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Partially dissipative 2D Boussinesq equations with Navier type boundary conditions

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Dedicated to Professor Edriss S. Titi on the occasion of his sixtieth birthday

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ABSTRACT

This paper concerns itself with two systems of the 2D Boussinesq equations with partial dissipation in bounded domains with the Navier type boundary conditions. We attempt to achieve two main goals: first, to prove the global existence and uniqueness under minimal regularity assumptions on the initial data; and second, to provide a direct and transparent approach that explicitly reveals the impacts of the Navier boundary conditions. The 2D Boussinesq equations with partial dissipation have attracted considerable interests in the last few years, although most of the results are aimed at sufficiently regular solutions in the whole space or periodic domains. Larios et al. (2013) made serious efforts to minimize the regularity assumptions necessary for the uniqueness of solutions in the spatially periodic setting. In contrast to the whole space and the periodic domains, the Navier boundary conditions generate boundary terms and require compatibility conditions. In addition, due to the lack of boundary conditions for the pressure, we resort to the existence and regularity result on the associated Stokes problem with Navier boundary conditions. The uniqueness relies on the Yudovich techniques and the introduction of a lower regularity counterpart of the temperature.

1. Introduction

The Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation (see, e.g., [1–3]). In addition, they play an important role in the study of Rayleigh–Benard convection (see, e.g., [4,5]). This paper is concerned with two systems of partially dissipated 2D Boussinesq equations: the Boussinesq system with only kinematic dissipation (without thermal diffusion)

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\,\Delta\mathbf{u} + \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{1.1}$$

and its counterpart with only partial kinematic dissipation. Instead of the full Laplacian dissipation, this partially dissipative system has only vertical dissipation in the horizontal velocity equation and horizontal dissipation in the vertical velocity equation

$$\begin{cases} \partial_t u_1 + \mathbf{u}\cdot\nabla u_1 = -\partial_1 p + \nu\,\partial_{22}u_1, \\ \partial_t u_2 + \mathbf{u}\cdot\nabla u_2 = -\partial_2 p + \nu\,\partial_{11}u_2 + \theta, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = 0, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.2}$$

In these equations $\mathbf{u} = \mathbf{u}(x,t)$ represents the 2D velocity with its horizontal and vertical components given by u_1 and u_2, respectively, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, $\mathbf{e}_2$ the unit vector in the vertical direction, and $\nu > 0$ represents the kinematic viscosity.

Our attention will be mainly focused on spatial domains $\Omega \subset \mathbb{R}^2$ that are bounded, connected and have smooth boundary, although the results presented here are also valid for $\Omega = \mathbb{R}^2$ and periodic domains, as explained later. We assume the velocity field $\mathbf{u}$ obeys the Navier boundary conditions. The Navier boundary conditions allow the fluid to slip along the boundary and require that the tangential component of the stress vector at the boundary be proportional to the tangential velocity. In the case of (1.1), the corresponding stress tensor $T = (T_{ij})$ is given by

$$T_{ij} = -\delta_{ij}p + 2\nu D_{ij}(\mathbf{u}), \qquad D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \text{or}$$

$$D(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

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and, if $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit normal and tangent vectors to the boundary $\partial\Omega$, respectively, the proportionality is then represented by

$$\sum_{i,j=1,2} \tau_i T_{ij} n_j = \sigma \sum_{k=1,2} u_k \tau_k \quad \text{on} \quad \partial\Omega$$

for a constant σ. Due to the orthogonality of $\mathbf{n}$ and $\boldsymbol{\tau}$,

$$\sum_{i,j=1,2} \tau_i \delta_{ij} p\, n_j = 0.$$

The Navier boundary conditions for (1.1) then become

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad 2\,\mathbf{n}\cdot D(\mathbf{u})\cdot\boldsymbol{\tau} + \alpha\mathbf{u}\cdot\boldsymbol{\tau} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\alpha > 0$ is a constant. For the system in (1.2), the kinematic dissipation is only partial. We follow the same principle to propose the corresponding Navier boundary condition. The corresponding stress tensor T associated with (1.2) is given by

$$T = -pI + 2\nu E(\mathbf{u})$$
$$\text{with} \quad E(\mathbf{u}) = \frac{1}{2}\begin{pmatrix} 0 & \partial_2 u_1 + \partial_1 u_2 \\ \partial_2 u_1 + \partial_1 u_2 & 0 \end{pmatrix}$$

and consequently, the Navier boundary conditions for (1.2) are

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad 2\,\mathbf{n}\cdot E(\mathbf{u})\cdot\boldsymbol{\tau} + \alpha\mathbf{u}\cdot\boldsymbol{\tau} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

As documented in many papers, the Navier boundary conditions are important in modeling many flows in the real world (see, e.g. [6–8]). Since the temperature is transported by the velocity field $\mathbf{u}$, no boundary condition should be imposed on θ.

A very important special case of the Navier boundary conditions in (1.3) or (1.4) is the stress-free boundary condition for which the vorticity $\omega = \nabla \times u$ vanishes on $\partial\Omega$,

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \omega = \partial_1 u_2 - \partial_2 u_1 = 0 \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

In addition, (1.1) and (1.2) will be supplemented with the initial data

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \Omega. \tag{1.6}$$

The goal of this paper is three fold: first, to establish the global existence and uniqueness of solutions to (1.1) and (1.2) with their corresponding Navier boundary conditions, second, to obtain the uniqueness of solutions with minimal regularity assumption on the initial data (u_0, θ_0), and third, to employ a direct approach from which one can clearly see the impacts of the Navier boundary conditions as opposed to those of the periodic boundary conditions and of the whole space case. Our main results are stated in the following two theorems.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$ (Lipschitz continuous second derivatives). Let $\nu > 0$. Consider the initial and boundary value problem (IBVP) in* (1.1), (1.3) and (1.6) *with $\alpha > 0$ being a constant and*

$$\mathbf{u}_0 \in H^1(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0$$

and

$$\theta_0 \in L^2(\Omega)\cap L^\infty(\Omega), \qquad \int_\Omega \theta_0(x)\,dx = 0.$$

Then the IBVP (1.1), (1.3) and (1.6) *has a unique global (in time) strong solution $(\mathbf{u}, \theta)$ satisfying, for any $T > 0$,*

$$\mathbf{u} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$
$$\theta \in L^\infty(0,\infty; L^2(\Omega)\cap L^\infty(\Omega)),$$
$$\int_\Omega \theta(x,t)\,dx = 0 \ \textit{for any}\ t \in [0,\infty). \tag{1.7}$$

When Ω is bounded, $\theta \in L^\infty(\Omega)$ automatically implies $\theta \in L^2(\Omega)$. We have kept $\theta \in L^2(\Omega)$ in the statement of Theorem 1.1 for the convenience of extension to the whole plane case below.

(1.2) involves only partial kinematic dissipation. The global well-posedness result obtained for this system is for the stress-free boundary condition

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \partial_1 u_2 = \partial_2 u_1 = 0 \quad \text{on} \quad \partial\Omega. \tag{1.8}$$

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Let $\nu > 0$. Consider the initial and boundary value problem (IBVP) in* (1.2), (1.6) and (1.8) *with $\alpha > 0$ being a constant and*

$$\mathbf{u}_0 \in H^1(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0$$

and

$$\theta_0 \in L^2(\Omega)\cap L^\infty(\Omega), \qquad \int_\Omega \theta_0(x)\,dx = 0.$$

Then the IBVP (1.1), (1.6) and (1.8) *has a unique global (in time) strong solution $(\mathbf{u}, \theta)$ satisfying, for any $T > 0$,*

$$\mathbf{u} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$
$$\theta \in L^\infty(0,\infty; L^2(\Omega)\cap L^\infty(\Omega)),$$
$$\int_\Omega \theta(x,t)\,dx = 0 \ \textit{for}\ t \in [0,\infty). \tag{1.9}$$

In contrast to the periodic boundary condition case or the whole space (with sufficient decay at ∞) case, the Navier type boundary conditions generate boundary terms and require compatibility conditions. In fact, the mean-zero assumption on θ_0 in Theorems 1.1 and 1.2, namely

$$\int_\Omega \theta_0(x)\,dx = 0 \tag{1.10}$$

is imposed to fulfill the compatibility condition in the proof of the uniqueness of the solutions. It is not difficult to understand that the results of Theorems 1.1 and 1.2 without (1.10) in the whole space or periodic domain case remain valid. More precisely, the following corollary (as consequences of the proofs of Theorems 1.1 and 1.2) holds.

