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Global Regularity for a 2D Tropical Climate Model with Fractional Dissipation

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Abstract
This paper examines the global regularity problem for a 2D tropical climate model with fractional dissipation. The inviscid version of this model was derived by Frierson, Majda and Pauluis for large-scale dynamics of precipitation fronts in the tropical atmosphere. The model considered here has some very special features. This nonlinear system involves interactions between a divergence-free vector field and a non-divergence-free vector field. In addition, the fractional dissipation not only models long-range interactions but also allows simultaneous investigations of a family of system. Our study leads to the global regularity of solutions when the indices of the fractional Laplacian are in two very broad ranges. In order to establish the global-in-time bounds, we introduce an efficient way to control the gradient of the non-divergence-free vector field and make sharp estimates by controlling the regularity of related quantities simultaneously.

Keywords Tropical climate model · Fractional dissipation · Global regularity

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1 Introduction

We aim at the existence and regularity of solutions to the 2D tropical climate model with fractional dissipation

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nu\Lambda^{2\alpha}u + \nabla p + \nabla \cdot (v \otimes v) = 0, \quad x \in \mathbb{R}^2, \ t > 0, \\ \partial_t v + (u \cdot \nabla)v - \mu\Delta v + \nabla\theta + (v \cdot \nabla)u = 0, \\ \partial_t \theta + (u \cdot \nabla)\theta + \eta\Lambda^{2\gamma}\theta + \nabla \cdot v = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where the vector fields $u = (u_1, u_2)$ and $v = (v_1, v_2)$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively, and the scalar p denotes the pressure and θ the temperature, and $\nu > 0$, $\mu > 0$, $\eta > 0$, $\alpha > 0$ and $\gamma > 0$ are real parameters. Here $v \otimes v$ denotes the tensor product of v with v, or equivalently $v \otimes v$ represents the matrix with the (i, j)-entry being $v_i v_j$. The fractional Laplacian operator Λ^α with $\alpha > 0$ is either defined through the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi)$$

or via the Riesz potential, for a constant $C = C_\alpha$,

$$\Lambda^\alpha f(x) = \text{p.v.}\, C_\alpha \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^{2+\alpha}}\, dy,$$

where p.v. means the principal value (see Córdoba and Córdoba 2004).

When $\nu = \mu = \eta = 0$, (1.1) reduces to the original tropical climate model derived by Frierson, Majda and Pauluis for large-scale dynamics of precipitation fronts in the tropical atmosphere (Frierson et al. 2004). More relevant background on the tropical climate model can be found in Gill (1980), Majda and Biello (2003), Matsuno (1966), Majda (2003), Stechmann and Majda (2006) and the references therein. We remark that the model considered in this article is actually a special case of the general framework developed in Frierson et al. (2004). The model equations they derived have the form of a shallow water equation and an equation for moisture coupled through a strongly nonlinear source term. When the precipitation rate P is approximately zero or when the moist region can be ignored, the tropical climate model reduces to the special model equation studied here. In addition, this simplified model and its β-plane linearization are useful in the study of many interesting types of waves such as the Rossby waves and mixed Rossby-gravity waves (Frierson et al. 2004). Li and Titi (2016a) investigated this reduced model in the case when the momentum equations contain the viscosities and the temperature equation involves no diffusivity. It is clear that the elimination of the moisture- related equations/terms would help improve the regularity of the solutions.

The original tropical climate model of Frierson, Majda and Pauluis involves only damping terms but no dissipation. The model studied here, namely (1.1), is appended with fractional dissipation terms. Mathematically, these extra dissipation terms increase the regularity for the system. These terms also significantly change the model properties and physics. The original tropical climate model is derived from the inviscid primitive equations (Frierson et al. 2004), and its viscous counterpart with the standard Laplacian can be derived by the same argument from the viscous primitive equations (see Khouider and Majda 2005; Li and Titi 2016a, b; Majda 2003; Stechmann and Majda 2006). The fractional dissipation terms may be relevant in the study of viscous flows in the thinning of atmosphere. Flows in the middle atmosphere traveling upward undergo changes due to the changes of atmospheric properties. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian. In addition, when α and γ are fractions, (1.1) with these fractional diffusion operators may also be relevant in modeling the so-called anomalous diffusion, a much studied topic in physics, probability and finance (see, e.g., Abe and Thurner 1905; Jara 2009; Mellet et al. 2011). Especially, (1.1) allows us to study long-range diffusive interactions.

