}\|v\|_{H^2}^{3/4}\|w\|_{L^2}^{1/4}\|w\|_{H^2}^{3/4} + c\|v\|_{L^2}^{1/2}\|v\|_{H^2}^{1/2}\|w\|_{H^2}, \quad v,w\in V_n^2(\Omega). \tag{4.20}
$$

Next, we estimate each term on the right-hand side of (4.19) by using (4.20). First, with the help of (4.10) and (4.20), we get

$$
\begin{aligned}
\int_0^T \|A^{1/2}\mathbb{P}(w\cdot\nabla w)\|_{L^2}\, dt &\le c\int_0^T \|w\|_{L^2}^{1/2}\|Aw\|_{L^2}^{3/2}\, dt \\
&\le c\left(\int_0^T \|w\|_{L^2}^2\, dt\right)^{1/4}\left(\int_0^T \|Aw\|_{L^2}^2\, dt\right)^{3/4} \\
&\le C(M_0,M_1)\left(\int_0^T \|Aw\|_{L^2}^2\, dt\right)^{3/4}.
\end{aligned} \tag{4.21}
$$

According to assumption (4.8), it follows that

$$
\begin{aligned}
&\int_0^T \|A^{1/2}\mathbb{P}(w\cdot\nabla(Ng))\|_{L^2}\, dt \\
&\quad\le c\int_0^T (\|w\|_{L^2}^{1/4}\|Aw\|_{L^2}^{3/4}\|Ng\|_{L^2}^{1/4}\|Ng\|_{H^2}^{3/4} + \|w\|_{L^2}^{1/2}\|Aw\|_{L^2}^{1/2}\|Ng\|_{H^2})\, dt \\
&\quad\le c\int_0^T \|Aw\|_{L^2}\|Ng\|_{H^2}\, dt \le c\left(\int_0^T \|Aw\|_{L^2}^2\, dt\right)^{1/2}\left(\int_0^T \|Ng\|_{H^2}^2\, dt\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\le c\left(\int_0^T \|Aw\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^T \|g\|_{H^{1/2}}^2\,dt\right)^{1/2}\\
&\le C(M_1)\left(\int_0^T \|Aw\|_{L^2}^2\,dt\right)^{1/2}.
\end{aligned}
\tag{4.22}
$$

Similarly,

$$
\begin{aligned}
\int_0^T \|A^{1/2}\mathbb{P}((Ng)\cdot\nabla w)\|_{L^2}\,dt &\le c\left(\int_0^T \|g\|_{L^2}^2\,dt\right)^{1/2}\left(\int_0^T \|Aw\|_{L^2}^2\,dt\right)^{1/2}\\
&\le C(M_1)\left(\int_0^T \|Aw\|_{L^2}^2\,dt\right)^{1/2}
\end{aligned}
\tag{4.23}
$$

and

$$
\begin{aligned}
\int_0^T \|A^{1/2}\mathbb{P}(Ng\cdot\nabla(Ng))\|_{L^2}\,dt &\le c\left(\int_0^T \|Ng\|_{L^2}^2\,dt\right)^{1/4}\left(\int_0^T \|Ng\|_{H^2}^2\,dt\right)^{3/4}\\
&\le c\left(\int_0^T \|g\|_{L^2(\Gamma)}^2\,dt\right)^{1/4}\left(\int_0^T \|g\|_{H^{1/2}(\Gamma)}^2\,dt\right)^{3/4} \le C(M_1).
\end{aligned}
\tag{4.24}
$$

As we can see from (4.21)–(4.24), it remains to estimate $\int_0^T \|Aw\|_{L^2}^2\,dt$. Taking the inner product of the velocity equation (4.3) with $Aw$ gives

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(w,Aw)+\nu\|Aw\|_{L^2}^2 &= -(\mathbb{P}(w\cdot\nabla(Ng)),Aw)-(\mathbb{P}((Ng)\cdot\nabla w),Aw)\\
&\quad -(\mathbb{P}(w\cdot\nabla w),Aw)-(\mathbb{P}((Ng)\cdot\nabla(Ng)),Aw)-(\mathbb{P}\frac{\partial(Ng)}{\partial t},Aw)\\
&\le \|w\|_{L^\infty}\|\nabla(Ng)\|_{L^2}\|Aw\|_{L^2}+\|Ng\|_{L^\infty}\|\nabla w\|_{L^2}\|Aw\|_{L^2}\\
&\quad +\|w\|_{L^2}^{1/2}\|A^{1/2}w\|_{L^2}\|Aw\|_{L^2}^{3/2}+\|Ng\|_{L^\infty}\|\nabla(Ng)\|_{L^2}\|Aw\|_{L^2}+\|N\dot g\|_{L^2}\|Aw\|_{L^2} \qquad (4.25)\\
&\le c\|w\|_{L^2}^{1/2}\|g\|_{L^2(\Gamma)}\|Aw\|_{L^2}^{3/2}+c\|Ng\|_{H^{1+\epsilon}}\|\nabla w\|_{L^2}\|Aw\|_{L^2}\\
&\quad +c\|w\|_{L^2}^{1/2}\|A^{1/2}w\|_{L^2}\|Aw\|_{L^2}^{3/2}+c\|Ng\|_{H^{1+\epsilon}}\|g\|_{L^2(\Gamma)}\|Aw\|_{L^2}+c\|N\dot g\|_{L^2}\|Aw\|_{L^2} \qquad (4.26)\\
&\le c\|w\|_{L^2}^2\|g\|_{L^2(\Gamma)}^4+\frac{\nu}{10}\|Aw\|_{L^2}^2+c\|g\|_{L^2(\Gamma)}^2\|A^{1/2}w\|_{L^2}^2+\frac{\nu}{10}\|Aw\|_{L^2}^2\\
&\quad +c\|w\|_{L^2}^2\|A^{1/2}w\|_{L^2}^4+\frac{\nu}{10}\|Aw\|_{L^2}^2+c\|g\|_{L^2(\Gamma)}^4+\frac{\nu}{10}\|Aw\|_{L^2}^2\\
&\quad +c\|\dot g\|_{L^2(\Gamma)}^2+\frac{\nu}{10}\|Aw\|_{L^2}^2,
\end{aligned}
$$

where, from (4.25) to (4.26), we used

$$
\|\psi\|_{L^\infty}\le c\|\psi\|_{H^{1+\epsilon}}, \text{ for } d=2 \quad\text{and}\quad 0<\epsilon<\frac{1}{2}.
\tag{4.27}
$$

Further, note that

$$
\frac{1}{2}\frac{d}{dt}(w,Aw)=\frac{1}{2}\frac{d}{dt}\left(2\|\mathbb{D}(w)\|_{L^2}^2+\frac{\alpha}{\nu}\|w\|_{L^2(\Gamma)}^2\right).
$$

Thus,

$$\begin{aligned}
\frac{d}{dt}\Big(\|\mathbb{D}(w)\|_{L^2}^2 + \frac{\alpha}{2\nu}\|w\|_{L^2(\Gamma)}^2\Big) + \nu\|Aw\|_{L^2}^2 &\le c\|w\|_{L^2}^2\|g\|_{L^2(\Gamma)}^4 + c\|g\|_{L^2(\Gamma)}^2\|A^{1/2}w\|_{L^2}^2 \\
&\quad + c\|w\|_{L^2}^2\|A^{1/2}w\|_{L^2}^4 + c\|g\|_{L^2(\Gamma)}^4 + c\|\dot g\|_{L^2(\Gamma)}^2 \\
&\le c\big[\|w\|_{L^2}^2\|g\|_{L^2(\Gamma)}^4 + \|g\|_{L^2(\Gamma)}^2(\|\mathbb{D}w\|_{L^2}^2 + \frac{\alpha}{2\nu}\|w\|_{L^2(\Gamma)}^2) \\
&\quad + \|w\|_{L^2}^2\|\mathbb{D}w\|_{L^2}^2(\|\mathbb{D}w\|_{L^2}^2 + \frac{\alpha}{2\nu}\|w\|_{L^2(\Gamma)}^2) + \|g\|_{L^2(\Gamma)}^4 + \|\dot g\|_{L^2(\Gamma)}^2\big]. \qquad (4.28)
\end{aligned}$$

With the help of (3.6), (4.9), and (4.11), we have

$$\int_t^{t+1} (\|g\|_{L^2(\Gamma)}^2 + \|w\|_{L^2}^2\|\mathbb{D}w\|_{L^2}^2)\,d\tau < C(M_0, M_1), \quad t \ge 0,$$

and

$$\int_t^{t+1} (\|w\|_{L^2}^2\|g\|_{L^2(\Gamma)}^4 + \|g\|_{L^2(\Gamma)}^4\|\dot g\|_{L^2(\Gamma)}^2)\,d\tau < C(M_0, M_1), \quad t \ge 0.$$

Moreover, note that $\sup_{t\in[0,T]}\|g(t)\|_{L^2(\Gamma)} \le c\|g\|_{H^1(0,T;L^2(\Gamma))}$. Thus,

$$\begin{aligned}
\|\mathbb{D}w_0\|_{L^2}^2 \le c\|A^{1/2}w_0\|_{L^2}^2 &\le c(\|A^{1/2}v_0\|_{L^2}^2 + \|A^{1/2}Ng(0)\|_{L^2}^2) \\
&\le c(\|v_0\|_{H^1}^2 + \|g(0)\|_{L^2(\Gamma)}^2) \le C(M_0, M_1).
\end{aligned}$$