Corollary 1.3. *Assume $\Omega = \mathbb{R}^2$ or $\Omega = [0, 2\pi]^2$ (periodic box). Assume $(\mathbf{u}_0, \theta_0)$ satisfies*

$$\mathbf{u}_0 \in H^1(\Omega), \quad \nabla\cdot\mathbf{u}_0 = 0, \qquad \theta_0 \in L^2(\Omega)\cap L^\infty(\Omega).$$

Then the initial value problem (IVP) (1.1) and (1.6) *or IVP* (1.2) and (1.6) *has a unique global strong solution $(\mathbf{u}, \theta)$ satisfying, for any $T > 0$,*

$$\mathbf{u} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$
$$\theta \in L^\infty(0,\infty; L^2(\Omega)\cap L^\infty(\Omega)).$$

The Navier–Stokes equations with Navier type boundary conditions have been studied extensively and there are many excellent references (see, e.g., [9–11]). The 2D Boussinesq equations with partial dissipation have recently attracted enormous attention, but most of the studies focus on the whole space or the periodic boundary. This paper is devoted to the partially dissipated Boussinesq equations with the Navier type boundary conditions. The theorems of this paper fill this gap. In addition, we strive to establish the uniqueness under minimal regularity assumptions on the initial data. [12] and [13] examined global solutions of (1.1) in the whole space $\mathbb{R}^2$ for $(\mathbf{u}_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$. [14] obtained the global existence and regularity of (1.2) in the whole space $\mathbb{R}^2$ for $(\mathbf{u}_0, \theta_0) \in H^3(\mathbb{R}^2)$. Larios, Lunasin and Titi [15] and Hu, Kukavica and Ziane [16] have made serious efforts to reduce the regularity assumption on the initial data. Their attention was focused on the periodic domain or whole space and established the uniqueness of solutions to (1.1) under reduced regularity condition on $(\mathbf{u}_0, \theta_0)$. We also mention the results of [17–19] on (1.1) with Dirichlet boundary conditions, of [15,20] on the Boussinesq equations with horizontal dissipation and of [14,21–23] on the Boussinesq equations with vertical dissipation in the whole space. We apologize for not being able to list all the relevant references.

We remark that the Navier type conditions are more delicate to handle. They generate boundary terms in the process of integration by parts and make the pressure term p hard to deal with due to the lack of boundary condition on p. In spite of these difficulties, we strive to provide a direct and transparent approach to the desired global bounds. To obtain a global bound for the H^1-norm of $\mathbf{u}$ to (1.1), we resort to the existence and regularity result of Beirao da Veiga on the associated Stokes problem with the Navier type boundary conditions (see [24] as well as Lemma 2.1 in Section 2). The global H^1 bound on $\mathbf{u}$ of (1.2) relies on the vorticity formulation. Due to the lack of the global bound for $\|\nabla \mathbf{u}\|_{L^\infty}$, the uniqueness relies on the Yudovich technique and the introduction of a lower regularity counterpart of θ.

The rest of this paper is divided into three sections. Section 2 makes several preparations including the result of Beirao da Veiga on the Stokes problem with the Navier type boundary conditions, the Calderon–Zygmund inequality for bounded domains with slip boundary conditions and several useful identities involving the Navier type boundary conditions. Section 3 proves Theorem 1.1 while Section 4 provides the proof of Theorem 1.2. For the sake of clarity, Sections 3 and 4 are further divided into subsections.

2. Preparations

This section makes several preparations for the proofs of Theorems 1.1 and 1.2. The first one states the existence and regularity result of Beirao da Veiga on the Stokes problem with the Navier type boundary conditions. The second one provides the Calderon–Zygmund type inequality for functions obeying no slip boundary conditions. The third preparation involves two lemmas presenting several identities on quantities that obey the Navier boundary conditions. In particular, these identities would facilitate the integration by parts process involving the dissipative term. We also provide an Osgood type inequality and its proof. Finally we define a weak formulation and a lemma stating the existence of solutions to this weak formulation.

As we know, regularity estimates for solutions of the Navier–Stokes equations with the classical no-slip boundary condition rely on the Stokes operator associated with the no slip boundary condition. For the Stokes problem with Navier type boundary conditions, H. Beirao da Veiga in [24] established a general existence and regularity theory. A special consequence of his theory is provided in the following lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Let $\alpha \geq 0$ be a constant and let $f \in L^2(\Omega)$. Consider the following Stokes problem with the Navier type boundary condition,*

$$\begin{cases} -\nu\,\Delta \mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{u}\cdot\mathbf{n} = 0, \quad 2\,\mathbf{n}\cdot D(\mathbf{u})\cdot\boldsymbol{\tau} + \alpha\mathbf{u}\cdot\boldsymbol{\tau} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Then (2.1) has a unique strong solution $(\mathbf{u}, p) \in H^2(\Omega) \times H^1(\Omega)$ (p is unique up to an additive constant). Moreover, for a constant $C = C(\Omega, \nu)$,

$$\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C\,\|f\|_{L^2(\Omega)}. \tag{2.2}$$

For notational convenience, we write

$$A\mathbf{u} \equiv -\Delta\mathbf{u} + \frac{1}{\nu}\nabla p. \tag{2.3}$$

(2.2) implies

$$\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C\,\|A\,\mathbf{u}\|_{L^2(\Omega)}. \tag{2.4}$$

The next lemma asserts the Calderon–Zygmund type inequality for divergence-free velocity fields that obey the slip boundary conditions (see, e.g., [11]).

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Let $\mathbf{n}$ denote the unit outnormal vector along $\partial\Omega$. Assume $\mathbf{u} \in L^2(\Omega)$ satisfies*

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \qquad \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega.$$

Let $q \in [2, \infty)$ and $\omega = \nabla \times \mathbf{u} \in L^q(\Omega)$. Then

$$\|\nabla\mathbf{u}\|_{L^q(\Omega)} \leq C_1(\Omega)\,q\|\omega\|_{L^q(\Omega)} + C_2(\Omega)\,\|\mathbf{u}\|_{L^2(\Omega)}.$$

In the special case when Ω is simply connected,

$$\|\nabla\mathbf{u}\|_{L^q(\Omega)} \leq C_1(\Omega)\,q\|\omega\|_{L^q(\Omega)}.$$

Due to the Navier boundary conditions, the integration by parts process in general generates boundary terms. The following two lemmas facilitate the integration by parts process. They are especially useful when we handle the dissipative term.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^2$. Let κ denote the curvature of $\partial\Omega$. As before, τ and $\mathbf{n}$ denote the unit tangential and outnormal vector along $\partial\Omega$. Assume $u \in C^1(\overline{\Omega})$.*

(1) *Assume $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Writing $\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n} = \tau_k\partial_k u_j\, n_j$ with Einstein's summation convention, we have*

$$\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n} + \kappa\,\mathbf{u}\cdot\tau = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

(2) *Assume $\mathbf{u} \in C^1(\overline{\Omega})$ satisfies the Navier boundary conditions*

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad 2\mathbf{n}\cdot D(\mathbf{u})\cdot\tau + \alpha\,\mathbf{u}\cdot\tau = 0 \text{ on } \partial\Omega. \tag{2.6}$$

Then,

$$\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau + (\alpha - \kappa)\,\mathbf{u}\cdot\tau = 0 \text{ on } \partial\Omega \tag{2.7}$$

and

$$\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau + \frac{\alpha}{2}(\mathbf{u}\cdot\tau) = \frac{\omega}{2} \text{ on } \partial\Omega. \tag{2.8}$$

Especially, $\omega = 0$ on $\partial\Omega$ if and only if $\kappa = \frac{\alpha}{2}$.

As we shall see in the subsequent sections, $\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau$ plays a crucial role in the handling of the dissipation and the identities stated here will be very handy. We alert that $\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n}$ differs from $\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau$ in general.

Lemma 2.4. *Assume Ω obeys the same conditions as in Lemma 2.3. Assume that $\mathbf{u}, \mathbf{v} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and they both satisfy the Navier boundary conditions, namely (2.6). Then*

$$\int_\Omega \Delta\mathbf{u}\cdot\mathbf{v}\,dx = -2\int_\Omega D(\mathbf{u})\cdot D(\mathbf{v})\,dx - \int_{\partial\Omega} \alpha(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x) \tag{2.9}$$

$$= -\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_{\partial\Omega}(\kappa - \alpha)(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x). \tag{2.10}$$

In particular, when $\mathbf{u}=\mathbf{v}$*, we have*

$$\int_\Omega \Delta\mathbf{u}\cdot\mathbf{u}\,dx = -2\int_\Omega |D(\mathbf{u})|^2\,dx - \int_{\partial\Omega}\alpha(\mathbf{u}\cdot\tau)^2\,dS(x) = -\int_\Omega |\nabla\mathbf{u}|^2\,dx + \int_{\partial\Omega}(\kappa-\alpha)(\mathbf{u}\cdot\tau)^2\,dS(x).$$

For the convenience of the readers, we provide the proofs of Lemmas 2.3 and 2.4. Some components can be found in [10,11].

