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Generalized 2D Euler–Boussinesq equations with a singular velocity

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Abstract

This paper studies the global (in time) regularity problem concerning a system of equations generalizing the two-dimensional incompressible Boussinesq equations. The velocity here is determined by the vorticity through a more singular relation than the standard Biot–Savart law and involves a Fourier multiplier operator. The temperature equation has a dissipative term given by the fractional Laplacian operator $\sqrt{-\Delta}$. We establish the global existence and uniqueness of solutions to the initial-value problem of this generalized Boussinesq equations when the velocity is "double logarithmically" more singular than the one given by the Biot–Savart law. This global regularity result goes beyond the critical case. In addition, we recover a result of Chae, Constantin and Wu [8] when the initial temperature is set to zero.
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1. Introduction

Attention here is focused on the generalized Euler–Boussinesq equations of the form

$$\begin{cases} \partial_t v + u\cdot\nabla v - \sum_{j=1}^{2} u_j \nabla v_j = -\nabla p + \theta \mathbf{e}_2, \\ \nabla\cdot v = 0, \quad u = \Lambda^\sigma P(\Lambda) v, \\ \partial_t \theta + u\cdot\nabla\theta + \Lambda\theta = 0, \end{cases} \tag{1.1}$$

where $v = v(x,t)$ and $u = u(x,t)$ are 2D vector fields depending on $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geqslant 0$, $p = p(x,t)$ and $\theta = \theta(x,t)$ are scalar functions, $\mathbf{e}_2$ is the unit vector in the x_2-direction and $\sigma \geqslant 0$ is a real parameter. Here the Zygmund operator $\Lambda = (-\Delta)^{1/2}$, Λ^σ and the Fourier multiplier operator $P(\Lambda)$ are defined through the Fourier transform, namely

$$\widehat{\Lambda^\sigma f}(\xi) = |\xi|^\sigma \widehat{f}(\xi) \quad \text{and} \quad \widehat{P(\Lambda) f}(\xi) = P\big(|\xi|\big)\widehat{f}(\xi).$$

More details on such operators can be found in many books and research papers such as [34] and [14]. We remark that (1.1) can be reformulated in terms of the vorticity $\omega = \nabla\times v$ as

$$\begin{cases} \partial_t \omega + u\cdot\nabla\omega = \partial_{x_1}\theta, \\ u = \nabla^\perp \psi, \quad \Delta\psi = \Lambda^\sigma P(\Lambda)\omega, \\ \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0, \end{cases} \tag{1.2}$$

where $\omega = \omega(x,t)$ is a scalar function and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. This paper will mostly work with the vorticity formulation.

The generalized Euler–Boussinesq system in (1.1) or (1.2) reduces to the generalized 2D Euler equations studied by [8] when $\theta = 0$. Furthermore, (1.1) or (1.2) generalizes the 2D incompressible Boussinesq equations. The standard velocity formulation of the 2D Boussinesq equations with fractional dissipation and fractional thermal diffusion is given by

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu\Lambda^\alpha u = -\nabla p + \theta\mathbf{e}_2, \\ \nabla\cdot u = 0, \\ \partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\beta\theta = 0 \end{cases} \tag{1.3}$$

with the corresponding vorticity $\omega = \nabla\times u$ satisfying

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega + \nu\Lambda^\alpha\omega = \partial_{x_1}\theta, \\ u = \nabla^\perp\psi, \quad \Delta\psi = \omega, \\ \partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\beta\theta = 0, \end{cases} \tag{1.4}$$

where $\nu \geqslant 0$, $\kappa \geqslant 0$, $\alpha \in (0,2]$ and $\beta \in (0,2]$ are real parameters. Clearly (1.2) with $\sigma = 0$ and $P(\Lambda) = I$ is simply (1.4) with $\nu = 0$ and $\beta = 1$, where I denotes the identity operator. The Boussinesq equations in (1.3) or (1.4) with $\alpha = 2$ and $\beta = 2$ model geophysical fluids and play a very important role in the study of Rayleigh–Bénard convection (see, e.g., [12,18,24,32]). Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the Boussinesq equations retain some key features of the 3D Navier–Stokes and the Euler equations such as the vortex stretching mechanism. As pointed out in [25], the inviscid Boussinesq equations, namely (1.3) with $\nu = \kappa = 0$ can be identified with the 3D Euler equations for axisymmetric flows (away from the symmetry axis).

The goal of this paper is to establish the global (in time) existence and uniqueness of solutions to the initial-value problem (IVP) for (1.2) supplemented with data

$$\omega(x,0) = \omega_0(x), \qquad \theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^2. \tag{1.5}$$

We work with a very general class of symbols $P(|\xi|)$ for the operator $P(\Lambda)$, as previously specified in the work of Chae, Constantin and Wu [8, p. 36]. More precisely, P is assumed to satisfy the following condition.

Condition 1.1. *The symbol $P(|\xi|)$ assumes the following properties:*

(1) *P is continuous on $\mathbb{R}^2$ and $P \in C^\infty(\mathbb{R}^2 \setminus \{0\})$;*
(2) *P is radially symmetric;*
(3) *$P = P(|\xi|)$ is nondecreasing in $|\xi|$;*
(4) *There exist two constants C and C_0 such that*

$$\sup_{2^{-1} \leqslant |\eta| \leqslant 2} \left| (I - \Delta_\eta)^n P\left(2^j |\eta|\right)\right| \leqslant C P\left(C_0 2^j\right)$$

for any integer j and $n = 1, 2$.

We remark that (4) in Condition 1.1 is a very natural condition on symbols of Fourier multiplier operators and is similar to the main condition in the Mihlin–Hörmander Multiplier Theorem (see, e.g., [34, p. 96]). For notational convenience, we also assume that $P \geqslant 0$. Some special examples of P are

$$P(\xi) = \left(\log\left(1 + |\xi|^2\right)\right)^\gamma \quad \text{with } \gamma \geqslant 0,$$
$$P(\xi) = \left(\log\left(1 + \log\left(1 + |\xi|^2\right)\right)\right)^\gamma \quad \text{with } \gamma \geqslant 0,$$
$$P(\xi) = |\xi|^\beta \quad \text{with } \beta \geqslant 0,$$
$$P(\xi) = \left(\log\left(1 + |\xi|^2\right)\right)^\gamma |\xi|^\beta \quad \text{with } \gamma \geqslant 0 \text{ and } \beta \geqslant 0.$$

Our motivation for this study is two fold: first, to extend the work of Chae, Constantin and Wu [8] on the generalized 2D Euler equations to the Euler–Boussinesq system of equations, and second, to explore how far one can go beyond the critical dissipation and still prove the global regularity for the Boussinesq equations with fractional dissipation. The 2D Boussinesq equations have recently attracted considerable attention and many important results on the global well-posedness issue concerning the partial dissipation case have been established (see, e.g., [1,2, 5–7,9,13,15–17,19–23,27–31]). One of the critical cases, (1.1) or (1.2) with $\sigma = 0$ and $P(\Lambda) = I$, was recently shown to be globally well-posed in [21]. This paper examines for what operator P obeying Condition 1.1, (1.1) or (1.2) is still globally well-posed. When $P(|\xi|)$ is not the identity operator, (1.1) involves a velocity field that is more singular than the standard velocity determined by the vorticity through the Biot–Savart law [25]. Our main result, Theorem 1.2, states that (1.1) is still globally well-posed when P satisfies two explicit conditions.

Theorem 1.2. *Let $\sigma = 0$. Assume the symbol $P(|\xi|)$ obeys Condition 1.1 and*

$$P(2^k) \leqslant C\sqrt{k} \quad \text{for a constant } C \text{ and any large integer } k > 0, \tag{1.6}$$

$$\int_1^\infty \frac{1}{r\log(1+r)P(r)}\,dr = \infty. \tag{1.7}$$

Let $q > 2$ and let $s > 2$. Consider the IVP (1.2) and (1.5) with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Then the IVP (1.2) and (1.5) has a unique global solution (ω, θ) satisfying, for any $T > 0$ and $t \leqslant T$,

$$\omega \in C\big([0,T]; B^s_{q,\infty}(\mathbb{R}^2)\big), \qquad \theta \in C\big([0,T]; B^s_{q,\infty}(\mathbb{R}^2)\big) \cap L^1\big([0,T]; B^{s+1}_{q,\infty}(\mathbb{R}^2)\big). \tag{1.8}$$

A special consequence is the global well-posedness of (1.2) when P is a "double logarithm", namely

$$P(|\xi|) = \big(\log\big(1+\log\big(1+|\xi|^2\big)\big)\big)^\gamma, \quad \gamma \in [0,1]. \tag{1.9}$$

It is easy to see that $P(|\xi|)$ given by (1.9) verifies Condition 1.1, (1.6) and (1.7). The global well-posedness of (1.2) with $P(|\xi|)$ given by (1.9) is given in the following corollary.