Using the uniform Gronwall inequality to (4.28) gives

$$\|A^{1/2}w\|_{L^2}^2 \le c\|\mathbb{D}w\|_{L^2}^2 \le C(M_0, M_1), \quad t \ge 0, \qquad (4.29)$$

and hence also

$$\int_0^T \|Aw\|_{L^2}^2\,dt \le C(M_0, M_1, T). \qquad (4.30)$$

Finally, in light of (4.18), (4.21)–(4.24), and (4.30), we get

$$\int_0^T \|\nabla w\|_{L^\infty}\,dt \le C(M_0, M_1, T) \qquad (4.31)$$

for every $T > 0$. This completes the proof. ∎

*Proof of Theorem* 4.1. First, according to (4.29), we have

$$\begin{aligned}
\|A^{1/2}v\|_{L^2}^2 &\le 2\|A^{1/2}w\|_{L^2}^2 + 2\|A^{1/2}Ng\|_{L^2}^2 \le 2\|A^{1/2}w\|_{L^2}^2 + 2c\|g\|_{L^2(\Gamma)}^2 \\
&\le C(M_0, M_1), \quad t \ge 0.
\end{aligned}$$

Furthermore, by (4.30),

$$\begin{aligned}
\int_0^T \|v\|_{H^2}^2\,dt &\le 2\int_0^T \|Aw\|_{L^2}^2\,dt + 2\int_0^T \|Ng\|_{H^2}^2\,dt \\
&\le 2\int_0^T \|Aw\|_{L^2}^2\,dt + 2c\int_0^T \|g\|_{H^{1/2}(\Gamma)}^2\,dt \\
&\le C(M_0, M_1, T). \qquad (4.32)
\end{aligned}$$

To estimate $\int_0^T \|\nabla v\|_{L^\infty}\, dt$, we apply (2.25) and (4.27) again and obtain

$$\int_0^T \|\nabla(Ng)\|_{L^\infty}\, dt \le c\int_0^T \|Ng\|_{H^{2+\epsilon}}\, dt \le c\int_0^T \|g\|_{H^{1/2+\epsilon}(\Gamma)}\, dt, \quad \epsilon > 0. \tag{4.33}$$

In light of (4.31) and (4.33), to simplify the notation, we still assume that

$$\|g\|_{U_{ad}} \le M_1.$$

Then

$$\begin{aligned}\int_0^T \|\nabla v\|_{L^\infty}\, dt &\le \int_0^T \|\nabla w\|_{L^\infty}\, dt + \int_0^T \|\nabla Ng\|_{L^\infty}\, dt\\ &\le C(M_0, M_1, T)\end{aligned}$$

for every $T > 0$. This completes the proof. ∎

In addition, based on (3.7), (3.9), (4.3), and (4.30), it is clear that

$$\begin{aligned}\int_0^T \|\frac{\partial v}{\partial t}\|_{L^2}^2\, dt &\le 2\int_0^T \|\frac{\partial w}{\partial t}\|_{L^2}^2\, dt + 2\int_0^T \|\frac{\partial Ng}{\partial t}\|_{L^2}^2\, dt\\ &\le 2\int_0^T (\nu\|Aw\|_{L^2}^2 + \|w\cdot\nabla(Ng)\|_{L^2}^2 + \|(Ng)\cdot\nabla w\|_{L^2}^2 + \|w\cdot\nabla w\|_{L^2}^2\\ &\quad + \|(Ng)\cdot\nabla(Ng)\|_{L^2}^2)\, dt + c\int_0^T \|\dot g\|_{L^2(\Gamma)}^2\, dt,\end{aligned}$$

where

$$\begin{aligned}&\int_0^T (\|w\cdot\nabla(Ng)\|_{L^2}^2 + \|(Ng)\cdot\nabla w\|_{L^2}^2 + \|w\cdot\nabla w\|_{L^2}^2 + \|(Ng)\cdot\nabla(Ng)\|_{L^2}^2)\, dt\\ &\quad \le c\int_0^T \|w\|_{L^4}^2\|\nabla(Ng)\|_{L^4}^2\, dt + c\int_0^T \|Ng\|_{L^4}^2\|\nabla w\|_{L^4}^2\, dt\\ &\qquad + c\int_0^T \|w\|_{L^4}^2\|\nabla w\|_{L^4}^2\, dt + c\int_0^T \|Ng\|_{L^\infty}^2\|\nabla(Ng)\|_{L^2}^2\, dt\\ &\quad \le c\int_0^T \|w\|_{L^2}\|\nabla w\|_{L^2}\|\nabla(Ng)\|_{L^2}\|Ng\|_{H^2}\, dt\\ &\qquad + c\int_0^T \|Ng\|_{L^2}\|Ng\|_{H^1}\|\nabla w\|_{L^2}\|Aw\|_{L^2}\, dt + c\int_0^T \|w\|_{L^2}\|\nabla w\|_{L^2}^2\|Aw\|_{L^2}\, dt\\ &\qquad + c\int_0^T \|Ng\|_{H^{1+\epsilon}}^2\|g\|_{L^2(\Gamma)}^2\, dt\\ &\quad \le c\sup_{t\in[0,T]}(\|w\|_{L^2}\|\nabla w\|_{L^2})\left(\int_0^T \|g\|_{L^2(\Gamma)}^2\, dt\right)^{1/2}\left(\int_0^T \|g\|_{H^{1/2}(\Gamma)}^2\, dt\right)^{1/2}\\ &\qquad + c\sup_{t\in[0,T]}(\|g\|_{L^2}^2\|\nabla w\|_{L^2})\int_0^T \|Aw\|_{L^2}\, dt + c\sup_{t\in[0,T]}(\|w\|_{L^2}\|\nabla w\|_{L^2}^2)\int_0^T \|Aw\|_{L^2}\, dt\\ &\qquad + c\sup_{t\in[0,T]}\|g\|_{L^2(\Gamma)}^2\int_0^T \|g\|_{L^2(\Gamma)}^2\, dt\\ &\quad \le C(M_0, M_1, T).\end{aligned}$$

Therefore, for $v_0 \in V_0^1(\Omega)$ and $g \in H^1(0,T;L^2(\Gamma)) \cap L^2(0,T;H^{1/2}(\Gamma))$,

$$\frac{\partial v}{\partial t} \in L^2(0,T;V_n^0(\Omega)). \tag{4.34}$$

**4.2. Existence of an optimal solution: Passive scalar case.** To prove the existence of an optimal solution, we first construct a weakly convergent sequence of the control inputs and then use the property of lower semicontinuity of the cost functional to derive an optimal solution by taking the limit of this sequence. The weak solution to the transport equation (1.1) is defined below.

DEFINITION 4.3. *For $\theta_0 \in L^\infty(\Omega)$, $\theta \in C([0,T],(H^1(\Omega))')$ is said to be a weak solution of* (1.1) *if $\theta$ satisfies*

$$\left(\frac{\partial \theta}{\partial t}, \phi\right) - (v\theta, \nabla\phi) = 0 \quad \forall \phi \in H^1(\Omega), \tag{4.35}$$

*where $v$ satisfies* (1.2)–(1.3) *and* (1.7)–(1.8) *with $v_0 \in V_n^0(\Omega)$ and $g \in L^2(0,T;V_n^0(\Gamma))$.*

THEOREM 4.4. *Consider the passive scalar field governed by* (1.1)–(1.3) *and* (1.7)–(1.8). *For $\theta_0 \in L^\infty(\Omega)$ and $v_0 \in V_n^0(\Omega)$, there exists at least one optimal solution $g \in U_{ad}$ to the problem (P).*

*Proof of Theorem* 4.4. Since $J$ is bounded from below, we can choose a minimizing sequence $\{g_n\} \subset U_{\mathrm{ad}}$ such that

$$\lim_{n\to\infty} J(g_n) = \inf_{g\in U_{\mathrm{ad}}} J(g).$$

By the definition of $J$, the sequences $\{g_m\}$ and $\{\frac{\partial g_m}{\partial t}\}$ are uniformly bounded in $U_{\mathrm{ad}}$, and hence there exists a weakly convergent subsequence, still denoted by $\{g_m\}$, such that

$$g_m \to g^* \quad \text{weakly in} \quad L^2(0,T;V_n^{1/2+\epsilon}(\Gamma))$$

and

$$\frac{\partial g_m}{\partial t} \to \frac{\partial g^*}{\partial t} \quad \text{weakly in} \quad L^2(0,T;V_n^0(\Gamma)).$$

For $v_0 \in V_n^1(\Omega)$, the corresponding $\{v_m\}$ and $v^*$ are bounded in $L^2(0,T;V_n^2(\Omega))$ based on (4.32). Thus,

$$v_m \to v^* \quad \text{weakly in} \quad L^2(0,T;V_n^2(\Omega)). \tag{4.36}$$

Moreover, according to (4.34), $\partial v_m/\partial t$ is bounded in $L^2(0,T;V_n^0(\Omega))$, and hence there exists a weakly convergent subsequence, still denoted by $\{v_m\}$, such that

$$\frac{\partial v_m}{\partial t} \to \frac{\partial v^*}{\partial t} \quad \text{weakly in } L^2(0,T;V_n^0(\Omega)). \tag{4.37}$$

Combining (4.36) and (4.37) yields

$$v_m \to v^* \quad \text{strongly in} \quad L^2(0,T;V_n^{2-\epsilon}(\Omega)) \quad \forall\, 0 < \epsilon \le 2. \tag{4.38}$$

Now, let sequence $\{\theta_m\}$ be the solutions corresponding to $\{v_m\}$ with $\theta_m(0) = \theta_0 \in L^\infty(\Omega)$. By (3.2), we have $\|\theta_m\|_{L^\infty} = \|\theta_0\|_{L^\infty}$ for any $t \ge 0$. Therefore, we may extract a subsequence, still denoted by $\{\theta_m\}$, such that

$$\theta_m \to \theta^* \quad \text{weak* in } L^\infty(0,T;L^\infty(\Omega)). \tag{4.39}$$