Proof of Lemma 2.3. Since $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$, the directional derivative of $\mathbf{u}\cdot\mathbf{n}$ along $\partial\Omega$ should also be zero, namely

$$\frac{d}{d\tau}(\mathbf{u}\cdot\mathbf{n}) = 0 \quad \text{on} \quad \partial\Omega.$$

The product rule then yields

$$\left(\frac{d}{d\tau}\mathbf{u}\right)\cdot\mathbf{n} + \mathbf{u}\cdot\left(\frac{d}{d\tau}\mathbf{n}\right) = 0 \quad \text{or}$$

$$\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n} + \mathbf{u}\cdot(\tau\cdot\nabla\mathbf{n}) = 0 \quad \text{on} \quad \partial\Omega.$$

Due to $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$,

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + (\mathbf{u}\cdot\tau)\tau = (\mathbf{u}\cdot\tau)\tau \quad \text{on} \quad \partial\Omega.$$

Therefore, due to $\kappa = \tau\cdot\nabla\mathbf{n}\cdot\tau$,

$$\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n} + (\tau\cdot\nabla\mathbf{n}\cdot\tau)(\mathbf{u}\cdot\tau) = 0 \quad \text{or}$$

$$\tau\cdot\nabla\mathbf{u}\cdot\mathbf{n} + \kappa\,\mathbf{u}\cdot\tau = 0 \quad \text{on} \quad \partial\Omega.$$

To prove (2.7), we recall $2D(\mathbf{u}) = \nabla\mathbf{u}+(\nabla\mathbf{u})^T$, and invoke (2.5) and (2.6)

$$\begin{aligned}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau &= 2\mathbf{n}\cdot D(\mathbf{u})\cdot\tau - \mathbf{n}\cdot(\nabla\mathbf{u})^T\cdot\tau\\ &= -\alpha(\mathbf{u}\cdot\tau) - \tau\cdot\nabla\mathbf{u}\cdot\mathbf{n}\\ &= (\kappa-\alpha)(\mathbf{u}\cdot\tau).\end{aligned}$$

To prove (2.8), we write

$$\nabla\mathbf{u} = D(\mathbf{u}) + \frac{1}{2}\left(\nabla u - (\nabla u)^T\right) = D(\mathbf{u}) + \frac{1}{2}\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}.$$

Thus

$$\begin{aligned}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau &= \mathbf{n}\cdot D(\mathbf{u})\cdot\tau + \mathbf{n}\cdot\frac{1}{2}\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}\cdot\tau\\ &= -\frac{\alpha}{2}(\mathbf{u}\cdot\tau) + \frac{\omega}{2}(-\tau_1 n_2 + n_1\tau_2)\\ &= -\frac{\alpha}{2}(\mathbf{u}\cdot\tau) + \frac{\omega}{2}\end{aligned}$$

due to $-\tau_1 n_2 + n_1\tau_2 = \tau_1^2+\tau_2^2 = 1$. This completes the proof of Lemma 2.3. □

Proofs of Lemma 2.4. Adopting Einstein's summation convention, we write

$$\begin{aligned}\int_\Omega \Delta\mathbf{u}\cdot\mathbf{v}\,dx &= \int_\Omega(\partial_k\partial_k u_j)v_j\,dx\\ &= \int_\Omega(\partial_k(\partial_k u_j v_j) - \partial_k u_j\partial_k v_j)\,dx\\ &= \int_{\partial\Omega} n_k\partial_k u_j v_j\,dx - \int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx\\ &= \int_{\partial\Omega}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{v}\,dS(x) - \int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx.\end{aligned}$$

Due to $\mathbf{v}\cdot\mathbf{n}=0$ on $\partial\Omega$, we write $\mathbf{v}=(\mathbf{v}\cdot\tau)\tau$ and obtain, by Lemma 2.3,

$$\begin{aligned}\int_{\partial\Omega}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{v}\,dS(x) &= \int_{\partial\Omega}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau(\mathbf{v}\cdot\tau)dS(x)\\ &= \int_{\partial\Omega}(\kappa-\alpha)(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)dS(x)\end{aligned}$$

Therefore, we have obtained (2.10),

$$\int_\Omega \Delta\mathbf{u}\cdot\mathbf{v}\,dx = -\int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_{\partial\Omega}(\kappa-\alpha)(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x).$$

To prove (2.9), we write out the terms in $D(\mathbf{u})\cdot D(\mathbf{v})$,

$$\begin{aligned}2\int_\Omega D(\mathbf{u})\cdot D(\mathbf{v})\,dx &= \int_\Omega\left(\nabla\mathbf{u}\cdot\nabla\mathbf{v} + \nabla\mathbf{u}\cdot(\nabla\mathbf{v})^T\right)dx\\ &= \int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_\Omega\partial_j u_k\partial_k v_j\,dx\\ &= \int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_\Omega\partial_k(\partial_j u_k v_j)\,dx\\ &= \int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_{\partial\Omega} n_k\partial_j u_k v_j\,dS(x)\\ &= \int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}dx + \int_{\partial\Omega}\mathbf{v}\cdot\nabla\mathbf{u}\cdot\mathbf{n}\,dS(x).\end{aligned}$$

Writing $\mathbf{v}=(\mathbf{v}\cdot\tau)\tau$ and applying Lemma 2.3, we have

$$2\int_\Omega D(\mathbf{u})\cdot D(\mathbf{v})\,dx = \int_\Omega\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx - \kappa\int_{\partial\Omega}(\mathbf{u}\cdot\tau)(\mathbf{v}\cdot\tau)\,dS(x). \tag{2.11}$$

Combining (2.10) and (2.11) yields (2.9). This completes the proof of Lemma 2.4. □

The following Osgood type inequality will also be used.

Lemma 2.5. *Let* $T>0$ *and* $I=[0,T)$*. Let* $f\geq 0$ *be a measurable function on* I*. Let* $A\geq 0$ *and* $B\geq 0$*, and* $A,B\in L^1(I)$*. Let* $M>0$ *be a fixed constant. Assume* f *satisfies, for* $t\in I$*,*

$$\frac{df}{dt} \leq Af + Bf(\ln M - \ln f).$$

Then, for $t\in I$*,*

$$f(t) \leq f(0)^{e^{-\int_0^t B(\tau)d\tau}} M^{1-e^{-\int_0^t B(\tau)d\tau}} e^{\int_0^t A(s)e^{\int_s^t B(\tau)d\tau}ds}.$$

Especially, $f(0)=0$ *implies* $f(t)=0$ *for* $t\in I$*.*

Proof. A quick proof is provided here for reader's convenience. We consider the case when $f\neq 0$. Dividing by f yields

$$\begin{aligned}\frac{d\ln f}{dt} &\leq A + B(\ln M - \ln f)\\ -\frac{d(\ln M - \ln f)}{dt} &\leq A + B(\ln M - \ln f)\\ -\frac{d}{dt}\left(e^{\int_0^t B(\tau)d\tau}(\ln M - \ln f)\right) &\leq e^{\int_0^t B(\tau)d\tau}A\\ e^{\int_0^t B(\tau)d\tau}\ln\frac{M}{f} &\geq \ln\frac{M}{f(0)} - \int_0^t e^{\int_0^s B(\tau)d\tau}A(s)ds\\ \ln\frac{M}{f} &\geq e^{-\int_0^t B(\tau)d\tau}\ln\frac{M}{f(0)}\\ &\quad - \int_0^t e^{\int_t^s B(\tau)d\tau}A(s)ds\\ \frac{M}{f} &\geq \left(\frac{M}{f(0)}\right)^{e^{-\int_0^t B(\tau)d\tau}} e^{-\int_0^t e^{\int_t^s B(\tau)d\tau}A(s)ds}\\ f &\leq f(0)^{e^{-\int_0^t B(\tau)d\tau}} M^{1-e^{-\int_0^t B(\tau)d\tau}}\\ &\quad \times e^{\int_0^t A(s)e^{\int_s^t B(\tau)d\tau}}ds.\end{aligned}$$

This completes the proof of Lemma 2.5. □

Finally we provide the local well-posedness theory for the IBVP (1.1), (1.3) and (1.6). To do so, we first define the functional setting and the weak formulation of this IBVP. We set

$$H = \left\{\mathbf{v} \in L^2(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\},$$
$$V = \left\{\mathbf{v} \in H^1(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\}.$$

Definition 2.6. Let $T > 0$. Assume $\mathbf{u}_0 \in H$ and $\theta_0 \in L^2 \cap L^\infty$. $(\mathbf{u}, \theta)$ with $\mathbf{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$ and $\theta \in L^\infty(0, T; L^2 \cap L^\infty)$ is a weak solution of the IBVP (1.1), (1.3) and (1.6) if, for $t \in [0, T]$,

$$\int_\Omega \mathbf{u}(t) \cdot \mathbf{v}\, dx - \int_0^t \int_\Omega \mathbf{u} \cdot \nabla\mathbf{u} \cdot \mathbf{v}\, dxds + \nu \int_0^t \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v}\, dxds$$
$$- \nu \int_0^t \int_{\partial\Omega} (\kappa - \alpha)(\mathbf{u} \cdot \tau)(\mathbf{v} \cdot \tau)\, dS(x)ds = \int_\Omega \mathbf{u}_0 \cdot \mathbf{v}\, dx$$
$$+ \int_0^t \int_\Omega \theta \mathbf{e}_2 \cdot \mathbf{v}\, dxds$$

for all $\mathbf{v} \in V$, and

$$\int_\Omega \theta(t)\, \psi\, dx - \int_0^t \int_\Omega \mathbf{u} \cdot \nabla\psi\, \theta\, dxds = \int_\Omega \theta_0\, \psi\, dx$$

for all $\psi \in H^1$.