Corollary 1.3. *Let $q > 2$ and let $s > 2$. Let $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Assume that $\sigma = 0$ and $P(|\xi|)$ is given by (1.9). Then the IVP (1.2) and (1.5) has a unique global solution.*

When $\theta \equiv 0$, the result in Corollary 1.3 reduces to Theorem 1.3 for the generalized 2D Euler in [8, p. 38]. When P is the identity operator, we reproduce the global well-posedness for one of the critical Boussinesq equations (see [21]). We now point out the key technical ingredients in the proof of Theorem 1.2. Since the solution class in (1.8) is very regular, the uniqueness part is not hard to obtain and our main effort is devoted to the global existence part. To show the global existence, we establish the desired global *a priori* bounds for (ω, θ). Direct energy estimates do not yield the desired estimates mainly due to the vortex stretching term $\partial_{x_1}\theta$. Our process of obtaining the global bounds for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$ is divided into two major steps. The first step establishes the global bound for $\|\omega\|_{L^q}$, $\|\theta\|_{B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty}$, where $B^{0,P}_{\infty,2}$ with P being the aforementioned operator denotes a generalized inhomogeneous Besov space and its definition is given in Appendix A. To prove the global bounds in this step, we follow the idea of [21] to consider the combined quantity

$$G = \omega + \mathcal{R}\theta \quad \text{with} \quad \mathcal{R} \equiv \Lambda^{-1}\partial_{x_1},$$

which satisfies

$$\partial_t G + u\cdot\nabla G = -[\mathcal{R}, u\cdot\nabla]\theta, \tag{1.10}$$

where the commutator $[\mathcal{R}, u\cdot\nabla]\theta = \mathcal{R}(u\cdot\nabla\theta) - u\cdot\nabla(\mathcal{R}\theta)$. The advantage of (1.10) is that it hides the vortex stretching term $\partial_{x_1}\theta$. The trade-off is that we need to obtain suitable bounds for the commutator. By proving suitable bounds for the commutator and making full use of the dissipation in the equation for θ, we are able to obtain the global bound for $\|\omega\|_{L^q}$, $\|\theta\|_{B^{0,p}_{\infty,2}}$ and consequently for $\|\omega\|_{L^\infty}$. We leave more details on these estimates in Section 2 and Section 3.

The second major step establishes the desired global bounds for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$ for $s>2$. This step is further divided into two sub-steps. The first sub-step proves the global bounds for $\|\omega\|_{B^\beta_{q,\infty}}$ and $\|\theta\|_{B^\beta_{q,\infty}}$ for β in the range $\frac{2}{q}<\beta<1$. It is necessary to use the equation for G in (1.10). As explained previously, (1.10) is more regular than the vorticity equation. The tools employed here include Besov space techniques and a logarithmic interpolation inequality bounding $\|\nabla u\|_{L^\infty}$ in terms of $\|\omega\|_{L^q\cap L^\infty}$ and $\|\omega\|_{B^\beta_{q,\infty}}$. The restriction of β in this range is due to one of paraproducts decomposed from the nonlinear terms. Osgood's inequality, together with the condition in (1.7), yields the bounds in the first sub-step. The second sub-step takes advantage of the regularity obtained in the first sub-step and establishes the bounds for the Besov index β_1 in the range $1<\beta_1<2-\frac{2}{q}$. A repetition of this sub-step allows us to reach any index $s>2$. More details are provided in Section 4.

The rest of this paper is divided into four regular sections with an appendix. The second section proves the aforementioned logarithmic interpolation inequality and provides estimates for the commutator $[\mathcal{R}, u\cdot\nabla]\theta$ in the Besov space $B^0_{p,r}$ and also in $B^0_{\infty,r}$. The third section establishes global *a priori* bounds for $\|\omega\|_{L^q}$, $\|\theta\|_{B^{0,p}_{\infty,2}}$ and $\omega\|_{L^\infty}$. The fourth section obtains global *a priori* bounds for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$. The fifth section outlines the proof of Theorem 1.2. Appendix A provides the definitions of Besov type spaces and related estimates such as the Bernstein inequality.

2. Preliminary estimates

This section presents several important estimates to be extensively used in the subsequent sections. First, we recall some fundamental estimates for $\|\Delta_j\nabla u\|_{L^p}$ and $\|S_N\nabla u\|_{L^p}$ from [8]. Here the Fourier localization operator Δ_j and the identity approximation operator S_j, along with other basic notions and techniques associate with Besov spaces, will be defined and explained in Appendix A. Second, we prove a logarithmic interpolation inequality bounding $\|\nabla u\|_{L^\infty}$ in terms of $\|\omega\|_{L^q\cap L^\infty}$ and the Besov norm $\|\omega\|_{B^\beta_{q,\infty}}$. Finally, we establish two estimates for the commutator $[\mathcal{R}, u\cdot\nabla]\theta$ in the Besov space $B^0_{p,r}$ and in $B^0_{\infty,r}$.

The velocity field u in (1.1) or (1.2) is determined by the vorticity ω through a Fourier multiplier operator, namely

$$u = \nabla^\perp\Delta^{-1}\Lambda^\sigma P(\Lambda)\omega.$$

In order to bound ω in Besov type spaces, we often need to bound ∇u in terms of ω and the basic ingredients involved are $\|\Delta_j\nabla u\|_{L^p}$ and $\|S_N\nabla u\|_{L^p}$. A recent work of Chae, Constantin and Wu [8] has provided bounds for these quantities associated with a very general Fourier multiplier operator (see Theorem 1.2 in [8, p. 37]). More precisely, they proved the following result.

Lemma 2.1. *Assume that the symbol Q satisfies Condition 1.1 and that u and ω are related through*

$$u = \nabla^\perp \Delta^{-1} Q(\Lambda)\omega.$$

Then, for any integer $j \geqslant 0$ and $N \geqslant 0$,

$$\|S_N \nabla u\|_{L^p} \leqslant C_p Q\big(C_0 2^N\big)\|S_N\omega\|_{L^p}, \quad 1 < p < \infty,$$
$$\|\Delta_j \nabla u\|_{L^q} \leqslant C Q\big(C_0 2^j\big)\|\Delta_j\omega\|_{L^q}, \quad 1 \leqslant q \leqslant \infty,$$

where C_p is a constant depending on p only, C_0 and C are pure constants.

The next proposition provides a logarithmic type interpolation inequality that bounds $\|\nabla u\|_{L^\infty}$. This inequality will be used in the subsequent sections.

Proposition 2.2. *Assume that the symbol Q satisfies Condition 1.1 and (1.6). Let u and ω be related through*

$$u = \nabla^\perp \Delta^{-1} Q(\Lambda)\omega.$$

Then, for any $1 \leqslant q \leqslant \infty$, $\beta > 2/q$, and $1 < p < \infty$,

$$\|\nabla u\|_{L^\infty} \leqslant C\big(1 + \|\omega\|_{L^p}\big) + C\|\omega\|_{L^\infty} \log\big(1 + \|\omega\|_{B^\beta_{q,\infty}}\big) Q\big(\|\omega\|_{B^\beta_{q,\infty}}^{\frac{2q}{q\beta-2}}\big),$$

where C's are constants that depend on p, q and β only.

Proof of Proposition 2.2. For any integer $N \geqslant 0$, we have

$$\|\nabla u\|_{L^\infty} \leqslant \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{k=0}^{N-1} \|\Delta_k \nabla u\|_{L^\infty} + \sum_{k=N}^{\infty} \|\Delta_k \nabla u\|_{L^\infty}.$$

By Bernstein's inequality and Lemma 2.1, we have

$$\|\nabla u\|_{L^\infty} \leqslant C\|\omega\|_{L^p} + CN Q\big(2^N\big)\|\omega\|_{L^\infty} + C\sum_{k=N}^{\infty} \big(2^k\big)^{\frac{2}{q}} \|\nabla \Delta_k u\|_{L^q}.$$

By Lemma 2.1,

$$\|\nabla u\|_{L^\infty} \leqslant C\|\omega\|_{L^p} + CN Q\big(2^N\big)\|\omega\|_{L^\infty} + C\sum_{k=N}^{\infty} \big(2^k\big)^{\frac{2}{q}} Q\big(2^k\big)\|\Delta_k \omega\|_{L^q}.$$

By the definition of Besov space $B^\beta_{q,\infty}$,

$$\|\Delta_k\omega\|_{L^q} \leqslant 2^{-\beta k}\|\omega\|_{B^\beta_{q,\infty}}.$$

Therefore,

$$\|\nabla u\|_{L^\infty} \leqslant C\|\omega\|_{L^p} + CNQ(2^N)\|\omega\|_{L^\infty} + C\|\omega\|_{B^\beta_{q,\infty}} \sum_{k=N}^{\infty} (2^k)^{(\frac{2}{q}-\beta)} Q(2^k).$$

Due to $\frac{2}{q} - \beta < 0$ and (1.6), we can choose $\epsilon > 0$ such that

$$\epsilon + \frac{2}{q} - \beta < 0 \quad \text{and} \quad Q(2^N) \leqslant 2^{\epsilon N}.$$

Especially, we take $\epsilon = \frac{1}{2}(\beta - \frac{2}{q})$ to get

$$\|\nabla u\|_{L^\infty} \leqslant C\|\omega\|_{L^p} + CNQ(2^N)\|\omega\|_{L^\infty} + C\|\omega\|_{B^\beta_{q,\infty}} (2^N)^{(\frac{1}{q}-\frac{\beta}{2})}.$$

If we choose N to be the largest integer satisfying

$$N \leqslant \frac{1}{\frac{\beta}{2} - \frac{1}{q}} \log_2\left(1 + \|\omega\|_{B^\beta_{q,\infty}}\right),$$

we then obtain the desired result in Proposition 2.2. □

We also use extensively an estimate for the commutator $[\mathcal{R}, u \cdot \nabla]\theta$. In order to prove this estimate, we first state a fact given by the following lemma.