Next, we verify that $\theta^*$ is the solution corresponding to $v^*$ based on Definition 4.3. Note that $g_m$ and $\theta_m$ satisfy

$$\begin{aligned}(4.40)\qquad \left(\frac{\partial\theta_m}{\partial t},\phi\right)-(v_m\theta_m,\nabla\phi)=0,\quad \phi\in H^1(\Omega),\\ \theta_m=\theta_0.\end{aligned}$$

Let $\varphi$ be a continuously differentiable function on $[0,T]$ with $\varphi(T)=0$. For each $\phi\in H^1(\Omega)$, we multiply (4.40) by $\varphi$ and integrate by parts. After integrating the first term by parts, we get

$$-\int_0^T(\theta_m,\phi\dot{\varphi})\,dt-\int_0^T(v_m\theta_m,\nabla\phi\varphi)\,dt=(\theta_0,\phi\varphi(0)). \tag{4.41}$$

Since $\phi\dot{\psi}\in L^1(0,T;L^1(\Omega))$, it is straightforward to pass to the limit in the first term of the left-hand side of (4.41) with the help of (4.39). To estimate the second term, we use the convergence results (4.38)–(4.39) and get

$$\begin{aligned}&\left|\int_0^T\int_\Omega(v_m\theta_m)\cdot\nabla(\phi\varphi)\,dx\,dt-\int_0^T\int_\Omega(v^*\theta^*)\cdot\nabla(\phi\varphi)\,dx\,dt\right|\\
&\quad\le\left|\int_0^T\int_\Omega(v_m\theta_m)\cdot\nabla(\phi\varphi)-(v^*\theta_m)\cdot\nabla(\phi\varphi)\,dx\,dt\right|\\
&\qquad+\left|\int_0^T\int_\Omega(v^*\theta_m)\cdot\nabla(\phi\varphi)-(v^*\theta^*)\cdot\nabla(\phi\varphi)\,dx\,dt\right|\\
&\quad\le\int_0^T\|v_m-v^*\|_{L^2}\|\theta_m\|_{L^\infty}\|\nabla\phi\|_{L^2}|\varphi|\,dt+\left|\int_0^T\int_\Omega(\theta_m-\theta^*)v^*\cdot\nabla(\phi\varphi)\,dx\,dt\right|,\end{aligned}$$

where

$$\begin{aligned}&\int_0^T\|v_m-v^*\|_{L^2}\|\theta_m\|_{L^\infty}\|\nabla\phi\|_{L^2}|\varphi|\,dt\\
&\qquad\le\|v_m-v^*\|_{L^2(0,T;V_n^0(\Omega))}\|\theta_0\|_{L^\infty}\|\nabla\phi\|_{L^2}\|\varphi\|_{L^2(0,T)}\to 0.\end{aligned} \tag{4.42}$$

Further note that $v^*\cdot\nabla(\phi\varphi)\in L^1(0,T;L^1(\Omega))$, and therefore

$$\left|\int_0^T\int_\Omega(\theta_m-\theta^*)v^*\cdot\nabla(\phi\varphi)\,dx\,dt\right|\to 0.$$

As a result, we pass to the limit in (4.40) to derive that

$$-\int_0^T(\theta^*,\phi\dot{\varphi})\,dt-\int_0^T(v^*\theta^*,\nabla\phi\varphi)\,dt=(\theta_0,\phi\varphi(0)),\quad \phi\in H^1(\Omega). \tag{4.43}$$

It remains to be shown that $\theta^*(0)=\theta_0$. Consider

$$\begin{aligned}(4.44)\qquad \left(\frac{\partial\theta^*}{\partial t},\phi\right)-(v^*\theta^*,\nabla\phi)=0,\quad \phi\in H^1(\Omega),\\ \theta^*(0)=\theta_0^*.\end{aligned}$$

Again multiplying (4.44) by a continuously differentiable function $\psi$ with $\psi(T) = 0$ and integrating by parts yields

$$-\int_0^T (\theta^*, \phi\dot{\varphi})\,dt - \int_0^T (v^*\theta^*, \nabla\phi\varphi)\,dt = (\theta_0^*, \phi\varphi(0)), \quad \phi \in H^1(\Omega). \tag{4.45}$$

Comparing (4.45) with (4.43) gives

$$(\theta_0^* - \theta_0, \phi\varphi(0)) = 0, \quad \phi \in H^1(\Omega). \tag{4.46}$$

We can choose $\psi$ with $\psi(0) = 1$. Then (4.46) becomes

$$(\theta_0^* - \theta_0, \phi) = 0, \quad \phi \in H^1(\Omega),$$

and thus $\theta_0^* = \theta_0$.

Finally, since the norm is lower semicontinuous, the cost functional $J$ is lower semicontinuous for all $g \in U_{\mathrm{ad}}$. Therefore,

$$J(g^*) \leq \lim_{m\to\infty} \inf J(g_m).$$

This completes the proof. $\square$

**4.3. Optimality conditions: Passive scalar case.** In this section, we derive the first-order necessary optimality conditions for problem (P) by using a variational inequality [35]; that is, if $g$ is an optimal solution of problem (P), then

$$J'(g) \cdot (f - g) \geq 0, \quad f \in U_{ad}.$$

We still let $z = \theta'(g) \cdot h$ denote the Gâteaux derivative of $\theta$ with respect to $g$ in every direction $h$ in $U_{ad}$. Then $\theta$ satisfies (3.11)–(3.12), where $y = v'(g) \cdot h$ is the Gâteaux derivative of $v$ with respect to $g$ in the direction $h$. The following lemma states the properties of $(y, z)$ and the adjoint problem.

LEMMA 4.5. *Assume* $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$ *and* $g \in U_{ad}$. *Then* $(y, z)$ *is the solution of the linearized problem*

$$\frac{\partial z}{\partial t} + y \cdot \nabla\theta + v \cdot \nabla z = 0 \quad \text{in} \quad \Omega \tag{4.47}$$

$$\frac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)v + (v \cdot \nabla)y + \nabla q = 0 \quad \text{in} \quad \Omega \tag{4.48}$$

$$\nabla \cdot y = 0 \quad \text{in} \quad \Omega \tag{4.49}$$

*with the Navier slip boundary conditions*

$$y \cdot n|_\Gamma = 0 \quad \text{and} \quad 2\nu n \cdot \mathbb{D}(y) \cdot \tau + \alpha y \cdot \tau)|_\Gamma = h \cdot \tau \tag{4.50}$$

*and initial condition* $(z(0), y(0)) = (0, 0)$, *where* $q = p'(g) \cdot h$. *Moreover,*

$$(z, y) \in L^\infty(0, T; L^2(\Omega)) \times \left(C([0, T]; V_n^1(\Omega)) \cap L^2(0, T; V_n^2(\Omega))\right).$$

*In addition, the adjoint state* $(\rho, \bar{y})$ *associated with the cost functional* $J$ *in (P) satisfies*

$$-\frac{\partial}{\partial t}\rho - v \cdot \nabla\rho = 0 \quad \text{in} \quad \Omega \tag{4.51}$$

$$-\frac{\partial \bar{y}}{\partial t} - \nu\Delta\bar{y} + (\nabla v)^T\bar{y} - (v \cdot \nabla)\bar{y} + \nabla\bar{q} = \theta\nabla\rho + \zeta\nabla^\perp(\nabla \times v) \quad \text{in} \quad \Omega \tag{4.52}$$

$$\nabla \cdot \bar{y} = 0 \quad \text{in} \quad \Omega \tag{4.53}$$

*with the Navier slip boundary conditions*

$$\bar{y}\cdot n|_\Gamma = 0 \quad \textit{and} \quad (2\nu n\cdot \mathbb{D}(\bar{y})\cdot\tau + \alpha\bar{y}\cdot\tau)|_\Gamma = -\zeta\nabla\times v \tag{4.54}$$

*and final condition*

$$(\rho(T),\bar{y}(T) = (\Lambda^{-2}\theta(T),0). \tag{4.55}$$

*Moreover,*

$$(\rho,\bar{y})\in L^\infty(0,T;H^1(\Omega))\times\left(C([0,T];V_n^1(\Omega))\cap L^2(0,T;V_n^2(\Omega))\right).$$

For solving the adjoint system, the one-way coupling allows us to first solve $\rho$. Note that $\theta\in C([0,T];H^1(\Omega))$, which gives $\rho(T)\in H^3(\Omega)$. Thanks to the regularity of $v$ obtained in Theorem 4.1, the existence of a unique solution $\rho\in L^\infty(0,T;H^1(\Omega))$ to (4.51) can be obtained by replacing $t = T-t$. Therefore, $\theta\nabla\rho\in L^2(0,T;L^2(\Omega))$ for $\theta\in L^\infty(0,T;L^\infty(\Omega))$. Further note that $\nabla^\perp(\nabla\times v)\in L^2(0,T;L^2(\Omega))$ and $\nabla\times v|_\Gamma\in L^2(0,T;V_n^{1/2}(\Gamma))$. The existence of a unique solution $\bar{y}\in C([0,T];V_n^1(\Omega))\cap L^2(0,T;V_n^2(\Omega))$ to (4.52)–(4.55) follows again by replacing $t=T-t$.

Since the norm $\|g\|_{U_{ad}}$ involves the fractal derivative on the boundary, it complicates the expression of the Gâteaux derivative of $J$. In the rest of our work, we restrict the control input function to be of the form

$$g(x,t) = \sum_{i=1}^M b_i(x)u_i(t), \tag{4.56}$$

where $M$ is a finite positive integer, $b_i\in V_n^{1/2+\epsilon}(\Gamma)$ are prescribed functions, and the controls are now $u_i\in H^1(0,T)$. From the point of view of applications, a finite number of control inputs is a more realistic assumption since it is not practical to create arbitrarily distributed force fields for stirring. Mathematically, this will also lead to a more transparent optimality system.