Mathematically, (1.1) possesses some special features. The first is that (1.1) involves the coupling of a divergence-free vector field u and a non-divergence-free vector field v. This mix poses mathematical challenges. In order to control the gradient of v, we need both the curl of v and the divergence of v. This paper demonstrates how to effectively bound ∇v. The second feature is that (1.1) allows us to examine two-parameter families of systems simultaneously and to understand how the regularity of the solutions is affected as the sizes of the parameters vary. Our aim here is to establish the global regularity for (1.1) with the smallest amount of dissipation and provide the sharpest global well-posedness results.

We establish the global existence and regularity for (1.1) for two ranges of the parameters α and γ. Different ranges of α require different treatments. The main result is stated in Theorem 1.1. To be concise, we set $\nu = \mu = \eta = 1$ from now on.

Theorem 1.1 *Consider* (1.1) *with* α *and* γ *satisfying*

$$\gamma \geq \begin{cases} \dfrac{4+\alpha-\sqrt{\alpha^2+8\alpha+8}}{2}, & 0 < \alpha < \dfrac{1}{2}, \\ 1-\alpha, & \dfrac{1}{2} \leq \alpha \leq 1. \end{cases} \tag{1.2}$$

Assume $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$. *Then,* (1.1) *admits a unique global solution satisfying, for any* $T > 0$,

$$\begin{aligned} u &\in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\alpha}(\mathbb{R}^2)), \\ v &\in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+1}(\mathbb{R}^2)), \\ \theta &\in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\gamma}(\mathbb{R}^2)). \end{aligned}$$

The result stated in Theorem 1.1 is new. There are several important previous global regularity results for different ranges of parameters. Li and Titi (2016a) dealt with the case when $\nu > 0$, $\mu > 0$, $\alpha = 1$ and $\eta = 0$. They introduced a combined quantity of

v and θ to establish the global (in time) H^1 bound. This case is covered by Theorem 1.1 when we set $\alpha = 1$ and $\gamma = 0$. Ye (2017) obtained the global regularity for (1.1) when $\alpha > 0$ and $\gamma = 1$. It is clear that Theorem 1.1 covers a bigger parameter range than Ye (2017). Dong et al. (2018) proved the global regularity for the climate model in the case when there is no thermal diffusion, and when $\alpha \le \frac{1}{2}$ and the total fractional dissipation in the equations of u and v is at the order of two Laplacians. We remark that some very interesting cases such as the model with both α and γ being small remain open.

The proof of Theorem 1.1 is not straightforward and demands new techniques. We describe the main difficulties and explain the techniques to overcome them. Since the local well-posedness of (1.1) follows from a standard procedure, the key to the global regularity is the global *a priori* bounds. For the sake of completeness, the local well-posedness part is presented in Appendix. The global L^2 bound for (u, v, θ), along with the time integrability of $\|\Lambda^\alpha u\|_{L^2}^2$, $\|\nabla v\|_{L^2}^2$ and $\|\Lambda^\gamma \theta\|_{L^2}^2$, is immediate due to the special structure of (1.1) and $\nabla \cdot u = 0$. By employing an inequality of Chamorro and Lemarié-Rieusset (2012), we turn the thermal diffusion of the θ equation into a global bound for $\|\theta\|_{L^{\frac{2}{1-\gamma}}}$, for any $t > 0$,

$$\|\theta(t)\|_{L^{\frac{2}{1-\gamma}}}^{\frac{2}{1-\gamma}} + \int_0^t \|\theta(\tau)\|_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}^{\frac{2}{1-\gamma}} \,\mathrm{d}\tau + \int_0^t \|\theta(\tau)\|_{L^{\frac{2}{(1-\gamma)^2}}}^{\frac{2}{1-\gamma}} \,\mathrm{d}\tau \le C_0,$$

where C_0 depends on the initial data only. These global bounds serve as a preparation for higher regularity estimates.