We have incorporated the result of Lemma 2.4 into this weak formulation. The global existence of solutions corresponding to the weak formulation defined in Definition 2.6 can be stated as follows.

Lemma 2.7. *Assume Ω obeys the same conditions as in Lemma 2.3. Let $\nu > 0$. Assume $\mathbf{u}_0 \in H$ and $\theta_0 \in L^2 \cap L^\infty$. Then the IBVP (1.1), (1.3) and (1.6) has a global solution $(\mathbf{u}, \theta)$ in the sense of Definition 2.6 satisfying, for any $T > 0$,*

$$u \in C([0, T]; H) \cap L^2(0, T; V), \quad \theta \in C_w(0, T; L^2) \cap L^\infty(0, T; L^\infty)$$

and

$$\frac{d\mathbf{u}}{dt} \in L^2(0, T; V'), \quad \frac{d\theta}{dt} \in L^2(0, T; H^{-1}).$$

Lemma 2.7 can be established following the Galerkin approximation approach as in [10,11,25] and [26]. As in the case of the Navier–Stokes equations in a bounded domain with the Navier type boundary conditions (see [10]), the boundedness and convergence of the approximation sequence for $\mathbf{u}$ can be similarly proven. The convergence of the approximation sequence for θ is in $L^\infty(0, T; H^{-1})$. We omit further details.

3. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. The proof is divided into two major parts. The first part establishes the global H^1 bound while the second shows the uniqueness. The proof for the global L^2-bound makes use of Lemmas 2.3 and 2.4. Due to the lack of the boundary condition for the pressure p, the global H^1 bound relies on the existence and regularity result on the Stokes system stated in Lemma 2.1. Since we do not know if $\|\nabla u\|_{L^\infty}$ is globally bounded, the uniqueness proof resorts to the Yudovich technique and the introduction of a lower regularity counterpart of θ. The rest is divided into three subsections.

3.1. Global L^2 bound

This subsection proves the *a priori* bounds stated in the following proposition.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^2$. Assume the initial data $(\mathbf{u}_0, \theta_0)$ satisfies the conditions stated in Theorem 1.1. Let $(\mathbf{u}, \theta)$ be the corresponding solution of the IBVP (1.1), (1.3) and (1.6). Then $(\mathbf{u}, \theta)$ obeys the global bounds, for any $t > 0$,*

$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q} \quad \text{for any} \quad 2 \le q \le \infty,$$
$$\|\mathbf{u}(t)\|_{L^2} \le \|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2},$$
$$\int_0^t \int_\Omega |\nabla\mathbf{u}|^2\, dx\, d\tau \le (\|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Proof of Proposition 3.1. For any $2 \le q < \infty$, we obtain by multiplying the equation of θ in (1.1) by $\theta|\theta|^{q-2}$,

$$\frac{1}{q}\frac{d}{dt}\|\theta\|_{L^q}^q = -\int_\Omega \theta|\theta|^{q-2}\mathbf{u} \cdot \nabla\theta\, dx.$$

Due to $\nabla \cdot \mathbf{u} = 0$, the divergence theorem and (1.3),

$$\int_\Omega \theta|\theta|^{q-2}\mathbf{u} \cdot \nabla\theta\, dx = \frac{1}{q}\int_{\partial\Omega} |\theta|^q \mathbf{u} \cdot \mathbf{n}\, dS(x) = 0.$$

As a consequence, for any $t > 0$,

$$\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q} \quad \text{and} \quad \|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}.$$

Taking the inner product of $\mathbf{u}$ with the equation of $\mathbf{u}$ in (1.1) yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 = \nu\int_\Omega \mathbf{u} \cdot \Delta\mathbf{u}\, dx + \int_\Omega \theta\, u_2\, dx, \tag{3.1}$$

where we have invoked the facts, due to $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$,

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u}\, dx = \frac{1}{2}\int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n})|\mathbf{u}|^2\, dS(x) = 0,$$
$$\int_\Omega \mathbf{u} \cdot \nabla p\, dx = 0.$$

According to Lemma 2.4,

$$\int_\Omega \mathbf{u} \cdot \Delta\mathbf{u}\, dx = -2\int_\Omega |D(\mathbf{u})|^2\, dx - \int_{\partial\Omega} \alpha\,(\mathbf{u} \cdot \tau)^2\, dS(x).$$

Therefore,

$$\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 + 4\nu\int_\Omega |D(\mathbf{u})|^2\, dx + 2\nu\int_{\partial\Omega} \alpha\,(\mathbf{u} \cdot \tau)^2\, dS(x)$$
$$\le 2\,\|\theta\|_{L^2}\|\mathbf{u}\|_{L^2},$$

which, in particular, implies

$$\|\mathbf{u}(t)\|_{L^2} \le \|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2}.$$

Furthermore, for any $t > 0$,

$$\nu\int_0^t \int_\Omega |D(\mathbf{u})|^2\, dxds,$$
$$\nu\int_0^t \int_{\partial\Omega} \alpha\,(\mathbf{u} \cdot \tau)^2\, dS(x)ds \le (\|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

By Lemma 2.4,

$$\int_0^t \int_\Omega |\nabla\mathbf{u}|^2\, dx\, d\tau \le (\|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

This completes the proof of Proposition 3.1. □

3.2. Global H^1 bound

This subsection establishes the global H^1-bound for $\mathbf{u}$.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with sufficient smooth boundary, say $\partial\Omega \in C^{2,1}$. Assume the initial data $(\mathbf{u}_0, \theta_0)$ satisfies the conditions stated in Theorem 1.1. Let $(\mathbf{u}, \theta)$*

be the corresponding solution of the IBVP (1.1), (1.3) and (1.6). *Then* $(\mathbf{u},\theta)$ *obeys the global* H^1 *bounds, for any* $t>0$,

$$\|\nabla \mathbf{u}(t)\|_{L^2}, \quad \int_0^t \|\mathbf{u}\|^2_{H^2(\Omega)}\,d\tau,$$

$$\int_0^t \|p\|^2_{H^1(\Omega)}\,d\tau \le C(t, \|\mathbf{u}_0\|_{H^1}, \|\theta_0\|_{L^2\cap L^\infty}).$$

Proof. Recall the definition of the operator A defined in (2.3). Dotting the velocity equation in (1.1) by $A\mathbf{u}$ yields

$$\nu\|A\mathbf{u}\|^2_{L^2} = -\int_\Omega \partial_t \mathbf{u}\cdot A\mathbf{u}\,dx - \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot A\mathbf{u}\,dx + \int_\Omega \theta e_2\cdot A\mathbf{u}\,dx. \tag{3.2}$$

By the definition of A in (2.3),

$$-\int_\Omega \partial_t\mathbf{u}\cdot A\mathbf{u}\,dx = \int_\Omega \partial_t\mathbf{u}\cdot\Delta\mathbf{u}\,dx - \frac{1}{\nu}\int_\Omega \partial_t\mathbf{u}\cdot\nabla p\,dx.$$

Writing the dot product in terms of the components and adopting Einstein's summation convention, we have

$$\begin{aligned}\int_\Omega \partial_t\mathbf{u}\cdot\Delta\mathbf{u}\,dx &= \int_\Omega \partial_k(\partial_t u_j\,\partial_k u_j)\,dx - \frac12\frac{d}{dt}\int_\Omega |\nabla\mathbf{u}|^2\,dx\\ &= -\frac12\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{L^2} + \int_{\partial\Omega} n_k\partial_k u_j\partial_t u_j\,dS(x).\end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$, we can write

$$\mathbf{u} = (\mathbf{u}\cdot\tau)\tau \quad \text{on } \partial\Omega.$$

By Lemmas 2.3 and 2.4,

$$\begin{aligned}\int_\Omega \partial_t\mathbf{u}\cdot\Delta\mathbf{u}\,dx &= -\frac12\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{L^2} + \int_{\partial\Omega}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\tau\,\partial_t(\mathbf{u}\cdot\tau)\,dS(x)\\ &= -\frac12\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{L^2}\\ &\quad +(\kappa-\alpha)\int_{\partial\Omega}(\mathbf{u}\cdot\tau)\,\partial_t(\mathbf{u}\cdot\tau)\,dS(x)\\ &= -\frac12\frac{d}{dt}\left(\|\nabla\mathbf{u}\|^2_{L^2} + (\alpha-\kappa)\|\mathbf{u}\cdot\tau\|^2_{L^2(\partial\Omega)}\right)\\ &= -\frac12\frac{d}{dt}\left(2\|D(\mathbf{u})\|^2_{L^2} + \alpha\,\|\mathbf{u}\cdot\tau\|^2_{L^2(\partial\Omega)}\right).\end{aligned}$$

By $\nabla\cdot\mathbf{u}=0$ in Ω and $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$,

$$\int_\Omega \partial_t\mathbf{u}\cdot\nabla p\,dx = \int_\Omega \nabla\cdot(p\,\partial_t\mathbf{u})\,dx = \int_{\partial\Omega} p\,\mathbf{n}\cdot\partial_t\mathbf{u}\,dS(x) = 0.$$