Let $\mathbf{u}(t) = [u_1(t),u_2(t),\dots,u_M(t)]^T$ and $\mathbf{b}(x) = [b_1(x),b_2(x),\dots,b_M(x)]^T$. Moreover, if we let $\eta = \|\mathbf{b}\|^2_{L^2}$ and $\beta = \|\mathbf{b}\|^2_{H^{1/2+\epsilon}}$, then the control problem (P) is now equivalent to

$$J(\mathbf{u}) = \frac{1}{2}(\Lambda^{-2}\theta(T),\theta(T)) + \frac{\gamma}{2}\int_0^T(\beta\mathbf{u}^T\mathbf{u}+\eta\dot{\mathbf{u}}^T\dot{\mathbf{u}})\,dt - \frac{\zeta}{2}\int_0^T(\nabla\times v,\nabla\times v)\,dt, \quad \text{(P')},$$

for $\mathbf{u}\in(H^1(0,T))^M$. The existence of an optimal solution of the form (4.56) can be obtained following the same approach as in the proof of Theorem 4.4 by replacing $U_{ad}$ by $(H^1(0,T))^M$ for each $M$. Because of (3.4), we derive

$$\begin{aligned}
\int_\Omega(\nabla\times v)\cdot(\nabla\times v)\,dx &= -\int_\Omega\omega\partial_2 v_1\,dx + \int_\Omega\omega\partial_1 v_2\,dx\\
&= -\int_\Gamma\omega n_2 v_1\,dx + \int_\Omega\partial_2\omega v_1\,dx + \int_\Gamma\omega n_1 v_2\,dx - \int_\Omega\partial_1\omega v_2\,dx\\
&= -\int_\Omega(\nabla^\perp\omega)\cdot v\,dx + \int_\Gamma\omega\tau_1 v_1\,dx + \int_\Gamma\omega\tau_2 v_2\,dx\\
&= -\int_\Omega(\nabla^\perp\omega)\cdot v\,dx + \int_\Gamma\omega(v\cdot\tau)\,dx\\
&= -\int_\Omega(\nabla^\perp(\nabla\times v))\cdot v\,dx + \int_\Gamma(2\kappa-\alpha)(v\cdot\tau)(v\cdot\tau)\,dx + \int_\Gamma(g\cdot\tau)(v\cdot\tau)\,dx.
\end{aligned}$$

Therefore, if $\mathbf{u}$ is the optimal solution of problem (P'), then for $\mathbf{h} \in (H^1(0,T))^M$,

$$
\begin{aligned}
J'(\mathbf{u})\cdot\mathbf{h} &= (\Lambda^{-2}\theta(T), (\theta'(\mathbf{u})\cdot\mathbf{h})(T)) + \gamma\int_0^T (\beta\mathbf{u}^T\mathbf{h} + \alpha\dot{\mathbf{u}}^T\dot{\mathbf{h}})\,dt \\
&\quad - \zeta\int_0^T (\nabla\times v, \nabla\times(v'(\mathbf{u})\cdot\mathbf{h}))\,dt \\
&= (\Lambda^{-2}\theta(T), (\theta'(\mathbf{u})\cdot\mathbf{h})(T)) + \gamma\int_0^T (\beta\mathbf{u}^T\mathbf{h} + \alpha\dot{\mathbf{u}}^T\dot{\mathbf{h}})\,dt \\
&\quad + \zeta\int_0^T (\nabla^\perp(\nabla\times v), v'(\mathbf{u})\cdot\mathbf{h})\,dt - \zeta(2\kappa-\alpha)\int_0^T \langle v\cdot\tau, (v'(\mathbf{u})\cdot\mathbf{h})\cdot\tau\rangle\,dt \\
&\quad - \zeta\int_0^T \langle(\mathbf{b}^T\mathbf{u})\cdot\tau, (v'(\mathbf{u})\cdot\mathbf{h})\cdot\tau\rangle\,dt,
\end{aligned}
\tag{4.57}
$$

where $\nabla^\perp\omega = (-\partial_2,\partial_1)\omega$, $z = \theta'(\mathbf{u})\cdot\mathbf{h}$, and $y = v'(\mathbf{u})\cdot\mathbf{h}$ satisfy the linearized equations (4.47)–(4.50).

The following theorem provides the first-order necessary conditions of optimality for solving an optimal solution of problem (P) for the passive scalar case.

THEOREM 4.6. *Assume* $(\theta_0, v_0) \in (L^\infty(\Omega)\cap H^1(\Omega))\times V_n^1(\Omega)$ *and* $g = \sum_{i=1}^M b_i(x)u_i(t)$ *with* $b_i \in V_n^{1/2+\epsilon}(\Gamma)$ *and* $u_i \in H^1(0,T)$. *Let* $(\theta, v)$ *be the solution of* (1.1)–(1.3) *and* (1.7)–(1.8). *Assume that* $\mathbf{u}^{opt} \in (H^1(0,T))^M$ *is an optimal controller of problem (P'). If* $(\rho,\bar{y})$ *is the solution of the adjoint system* (4.51)–(4.55) *associated with* $(\theta, v)$, *then*

$$
\begin{aligned}
&\gamma\eta\ddot{\mathbf{u}}^{opt} - \gamma\beta\mathbf{u}^{opt} = [\langle b_1,\bar{y}\rangle,\ldots,\langle b_M,\bar{y}\rangle]^T, \\
&\dot{\mathbf{u}}^{opt}(0) = \dot{\mathbf{u}}^{opt}(T) = 0.
\end{aligned}
$$

*Proof of Theorem* 4.6. To obtain the desired result, we first analyze the term in the Gâteaux derivative of $J$ in (4.57). We first multiply (4.47) by $\rho$ and get

$$
\int_0^T \left(\frac{\partial}{\partial t}z, \rho\right)dt + \int_0^T (y\cdot\nabla\theta, \rho)\,dt + \int_0^T (v\cdot\nabla z, \rho)\,dt = 0.
$$

Integrating the first term with respect to $t$ and the third term with respect to $x$ yields

$$
-\int_0^T \left(\frac{\partial}{\partial t}\rho, z\right)dt + (\rho(T), z(T)) + \int_0^T (y\cdot\nabla\theta,\rho)\,dt - \int_0^T (v\cdot\nabla\rho, z)\,dt = 0.
$$

Due to the adjoint equation (4.51) and the final condition (4.55), we obtain

$$
(\Lambda^{-2}\theta(T), z(T)) = (\rho(T), z(T)) = -\int_0^T (y\cdot\nabla\theta,\rho)\,dt = \int_0^T (y, \theta\nabla\rho)\,dt.
$$

Therefore, (4.57) becomes

$$
\begin{aligned}
J'(\mathbf{u})\cdot\mathbf{h} &= \int_0^T (y, \theta\nabla\rho + \zeta\nabla^\perp(\nabla\times v))\,dt + \gamma\int_0^T (\beta\mathbf{u}^T\mathbf{h} + \eta\dot{\mathbf{u}}^T\dot{\mathbf{h}})\,dt \\
&\quad - \zeta(2\kappa-\alpha)\int_0^T \langle v\cdot\tau, y\cdot\tau\rangle\,dt - \zeta\int_0^T \langle(\mathbf{b}^T\mathbf{u})\cdot\tau, y\cdot\tau\rangle\,dt \\
&= \int_0^T (y, \theta\nabla\rho + \zeta\nabla^\perp(\nabla\times v))\,dt + \gamma\beta\int_0^T \mathbf{u}^T\mathbf{h}\,dt + \gamma\eta\dot{\mathbf{u}}^T\mathbf{h}|_0^T - \gamma\eta\int_0^T \mathbf{h}^T\ddot{\mathbf{u}}\,dt \\
&\quad - \zeta(2\kappa-\alpha)\int_0^T \langle v\cdot\tau, y\cdot\tau\rangle\,dt - \zeta\int_0^T \langle(\mathbf{b}^T\mathbf{u})\cdot\tau, y\cdot\tau\rangle\,dt.
\end{aligned}
\tag{4.58}
$$

Replacing $h$ by $\mathbf{b}^T\mathbf{h}$ in (4.50) and using (2.12) and (4.52)–(4.53), we have

$$\int_0^T (y, \theta\nabla\rho + \zeta\nabla^\perp(\nabla\times v))\,dt = \int_0^T \left(y, -\frac{\partial \bar{y}}{\partial t} - \nu\Delta\bar{y} + (\nabla v)^T\bar{y} - (v\cdot\nabla)\bar{y} + \nabla\bar{q}\right)dt$$

$$= -[(y(T),\bar{y}(T)) - (y(0),\bar{y}(0))] + \int_0^T \left[\left(\frac{dy}{dt},\bar{y}\right) + 2\nu(\mathbb{D}y,\mathbb{D}\bar{y}) + \langle\alpha y,\bar{y}\rangle\right.$$
$$\left. + \langle y\cdot\tau, \zeta\nabla\times v\rangle + ((y\cdot\nabla)v + (v\cdot\nabla)y,\bar{y})\right]dt$$
$$= \int_0^T (\frac{dy}{dt},\bar{y})\,dt + \int_0^T [(-\nu\Delta y,\bar{y}) + \langle\mathbf{b}^T\mathbf{h},\bar{y}\rangle + \langle y\cdot\tau,\zeta((2\kappa-\alpha)v\cdot\tau + (\mathbf{b}^T\mathbf{u})\cdot\tau)\rangle$$
$$+ ((y\cdot\nabla)v + (v\cdot\nabla)y,\bar{y}) + (\nabla q,\bar{y})]\,dt$$