Lemma 2.3. *Consider two different cases: $\delta \in (0, 1)$ and $\delta = 1$.*

(1) *Let $\delta \in (0, 1)$ and $q \in [1, \infty]$. If $|x|^\delta h \in L^1$, $f \in \mathring{B}^\delta_{q,\infty}$ and $g \in L^\infty$, then*

$$\left\|h * (fg) - f(h * g)\right\|_{L^q} \leqslant C\left\||x|^\delta h\right\|_{L^1} \|f\|_{\mathring{B}^\delta_{q,\infty}} \|g\|_{L^\infty}, \tag{2.1}$$

where C is a constant independent of f, g and h.

(2) *Let $\delta = 1$. Let $q \in [1, \infty]$. Let $r_1 \in [1, q]$ and $r_2 \in [1, \infty]$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then*

$$\left\|h * (fg) - f(h * g)\right\|_{L^q} \leqslant C\left\||x|h\right\|_{L^{r_1}} \|\nabla f\|_{L^q} \|g\|_{L^{r_2}}. \tag{2.2}$$

Here $\mathring{B}^\delta_{q,\infty}$ denotes a homogeneous Besov space, as defined in Appendix A. (2.1) is taken from [9] while (2.2) was obtained in [20, p. 426]. With these notions at our disposal, we are ready to state and prove the commutator estimate.

Proposition 2.4. *Let $\mathcal{R} = \Lambda^{-1}\partial_{x_1}$ denote the Riesz transform. Assume that the symbol P satisfies Condition 1.1 and*

$$\textit{for any } \epsilon > 0, \quad \lim_{|\xi| \to \infty} \frac{P(|\xi|)}{|\xi|^\epsilon} = 0. \tag{2.3}$$

Assume that u and ω are related by

$$u = \nabla^{\perp}\Delta^{-1}\Lambda^{\sigma}P(\Lambda)\omega$$

with $\sigma \in [0, 1)$. *Then, for any* $p \in (1, \infty)$ *and* $r \in [1, \infty]$,

$$\big\| [\mathcal{R}, u \cdot \nabla]\theta \big\|_{B^0_{p,r}} \leqslant C\|\omega\|_{L^p}\|\theta\|_{B^{\sigma,P}_{\infty,r}} + C\|\omega\|_{L^p}\|\theta\|_{L^p} \tag{2.4}$$

and, for any $r \in [1, \infty]$, $q \in (1, \infty)$ *and any* $\epsilon > 0$,

$$\big\| [\mathcal{R}, u \cdot \nabla]\theta \big\|_{B^0_{\infty,r}} \leqslant C\big(\|\omega\|_{L^q} + \|\omega\|_{L^\infty}\big)\|\theta\|_{B^{\sigma+\epsilon}_{\infty,r}} + C\|\omega\|_{L^q}\|\theta\|_{L^q} \tag{2.5}$$

for some constant C, where the generalized Besov space $B^{\sigma,P}_{\infty,r}$ *with P being the symbol of the operator P is defined in Appendix A, and* $B^{\sigma+\epsilon}_{\infty,r}$ *is a standard Besov space whose definition is also provided in Appendix A.*

Proof of Proposition 2.4. By the definition of $B^0_{p,r}$,

$$\big\| [\mathcal{R}, u \cdot \nabla]\theta \big\|_{B^0_{p,r}} = \left[\sum_{j=-1}^{\infty} \big\| \Delta_j[\mathcal{R}, u \cdot \nabla]\theta \big\|^r_{L^p} \right]^{\frac{1}{r}}.$$

Using the notion of paraproducts, we decompose $\Delta_j[\mathcal{R}, u \cdot \nabla]\theta$ into three parts,

$$\Delta_j[\mathcal{R}, u \cdot \nabla]\theta = J_1 + J_2 + J_3,$$

where

$$J_1 = \sum_{|k-j|\leqslant 2} \Delta_j\big(\mathcal{R}(S_{k-1}u \cdot \nabla\Delta_k\theta) - S_{k-1}u \cdot \nabla\mathcal{R}\Delta_k\theta\big),$$

$$J_2 = \sum_{|k-j|\leqslant 2} \Delta_j\big(\mathcal{R}(\Delta_k u \cdot \nabla S_{k-1}\theta) - \Delta_k u \cdot \nabla\mathcal{R}S_{k-1}\theta\big),$$

$$J_3 = \sum_{k\geqslant j-1} \Delta_j\big(\mathcal{R}(\Delta_k u \cdot \nabla\widetilde{\Delta}_k\theta) - \Delta_k u \cdot \nabla\mathcal{R}\widetilde{\Delta}_k\theta\big)$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. The Fourier transform of $S_{k-1}u \cdot \nabla\Delta_k\theta$ is supported in the annulus $2^k A$, where A denotes a fixed annulus. $\mathcal{R}$ acting on this term can be represented as a convolution with the kernel $h_k(x) = 2^{dk}h(2^k x)$ with $d = 2$, where h is a smooth function with compact support. That is,

$$\begin{aligned}
&\mathcal{R}(S_{k-1}u \cdot \nabla\Delta_k\theta) - S_{k-1}u \cdot \nabla\mathcal{R}\Delta_k\theta \\
&\quad = h_k * (S_{k-1}u \cdot \nabla\Delta_k\theta) - S_{k-1}u \cdot \nabla(h_k * \Delta_k\theta).
\end{aligned}$$

Therefore, according to Lemma 2.3,

$$\|J_1\|_{L^p} \leqslant C\big\| |x|h_j \big\|_{L^1}\|\nabla S_{j-1}u\|_{L^p}\|\nabla\Delta_j\theta\|_{L^\infty}. \tag{2.6}$$

Applying Lemma 2.1, Bernstein's inequality and the equality

$$\big\| |x| h_j \big\|_{L^1} = 2^{-j} \big\| |x| h(x) \big\|_{L^1} = C 2^{-j},$$

we have

$$\begin{aligned} \|J_1\|_{L^p} &\leqslant C 2^{\sigma j} P\big(2^j\big) \|S_{j-1}\omega\|_{L^p} \|\Delta_j \theta\|_{L^\infty} \\ &\leqslant C 2^{\sigma j} P\big(2^j\big) \|\omega\|_{L^p} \|\Delta_j \theta\|_{L^\infty}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|J_2\|_{L^p} &\leqslant C 2^{-j} 2^{\sigma j} P\big(2^j\big) \|\Delta_j \omega\|_{L^p} \|\nabla S_{j-1}\theta\|_{L^\infty} \\ &\leqslant C 2^{-(1-\sigma) j} P\big(2^j\big) \|\Delta_j \omega\|_{L^p} \sum_{m \leqslant j-1} 2^m \|\Delta_m \theta\|_{L^\infty} \\ &\leqslant C \|\Delta_j \omega\|_{L^p} \sum_{m \leqslant j-1} \frac{2^{(1-\sigma)m} P(2^j)}{2^{(1-\sigma)j} P(2^m)} 2^{\sigma m} P\big(2^m\big) \|\Delta_m \theta\|_{L^\infty}. \end{aligned}$$

The estimate of $\|J_3\|_{L^p}$ is different. We need to distinguish between low frequency and high frequency terms. For the high frequency terms, we do not need the commutator structure. For $j = 0, 1$, the terms in J_3 with $k = -1, 0, 1$ have Fourier transforms containing the origin in their support and the lower bound part of Bernstein's inequality does not apply. To deal with these low frequency terms, we take advantage of the commutator structure and bound them by Lemma 2.3. More precisely, for $j = 0, 1$ and $k = -1, 0, 1$,

$$\begin{aligned} &\big\| \Delta_j \big(\mathcal{R}(\Delta_k u \cdot \nabla \widetilde{\Delta}_k \theta) - \Delta_k u \cdot \nabla \mathcal{R} \widetilde{\Delta}_k \theta \big) \big\|_{L^p} \\ &\quad \leqslant C \|\nabla \Delta_k u\|_{L^p} \|\Delta_k \theta\|_{L^p} \leqslant C \|\omega\|_{L^p} \|\theta\|_{L^p}. \end{aligned}$$