By Hölder's inequality,

$$\left|\int_\Omega \theta e_2\cdot A\mathbf{u}\,dx\right| \le \|\theta\|_{L^2}\,\|A\mathbf{u}\|_{L^2} \le \frac{\nu}{4}\|A\mathbf{u}\|^2_{L^2} + C\,\|\theta_0\|^2_{L^2}.$$

By Hölder's inequality, Ladyzhenskaya's inequality and (2.4),

$$\begin{aligned}\left|\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot A\mathbf{u}\,dx\right| &\le \|A\mathbf{u}\|_{L^2}\,\|\mathbf{u}\|_{L^4}\,\|\nabla\mathbf{u}\|_{L^4}\\ &\le C\,\|A\mathbf{u}\|_{L^2}\,\|\mathbf{u}\|^{\frac12}_{L^2}\|\nabla\mathbf{u}\|_{L^2}\,\|\nabla(\nabla\mathbf{u})\|^{\frac12}_{L^2}\\ &\le C\,\|A\mathbf{u}\|^{\frac32}_{L^2}\,\|\mathbf{u}\|^{\frac12}_{L^2}\|\nabla\mathbf{u}\|_{L^2}\\ &\le \frac{\nu}{4}\|A\mathbf{u}\|^2_{L^2} + C\,\|\mathbf{u}\|^2_{L^2}\|\nabla\mathbf{u}\|^4_{L^2}.\end{aligned}$$

By Lemma 2.4,

$$\begin{aligned}\|\nabla\mathbf{u}\|^2_{L^2} &= 2\|D(\mathbf{u})\|^2_{L^2} + \kappa\,\|\mathbf{u}\cdot\tau\|^2_{L^2(\partial\Omega)}\\ &\le \left(1+\frac{|\kappa|}{\alpha}\right)\left(2\|D(\mathbf{u})\|^2_{L^2} + \alpha\,\|\mathbf{u}\cdot\tau\|^2_{L^2(\partial\Omega)}\right).\end{aligned} \tag{3.3}$$

Inserting the estimates above in (3.2) and writing

$$Y(t) \equiv 2\|D(\mathbf{u})\|^2_{L^2} + \alpha\,\|\mathbf{u}\cdot\tau\|^2_{L^2(\partial\Omega)},$$

we obtain, after integrating in time,

$$Y(t) + \nu\int_0^t \|A\mathbf{u}(\tau)\|^2_{L^2}\,d\tau \le C\,\|\theta_0\|^2_{L^2}\,t + C\int_0^t \|\mathbf{u}\|^2_{L^2}\|\nabla\mathbf{u}\|^2_{L^2}\,Y(\tau)\,d\tau.$$

Gronwall's inequality and the global bound in Proposition 3.1 imply, for any $t>0$,

$$\|D(\mathbf{u})\|^2_{L^2}, \quad \|\mathbf{u}\cdot\tau\|^2_{L^2(\partial\Omega)}, \quad \int_0^t \|A\mathbf{u}(\tau)\|^2_{L^2}\,d\tau \le C(t, \|\mathbf{u}_0\|_{H^1}, \|\theta_0\|_{L^2\cap L^\infty}). \tag{3.4}$$

Then, (2.4) and (3.3) lead to the desired global bound in Proposition 3.2. □

3.3. Uniqueness

This subsection proves the uniqueness part of Theorem 1.1. More precisely, we establish the following proposition. We follow the idea of Larios, Lunasin, and Titi [15], who write $\theta = \Delta h$ for a function h.

Proposition 3.3. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded and connected domain with sufficient smooth boundary, say* $\partial\Omega\in C^{2,1}$. *Assume the initial and boundary conditions as stated in* Theorem 1.1. *Let* $(\mathbf{u}^{(1)},\theta^{(1)})$ *and* $(\mathbf{u}^{(2)},\theta^{(2)})$ *be two solutions of the IBVP* (1.1), (1.3) *and* (1.6) *satisfying* (1.7). *Then* $(\mathbf{u}^{(1)},\theta^{(1)}) = (\mathbf{u}^{(2)},\theta^{(2)})$.

We need the following existence and regularity result on solutions of the Poisson equation with a Neumann boundary condition. This result can be found in [25] or [27].

Lemma 3.4. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded and connected domain with sufficient smooth boundary, say* $\partial\Omega\in C^2$. *Let* $1<p<\infty$. *Assume* $f\in L^p(\Omega)$ *satisfies*

$$\int_\Omega f(x)\,dx = 0.$$

Then the Poisson equation with a pure Neumann boundary condition

$$\Delta g = f \quad \text{in } \Omega, \qquad \frac{dg}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega$$

has a unique solution g (up to an additive constant) satisfying

$$\|g\|_{W^{2,p}(\Omega)} \le C(\Omega,p)\,\|f\|_{L^p(\Omega)}.$$

We now prove Proposition 3.3.

Proof of Proposition 3.3. Let $(\mathbf{u}^{(1)},\theta^{(1)})$ and $(\mathbf{u}^{(2)},\theta^{(2)})$ be two solutions of the IBVP (1.1), (1.3) and (1.6) satisfying (1.7). Define $h^{(1)}$ and $h^{(2)}$ by

$$\Delta h^{(1)} = \theta^{(1)} \quad \text{in } \Omega, \qquad \frac{dh^{(1)}}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{3.5}$$

$$\Delta h^{(2)} = \theta^{(2)} \quad \text{in } \Omega, \qquad \frac{dh^{(2)}}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega. \tag{3.6}$$

According to Lemma 3.4, $h^{(1)}$ and $h^{(2)}$ exist and are unique (up to additive constants). Denote by $p^{(1)}$ and $p^{(2)}$ the associated pressures. Then the differences

$$\tilde{\mathbf{u}} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \quad \tilde{p} = p^{(1)} - p^{(2)}, \quad \tilde{\theta} = \theta^{(1)} - \theta^{(2)},$$
$$\tilde{h} = h^{(1)} - h^{(2)}, \quad \tilde{\theta} = \Delta\tilde{h},$$

satisfy

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + (\mathbf{u}^{(1)} \cdot \nabla)\tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} = -\nabla \tilde{p} + \nu \Delta \tilde{\mathbf{u}} + \Delta \tilde{h}\, \mathbf{e}_2, \\ \nabla \cdot \tilde{\mathbf{u}} = 0, \\ \partial_t \Delta \tilde{h} + \mathbf{u}^{(1)} \cdot \nabla(\Delta \tilde{h}) + \tilde{\mathbf{u}} \cdot \nabla \theta^{(2)} = 0, \\ \tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x) = 0, \quad \tilde{\theta}(x,0) = \tilde{\theta}_0(x) = 0. \end{cases} \tag{3.7}$$

Dotting the first equation of (3.7) with $\tilde{\mathbf{u}}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\mathbf{u}}\|_{L^2}^2 = \nu \int_\Omega \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}\, dx - \int_\Omega \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \tilde{\mathbf{u}}\, dx + \int_\Omega \tilde{u}_2 \Delta \tilde{h}\, dx, \tag{3.8}$$

where, we have invoked the facts, due to $\mathbf{u}^{(1)} \cdot \mathbf{n} = 0$ and $\tilde{\mathbf{u}} \cdot \mathbf{n} = 0$,

$$\int_\Omega \mathbf{u}^{(1)} \cdot \nabla \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}\, dx = 0, \qquad -\int_\Omega \nabla \tilde{p} \cdot \tilde{\mathbf{u}}\, dx = 0.$$

According to Lemma 2.4,

$$\int_\Omega \Delta \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}\, dx = -2\int_\Omega |D(\tilde{\mathbf{u}})|^2\, dx - \int_{\partial\Omega} \alpha(\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x).$$

By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned} \left|\int_\Omega \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \tilde{\mathbf{u}}\, dx\right| &\le \|\nabla \mathbf{u}^{(2)}\|_{L^2} \|\tilde{\mathbf{u}}\|_{L^4}^2 \\ &\le C\, \|\nabla \mathbf{u}^{(2)}\|_{L^2} \|\tilde{\mathbf{u}}\|_{L^2} \|\nabla \tilde{\mathbf{u}}\|_{L^2}. \end{aligned}$$

By Young's inequality and (3.3)

$$\begin{aligned} &\left|\int_\Omega \tilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \tilde{\mathbf{u}}\, dx\right| \\ &\le \frac{1}{4}\left(2\nu \int_\Omega |D(\tilde{\mathbf{u}})|^2\, dx + \nu \int_{\partial\Omega} \alpha(\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x)\right) \\ &\quad + C\, \|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\tilde{\mathbf{u}}\|_{L^2}^2. \end{aligned}$$

By the divergence theorem and the definitions of h_1 in (3.5) and h_2 in (3.6),

$$\begin{aligned} \int_\Omega \tilde{u}_2\, \Delta \tilde{h}\, dx &= \int_{\partial\Omega} \frac{d\tilde{h}}{d\mathbf{n}}\, \tilde{u}_2\, dS(x) - \int_\Omega \nabla \tilde{h} \cdot \nabla \tilde{u}_2\, dx \\ &= -\int_\Omega \nabla \tilde{h} \cdot \nabla \tilde{u}_2\, dx. \end{aligned}$$