(4.59)
$$= \int_0^T [\langle b_1,\bar{y}\rangle,\ldots,\langle b_M,\bar{y}\rangle]\mathbf{h}\,dt + \int_0^T \langle y\cdot\tau,\zeta((2\kappa-\alpha)v\cdot\tau + (\mathbf{b}^T\mathbf{u})\cdot\tau)\rangle\,dt.$$

Set $\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}(T) = 0$. Combining (4.58) with (4.59) yields

$$\int_0^T ([\langle b_1,\bar{y}\rangle,\ldots,\langle b_M,\bar{y}\rangle]^T + \gamma\beta\mathbf{u} - \gamma\eta\ddot{\mathbf{u}},\mathbf{h})) \geq 0, \quad \mathbf{h}\in(H^1(0,T))^M,$$

which implies

$$\gamma\eta\ddot{\mathbf{u}}^{opt} - \gamma\beta\mathbf{u}^{opt} = [\langle b_1,\bar{y}\rangle,\ldots,\langle b_M,\bar{y}\rangle]^T.$$

This also indicates that control $\mathbf{u}^{opt}\in(H^2(0,T))^M$. The proof is complete. ∎

## 5. Case II: Optimal mixing of an active scalar via Navier–Stokes flow.

To address the optimal control problem of the active scalar case described by Model II, we first prove its global well-posedness for $(\theta_0, v_0)\in H^1(\Omega)\times V_n^1(\Omega)$.

### 5.1. Well-posedness of Model II in $H^1(\Omega)\times V_n^1(\Omega)$.

DEFINITION 5.1. *For $(\theta_0,v_0)\in L^\infty(\Omega)\times V_n^1(\Omega)$ and $g\in L^2(0,T;V_n^0(\Gamma))$, $(\theta,v)\in C([0,T],(H^1(\Omega))')\times(C([0,T];V_n^0(\Omega))\cap L^2(0,T;V_n^1(\Omega)))$ is said to be a weak solution of (1.4)–(1.8) if $(\theta,v)$ satisfies*

(5.1)
$$\left(\frac{\partial\theta}{\partial t},\phi\right) - (v\theta,\nabla\phi) = 0 \quad \forall\phi\in H^1(\Omega)$$

(5.2)
$$\left(\frac{\partial v}{\partial t},\psi\right) + 2(\mathbb{D}(v),\mathbb{D}(\psi)) + \alpha\langle v,\psi\rangle + (v\cdot\nabla v,\psi) = \langle g,\psi\rangle + (\theta e_2,\psi) \quad \forall\psi\in V_n^1(\Omega).$$

The global well-posedness result for $(\theta_0,v_0)\in L^\infty(\Omega)\times V_n^1(\Omega)$ with homogeneous Navier slip boundary conditions has been obtained in [27, Theorem 1.1]. However, to have the differentiability of Model II, we establish the following result to address the well-posedness and regularity issues for $(\theta_0,v_0)\in H^1(\Omega)\times V_n^1(\Omega)$.

THEOREM 5.2. *Assume that $(\theta_0,v_0)\in H^1(\Omega)\times V_n^1(\Omega)$ and $g\in U_{ad}$ with $\|\theta_0\|_{H^1}\leq\Theta_0$, $\|v_0\|_{H^1}\leq M_0$, and $\|g\|_{U_{ad}}\leq M_1$, where $\Theta_0, M_0, M_1 > 0$ are arbitrary. Then there exists a unique global solution $(\theta,v)$ such that $\theta\in L^\infty(0,\infty;H^1(\Omega))$ and $v\in L^\infty(0,\infty;H^1(\Omega))\cap L^2_{loc}(0,\infty;H^2(\Omega))$. Moreover, for every $T>0$, we have*

$$\int_0^T \|\nabla v(t)\|_{L^\infty}\, dt \le C(\Theta_0, M_0, M_1, T)$$

*and*

$$\|\theta(t)\|_{H^1} \le C(\Theta_0, M_0, M_1, T)$$

*for* $t \in [0, T]$.

*Proof of Theorem* 5.2. Slightly modifying the $L^2$-estimate for the passive case and recalling (3.1) that $\|\theta(t)\|_{L^2} = \|\theta_0\|_{L^2}$ for all $t \ge 0$, we get

$$\frac{d\|v\|_{L^2}^2}{dt} + c\nu\|A^{1/2}v\|_{L^2}^2 + 2\alpha\|v\|_{L^2(\Gamma)}^2 \le c\lambda_1^{-1/2}\|g\|_{L^2(\Gamma)}^2 + c\|\theta_0\|_{L^2}^2$$

and

$$\frac{d\|v\|_{L^2}^2}{dt} + c\nu\lambda_1\|v\|_{L^2}^2 + 2\alpha\|v\|_{L^2(\Gamma)}^2 \le c\lambda_1^{-1/2}\|g\|_{L^2(\Gamma)}^2 + c\|\theta_0\|_{L^2}^2.$$

Similarly, we can adjust the initial datum $\|v_0\|_{H^1}$ so that

$$\sup_{t\in[0,T]} \|v\|_{L^2} \le M_0 = c\left(\frac{\lambda_1^{-1/2}M_1^2 + \Theta_0^2}{\nu\lambda_1}\right)^{1/2} \tag{5.3}$$

$$\int_0^t \|v\|_{L^2}^2\, d\tau \le C(M_0, M_1) + \Theta_0^2 t, \quad t > 0, \tag{5.4}$$

and

$$\int_t^{t+1} \|A^{1/2}v\|_{L^2}^2\, d\tau \le C(M_0, M_1, \Theta_0), \quad t \ge 0.$$

Note that in the active scalar case, $\int_0^t \|v\|_{L^2}^2\, dt$ may grow in time due to the constant term $\|\theta_0\|_{L^2}$.

Again making change of variable, we let $w = v - Ng$; then

$$\begin{aligned}
&\partial_t\theta + w\cdot\nabla\theta + Ng\cdot\nabla\theta = 0,\\
&\frac{\partial w}{\partial t} = \nu\Delta w - w\cdot\nabla(Ng) - (Ng)\cdot\nabla w - w\cdot\nabla w - \nabla p\\
&\qquad - (Ng)\cdot\nabla(Ng) - \frac{\partial(Ng)}{\partial t} + \theta e_2,\\
&\nabla\cdot w = 0,
\end{aligned} \tag{5.5}$$

where $w$ also satisfies the homogeneous Navier slip boundary condition (4.5) and $(\theta(0), w(0)) = (\theta_0, v_0 - Ng(0))$. According to (5.3)–(5.4) and $\|g\|_{U_{ad}} \le M_1$, it follows that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|w\|_{L^2} \le \sup_{t\in[0,T]} \|v\|_{L^2} + \sup_{t\in[0,T]} \|Ng\|_{L^2} \le C(M_0, M_1, \Theta_0),\\
&\int_0^t \|w\|_{L^2}^2\, d\tau \le 2\int_0^t \|v\|_{L^2}^2\, d\tau + 2\int_0^t \|Ng\|_{L^2}^2\, d\tau \le C(M_0, M_1) + c\Theta_0^2 t
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
\int_t^{t+1} \|A^{1/2}w\|_{L^2}\, d\tau &\le \int_t^{t+1} \|A^{1/2}v\|_{L^2}\, d\tau + \int_t^{t+1} \|A^{1/2}Ng\|_{L^2}\, d\tau\\
&\le C(M_0, M_1, \Theta_0).
\end{aligned}$$

Next, taking the inner product of the velocity equation (5.5) with $Aw$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(w, Aw) + \nu\|Aw\|_{L^2}^2 = &-(\mathbb{P}(w\cdot\nabla(Ng)), Aw) - (\mathbb{P}((Ng)\cdot\nabla w), Aw)\\ &-(\mathbb{P}(w\cdot\nabla w), Aw) - (\mathbb{P}((Ng)\cdot\nabla(Ng)), Aw)\\ &-(\mathbb{P}\frac{\partial(Ng)}{\partial t}, Aw) + (\mathbb{P}\theta e_2, Aw).\end{aligned}$$

Following the same approach as in the proof of Lemma 4.2, we find that (4.28) becomes

$$\begin{aligned}&\frac{d}{dt}(\|\mathbb{D}(w)\|_{L^2}^2 + \frac{\alpha}{2\nu}\|w\|_{L^2(\Gamma)}^2) + \nu\|Aw\|_{L^2}^2\\ &\quad\le c\big[\|w\|_{L^2}^2\|g\|_{L^2(\Gamma)}^4 + \|g\|_{L^2(\Gamma)}^2(\|\mathbb{D}w\|_{L^2}^2 + \frac{\alpha}{2}\|w\|_{L^2(\Gamma)}^2)\\ &\qquad + \|w\|_{L^2}^2\|\mathbb{D}w\|_{L^2}^2(\|\mathbb{D}w\|_{L^2}^2 + \frac{\alpha}{2}\|w\|_{L^2(\Gamma)}^2) + \|g\|_{L^2(\Gamma)}^4 + \|\dot g\|_{L^2(\Gamma)}^2 + \|\theta_0\|_{L^2}\big].\end{aligned} \tag{5.7}$$