For higher frequency terms, we first apply Bernstein's inequality to obtain

$$\begin{aligned} \|J_3\|_{L^p} &\leqslant C \sum_{k \geqslant j-1} 2^j \big\| \mathcal{R}(\Delta_k u \cdot \widetilde{\Delta}_k \theta) \big\|_{L^p} + C \sum_{k \geqslant j-1} 2^j \|\Delta_k u \cdot \mathcal{R} \widetilde{\Delta}_k \theta\|_{L^p} \\ &\leqslant C \sum_{k \geqslant j-1} 2^{j-k} \|\nabla \Delta_k u\|_{L^p} \|\Delta_k \theta\|_{L^\infty} \\ &\leqslant C \sum_{k \geqslant j-1} 2^{j-k} \|\Delta_k \omega\|_{L^p} 2^{\sigma k} P\big(2^k\big) \|\Delta_k \theta\|_{L^\infty}. \end{aligned}$$

Thanks to $\sigma \in [0, 1)$ and the assumption on P in (2.3), we obtain, by Young's inequality for series convolution,

$$\begin{aligned} \big\| [\mathcal{R}, u \cdot \nabla] \theta \big\|_{B^0_{p,r}} &= C \Bigg[\sum_{j=-1}^{\infty} \big(\|J_1\|_{L^p}^r + \|J_2\|_{L^p}^r + \|J_3\|_{L^p}^r \big) \Bigg]^{\frac{1}{r}} \\ &= C \|\omega\|_{L^p} \|\theta\|_{B^{\sigma,P}_{\infty,r}} + C \|\omega\|_{L^p} \|\theta\|_{L^p}. \end{aligned}$$

This completes the proof of (2.4). We now prove (2.5). We shall only provide those estimates that are different from the previous ones. As in (2.6), we still have

$$\|J_1\|_{L^\infty} \leqslant C\big\||x|h_j\big\|_{L^1}\|\nabla S_{j-1}u\|_{L^\infty}\|\nabla\Delta_j\theta\|_{L^\infty}.$$

But $\|\nabla S_{j-1}u\|_{L^\infty}$ is bounded differently here. By Lemma 2.3 and the assumption in (2.3), we obtain, for $\sigma \in [0,1)$ and for any $\epsilon > 0$,

$$\begin{aligned}\|\nabla S_{j-1}u\|_{L^\infty} &\leqslant \|\nabla\Delta_{-1}u\|_{L^\infty} + \sum_{0\leqslant m\leqslant j-2}\|\Delta_m\nabla u\|_{L^\infty}\\ &\leqslant C\|w\|_{L^q} + \sum_{0\leqslant m\leqslant j-2}2^{\sigma m}P\big(2^m\big)\|\Delta_m\omega\|_{L^\infty}\\ &\leqslant C\|w\|_{L^q} + C2^{(\sigma+\epsilon)j}\|\omega\|_{L^\infty}.\end{aligned}$$

Consequently,

$$\|J_1\|_{L^\infty} \leqslant C\big(\|w\|_{L^q} + \|\omega\|_{L^\infty}\big)2^{(\sigma+\epsilon)j}\|\Delta_j\theta\|_{L^\infty}.$$

The bounds for J_2 and J_3 can be obtained by simply setting $p=\infty$ in the corresponding bounds for $\|J_2\|_{L^p}$ and $\|J_3\|_{L^p}$ above. This completes the proof of Proposition 2.4. □

3. Global *a priori* bounds for $\|\omega\|_{L_t^\infty L^q}$, $\|\theta\|_{L_t^1 B_{\infty,2}^{0,P}}$ and $\|\omega\|_{L_t^\infty L^\infty}$

This section establishes global bounds for $\|\omega\|_{L_t^\infty L^q}$, $\|\theta\|_{L_t^1 B_{\infty,2}^{0,P}}$ and $\|\omega\|_{L_t^\infty L^\infty}$. In order to obtain these bounds, we have to set $\sigma=0$ and assume P satisfies Condition 1.1 and also (1.6), even though some of the intermediate results in this section hold without $\sigma=0$ and with P satisfying milder conditions.

Proposition 3.1. *Let $\sigma=0$ and $q>2$. Assume the symbol P satisfies Condition 1.1 and (1.6). Let (ω,θ) be a smooth solution of (1.2) with $\omega_0 \in B_{q,\infty}^s$ and $\theta_0 \in B_{q,\infty}^s$. Then, for any $T>0$ and $0<t\leqslant T$,*

$$\big\|\omega(t)\big\|_{L^q} \leqslant C(T), \qquad \|\theta\|_{L_t^1 B_{\infty,2}^{0,P}} \leqslant C(T), \qquad \big\|\omega(t)\big\|_{L^\infty} \leqslant C(T)$$

for some constant C depending T and the initial norms of ω_0 and θ_0.

We remark that (1.6) clearly implies (2.3). In order to prove this proposition, we need the following two lemmas.

Lemma 3.2. *Let $\sigma \in [0,1)$. Assume that the symbol P satisfies Condition 1.1 and (2.3). Let (ω,θ) be a smooth solution of (1.2). Then, for any $q\in[2,\infty)$ and for any $t>0$,*

$$\big\|\omega(t)\big\|_{L^q} \leqslant C\big(\|\omega_0\|_{L^q} + \|\theta_0\|_{L^q}\big)e^{Ct\|\theta_0\|_{L^q}}e^{C\int_0^t\|\theta(\tau)\|_{B_{\infty,2}^{\sigma,P}}d\tau}, \tag{3.1}$$

where C's are pure constants.

Proof of Lemma 3.2. We start with the equations satisfied by G and $\mathcal{R}\theta$,

$$\partial_t G + u\cdot\nabla G = -[\mathcal{R}, u\cdot\nabla]\theta,$$
$$\partial_t \mathcal{R}\theta + u\cdot\nabla\mathcal{R}\theta + \Lambda\mathcal{R}\theta = -[\mathcal{R}, u\cdot\nabla]\theta. \tag{3.2}$$

By the embedding $B^0_{q,2} \hookrightarrow L^q$ for $q \geqslant 2$ and Lemma 2.4,

$$\begin{aligned}\|\omega(t)\|_{L^q} &\leqslant \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \big\|[\mathcal{R}, u\cdot\nabla]\theta\big\|_{L^q}\,d\tau \\ &\leqslant \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \big\|[\mathcal{R}, u\cdot\nabla]\theta\big\|_{B^0_{q,2}}\,d\tau \\ &\leqslant \|G_0\|_{L^q} + \|\theta_0\|_{L^q} + C\int_0^t \big[\|\omega(\tau)\|_{L^q}\big(\|\theta(\tau)\|_{B^{\sigma,P}_{\infty,2}} + \|\theta_0\|_{L^q}\big)\big]\,d\tau,\end{aligned}$$

which implies (3.1), by Gronwall's inequality. □

The second lemma makes use of the dissipation in the θ-equation,

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0,\\ u = \nabla^\perp\psi, \quad \Delta\psi = \Lambda^\sigma P(\Lambda)\omega,\\ \theta(x,0) = \theta_0(x).\end{cases} \tag{3.3}$$

Lemma 3.3. *Let $\sigma \in [0,1)$. Assume that the symbol P satisfies Condition 1.1 and (2.3). Let $q \in (1,\infty)$. Then, any smooth solution (ω,θ) solving (3.3) satisfies, for each integer $j \geqslant 0$,*

$$2^{j(1-\sigma)}\|\Delta_j\theta\|_{L^1_t L^q} \leqslant 2^{-j\sigma}\|\Delta_j\theta_0\|_{L^q} + C P\big(2^j\big)\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q}\,d\tau, \tag{3.4}$$

where C is a pure constant.