By Hölder's inequality and (3.3),

$$\begin{aligned} \left|\int_\Omega \tilde{u}_2\, \Delta \tilde{h}\, dx\right| &\le \|\nabla \tilde{\mathbf{u}}\|_{L^2}\, \|\nabla \tilde{h}\|_{L^2} \\ &\le \frac{1}{4}\left(2\nu \int_\Omega |D(\tilde{\mathbf{u}})|^2\, dx + \nu \int_{\partial\Omega} \alpha(\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x)\right) \\ &\quad + C\, \|\nabla \tilde{h}\|_{L^2}^2. \end{aligned}$$

Combining the estimates above with (3.8), we obtain

$$\begin{aligned} &\frac{d}{dt}\|\tilde{\mathbf{u}}\|_{L^2}^2 + 2\nu \int_\Omega |D(\tilde{\mathbf{u}})|^2\, dx + \nu\, \alpha \int_{\partial\Omega} (\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x) \\ &\le C\, \|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\tilde{\mathbf{u}}\|_{L^2}^2 + C\, \|\nabla \tilde{h}\|_{L^2}^2. \end{aligned} \tag{3.9}$$

Multiplying the equation of $\tilde{h}$ in (3.7) by $\tilde{h}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla \tilde{h}\|_{L^2}^2 = \int_\Omega \mathbf{u}^{(1)} \cdot \nabla(\Delta \tilde{h})\tilde{h}\, dx + \int_\Omega \tilde{\mathbf{u}} \cdot \nabla \theta^{(2)} \tilde{h}\, dx. \tag{3.10}$$

By integration by parts and Hölder's inequality,

$$\begin{aligned} \left|\int_\Omega \tilde{\mathbf{u}} \cdot \nabla \theta^{(2)} \tilde{h}\, dx\right| &= \left|\int_\Omega \theta^{(2)} \tilde{\mathbf{u}} \cdot \nabla \tilde{h}\, dx\right| \\ &\le \|\theta^{(2)}\|_{L^\infty} \left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla \tilde{h}\|_{L^2}^2\right). \end{aligned}$$

The first term on the right of (3.10) is more difficult to handle. By integrating by parts and invoking the boundary conditions for $\mathbf{u}^{(1)}$ and $\tilde{h}$, we have

$$\begin{aligned} &\int_\Omega \mathbf{u}^{(1)} \cdot \nabla(\Delta \tilde{h})\tilde{h}\, dx \\ &= -\int_\Omega \Delta \tilde{h}\, \mathbf{u}^{(1)} \cdot \nabla \tilde{h}\, dx + \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{u}^{(1)}\, \Delta \tilde{h}\, \tilde{h}\, dS(x) \\ &= \int_\Omega \partial_k \partial_k \tilde{h}\, \mathbf{u}^{(1)} \cdot \nabla \tilde{h}\, dx \\ &= -\int_\Omega \partial_k \tilde{h} (\partial_k \mathbf{u}^{(1)} \cdot \nabla \tilde{h} + \mathbf{u}^{(1)} \cdot \nabla \partial_k \tilde{h})\, dx \\ &\quad + \int_{\partial\Omega} \frac{d\tilde{h}}{d\mathbf{n}}\, \mathbf{u}^{(1)} \cdot \nabla \tilde{h}\, dS(x) \\ &= -\int_\Omega \partial_k \tilde{h} \partial_k \mathbf{u}^{(1)} \cdot \nabla \tilde{h}\, dx \\ &= -\int_\Omega \nabla \tilde{h} \cdot \nabla \mathbf{u}^{(1)} \cdot \nabla \tilde{h}\, dx, \end{aligned} \tag{3.11}$$

where we have invoked the fact that $\frac{d\tilde{h}}{d\mathbf{n}} = 0$ on $\partial\Omega$ due to (3.5) and (3.6). We employ Yudovich's method to estimate the term on (3.11). For notational convenience, we denote it by I

$$I = \int_\Omega \nabla \tilde{h} \cdot \nabla \mathbf{u}^{(1)} \cdot \nabla \tilde{h}\, dx.$$

The Yudovich approach applies to the situation when the bound

$$\|\nabla \mathbf{u}^{(1)}\|_{L^\infty} < \infty$$

is unknown, but any L^q bound of $\nabla \mathbf{u}^{(1)}$ does not grow faster than $O(q)$, namely

$$\sup_{q \ge 2} \frac{\|\nabla \mathbf{u}^{(1)}\|_{L^q}}{q} < \infty. \tag{3.12}$$

Recall that $\mathbf{u}^{(1)}$ satisfies the conditions of (1.7), namely, for any $T > 0$,

$$\mathbf{u}^{(1)} \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; \dot{H}^2(\Omega)),$$

which allows us to verify (3.12). In fact, by Sobolev's embedding inequality, for any $2 \le q < \infty$,

$$\begin{aligned} \|\nabla \mathbf{u}^{(1)}\|_{L^q(\Omega)} &\le C(\Omega)\, q\, \|\nabla \mathbf{u}^{(1)}\|_{L^2(\Omega)} + C(\Omega)\, q\, \|\nabla\nabla \mathbf{u}^{(1)}\|_{L^2(\Omega)} \\ &\le C(\Omega)\, q\, \|\nabla \mathbf{u}^{(1)}\|_{L^2(\Omega)} + C(\Omega)\, q\, \|\mathbf{u}^{(1)}\|_{\dot{H}^2(\Omega)}. \end{aligned}$$

That is,

$$\sup_{q \ge 2} \frac{\|\nabla \mathbf{u}^{(1)}\|_{L^q}}{q} \le C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}. \tag{3.13}$$

For any $2 < q < \infty$, by Hölder's inequality,

$$\begin{aligned} |I| &\le \|\nabla \tilde{h}\|_{L^2}\, \|\nabla \mathbf{u}^{(1)}\|_{L^q}\, \|\nabla \tilde{h}\|_{L^{\frac{2q}{q-2}}} \\ &\le \|\nabla \tilde{h}\|_{L^2}\, \|\nabla \mathbf{u}^{(1)}\|_{L^q}\, \|\nabla \tilde{h}\|_{L^2}^{1-\frac{2}{q}}\, \|\nabla \tilde{h}\|_{L^\infty}^{\frac{2}{q}}. \end{aligned}$$

Since $\theta^{(1)}$ and $\theta^{(2)}$ are in the class (1.7), Lemma 3.4 states that, for any $2 < r < \infty$,

$$M \equiv \|\nabla \tilde{h}\|_{L^\infty} \le C\, \|\nabla\nabla \tilde{h}\|_{L^r} \le C\|\tilde{\theta}\|_{L^r} < \infty.$$

Therefore, by (3.13), for any $2 < q < \infty$,

$$\begin{aligned} |I| &\le C\, M^{\frac{2}{q}}\, \|\nabla \tilde{h}\|_{L^2}^{2(1-\frac{1}{q})}\, \|\nabla \mathbf{u}^{(1)}\|_{L^q} \\ &\le C\, q \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, M^{\frac{2}{q}}\, \|\nabla \tilde{h}\|_{L^2}^{2-\frac{2}{q}} \\ &= C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, \|\nabla \tilde{h}\|_{L^2}^2 \left(q M^{\frac{2}{q}}\, \|\nabla \tilde{h}\|_{L^2}^{-\frac{2}{q}}\right). \end{aligned}$$

By taking $q = 2\ln(M/\|\nabla\tilde{h}\|_{L^2})$, we obtain the minimizer of $qM^{\frac{2}{q}}\|\nabla\tilde{h}\|_{L^2}^{-\frac{2}{q}}$, namely

$$\min_{2\le q<\infty} qM^{\frac{2}{q}}\|\nabla\tilde{h}\|_{L^2}^{-\frac{2}{q}} = 2e\left(\ln M - \ln\|\nabla\tilde{h}\|_{L^2}\right).$$

Consequently,

$$|I| \le C\,\|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\,\|\nabla\tilde{h}\|_{L^2}^2\,(\ln M - \ln\|\nabla\tilde{h}\|_{L^2}).$$

Inserting these bounds in (3.10) leads to

$$\begin{aligned}\frac{d}{dt}&\|\nabla\tilde{h}\|_{L^2}^2\\ &\le \|\theta^{(2)}\|_{L^\infty}\left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\tilde{h}\|_{L^2}^2\right)\\ &\quad + C\,\|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\,\|\nabla\tilde{h}\|_{L^2}^2\,(\ln M - \ln\|\nabla\tilde{h}\|_{L^2}),\end{aligned}$$

which, together with (3.9), yields

$$\begin{aligned}\frac{d}{dt}&\left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\tilde{h}\|_{L^2}^2\right) + 2\nu\int_\Omega |D(\mathbf{u})|^2\,dx + \nu\,\alpha\int_{\partial\Omega}(\mathbf{u}\cdot\tau)^2\,dS(x)\\ &\le C\,\|\nabla\mathbf{u}^{(2)}\|_{L^2}^2\|\tilde{\mathbf{u}}\|_{L^2}^2 + C\,\|\nabla\tilde{h}\|_{L^2}^2 + \|\theta^{(2)}\|_{L^\infty}\left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\tilde{h}\|_{L^2}^2\right)\\ &\quad + C\,\|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\,\|\nabla\tilde{h}\|_{L^2}^2\,(\ln M - \ln\|\nabla\tilde{h}\|_{L^2}).\end{aligned}$$

Especially, $Y(t) \equiv \delta + \|\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{h}\|_{L^2}^2$ with any small $\delta > 0$ satisfies

$$\begin{aligned}\frac{d}{dt}Y &\le C(1 + \|\nabla u^{(2)}\|_{L^2}^2 + \|\theta^{(2)}\|_{L^\infty})Y\\ &\quad + C\,\|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\,Y\,(\ln M - \ln Y)\end{aligned}$$

where we have used the fact $z \to z\,(\ln M - \ln z)$ is an increasing function for $0 < z < M/e$. Applying the Osgood inequality in Lemma 2.5 and letting $\delta \to 0$ yield the desired uniqueness

$$\|\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{h}\|_{L^2}^2 \equiv 0.$$

This completes the proof of Proposition 3.3. □

4. Proof of Theorem 1.2

This section proves Theorem 1.2. As we explained in Section 3, it suffices to prove the global *a priori* bounds and the uniqueness in the functional setting in (1.9). Naturally we divide the rest of this section into two parts with the first part devoted to the global *a priori* bounds and the second to the uniqueness.