Again employing uniform Gronwall inequality to (5.7), we obtain

$$\|A^{1/2}w\|_{L^2}^2 \le c\|\mathbb{D}w\|_{L^2}^2 \le C(M_0, M_1, \Theta_0), \quad t \ge 0,$$

and

$$\int_0^t \|Aw\|_{L^2}^2\, d\tau \le C(M_0, M_1, \Theta_0)t. \tag{5.8}$$

By virtue of (4.18), (4.21)–(4.24), (5.6), and (5.8), we have

$$\begin{aligned}\int_0^t \|\nabla w\|_{L^\infty}\, d\tau &\le c\left(\|A^{1/2}w_0\|_{L^2} + \int_0^t \|A^{1/2}\mathbb{P}I\|_{L^2}\, d\tau + \int_0^t \|A^{1/2}N\dot g\|_{L^2}\, d\tau\right.\\ &\qquad\left. + \int_0^t \|A^{1/2}\theta\|_{L^2}\, d\tau\right)\\ &\le c\left(\|A^{1/2}w_0\|_{L^2} + C(M_0, M_1, \Theta_0)(t + t^{1/2}) + \int_0^t \|A^{1/2}\theta\|_{L^2}\, d\tau\right).\end{aligned}$$

If $t \in (0,1)$, then $t + t^{1/2} < 2$, and hence

$$\int_0^t \|\nabla w\|_{L^\infty}\, d\tau \le c\int_0^t \|\nabla\theta\|_{L^2}\, d\tau + C(M_0, M_1, \Theta_0). \tag{5.9}$$

Combining (3.13) with (4.33) and (5.9) yields

$$\begin{aligned}\|\nabla\theta\|_{L^2} &\le \|\nabla\theta_0\|_{L^2} e^{\int_0^t (\|\nabla w\|_{L^\infty} + \|\nabla(Ng)\|_{L^\infty})\, d\tau}\\ &\le C(M_0, M_1, \Theta_0)e^{c\int_0^t \|\nabla\theta\|_{L^2}\, d\tau}.\end{aligned} \tag{5.10}$$

Let $X = c\int_0^t \|\nabla\theta\|_{L^2}\, d\tau$. Then, by (5.10),

$$X' \le C(M_0, M_1, \Theta_0)e^X. \tag{5.11}$$

If $t \in [1,\infty)$, then $t + t^{1/2} \le 2t$, and hence we can derive the same inequality (5.11) by setting $X = c\int_0^t (1 + \|\nabla\theta\|_{L^2})\, d\tau$. Solving (5.11), we derive that there exists a $\tilde t > 0$ such that

$$X \le \ln\left[\frac{1}{1 - C(M_0, M_1, \Theta_0)t}\right] < \infty, \quad t \in [0, \tilde{t}].$$

Therefore, from (5.10), it follows that

$$\|\nabla\theta\|_{L^2} \le \frac{C(M_0, M_1, \Theta_0)}{1 - C(M_0, M_1, \Theta_0)t} < \infty, \quad t \in [0, \tilde{t}],$$

which shows that if $(\theta_0, w_0) \in H^1(\Omega) \times V_n^1(\Omega)$, then there exists a $\tilde{t} > 0$ such that $(\theta, w) \in L^\infty(0, \tilde{t}; H^1(\Omega)) \times L^1(0, \tilde{t}; W^{1,\infty}(\Omega))$. Moreover, from (5.8),

$$\int_0^{\tilde{t}} \|Aw\|_{L^2}^2 \, d\tau \le C(M_0, M_1, \Theta_0)\tilde{t}.$$

Thus, there exists a $t^* \in [0, \tilde{t}]$ such that $w(t^*) \in D(A)$. Finally, for $(\theta(t^*), w(t^*)) \in H^1(\Omega) \times D(A)$, slightly modifying the proof of Theorem 2.1 in [25] and estimating $\|A^{1+\epsilon/2} w\|_{L^2(t^*,\infty;L^2(\Omega))}$, $\epsilon > 0$, by utilizing the variation of parameters formula together with the properties of the analytic $C_0$-semigroup generated by $A$, we can show that there exists a unique global solution $(\theta, w)$ such that

$$\theta \in L^\infty(t^*, \infty; H^1(\Omega)) \quad \text{and} \quad w \in L^\infty(t^*, \infty; H^2(\Omega)) \cap L^2_{loc}(t^*, \infty; H^{2+\epsilon}(\Omega)).$$

As a result,

$$\theta \in L^\infty(0, \infty; H^1(\Omega)) \quad \text{and} \quad w \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2_{loc}(0, \infty; H^2(\Omega))$$

and, by (5.9),

$$\int_0^T \|\nabla w\|_{L^\infty} \, dt \le C(M_0, M_1, \Theta_0, T).$$

Consequently,

$$\int_0^T \|\nabla v\|_{L^\infty} \, dt \le C(M_0, M_1, \Theta_0, T) \quad \text{and} \quad \|\theta(t)\|_{H^1} \le C(M_0, M_1, \Theta_0, T), \quad t \in [0, T].$$

This completes the proof. ∎

**5.2. Existence of an optimal solution: Active scalar case.** Following the same procedures as in Theorem 4.4, we obtain the existence of an optimal solution to the active scalar case.

THEOREM 5.3. *Consider the active scalar field governed* (1.4)–(1.8)*. For* $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$*, there exists at least one optimal solution* $g \in U_{ad}$ *to the problem (P).*

*Proof.* As shown in the proof of Theorem 4.4, there exists a sequence $\{g_m\} \subset U_{ad}$ satisfying

$$g_m \to g^* \quad \text{weakly in} \quad L^2(0, T; V_n^{1/2+\epsilon}(\Gamma))$$

and

$$\frac{\partial g_m}{\partial t} \to \frac{\partial g^*}{\partial t} \quad \text{weakly in} \quad L^2(0, T; V_n^0(\Gamma)).$$

For $(\theta_0, v_0) \in (L^\infty(\Omega) \times H^1(\Omega)) \times V_n^1(\Omega)$, the corresponding sequences $\{\theta_m\}$ and $\{v_m\}$ satisfy

$$\theta_m \to \theta^* \quad \text{weak* in } L^\infty(0, T; L^\infty(\Omega)) \tag{5.12}$$

and

$$v_m \to v^* \quad \text{strongly in} \quad L^2(0,T;V_n^{2-\epsilon}(\Omega)) \quad \forall 0<\epsilon\le 2. \tag{5.13}$$

Now we verify that $(\theta^*, v^*)$ is the weak solution based on Definition 5.1. Note that $g_m$ and $(\theta_m, v_m)$ satisfy

$$\left(\frac{\partial \theta_m}{\partial t}, \phi\right) - (v_m\theta_m, \nabla\phi) = 0, \quad \phi \in H^1(\Omega), \tag{5.14}$$

$$\begin{aligned}&\left(\frac{\partial v_m}{\partial t}, \psi\right) + 2\nu(\mathbb{D}(v_m), \mathbb{D}(\psi)) + \nu\alpha\langle v_m, \psi\rangle + (v_m\cdot\nabla v_m, \psi)\\ &\qquad = \nu\langle g, \psi\rangle + (\theta_m e_2, \psi), \quad \psi\in V_n^1(\Omega),\end{aligned} \tag{5.15}$$

with $(\theta_m, v_m) = (\theta_0, v_0)$. Let $(\varphi, \Psi)$ be a vector of continuously differentiable function on $[0,T]$ with $(\varphi(T), \Psi(T)) = (0,0)$. For each $(\phi,\psi) \in H^1(\Omega)\times V_n^1(\Omega)$, we multiply (5.14) by $\varphi$ and (5.15) by $\Psi$, respectively, and then integrate by parts. After integrating the first term by parts for each equation, we get

$$\begin{aligned}&-\int_0^T(\theta_m, \phi\dot\varphi)\,dt - \int_0^T (v_m\theta_m, \nabla\phi\varphi)\,dt = (\theta_0, \phi\varphi(0)).\\ &-\int_0^T(v_m,\psi\dot\Psi)\,dt + 2\nu\int_0^T(\mathbb{D}(v_m),\mathbb{D}(\psi)\Psi)\,dt + \nu\alpha\int_0^T\langle v_m,\psi\Psi\rangle\,dt\\ &\quad+\int_0^T(v_m\cdot\nabla v_m,\psi\Psi)\,dt\\ &\qquad = \nu\int_0^T\langle g_m,\psi\Psi\rangle\,dt + \int_0^T(\theta_m e_2,\psi\Psi)\,dt + (v_0,\psi\Psi(0)).\end{aligned}$$

Based on the results we have established in the proof of Theorem 4.4, it suffices to show that

$$\int_0^T(v_m\cdot\nabla v_m,\psi\Psi)\,dt \to \int_0^T(v^*\cdot\nabla v^*,\psi\Psi)\,dt, \quad \psi\in V_n^1(\Omega).$$

We first write

$$\begin{aligned}(v_m\cdot\nabla v_m,\psi\Psi) &= \int_\Omega v_{im}\partial_i(v_{jm}\psi_j\Psi)\,dx - \int_\Omega v_{im}v_{jm}\partial_i(\psi_j\Psi)\,dx\\ &= -\int_\Omega v_{im}v_{jm}\partial_i(\psi_j\Psi)\,dx.\end{aligned}$$

Using the convergence results (5.12)–(5.13), we have

$$\begin{aligned}&\left|\int_0^T\int_\Omega v_{im}v_{jm}\partial_i(\psi_j\Psi)\,dx\,dt - \int_0^T\int_\Omega v_i^*v_j^*\partial_i(\psi_j\Psi)\,dx\,dt\right|\\ &\quad\le\left|\int_0^T\int_\Omega(v_{im}-v_i^*)v_{jm}\partial_i(\psi_j\Psi)\,dx\,dt\right| + \left|\int_0^T\int_\Omega v_i^*(v_{jm}-v_j^*)\partial_i(\psi_j\Psi)\,dx\,dt\right|\\ &\quad\le\int_0^T\|v_m-v^*\|_{L^\infty}\|v_m\|_{L^2}\|\nabla\psi\|_{L^2}|\Psi|\,dt + \int_0^T\|v^*\|_{L^2}\|v_m-v^*\|_{L^\infty}\|\nabla\psi\|_{L^2}|\Psi|\,dt\\ &\quad\le\|v_m-v^*\|_{L^2(0,T;V_n^{2-\epsilon}(\Omega))}\|v_m\|_{L^2(0,T;L^2(\Omega))}\|\nabla\psi\|_{L^2}\|\Psi\|_{L^\infty(0,T)}\\ &\qquad+\|v^*\|_{L^2(0,T;L^2(\Omega))}\|v_m-v^*\|_{L^2(0,T;V_n^{2-\epsilon}(\Omega))}\|\nabla\psi\|_{L^2}\|\Psi\|_{L^\infty(0,T)}\to 0\end{aligned}$$

for $0<\epsilon<1$. The rest of the proof follows the same procedures as in Theorem 4.4. ∎

**5.3. Optimality conditions: Active scalar case.** In this section, we derive the first-order optimality conditions for the active scalar case with a finite number of control inputs.