The lemma will be proven at the end of this section. With these two lemmas at our disposal, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. The proof uses the bounds in Lemmas 3.2 and 3.3 with $\sigma = 0$. By the definition of $B^{0,P}_{\infty,2}$ and the embedding $B^{0,P}_{\infty,1} \hookrightarrow B^{0,P}_{\infty,2}$,

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leqslant \int_0^t \left[\sum_{j=-1}^{N-1}\big(P\big(2^j\big)\big)^2\|\Delta_j\theta\|^2_{L^\infty}\right]^{\frac12} d\tau + \int_0^t \sum_{j=N}^{\infty} P\big(2^j\big)\|\Delta_j\theta\|_{L^\infty}\,d\tau.$$

Thanks to the condition on P in (1.6),

$$\|\theta\|_{L_t^1 B_{\infty,2}^{0,P}} \leqslant t\|\theta_0\|_{L^\infty} N + \sum_{j\geqslant N} P\big(2^j\big)\|\Delta_j\theta\|_{L_t^1 L^\infty}. \tag{3.5}$$

Since $q \in (2, \infty)$ and P satisfies (1.6), we choose $\epsilon > 0$ such that

$$-1+\epsilon+\frac{2}{q} < 0, \qquad \big(P\big(2^j\big)\big)^2 2^{-j\epsilon} \leqslant 1.$$

By Bernstein's inequality and Lemma 3.3 with $\sigma = 0$,

$$\begin{aligned}
\sum_{j\geqslant N} P\big(2^j\big)\|\Delta_j\theta\|_{L_t^1 L^\infty} &\leqslant \sum_{j\geqslant N} P\big(2^j\big)2^{j\frac{2}{q}}\|\Delta_j\theta\|_{L_t^1 L^q} \\
&\leqslant C\sum_{j\geqslant N}\big(P\big(2^j\big)\big)^2 2^{j(\frac{2}{q}-1)}\big(\|\theta_0\|_{L^q}+\|\theta_0\|_{L^\infty}\|\omega\|_{L_t^1 L^q}\big) \\
&\leqslant C\sum_{j\geqslant N} 2^{j(\frac{2}{q}+\epsilon-1)}\big(\|\theta_0\|_{L^q}+\|\theta_0\|_{L^\infty}\|\omega\|_{L_t^1 L^q}\big) \\
&\leqslant C\|\theta_0\|_{L^q} + C2^{N(-1+\epsilon+\frac{2}{q})}\|\theta_0\|_{L^\infty}\|\omega\|_{L_t^1 L^q}.
\end{aligned}$$

Inserting the estimates above in (3.5) and choosing N to be the largest integer satisfying

$$N \leqslant \frac{\log(1+\|\omega\|_{L_t^1 L^q})}{(1-\epsilon-\frac{2}{q})}+1$$

leads to

$$\|\theta\|_{L_t^1 B_{\infty,2}^{0,P}} \leqslant C\|\theta_0\|_{L^\infty\cap L^q} + C\|\theta_0\|_{L^\infty} t\log\left(1+\int_0^t \big\|\omega(\tau)\big\|_{L^q}\,d\tau\right).$$

It then follows from this estimate and (3.1) with $\sigma = 0$ that

$$\|\theta\|_{L_t^1 B_{\infty,2}^{0,P}} \leqslant Ct\log(1+Ct) + Ct\|\theta\|_{L_t^1 B_{\infty,2}^{0,P}}, \tag{3.6}$$

where C's are constants depending on $\|\theta_0\|_{L^q}$ and $\|\theta_0\|_{L^\infty}$. This inequality allows us to conclude that, for any $T > 0$ and $t \leqslant T$,

$$\|\theta\|_{L_t^1 B_{\infty,2}^{0,P}} \leqslant C\big(T, \|\omega_0\|_{L^q}, \|\theta_0\|_{L^q\cap L^\infty}\big). \tag{3.7}$$

In fact, (3.7) is first obtained on a finite-time interval and the global bound is then obtained through an iterative process. Finally we prove the global bound for $\|\omega\|_{L^\infty}$. By (3.4) with $\sigma = 0$ and (1.6), we have, for any integer $j \geqslant 0$ and any $\epsilon > 0$,

$$2^{j(1-\epsilon)}\|\Delta_j\theta\|_{L_t^1 L^q} \leqslant \|\theta_0\|_{L^q} + C\|\theta_0\|_{L^\infty}\int_0^t \big\|\omega(\tau)\big\|_{L^q}\,d\tau \leqslant C(T). \tag{3.8}$$

Since $q \in (2, \infty)$, we can choose $\epsilon > 0$ such that

$$2\epsilon + \frac{2}{q} - 1 < 0.$$

By Bernstein's inequality,

$$\|\theta\|_{B^{\epsilon}_{\infty,1}} \leqslant \sum_{j \geqslant -1} 2^{(2\epsilon + \frac{2}{q} - 1)j} 2^{(1-\epsilon)j} \|\Delta_j \theta\|_{L^q} \leqslant C \sup_{j \geqslant -1} 2^{j(1-\epsilon)} \|\Delta_j \theta\|_{L^q}.$$

It then follows from (3.8) that, for any $t \leqslant T$,

$$\|\theta\|_{L^1_t B^{\epsilon}_{\infty,1}} \leqslant C(T). \tag{3.9}$$

Starting with the equations of G and $\mathcal{R}\theta$, namely (3.2), and applying Lemma 2.4, we have, for any $\epsilon > 0$,

$$\begin{aligned}
\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} &\leqslant \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + 2\int_0^t \big\|[\mathcal{R}, u \cdot \nabla]\theta\big\|_{B^0_{\infty,1}} \, d\tau \\
&\leqslant \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} \\
&\quad + \int_0^t \Big(\big(\|\omega\|_{L^q} + \|\omega\|_{L^\infty}\big)\|\theta\|_{B^{\epsilon}_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q}\Big) \, d\tau \\
&\leqslant \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + \int_0^t \big(\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty}\big)\|\theta\|_{B^{\epsilon}_{\infty,1}} \, d\tau \\
&\quad + \int_0^t \big(\|\omega\|_{L^q}\|\theta\|_{B^{\epsilon}_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q}\big) \, d\tau.
\end{aligned}$$

By Gronwall's inequality, (3.9) and the global bound for $\|\omega\|_{L^q}$, we have

$$\|\omega\|_{L^\infty} \leqslant \|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} \leqslant C(T).$$

This completes the proof of Proposition 3.1. □

We now provide the proof of Lemma 3.3.

Proof of Lemma 3.3. Letting $j \geqslant 0$ and applying Δ_j to (3.3), multiplying by $\Delta_j\theta|\Delta_j\theta|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|^q_{L^q} + \int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta \, dx = -\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(u \cdot \nabla\theta) \, dx.$$

Due to the lower bound (see, e.g., [11,36])

$$\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx \geqslant C2^j\|\Delta_j\theta\|_{L^q}^q$$

and the decomposition of $[\Delta_j, u\cdot\nabla]\theta$ into five parts,

$$\Delta_j(u\cdot\nabla\theta) = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$\begin{aligned}
J_1 &= \sum_{|j-k|\leqslant 2} [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta,\\
J_2 &= \sum_{|j-k|\leqslant 2} (S_{k-1}u - S_ju)\cdot\nabla\Delta_j\Delta_k\theta,\\
J_3 &= S_ju\cdot\nabla\Delta_j\theta,\\
J_4 &= \sum_{|j-k|\leqslant 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta),\\
J_5 &= \sum_{k\geqslant j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta),
\end{aligned}$$

we obtain, by Hölder's inequality,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|_{L^q}^q + C2^j\|\Delta_j\theta\|_{L^q}^q \leqslant \|\Delta_j\theta\|_{L^q}^{q-1}\big(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q}\big).$$

The integral involving J_3 becomes zero due to the divergence-free condition $\nabla\cdot S_ju = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta$ as an integral,

$$[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta = \int \Phi_j(x-y)\big(S_{k-1}u(y) - S_{k-1}u(x)\big)\cdot\nabla\Delta_k\theta(y)\,dy,$$

where Φ_j is the kernel associated with the operator Δ_j (see Appendix A for more details). By Lemma 2.3 and the inequality

$$\big\|\Phi_j(x)|x|^{1-\sigma}\big\|_{L^1} \leqslant 2^{-j(1-\sigma)}\big\|\Phi_0(x)|x|^{1-\sigma}\big\|_{L^1} \leqslant C2^{-j(1-\sigma)},$$

we have

$$\begin{aligned}
\|J_1\|_{L^q} &\leqslant \sum_{|j-k|\leqslant 2} \big\|\Phi_j(x)|x|^{1-\sigma}\big\|_{L^1}\|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}}\|\nabla\Delta_k\theta\|_{L^\infty}\\
&\leqslant C\sum_{|j-k|\leqslant 2} 2^{-j(1-\sigma)}\|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}}2^k\|\Delta_k\theta\|_{L^\infty}.
\end{aligned}$$

Recalling that $\Lambda^{1-\sigma}u = \nabla^\perp \Delta^{-1}\Lambda P(\Lambda)\omega$ and applying Lemma 2.1, we obtain

$$\|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}} \leqslant C\big\|\Lambda^{1-\sigma}S_{k-1}u\big\|_{L^q} \leqslant CP\big(2^j\big)\|S_{k-1}\omega\|_{L^q} \leqslant CP\big(2^j\big)\|\omega\|_{L^q}.$$

Therefore,

$$\|J_1\|_{L^q} \leqslant C2^{j\sigma}P\big(2^j\big)\|\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}.$$

By Bernstein's inequality,

$$\begin{aligned}\|J_2\|_{L^q} &\leqslant \sum_{|j-k|\leqslant 2}\|S_ju - S_{k-1}u\|_{L^q}\|\nabla\Delta_j\theta\|_{L^\infty} \leqslant C\|\Delta_ju\|_{L^q}2^j\|\Delta_j\theta\|_{L^\infty}\\ &\leqslant C\|\nabla\Delta_ju\|_{L^q}\|\Delta_j\theta\|_{L^\infty}\\ &\leqslant C2^{j\sigma}P\big(2^j\big)\|\Delta_j\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}.\end{aligned}$$