4.1. Global a priori bounds

The global L^q-bound for θ is obtained as in the proof of Proposition 3.1. To prove the global L^2 bound for $\mathbf{u}$, we take the inner product of $\mathbf{u}$ with the equation of $\mathbf{u}$ in (1.2) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2 = \nu\int_\Omega (u_1\partial_{22}u_1 + u_2\partial_{11}u_2)\,dx + \int_\Omega \theta\,u_2\,dx, \tag{4.1}$$

where we have invoked the facts, due to $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$,

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,dx = 0, \qquad \int_\Omega \mathbf{u}\cdot\nabla p\,dx = 0.$$

Integrating by parts and applying the divergence theorem lead to

$$\begin{aligned}\nu\int_\Omega (u_1\partial_{22}u_1 + u_2\partial_{11}u_2)\,dx &= -\nu\int_\Omega ((\partial_2u_1)^2 + (\partial_1u_2)^2)\,dx\\ &\quad + \nu\int_{\partial\Omega}(n_2u_1\partial_2u_1 + n_1u_2\partial_1u_2)\,dS(x).\end{aligned} \tag{4.2}$$

Clearly, for $\omega = \partial_1u_2 - \partial_2u_1$,

$$\|\omega\|_{L^2(\Omega)}^2 \le 2\int_\Omega ((\partial_2u_1)^2 + (\partial_1u_2)^2)\,dx.$$

By Lemma 2.2, for a constant $C > 0$,

$$\|\nabla\mathbf{u}\|_{L^2(\Omega)} \le C\,\|\mathbf{u}\|_{L^2(\Omega)} + C\,\|\omega\|_{L^2(\Omega)}$$

and thus

$$\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2 \le C\,\|\mathbf{u}\|_{L^2(\Omega)}^2 + C\int_\Omega ((\partial_2u_1)^2 + (\partial_1u_2)^2)\,dx.$$

In the form of matrices, the second term in (4.2) can be written as

$$\begin{aligned}\nu&\int_{\partial\Omega}(n_2u_1\partial_2u_1 + n_1u_2\partial_1u_2)\,dS(x)\\ &= \nu\int_{\partial\Omega}\mathbf{n}\cdot\begin{pmatrix}0 & \partial_2u_1\\ \partial_1u_2 & 0\end{pmatrix}\cdot\mathbf{u}\,dS(x)\\ &= \nu\int_{\partial\Omega}\mathbf{n}\cdot E(u)\cdot\tau\,(\mathbf{u}\cdot\tau)\,dS(x)\\ &\quad + \frac{1}{2}\nu\int_{\partial\Omega}\mathbf{n}\cdot\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}\cdot\tau\,(\mathbf{u}\cdot\tau)\,dS(x)\\ &= -\frac{\nu\alpha}{2}\int_{\partial\Omega}(\mathbf{u}\cdot\tau)^2\,dS(x)\\ &\quad - \frac{1}{2}\nu\int_{\partial\Omega}(-n_2\tau_1 + n_1\tau_2)\,\omega\,(\mathbf{u}\cdot\tau)\,dS(x)\\ &= -\frac{\nu\alpha}{2}\int_{\partial\Omega}(\mathbf{u}\cdot\tau)^2\,dS(x)\end{aligned}$$

where we have invoked the boundary condition in (1.4). Inserting the estimates for the terms of (4.2) in (4.1), we have, for $C_0 > 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2(\Omega)}^2 + C_0\,\nu\,\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2 + \frac{\nu\alpha}{2}\int_{\partial\Omega}(\mathbf{u}\cdot\tau)^2\,dS(x)\\ \le C\,(\|\mathbf{u}\|_{L^2(\Omega)}^2 + \|\theta_0\|_{L^2(\Omega)})\,\|\mathbf{u}\|_{L^2(\Omega)}.\end{aligned}$$

Gronwall's inequality then yields the desired global bound, for any $t > 0$,

$$\begin{aligned}\|\mathbf{u}(t)\|_{L^2(\Omega)}^2 + C_0\,\nu\int_0^t \|\nabla\mathbf{u}(\tau)\|_{L^2(\Omega)}^2\,d\tau\\ \le C(t,\,\|\mathbf{u}_0\|_{L^2(\Omega)},\,\|\theta_0\|_{L^2(\Omega)}).\end{aligned} \tag{4.3}$$

We now turn to the global H^1 bound. We resort to the vorticity equation

$$\partial_t\omega + \mathbf{u}\cdot\nabla\omega = \nu(\partial_{111}u_2 - \partial_{222}u_1) + \partial_1\theta. \tag{4.4}$$

Taking the inner product of ω with (4.4) yields

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2(\Omega)}^2 = \int_\Omega \omega\,\partial_1\theta\,dx + \nu\int_\Omega (\partial_{111}u_2 - \partial_{222}u_1)\,\omega\,dx. \tag{4.5}$$

Invoking the stress free boundary condition in (1.8), we have

$$\int_\Omega \omega\,\partial_1\theta\,dx = -\int_\Omega \theta\,\partial_1\omega\,dx \le \frac{\nu}{4}\|\partial_1\omega\|_{L^2}^2 + C\,\|\theta_0\|_{L^2}^2. \tag{4.6}$$

Again, due to the boundary condition in (1.8),

$$\begin{aligned}\int_\Omega &(\partial_{111}u_2 - \partial_{222}u_1)\,\omega\,dx\\ &= -\int_\Omega \partial_{11}u_2\partial_1\omega\,dx + \int_\Omega \partial_{22}u_1\partial_2\omega\,dx\\ &= -\int_\Omega (\partial_{11}u_2)^2\,dx + \int_\Omega \partial_{11}u_2\,\partial_{12}u_1\,dx - \int_\Omega (\partial_{22}u_1)^2\,dx\\ &\quad + \int_\Omega \partial_{22}u_1\,\partial_{12}u_2\,dx.\end{aligned}$$

By $\nabla \cdot u = 0$ and the divergence theorem,

$$\begin{aligned}
&\int_\Omega \partial_{11}u_2\, \partial_{12}u_1\, dx \\
&= \int_\Omega \partial_2(\partial_{11}u_2\, \partial_1 u_1)\, dx - \int_\Omega \partial_{112}u_2\, \partial_1 u_1\, dx \\
&= \int_\Omega \partial_2(\partial_{11}u_2\, \partial_1 u_1)\, dx - \int_\Omega \partial_1(\partial_{12}u_2\, \partial_1 u_1)\, dx \\
&\quad + \int_\Omega \partial_{12}u_2\, \partial_{11}u_1\, dx \\
&= -\int_\Omega (\partial_{11}u_1)^2\, dx + \int_{\partial\Omega} n_2\, \partial_{11}u_2\, \partial_1 u_1\, dS(x) \\
&\quad - \int_{\partial\Omega} n_1\, \partial_{12}u_2\, \partial_1 u_1\, dS(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_\Omega \partial_{22}u_1\, \partial_{12}u_2\, dx \\
&= -\int_\Omega (\partial_{22}u_2)^2\, dx + \int_\Omega \partial_1(\partial_{22}u_1\, \partial_2 u_2)\, dx \\
&\quad - \int_\Omega \partial_2(\partial_{12}u_1\, \partial_2 u_2)\, dx \\
&= -\int_\Omega (\partial_{22}u_2)^2\, dx + \int_{\partial\Omega} n_1\, \partial_{22}u_1\, \partial_2 u_2\, dS(x) \\
&\quad - \int_{\partial\Omega} n_2\, \partial_{12}u_1\, \partial_2 u_2\, dS(x).
\end{aligned}$$