LEMMA 5.4. *Consider the active scalar field governed* (1.4)–(1.8). *Assume* $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$ *and* $g \in U_{ad}$. *Let* $y = (v'(g) \cdot h)$ *and* $z = \theta'(g) \cdot h$ *be the Gâteaux derivatives of* $v$ *and* $\theta$ *with respect to* $g$ *in every direction* $h$ *in* $U_{ad}$, *respectively. Then* $(y, z)$ *is the solution of the linearized problem*

$$\frac{\partial z}{\partial t} + y \cdot \nabla\theta + v \cdot \nabla z = 0 \quad \text{in} \quad \Omega \tag{5.16}$$

$$\frac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)v + (v \cdot \nabla)y + \nabla q = ze_2 \quad \text{in} \quad \Omega \tag{5.17}$$

$$\nabla \cdot y = 0 \quad \text{in} \quad \Omega \tag{5.18}$$

*with the Navier slip boundary conditions*

$$y \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(y) \cdot \tau + \alpha y \cdot \tau)|_\Gamma = h \cdot \tau$$

*and initial condition* $(z(0), y(0)) = (0, 0)$, *where* $q = p(g)' \cdot h$. *The adjoint state* $(\rho, \bar{y}, \bar{q})$ *associated with the cost functional* $J$ *in (P) satisfies*

$$-\frac{\partial \rho}{\partial t} - v \cdot \nabla\rho - \bar{y} \cdot e_2 = 0 \quad \text{in} \quad \Omega \tag{5.19}$$

$$-\frac{\partial \bar{y}}{\partial t} - \nu\Delta\bar{y} + (\nabla v)^T\bar{y} - (v \cdot \nabla)\bar{y} + \nabla\bar{q} = \theta\nabla\rho + \zeta\nabla^\perp(\nabla \times v) \quad \text{in} \quad \Omega \tag{5.20}$$

$$\nabla \cdot \bar{y} = 0 \quad \text{in} \quad \Omega \tag{5.21}$$

*with the Navier slip boundary conditions*

$$\bar{y} \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(\bar{y}) \cdot \tau + \alpha\bar{y} \cdot \tau)|_\Gamma = -\zeta\nabla \times v \tag{5.22}$$

*and final condition*

$$(\rho(T), \bar{y}(T)) = (\Lambda^{-2}\theta(T), 0). \tag{5.23}$$

*Moreover,*

$$(\rho, \bar{y}) \in L^\infty(0, T; H^1(\Omega)) \times \left(C([0, T]; V_n^1(\Omega)) \cap L^2(0, T; V_n^2(\Omega))\right).$$

THEOREM 5.5. *Assume* $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$ *and* $g = \sum_{i=1}^M b_i(x)u_i(t)$ *with* $b_i \in V_n^{1/2+\epsilon}(\Gamma)$ *and* $u_i \in H^1(0, T)$. *Let* $(\theta, v)$ *be the solution of* (1.4)–(1.8). *Assume that* $\mathbf{u}^{opt} \in (H^1(0, T))^M$ *is an optimal controller of problem (P'). If* $(\rho, \bar{y})$ *is the solution of the adjoint system* (5.19)–(5.23) *associated with* $(\theta, v)$, *then*

$$\gamma\eta\ddot{\mathbf{u}}^{opt} - \gamma\beta\mathbf{u}^{opt} = [\langle b_1, \bar{y}\rangle, \ldots, \langle b_M, \bar{y}\rangle]^T,$$
$$\dot{\mathbf{u}}^{opt}(0) = \dot{\mathbf{u}}^{opt}(T) = 0.$$

*Proof of Theorem* 5.5. Following the same approach as in the proof of Theorem 4.6, we multiply (5.16) by $\rho$ and integrate the first term with respect to $t$. This gives

$$-\int_0^T \left(\frac{\partial}{\partial t}\rho, z\right) dt + (\rho(T), z(T)) + \int_0^T (y \cdot \nabla\theta, \rho)\, dt - \int_0^T (v \cdot \nabla\rho, z)\, dt = 0.$$

Due to the adjoint equation (5.19) and the final condition (5.23), we have

$$\begin{aligned}(\Lambda^{-2}\theta(T), z(T)) = (\rho(T), z(T)) &= -\int_0^T (\bar{y}\cdot e_2, z)\,dt - \int_0^T (y\cdot\nabla\theta, \rho)\,dt \\ &= -\int_0^T (ze_2, \bar{y})\,dt + \int_0^T (y, \theta\nabla\rho)\,dt. \end{aligned}\tag{5.24}$$

Thus, (4.57) becomes

$$\begin{aligned}J'(\mathbf{u})\cdot\mathbf{h} = &-\int_0^T (ze_2, \bar{y})\,dt + \int_0^T (y, \theta\nabla\rho + \zeta\nabla^{\perp}(\nabla\times v))\,dt + \gamma\beta\int_0^T \mathbf{u}^T\mathbf{h}\,dt \\ &+ \gamma\eta\dot{\mathbf{u}}^T\mathbf{h}|_0^T - \gamma\eta\int_0^T \mathbf{h}^T\ddot{\mathbf{u}}\,dt - \zeta(2\kappa-\alpha)\int_0^T \langle v\cdot\tau, y\cdot\tau\rangle\,dt \\ &- \zeta\int_0^T \langle(\mathbf{b}^T\mathbf{u})\cdot\tau, y\cdot\tau\rangle\,dt,\end{aligned}\tag{5.25}$$

where the difference compared to (4.58) is by the term $\int_0^T (ze_2, \bar{y})\,dt$. By virtue of (4.59) and (5.17), we get

$$\begin{aligned}&\int_0^T (y, \theta\nabla\rho + \zeta\nabla^{\perp}(\nabla\times v))\,dt \\ &\quad= \int_0^T \left(\frac{dy}{dt}, \bar{y}\right)\,dt + \int_0^T [(-\nu\Delta y, \bar{y}) + \langle\mathbf{b}^T\mathbf{h}, \bar{y}\rangle + \langle y\cdot\tau, \zeta((2\kappa-\alpha)v\cdot\tau + (\mathbf{b}^T\mathbf{u})\cdot\tau)\rangle \\ &\qquad + ((y\cdot\nabla)v + (v\cdot\nabla)y, \bar{y}) + (\nabla q, \bar{y})]\,dt \\ &\quad= \int_0^T (ze_2, \bar{y})\,dt + \int_0^T [\langle b_1, \bar{y}\rangle, \ldots, \langle b_M, \bar{y}\rangle]\mathbf{h}\,dt \\ &\qquad + \int_0^T \langle y\cdot\tau, \zeta((2\kappa-\alpha)v\cdot\tau + (\mathbf{b}^T\mathbf{u})\cdot\tau)\rangle\,dt.\end{aligned}\tag{5.26}$$

Therefore, combining (5.25) with (5.26) gives

$$\begin{aligned}J'(\mathbf{u})\cdot\mathbf{h} = &\int_0^T [\langle b_1, \bar{y}\rangle, \ldots, \langle b_M, \bar{y}\rangle]\mathbf{h}\,dt + \gamma\beta\int_0^T \mathbf{u}^T\mathbf{h}\,dt \\ &+ \gamma\eta\dot{\mathbf{u}}^T\mathbf{h}|_0^T - \gamma\eta\int_0^T \mathbf{h}^T\ddot{\mathbf{u}}\,dt.\end{aligned}\tag{5.27}$$

Again, set $\dot{u}(0) = \dot{u}(T) = 0$. Then

$$J'(\mathbf{u})\cdot\mathbf{h} = \int_0^T ([\langle b_1, \bar{y}\rangle, \ldots, \langle b_M, \bar{y}\rangle]^T + \gamma\beta\mathbf{u} - \gamma\eta\ddot{\mathbf{u}}, \mathbf{h}) \geq 0, \quad \mathbf{h}\in(H^1(0,T))^M,\tag{5.28}$$

which implies that

$$\gamma\eta\ddot{\mathbf{u}}^{opt} - \gamma\beta\mathbf{u}^{opt} = [\langle b_1, \bar{y}\rangle, \ldots, \langle b_M, \bar{y}\rangle]^T.\tag{5.29}$$

This completes the proof. ∎

*Remark* 5.6. Thanks to Proposition 2.4, compatibility conditions will not be required for $v_0 \in V_n^1(\Omega)$ in both passive and active scalar cases; thus, the controls can be localized on a portion of the boundary $\Gamma_c \subset \Gamma$.