We remark that we have applied the lower bound part of Bernstein's inequality in the second inequality above. This is valid for $j \geqslant 0$. Similarly,

$$\begin{aligned}\|J_4\|_{L^q} &\leqslant C\|\Delta_ju\|_{L^q}\|\nabla S_{j-1}\theta\|_{L^\infty} \leqslant C\|\Delta_ju\|_{L^q}2^j\|S_j\theta\|_{L^\infty}\\ &\leqslant C\|\nabla\Delta_ju\|_{L^q}\|\theta\|_{L^\infty} \leqslant C2^{j\sigma}P\big(2^j\big)\|\Delta_j\omega\|_{L^q}\|\theta\|_{L^\infty}.\end{aligned}$$

Thanks to $\sigma \in [0, 1)$ and the condition on P in (2.3),

$$\begin{aligned}\|J_5\|_{L^q} &\leqslant C\sum_{k\geqslant j-1}2^j\|\Delta_ku\|_{L^q}\|\widetilde{\Delta}_k\theta\|_{L^\infty}\\ &\leqslant C\sum_{k\geqslant j-1}2^{j-k}\|\nabla\Delta_ku\|_{L^q}\|\Delta_k\theta\|_{L^\infty}\\ &\leqslant 2^{j\sigma}\sum_{k\geqslant j-1}2^{(j-k)(1-\sigma)}P\big(2^k\big)\|\Delta_k\omega\|_{L^q}\|\Delta_k\theta\|_{L^\infty}\\ &\leqslant C2^{j\sigma}P\big(2^j\big)\|\omega\|_{L^q}\|\theta\|_{L^\infty}.\end{aligned}$$

Collecting the estimates above, we obtain

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} + C2^j\|\Delta_j\theta\|_{L^q} \leqslant C2^{j\sigma}P\big(2^j\big)\|\omega\|_{L^q}\|\theta_0\|_{L^\infty}.$$

Integrating with respect to time yields

$$\big\|\Delta_j\theta(t)\big\|_{L^q} \leqslant e^{-C2^jt}\|\Delta_j\theta_0\|_{L^q} + C2^{j\sigma}P\big(2^j\big)\|\theta_0\|_{L^\infty}\int_0^t e^{-C2^j(t-\tau)}\big\|\omega(\tau)\big\|_{L^q}\,d\tau.$$

We further take the L^1-norm in time to obtain

$$2^j \|\Delta_j \theta\|_{L_t^1 L^q} \leqslant \|\Delta_j \theta_0\|_{L^q} + C 2^{j\sigma} P\big(2^j\big) \|\theta_0\|_{L^\infty} \int_0^t \big\|\omega(\tau)\big\|_{L^q}\, d\tau,$$

which is the desired result. This completes the proof of Lemma 3.3. □

4. Global bound for $\|(\omega, \theta)\|_{B^s_{q,\infty}}$

This section establishes a global bound for $\|(\omega, \theta)\|_{B^s_{q,\infty}}$.

Proposition 4.1. *Assume that $\sigma = 0$ and the symbol $P(|\xi|)$ obeys Condition 1.1, (1.6) and (1.7). Let $q > 2$ and let $s > 2$. Consider the IVP (1.2) and (1.5) with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Let (ω, θ) be a smooth solution of (1.2). Then (ω, θ) admits a global a priori bound. More precisely, for any $T > 0$ and $t \leqslant T$,*

$$\big\|\big(\omega(t), \theta(t)\big)\big\|_{B^s_{q,\infty}} \leqslant C\big(s, q, T, \big\|(\omega_0, \theta_0)\big\|_{B^s_{q,\infty}}\big),$$

where C is a constant depending on s, q, T and the initial norm.

Proof of Proposition 4.1. The proof is divided into two main steps. The first step provides bounds for $\|\omega\|_{B^\beta_{q,\infty}}$ and $\|\theta\||_{B^\beta_{q,\infty}}$ for β in the range $\frac{2}{q} < \beta < 1$ while the second step proves the global bounds for $\|\omega\|_{B^{\beta_1}_{q,\infty}}$ and $\|\theta\|_{B^{\beta_1}_{q,\infty}}$ for $1 \leqslant \beta_1 < 2 - \frac{2}{q}$. The desired bounds in $B^s_{q,\infty}$ with $s > 2$ can be obtained by a repetition of the second step.

Let $j \geqslant -1$ be an integer. Applying Δ_j to the equation of G, namely (3.2), multiplying by $\Delta_j G |\Delta_j G|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} \|\Delta_j G\|^q_{L^q} = &- \int \Delta_j G |\Delta_j G|^{q-2} \Delta_j (u \cdot \nabla G)\, dx \\ &- \int \Delta_j [\mathcal{R}, u \cdot \nabla] \theta \Delta_j G |\Delta_j G|^{q-2}\, dx. \end{aligned}$$

Following the notion of paraproducts, we decompose $\Delta_j(u \cdot \nabla G)$ into five parts,

$$\Delta_j(u \cdot \nabla G) = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$\begin{aligned} J_1 &= \sum_{|j-k| \leqslant 2} [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k G, \\ J_2 &= \sum_{|j-k| \leqslant 2} (S_{k-1} u - S_j u) \cdot \nabla \Delta_j \Delta_k G, \\ J_3 &= S_j u \cdot \nabla \Delta_j G, \\ J_4 &= \sum_{|j-k| \leqslant 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1} G), \end{aligned}$$

$$J_5 = \sum_{k \geqslant j-1} \Delta_j(\Delta_k u \cdot \nabla \widetilde{\Delta}_k G).$$

By Hölder's inequality,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|_{L^q}^q \leqslant \|\Delta_j G\|_{L^q}^{q-1}\big(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q} + \|J_6\|_{L^q}\big),$$

where $J_6 = \Delta_j[\mathcal{R}, u \cdot \nabla]\theta$. The integral involving J_3 becomes zero due to the divergence-free condition $\nabla \cdot S_j u = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j, S_{k-1}u \cdot \nabla]\Delta_k G$ as an integral,

$$[\Delta_j, S_{k-1}u \cdot \nabla]\Delta_k G = \int \Phi_j(x-y)\big(S_{k-1}u(y) - S_{k-1}u(x)\big) \cdot \nabla \Delta_k G(y)\, dy,$$

where Φ_j is the kernel associated with the operator Δ_j (see Appendix A for more details). By a standard commutator estimate (see, e.g., [10, p. 39], [36, p. 814–815]),

$$\|J_1\|_{L^q} \leqslant C \sum_{|j-k| \leqslant 2} \|\nabla S_{k-1}u\|_{L^\infty} \|\Delta_k G\|_{L^q}.$$

By Hölder's and Bernstein's inequalities,

$$\|J_2\|_{L^q} \leqslant C\|\nabla \Delta_j u\|_{L^\infty} \|\Delta_j G\|_{L^q}.$$

We have especially applied the lower bound part in Bernstein's inequalities (see Proposition A.6). The purpose is to shift the derivative ∇ from G to u. It is worth pointing out that the lower bound does not apply when $j = -1$. In the case when $j = -1$, J_2 involves only low modes and there is no need to shift the derivative from G to u. J_2 is bounded differently. When $j = -1$, J_2 becomes

$$J_2 = -S_0(u) \cdot \nabla \Delta_1 \Delta_{-1} G = -\Delta_{-1} u \cdot \nabla \Delta_1 \Delta_{-1} G,$$

whose L^q-norm can be bounded by

$$\|J_2\|_{L^q} \leqslant C\|\Delta_{-1}u\|_{L^\infty}\|\Delta_{-1}G\|_{L^q} \leqslant C\|\omega\|_{L^q}\|G\|_{L^q}.$$

For J_4 and J_5, we have, by Bernstein's inequality,

$$\begin{aligned}
\|J_4\|_{L^q} &\leqslant C \sum_{|j-k| \leqslant 2} \|\Delta_k u\|_{L^\infty} \|\nabla S_{k-1} G\|_{L^q} \\
&\leqslant C \sum_{|j-k| \leqslant 2} \|\nabla \Delta_k u\|_{L^\infty} \sum_{m \leqslant k-1} 2^{m-k} \|\Delta_m G\|_{L^q}, \\
\|J_5\|_{L^q} &\leqslant C \sum_{k \geqslant j-1} 2^j \|\Delta_k u\|_{L^\infty} \|\widetilde{\Delta}_k G\|_{L^q} \\
&\leqslant C \sum_{k \geqslant j-1} 2^{j-k} \|\nabla \Delta_k u\|_{L^\infty} \|\widetilde{\Delta}_k G\|_{L^q}.
\end{aligned}$$