Therefore, thanks to $\tau = \mathbf{n}^\perp$ or $(\tau_1, \tau_2) = (-n_2, n_1)$,

$$\begin{aligned}
&\int_\Omega (\partial_{111}u_2 - \partial_{222}u_1)\, \omega\, dx \\
&= -\int_\Omega (\partial_{11}u_1)^2\, dx - \int_\Omega (\partial_{22}u_1)^2\, dx - \int_\Omega (\partial_{11}u_2)^2\, dx \\
&\quad - \int_\Omega (\partial_{22}u_2)^2\, dx \\
&\quad - \int_{\partial\Omega} \tau_1\, \partial_{11}u_2\, \partial_1 u_1\, dS(x) - \int_{\partial\Omega} \tau_2\, \partial_{12}u_2\, \partial_1 u_1\, dS(x) \qquad (4.7) \\
&\quad + \int_{\partial\Omega} \tau_2\, \partial_{22}u_1\, \partial_2 u_2\, dS(x) + \int_{\partial\Omega} \tau_1\, \partial_{12}u_1\, \partial_2 u_2\, dS(x). \qquad (4.8)
\end{aligned}$$

The terms in (4.7) and (4.8) can be written as directional derivatives along τ. In fact,

$$\begin{aligned}
&-\int_{\partial\Omega} \tau_1\, \partial_{11}u_2\, \partial_1 u_1\, dS(x) - \int_{\partial\Omega} \tau_2\, \partial_{12}u_2\, \partial_1 u_1\, dS(x) \\
&= -\int_{\partial\Omega} \partial_1 u_1 \frac{d}{d\tau} \partial_1 u_2\, dS(x), \\
&\int_{\partial\Omega} \tau_2\, \partial_{22}u_1\, \partial_2 u_2\, dS(x) + \int_{\partial\Omega} \tau_1\, \partial_{12}u_1\, \partial_2 u_2\, dS(x) \\
&= \int_{\partial\Omega} \partial_2 u_2 \frac{d}{d\tau} \partial_2 u_1\, dS(x).
\end{aligned}$$

According to the boundary condition in (1.8), $\partial_1 u_2 = \partial_2 u_1 = 0$ on $\partial\Omega$, their directional derivatives along τ should also be zero,

$$\frac{d}{d\tau}\partial_1 u_2 = \frac{d}{d\tau}\partial_2 u_1 = 0 \quad \text{on} \quad \partial\Omega.$$

As a consequence,

$$\begin{aligned}
&\int_\Omega (\partial_{111}u_2 - \partial_{222}u_1)\, \omega\, dx \\
&= -\int_\Omega (\partial_{11}u_1)^2\, dx - \int_\Omega (\partial_{22}u_1)^2\, dx - \int_\Omega (\partial_{11}u_2)^2\, dx \\
&\quad - \int_\Omega (\partial_{22}u_2)^2\, dx. \qquad (4.9)
\end{aligned}$$

Inserting (4.6) and (4.9) in (4.5) yields

$$\begin{aligned}
&\frac{d}{dt}\|\omega\|^2_{L^2(\Omega)} + 2\nu \int_\Omega \Big((\partial_{11}u_1)^2 + (\partial_{22}u_1)^2 \\
&\quad + (\partial_{11}u_2)^2 + (\partial_{22}u_2)^2\Big)\, dx \\
&\le \frac{\nu}{2}\, \|\partial_1\omega\|^2_{L^2} + C\, \|\theta_0\|^2_{L^2}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\|\partial_1\omega\|^2_{L^2} \le 2\int_\Omega \Big((\partial_{11}u_2)^2 + (\partial_{22}u_2)^2\Big)\, dx, \\
&\|\partial_2\omega\|^2_{L^2} \le 2\int_\Omega \Big((\partial_{11}u_1)^2 + (\partial_{22}u_1)^2\Big)\, dx.
\end{aligned}$$

Therefore,

$$\|\omega\|^2_{L^2(\Omega)} + \nu\int_0^t \int_\Omega |\nabla\omega|^2\, dx\, d\tau \le \|\omega_0\|^2_{L^2} + C\, t\|\theta_0\|^2_{L^2},$$

which yields the desired global H^1 bound.

4.2. Uniqueness

This subsection proves the uniqueness part of Theorem 1.2. Assume $(\mathbf{u}^{(1)}, \theta^{(1)})$ and $(\mathbf{u}^{(2)}, \theta^{(2)})$ are two solutions of the IBVP (1.2), (1.6) and (1.8) satisfying (1.9). Define $h^{(1)}$ and $h^{(2)}$ by

$$\Delta h^{(1)} = \theta^{(1)} \quad \text{in } \Omega, \qquad \frac{dh^{(1)}}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \qquad (4.10)$$

$$\Delta h^{(2)} = \theta^{(2)} \quad \text{in } \Omega, \qquad \frac{dh^{(2)}}{d\mathbf{n}} = 0 \quad \text{on } \partial\Omega. \qquad (4.11)$$

According to Lemma 3.4, $h^{(1)}$ and $h^{(2)}$ exist and are unique (up to additive constants). Denote by $p^{(1)}$ and $p^{(2)}$ the associated pressures. Then the differences

$$\begin{aligned}
&\tilde{\mathbf{u}} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \quad \tilde{p} = p^{(1)} - p^{(2)}, \quad \tilde{\theta} = \theta^{(1)} - \theta^{(2)}, \\
&\tilde{h} = h^{(1)} - h^{(2)}, \quad \tilde{\theta} = \Delta\tilde{h}
\end{aligned}$$

satisfy

$$\begin{cases}
\partial_t \tilde{u}_1 + (\mathbf{u}^{(1)} \cdot \nabla)\tilde{u}_1 + \tilde{\mathbf{u}} \cdot \nabla u_1^{(2)} = -\partial_1\tilde{p} + \nu\partial_{22}\tilde{u}_1, \\
\partial_t \tilde{u}_2 + (\mathbf{u}^{(1)} \cdot \nabla)\tilde{u}_2 + \tilde{\mathbf{u}} \cdot \nabla u_2^{(2)} = -\partial_2\tilde{p} + \nu\partial_{11}\tilde{u}_2 + \Delta\tilde{h}, \\
\nabla \cdot \tilde{\mathbf{u}} = 0, \\
\partial_t \Delta\tilde{h} + \mathbf{u}^{(1)} \cdot \nabla(\Delta\tilde{h}) + \tilde{\mathbf{u}} \cdot \nabla\theta^{(2)} = 0, \\
\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x) = 0, \quad \tilde{\theta}(x,0) = \tilde{\theta}_0(x) = 0.
\end{cases} \qquad (4.12)$$

Taking the inner product of (4.12) with $(\tilde{u}_1, \tilde{u}_2, \tilde{h})$ and invoking the boundary conditions for $\tilde{\mathbf{u}}$ and $\tilde{h}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\mathbf{u}}\|^2_{L^2} + \|\nabla\tilde{h}\|^2_{L^2}\Big) \\
&= \nu\int_\Omega (\tilde{u}_1\partial_{22}\tilde{u}_1 + \tilde{u}_2\partial_{11}\tilde{u}_2)\, dx - \int_\Omega \tilde{\mathbf{u}} \cdot \nabla\mathbf{u}^{(2)} \cdot \tilde{\mathbf{u}}\, dx \\
&\quad + \int_\Omega \tilde{u}_2 \Delta\tilde{h}\, dx \\
&\quad + \int_\Omega \mathbf{u}^{(1)} \cdot \nabla(\Delta\tilde{h})\tilde{h}\, dx + \int_\Omega \tilde{\mathbf{u}} \cdot \nabla\theta^{(2)}\tilde{h}\, dx. \qquad (4.13)
\end{aligned}$$

The term associated with dissipation can be handled as in Section 4.1 and we have, for constants $C_0 > 0$ and $C > 0$,

$$- \nu \int_{\Omega} (\tilde{u}_1 \partial_{22} \tilde{u}_1 + \tilde{u}_2 \partial_{11} \tilde{u}_2)\, dx$$
$$\geq C_0\, \nu\, \|\nabla \tilde{\mathbf{u}}(\tau)\|_{L^2(\Omega)}^2 - C\, \|\tilde{\mathbf{u}}\|_{L^2}^2 + \frac{\nu\alpha}{2} \int_{\partial\Omega} (\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x).$$

The other four terms on the right of (4.13) can be handled as in the proof of Proposition 3.3. Invoking those bounds yields

$$\frac{d}{dt} \left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla \tilde{h}\|_{L^2}^2 \right) + C_0\, \nu\, \|\nabla \tilde{\mathbf{u}}(\tau)\|_{L^2(\Omega)}^2$$
$$+ \frac{\nu\alpha}{2} \int_{\partial\Omega} (\tilde{\mathbf{u}} \cdot \tau)^2\, dS(x)$$
$$\leq C\, \|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\tilde{\mathbf{u}}\|_{L^2}^2 + C\, (1 + \|\theta^{(2)}\|_{L^\infty}) \left(\|\tilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla \tilde{h}\|_{L^2}^2 \right)$$
$$+ C\, \|\mathbf{u}^{(1)}\|_{H^2(\Omega)}\, \|\nabla \tilde{h}\|_{L^2}^2\, (\ln M - \ln \|\nabla \tilde{h}\|_{L^2}).$$

Osgood's inequality in Lemma 2.5 then yields the desired uniqueness.

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