**6. Conclusion.** In this paper, we presented a rigorous mathematical framework for optimizing mixing of nondissipative scalars via an active control of incompressible Navier–Stokes flows through Navier slip boundary conditions. The first-order necessary optimality conditions are derived by using a variational inequality. The main challenge arises in the analysis of differentiability of the semidissipative systems. Establishing the Gâteaux differentiability of the scalar field with respect to the control input requires that the gradient of the scalar field is well-defined, which results in demanding high regularity on the velocity field due to vanishing diffusivity together with the nonlinear coupling. As a consequence, the time regularity on the boundary data of the velocity field is needed. However, the optimality conditions can be implemented via the gradient-based iterative schemes to obtain the optimal solution explicitly. The optimal control synthesis established in this paper will also enable the study of optimal transport via fluid flows, tracking the moving fluid interfaces, or steering the scalar field to a desired distribution by formulating appropriate cost functionals. It is hoped that the results of this paper will stimulate further investigation in nonlinear control and optimization and lead to practical applications. Moreover, there are several interesting directions that merit further investigation in our future work, such as establishing the relation between the mixing scale $\|\theta\|_{(H^1(\Omega))'}$ and the optimal control actuation in terms of different boundary input profiles $b_i$ and establishing the sensitivity analysis of mixing rate with respect to vorticity.

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REFERENCES

[1] G. Alberti, G. Crippa, and A. L. Mazzucato, *Exponential Self-Similar Mixing by Incompressible Flows*, preprint, arXiv:1605.02090, 2016.

[2] H. Beirão Da Veiga, *Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions*, Adv. Differential Equations, 9 (2004), pp. 1079–1114.

[3] A. Biswasa, C. Foias, and A. Larios, *On the attractor for the semi-dissipative Boussinesq equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 34 (2017), pp. 381–405.

[4] C. Cao and J. Wu, *Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal., 208 (2013), pp. 985–1004.

[5] A. Celani, M. Cencini, A. Mazzino, and M. Vergassola, *Active and passive fields face to face*, New J. Phys., 6 (2004), p. 72.

[6] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math., 203 (2006), pp. 497–513.

[7] R.-H. Chen and J. F. Driscoll, *The role of the recirculation vortex in improving fuel-air mixing within swirling flames*, Sympos. Combust., 22 (1989), pp. 531–540.

[8] T. Clopeau, A. Mikelic, and R. Robert, *On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions*, Nonlinearity, 11 (1998), pp. 1625–1636.

[9] F. Coron, *Derivation of slip boundary conditions for the Navier-Stokes system from the Boltzmann equation*, J. Stat. Phys., 54 (1989), pp. 829–857.

[10] J.-M. Coron, *On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions*, ESAIM Control Optim. Calc. Var., 1 (1996), pp. 35–75.

[11] R. Danchin and M. Paicu, *Global existence results for the anisotropic Boussinesq system in dimension two*, Math. Models Methods Appl. Sci., 21 (2011), pp. 421–457.

[12] P. V. Danckwerts, *The definition and measurement of some characteristics of mixtures*, Appl. Sci. Res. Sect. A, 3 (1952), pp. 279–296.

[13] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., 98 (1989), pp. 511–547.

[14] M. C. L. Filho, H. J. N. Lopes, and G. Planas, *On the inviscid limit for two-dimensional incompressible flow with Navier friction condition*, SIAM J. Math. Anal., 34 (2005), pp. 1130–1141.

[15] D. P. G. Foures, C. P. Caulfield, and P. J. Schmid, *Optimal mixing in two-dimensional plane Poiseuille flow at finite Péclet number*, J. Fluid Mech., 748 (2014), pp. 241–277.
[16] I. G. Girgis and J. T. C. Liu, *Mixing enhancement via the release of strongly nonlinear longitudinal Görtler vortices and their secondary instabilities into the mixing region*, J. Fluid Mech., 468 (2002), pp. 29–75.
[17] E. Gouillart, O. Dauchot, B. Dubrulle, S. Roux, and J.-L. Thiffeault, *Slow decay of concentration variance due to no-slip walls in chaotic mixing*, Phys. Rev. E, 78 (2008), p. 026211.
[18] E. Gouillart, N. Kuncio, O. Dauchot, B. Dubrulle, S. Roux, and J.-L. Thiffeault, *Walls inhibit chaotic mixing*, Phys. Rev. Lett., 99 (2007), p. 114501.
[19] E. Gouillart, J.-L. Thiffeault, and O. Dauchot, *Rotation shields chaotic mixing regions from no-slip walls*, Phys. Rev. Lett., 104 (2010), p. 204502.
[20] O. Gubanov and L. Cortelezzi, *Towards the design of an optimal mixer*, J. Fluid Mech., 651 (2010), pp. 27–53.
[21] P. Hassanzadeh, G. P. Chini, and C. R. Doering, *Wall to wall optimal transport*, J. Fluid Mech., 751 (2014), pp. 627–662.
[22] T. Y. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete Contin. Dyn. Syst., 12 (2005), pp. 1–12.
[23] W. Hu, *Boundary control for optimal mixing by Stokes flows*, Appl. Math. Optim., 2017. https://doi.org/10.1007/s00245-017-9404-6.
[24] W. Hu, I. Kukavica, F. Wang, and M. Ziane, *Boussinesq equations with zero viscosity or zero diffusivity: A review*, in Recent Progress in the Theory of the Euler and Navier-Stokes Equations, Part of London Mathematical Society Lecture Note Series, 430, 2016.
[25] W. Hu, I. Kukavica, and M. Ziane, *On the regularity for the Boussinesq equations in a bounded domain*, J. Math. Phys., 54 (2013), p. 081507.
[26] W. Hu, I. Kukavica, and M. Ziane, *Persistence of regularity for the viscous Boussinesq equations with zero diffusivity*, Asymptot. Anal., 91 (2015), pp. 111–124.
[27] W. Hu, Y. Wang, J. Wu, B. Xiao, and J. Yuan, *Partially dissipative 2D Boussinesq equations with Navier type boundary conditions*, Phys. D, 376–377 (2018), pp. 39–48.
[28] G. Iyer, A. Kiselev, and X. Xu, *Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows*, Nonlinearity, 27 (2014), p. 973.
[29] N. Ju, *Global regularity and long-time behavior of the solutions to the 2D Boussinesq equations without diffusivity in a bounded domain*, J. Math. Fluid Mech., (2016), pp. 1–17.
[30] J. P. Kelliher, *Navier–Stokes equations with Navier boundary conditions for a bounded domain in the plane*, SIAM J. Math. Anal., 38 (2006), pp. 210–232.
[31] M. Lai, R. Pan, and K. Zhao, *Initial boundary value problem for 2D viscous Boussinesq equations*, Arch. Ration. Mech. Anal., 199 (2011), pp. 739–760.
[32] A. Larios, E. Lunasin, and E. S. Titi, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Differential Equations, 255 (2013), pp. 2636–2654.
[33] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, Vol. 1, Abstract Parabolic Systems, 2000.
[34] Z. Lin, J.-L. Thiffeault, and C. R. Doering, *Optimal stirring strategies for passive scalar mixing*, J. Fluid Mech., 675 (2011), pp. 465–476.
[35] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations Problèmes aux Limites*, Springer, New York, 1971.
[36] W. Liu, *Mixing enhancement by optimal flow advection*, SIAM J. Control Optim., 47 (2008), pp. 624–638.
[37] E. Lunasin, Z. Lin, A. Novikov, A. Mazzucato, and C. R. Doering, *Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows*, J. Math. Phys., 53 (2012), p. 115611.
[38] G. Mathew, I. Mezić, S. Grivopoulos, U. Vaidya, and L. Petzold, *Optimal control of mixing in Stokes fluid flows*, J. Fluid Mech., 580 (2007), pp. 261–281.
[39] G. Mathew, I. Mezić, and L. Petzold, *A multiscale measure for mixing*, Phys. D, 211 (2005), pp. 23–46.
[40] J. C. Maxwell, *On stresses in rarified gases arising from inequalities of temperature*, Philos. Trans. Roy. Soc. London, 170 (1879), pp. 231–256.
[41] H. K. Moffatt, *Some remarks on topological fluid mechanics*, in An Introduction to the Geometry and Topology of Fluid Flows, Springer, New York, 2001, pp. 3–10.
[42] C.-L. Navier, *Mémoire sur les lois du mouvement des fluides*, Mem. Acad. Roy. Sci. Inst. France, 6 (1823), pp. 389–440.
[43] A. Sharma and N. Gupte, *Control Methods for Problems of Mixing and Coherence in Chaotic Maps and Flows*, Vol. 48, Springer, New York, 1997, pp. 231–248.

[44] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis, Studies in Mathematics and Its Applications*, Vol. 2, North-Holland, Amsterdam, 1997.

[45] J.-L. Thiffeault, *Using multiscale norms to quantify mixing and transport*, Nonlinearity, 25 (2012), p. R1.

[46] J.-L. Thiffeault, E. Gouillart, and O. Dauchot, *Moving walls accelerate mixing*, Phys. Rev. E, 84 (2011), p. 036313.

[47] A. Vikhansky, *Enhancement of laminar mixing by optimal control methods*, Chem. Eng. Sci., 57 (2002), pp. 2719–2725.

[48] I. A. Waitz, Y. J. Qiu, T. A. Manning, A. K. S. Fung, J. K. Elliot, J. M. Kerwin, J. K. Krasnodebski, M. N. O'Sullivan, D. E. Tew, E. M. Greitzer, F. E. Marble, C. S. Tan, and T. G. Tillman, *Enhanced mixing with streamwise vorticity*, Prog. Aerosp. Sci., 33 (1997), pp. 323–351.

[49] Z. Warhaft, *Passive scalars in turbulent flows*, Annu. Rev. Fluid Mech., 32 (2000), pp. 203–240.

[50] Y. Yao and A. Zlatoš, *Mixing and un-mixing by incompressible flows*, J. Eur. Math. Soc., 19 (2017), pp. 1911–1948.