Furthermore, for any $\beta \in \mathbb{R}$,

$$\|J_1\|_{L^q} \leqslant C \sum_{|j-k|\leqslant 2} \|\nabla u\|_{L^\infty} 2^{-\beta(k+1)} 2^{\beta(k+1)} \|\Delta_k G\|_{L^q} \tag{4.1}$$

$$\leqslant C 2^{-\beta(j+1)} \|G\|_{B^\beta_{q,\infty}} \|\nabla u\|_{L^\infty} \sum_{|j-k|\leqslant 2} 2^{\beta(j-k)} \tag{4.2}$$

$$\leqslant C 2^{-\beta(j+1)} \|G\|_{B^\beta_{q,\infty}} \|\nabla u\|_{L^\infty}, \tag{4.3}$$

where C is a constant depending on β only. It is clear that $\|J_2\|_{L^q}$ admits the same bound. For any $\beta < 1$, we have

$$\begin{aligned}
\|J_4\|_{L^q} &\leqslant C\|\nabla u\|_{L^\infty} \sum_{|j-k|\leqslant 2} \sum_{m<k-1} 2^{m-k} 2^{-\beta(m+1)} 2^{\beta(m+1)} \|\Delta_m G\|_{L^q} \\
&\leqslant C\|\nabla u\|_{L^\infty} \|G\|_{B^\beta_{q,\infty}} \sum_{|j-k|\leqslant 2} \sum_{m<k-1} 2^{m-k} 2^{-\beta(m+1)} \\
&= C 2^{-\beta(j+1)} \|G\|_{B^\beta_{q,\infty}} \|\nabla u\|_{L^\infty} \sum_{|j-k|\leqslant 2} 2^{\beta(j-k)} \sum_{m<k-1} 2^{(m-k)(1-\beta)} \\
&\leqslant C 2^{-\beta(j+1)} \|G\|_{B^\beta_{q,\infty}} \|\nabla u\|_{L^\infty},
\end{aligned}$$

where C is a constant depending on β only and the condition $\beta < 1$ is used to guarantee that $(m-k)(1-\beta) < 0$. For any $\beta > -1$,

$$\begin{aligned}
\|J_5\|_{L^q} &\leqslant C\|\nabla u\|_{L^\infty} 2^{-\beta(j+1)} \sum_{k\geqslant j-1} 2^{(\beta+1)(j-k)} 2^{\beta(k+1)} \|\widetilde{\Delta}_k G\|_{L^q} \\
&\leqslant C 2^{-\beta(j+1)} \|G\|_{B^\beta_{q,\infty}} \|\nabla u\|_{L^\infty}.
\end{aligned}$$

$\|J_6\|_{L^q} = \|\Delta_j[\mathcal{R}, u\cdot\nabla]\theta\|_{L^q}$ can be estimated as in the proof of Proposition 2.4,

$$\|J_6\|_{L^q} \leqslant C\big(\|\omega\|_{L^q} + \|\omega\|_{L^\infty}\big) 2^{\epsilon j} \|\Delta_j\theta\|_{L^q}$$

for any fixed $\epsilon > 0$, where C is a constant depending on ϵ. For the purpose to be specified later, we choose

$$\epsilon > 0, \quad \beta + \epsilon < 1.$$

Collecting these estimates and invoking the global bounds for $\|\omega\|_{L^q\cap L^\infty}$, we obtain, for any $-1 < \beta < 1$,

$$\frac{d}{dt}\|\Delta_j G\|_{L^q} \leqslant C 2^{-\beta(j+1)} \|G\|_{B^\beta_{q,\infty}} \|\nabla u\|_{L^\infty} + C 2^{\epsilon j} \|\Delta_j\theta\|_{L^q} + C.$$

Let $\widetilde{\beta} = \beta + \epsilon < 1$. By applying the process above to the equation for θ and making use of the fact that

$$\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx \geqslant 0,$$

we obtain

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} \leqslant C2^{-\widetilde{\beta}(j+1)}\|\theta\|_{B^{\widetilde{\beta}}_{q,\infty}}\|\nabla u\|_{L^\infty}.$$

Integrating the inequalities in time and adding them up, we obtain

$$X(t) \leqslant C + X(0) + C\int_0^t \left(1+\left\|\nabla u(\tau)\right\|_{L^\infty}\right)X(\tau)\,d\tau, \tag{4.4}$$

where we have set

$$X(t) \equiv \left\|G(t)\right\|_{B^{\beta}_{q,\infty}} + \left\|\theta(t)\right\|_{B^{\widetilde{\beta}}_{q,\infty}}.$$

By Proposition 2.2, for any $\frac{2}{q} < \beta$,

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\leqslant C\left(1+\|\omega\|_{L^p}\right) + C\|\omega\|_{L^\infty} P\left(\|\omega\|^{\frac{2q}{q\beta-2}}_{B^{\beta}_{q,\infty}}\right)\log\left(1+\|\omega\|_{B^{\beta}_{q,\infty}}\right) \\ &\leqslant C\left(1+\|\omega\|_{L^p}\right) + C\|\omega\|_{L^\infty} P\left(X(t)^{\frac{2q}{q\beta-2}}\right)\log\left(1+X(t)\right).\end{aligned}$$

Inserting this inequality in (4.4) and applying Osgood's inequality, we obtain desired bound, for $t \leqslant T$,

$$\left\|\omega(t)\right\|_{B^{\beta}_{q,\infty}} \leqslant \left\|G(t)\right\|_{B^{\beta}_{q,\infty}} + \left\|\theta(t)\right\|_{B^{\widetilde{\beta}}_{q,\infty}} = X(t) \leqslant C(T).$$

We now proceed to show that, for any $t \leqslant T$,

$$\left\|\omega(t)\right\|_{B^{\beta_1}_{q,\infty}} \leqslant C(T) \quad \text{for any } \beta_1 \text{ satisfying } 1 < \beta_1 < 2 - \frac{2}{q}.$$

The strategy is first to get the global bound for $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$ from the equation for θ and then get the global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$. As we have seen from the previous part, J_4 is the only term that requires $\beta < 1$. In the process of estimating $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$, the corresponding terms $\widetilde{J}_1$, $\widetilde{J}_2$, $\widetilde{J}_5$ can be bounded the same way as before, namely

$$\|\widetilde{J}_1\|_{L^q}, \|\widetilde{J}_2\|_{L^q}, \|\widetilde{J}_5\|_{L^q} \leqslant C2^{-\beta_1(j+1)}\|\theta\|_{B^{\beta_1}_{q,\infty}}\|\nabla u\|_{L^\infty}. \tag{4.5}$$

$\|\widetilde{J}_4\|_{L^q}$ is estimated differently. We start with the basic bound

$$\|\widetilde{J}_4\|_{L^q} \leqslant C\sum_{|j-k|\leqslant 2}\|\nabla\Delta_k u\|_{L^\infty}\sum_{m<k-1}2^{m-k}\|\Delta_m\theta\|_{L^q}.$$

Since $\beta_1 + \frac{2}{q} < 2$, we can choose $\frac{2}{q} < \beta < 1$ and $\epsilon > 0$ such that

$$\beta_1 + \frac{2}{q} + \epsilon < 2\beta. \tag{4.6}$$

By Berntsein's inequality and Lemma 2.1,

$$\begin{aligned}\|\nabla\Delta_k u\|_{L^\infty} &\leqslant C2^{\frac{2k}{q}}\|\nabla\Delta_k u\|_{L^q} \leqslant C2^{\frac{2k}{q}} P\big(2^k\big)\|\Delta_k\omega\|_{L^q} \\ &\leqslant C2^{k(\frac{2}{q}+\epsilon)}\|\Delta_k\omega\|_{L^q} \leqslant C2^{k(\frac{2}{q}+\epsilon-\beta)}\|\omega\|_{B^\beta_{q,\infty}}.\end{aligned}$$

Clearly, for any $\beta < 1$,

$$\begin{aligned}\sum_{m<k-1} 2^{m-k}\|\Delta_m\theta\|_{L^q} &= 2^{-\beta k}\sum_{m<k-1} 2^{(m-k)(1-\beta)}2^{\beta m}\|\Delta_m\theta\|_{L^q} \\ &\leqslant C2^{-\beta k}\|\theta\|_{B^\beta_{q,\infty}}.\end{aligned}$$

Therefore, according to (4.6) and the global bound in the first step,

$$\|\widetilde{J}_4\|_{L^q} \leqslant C2^{-\beta_1(j+1)}\|\omega\|_{B^\beta_{q,\infty}}\|\theta\|_{B^\beta_{q,\infty}} 2^{(\beta_1+\frac{2}{q}+\epsilon-2\beta)j} \leqslant C2^{-\beta_1(j+1)}. \tag{4.7}$$

Collecting the estimates in (4.5) and (4.7), we have

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} \leqslant C2^{-\beta_1(j+1)}\|\theta\|_{B^{\beta_1}_{q,\infty}}\|\nabla u\|_{L^\infty} + C2^{-\beta_1(j+1)}.$$

Bounding $\|\nabla u\|_{L^\infty}$ by the interpolation inequality in Proposition 2.2 and applying Osgood inequality lead to the desired global bound for $\|\theta\|_{B^{\beta_1}_{q,\infty}}$. With this bound at our disposal, we then obtain a global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$ by going through a similar process on the equation of G. Therefore, for any $t \leqslant T$,

$$\|\omega\|_{B^{\beta_1}_{q,\infty}} \leqslant \|\theta\|_{B^{\beta_1}_{q,\infty}} + \|G\|_{B^{\beta_1}_{q,\infty}} \leqslant C(T).$$

If necessary, we can repeat the second step a few times to achieve the global bound for ω and θ in $B^s_{q,\infty}$ for any $s > 2$. This completes the proof of Proposition 4.1. □

5. Proof of Theorem 1.2

This section proves Theorem 1.2.

Proof of Theorem 1.2. Due to the high regularity in the class (1.8), the uniqueness of solutions satisfying (1.8) is easy to prove. Attention here will be focused on establishing the existence of solutions.

The existence part starts with the construction of a local solution through the method of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)}, \theta^{(n)})\}$ solving

$$\begin{cases} \omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0, \\ u^{(n)} = \nabla^\perp \Delta^{-1} P(\Lambda)\omega^{(n)}, \\ \partial_t \omega^{(n+1)} + u^{(n)} \cdot \nabla \omega^{(n+1)} = \partial_{x_1} \theta^{(n+1)}, \\ \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} + \Lambda \theta^{(n+1)} = 0, \\ \omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x). \end{cases} \tag{5.1}$$

In order to show that $\{(\omega^{(n)}, \theta^{(n)})\}$ converges to a solution of (1.2), it suffices to prove that $\{(\omega^{(n)}, \theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0, T_1]$ over which $\{(\omega^{(n)}, \theta^{(n)})\}$ are bounded uniformly in terms of n. More precisely, we show that

$$\big\|\big(\omega^{(n)}, \theta^{(n)}\big)\big\|_{B^s_{q,\infty}} \leqslant C\big(T_1, \big\|(\omega_0, \theta_0)\big\|_{B^s_{q,\infty}}\big),$$

for a constant depending on T_1 and the initial norm only.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ are Cauchy sequence in $B^{s-1}_{q,\infty}$, namely

$$\big\|\omega^{(n+1)} - \omega^{(n)}\big\|_{B^{s-1}_{q,\infty}} \leqslant C(T_2)2^{-n}, \qquad \big\|\theta^{(n+1)} - \theta^{(n)}\big\|_{B^{s-1}_{q,\infty}} \leqslant C(T_2)2^{-n}$$

for any $t \in [0, T_2]$, where $C(T_2)$ is independent of n.

If the properties stated in (1) and (2) hold, then there exists (ω, θ) satisfying, for $T = \min\{T_1, T_2\}$,

$$\omega(\cdot,t) \in B^s_{q,\infty}, \qquad \theta(\cdot,t) \in B^s_{q,\infty} \quad \text{for } 0 \leqslant t \leqslant T,$$
$$\omega^{(n)}(\cdot,t) \to \omega(\cdot,t) \quad \text{in } B^{s-1}_{q,\infty}, \qquad \theta^{(n)}(\cdot,t) \to \theta(\cdot,t) \quad \text{in } B^{s-1}_{q,\infty}.$$

It is then easy to show that (ω, θ) solves (1.2) and we thus obtain a local solution and the global bounds in Sections 3 and 4 allow us to extend it into a global solution. It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in Sections 3 and 4. To verify property (2), we consider the equations for the differences $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ and prove property (2) inductively in n. The bounds can be achieved in a similar fashion in Sections 3 and 4. We thus omit further details. This completes the proof of Theorem 1.2. □

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Appendix A. Functional spaces and Osgood inequality

This appendix provides the definitions of some of the functional spaces and related facts used in the previous sections. In addition, the Osgood inequality used in the proof of Proposition 4.1 is also provided here for the convenience of readers. Materials presented in this appendix can be found in several books and many papers (see, e.g., [3,4,26,33,35]).

We start with several notation. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{\phi \in \mathcal{S}: \int_{\mathbb{R}^d} \phi(x) x^\gamma \, dx = 0, \ |\gamma| = 0, 1, 2, \ldots\right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials.

To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{\xi \in \mathbb{R}^d: \ 2^{j-1} \leqslant |\xi| < 2^{j+1}\right\}. \tag{A.1}$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\operatorname{supp} \widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0\left(2^{-j}\xi\right) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0\left(2^j x\right),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \quad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \quad j \in \mathbb{Z}. \tag{A.2}$$

Definition A.1. For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \big\| 2^{js} \|\mathring{\Delta}_j f\|_{L^p} \big\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{A.3}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leqslant -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \ldots. \end{cases} \tag{A.4}$$

Definition A.2. The inhomogeneous Besov space $B^s_{p,q}$ with $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \big\| 2^{js} \|\Delta_j f\|_{L^p} \big\|_{l^q} < \infty.$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0, 1)$ and $1 \leqslant p, q \leqslant \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int\limits_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}}\, dt \right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int\limits_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}}\, dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way. We have also used the following generalized version of Besov spaces.

Definition A.3. Let $P = P(|x|) : (0, \infty) \to (0, \infty)$ be a non-decreasing function satisfying Condition 1.1 and

$$\lim_{|x| \to \infty} \frac{P(|x|)}{|x|^\epsilon} = 0, \qquad \forall \epsilon > 0.$$

For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the generalized Besov spaces $\mathring{B}^{s,P}_{p,q}$ and $B^{s,P}_{p,q}$ are defined through the norms

$$\|f\|_{\mathring{B}^{s,P}_{p,q}} \equiv \left\| 2^{js} P(2^j) \|\mathring{\Delta}_j f\|_{L^p} \right\|_{l^q} < \infty,$$
$$\|f\|_{B^{s,P}_{p,q}} \equiv \left\| 2^{js} P(2^j) \|\Delta_j f\|_{L^p} \right\|_{l^q} < \infty. \tag{A.5}$$

We have also used the space–time spaces defined below.

Definition A.4. For $t > 0$, $s \in \mathbb{R}$ and $1 \leqslant p, q, r \leqslant \infty$, the space–time spaces $\widetilde{L}^r_t \mathring{B}^s_{p,q}$ and $\widetilde{L}^r_t B^s_{p,q}$ are defined through the norms

$$\|f\|_{\widetilde{L}^r_t \mathring{B}^s_{p,q}} \equiv \left\| 2^{js} \|\mathring{\Delta}_j f\|_{L^r_t L^p} \right\|_{l^q},$$
$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} \equiv \left\| 2^{js} \|\Delta_j f\|_{L^r_t L^p} \right\|_{l^q}.$$

$\widetilde{L}^r_t \mathring{B}^{s,P}_{p,q}$ and $\widetilde{L}^r_t B^{s,P}_{p,q}$ are similarly defined.

These spaces are related to the classical space–time spaces $L^r_t \mathring{B}^s_{p,q}$, $L^r_t \mathring{B}^{s,P}_{p,q}$ and $L^r_t B^{s,P}_{p,q}$ via the Minkowski inequality.

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.5. *For any* $s \in \mathbb{R}$,

$$\mathring{H}^s \sim \mathring{B}^s_{2,2}, \qquad H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ *and* $1 < q < \infty$,

$$\mathring{B}^s_{q,\min\{q,2\}} \hookrightarrow \mathring{W}^s_q \hookrightarrow \mathring{B}^s_{q,\max\{q,2\}}.$$

In particular, $\mathring{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \mathring{B}^0_{q,\max\{q,2\}}$.

For notational convenience, we write Δ_j for $\mathring{\Delta}_j$. There will be no confusion if we keep in mind that Δ_j's associated with the homogeneous Besov spaces is defined in (A.2) while those associated with the inhomogeneous Besov spaces are defined in (A.4). Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (A.4). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition A.6. *Let $\alpha \geqslant 0$. Let $1 \leqslant p \leqslant q \leqslant \infty$.*

1) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d: |\xi| \leqslant K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_1 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If f satisfies*

$$\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^d: K_1 2^j \leqslant |\xi| \leqslant K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leqslant K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leqslant \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leqslant C_2 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

Finally we recall the Osgood inequality.

Proposition A.7. *Let $\alpha(t) > 0$ be a locally integrable function. Assume $\omega(t) \geqslant 0$ be a continuous and nondecreasing function on $(0, \infty)$ satisfying*

$$\int_1^\infty \frac{1}{\omega(r)}\, dr = \infty.$$

Suppose that $\rho(t) > 0$ satisfies

$$\rho(t) \leqslant a + \int_{t_0}^{t} \alpha(s)\omega\big(\rho(s)\big)\,ds$$

for some constant $a \geqslant 0$*. Then if* $a = 0$*, then* $\rho \equiv 0$*; if* $a > 0$*, then*

$$-\Omega\big(\rho(t)\big) + \Omega(a) \leqslant \int_{t_0}^{t} \alpha(\tau)\,d\tau,$$

where

$$\Omega(x) = \int_{x}^{1} \frac{dr}{\omega(r)}.$$

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