The major step is the global H^1 bound. As demonstrated in [7], there is an advantage to make use of the equations of

$$\omega = \nabla \times u, \quad j = \nabla \times v, \quad h = \nabla \cdot v,$$

which satisfy

$$\begin{cases} \partial_t \omega + (u\cdot\nabla)\omega + \Lambda^{2\alpha}\omega + (v\cdot\nabla)j + 2hj - (v_1\partial_2 h - v_2\partial_1 h) = 0, \\ \partial_t j + (u\cdot\nabla)j - \Delta j + (v\cdot\nabla)\omega + h\omega = 0, \\ \partial_t h + (u\cdot\nabla)h - \Delta h + \Delta\theta + Q(\nabla u, \nabla v) = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{1.3}$$

where $Q(\nabla u, \nabla v)$ is given by

$$Q(\nabla u, \nabla v) = 2\partial_1 u_1(\partial_1 v_1 - \partial_2 v_2) + 2\partial_1 u_2 \partial_2 v_1 + 2\partial_2 u_1 \partial_1 v_2.$$

The equation of j completely eliminates θ. In order to remove the most regularity-demanding term $\Delta\theta$ in the equation of h, we form the combined quantity $H = h - \theta$, which satisfies

$$\partial_t H + (u\cdot\nabla)H - \Delta H + Q(\nabla u, \nabla v) = \Lambda^{2\gamma}\theta + H + \theta.$$

The global H^1 bound for (u, v, θ) is established by estimating $\|(\omega, j, H)\|_{L^2}$. In this process, the most difficult terms are generated by the last two terms in the vorticity equation, namely $v_1\partial_2 h - v_2\partial_1 h$. They are handled differently when α is in different ranges. For $0 < \alpha < \frac{1}{2}$, we replace h by $H + \theta$ and control the derivative of θ by $\|\nabla\theta\|_{L^2}$, which requires the estimate of $\|\Lambda^{1-\gamma}\theta\|_{L^2}$ simultaneously. When estimating $\|\Lambda^{1-\gamma}\theta\|_{L^2}$ via the equation of θ, we need to bound the nonlinearity of the θ-equation suitably. The condition that

$$\gamma \geq \frac{4 + \alpha - \sqrt{\alpha^2 + 8\alpha + 8}}{2}$$

is necessary in order to form a closed differential inequality. More details can be found in the proof of Lemma 4.1. For α in the range $\frac{1}{2} \leq \alpha \leq 1$, the terms generated by $v_1\partial_2 h - v_2\partial_1 h$ in the estimates of $\|(\omega, j, H)\|_{L^2}$ remain the most regularity-demanding terms. But the estimation process is different and we take advantage of the fact that $\alpha \geq \frac{1}{2}$. We further split this case into two subcases

$$\text{Subcase 1: } \frac{1}{2} \leq \alpha < 1, \quad \gamma \geq 1 - \alpha; \qquad \text{Subcase 2: } \alpha = 1, \quad \gamma = 0.$$

We still substitute h by $H + \theta$, but we only need to control $\|\Lambda^{1-\alpha}\theta\|_{L^2}$ after shifting the derivative to the vorticity. The condition that $\gamma \geq 1-\alpha$ is sufficient for subsequent estimates. More details can be found in the proof of Lemma 5.1. Subcase 2 is special and is treated separately.

In order to establish the global H^s-bound for the solutions, a few more steps are necessary. In the case when $0 < \alpha < \frac{1}{2}$, we consecutively establish more and more regular global bounds. Using the global H^1-level bounds, we further control $\|\nabla\theta\|_{L^2}$. The next tier of bounds is time integrability of $\|\nabla v\|_{L^\infty}$, $\|\Delta v\|_{L^{\tilde{q}}}$ (for certain range of $\tilde{q}$) and of $\|\Delta H\|_{L^2}$ shown via the maximal regularity of parabolic type equations. These bounds further lead to the global bound for $\|\nabla u\|_{L^q}$ and $\|\nabla\theta\|_{L^q}$ for any $2 \leq q < \infty$, which are sufficient for the global H^s-bound. In the case when $\frac{1}{2} \leq \alpha < 1$ and $\gamma \geq 1 - \alpha$, we make use of the global H^1 bound to further establish the bounds for $\|\nabla\theta\|_{L^2}$ and $\|v\|_{L^\infty(0,T;L^\infty)}$, which are then sufficient in proving any higher regularity estimates.

The rest of this paper is divided into four sections. The Sect. 2 lists some of the tools to be used for the proof of Theorem 1.1. The Sect. 3 provides the global L^2-level bounds as well as a global bound for $\|\theta\|_{L^{\frac{2}{1-\gamma}}}$. The Sect. 4 proves Theorem 1.1 for the first case, while the Sect. 5 proves Theorem 1.1 for the second case, which is further split into two subcases.

2 Several Tools

For the sake of clarity, this section lists some of the tools to be used in the proof of Theorem 1.1. We begin with the following fractional type Gagliardo–Nirenberg inequality due to Hajaiej et al. (2011).

Lemma 2.1 *Let* $0 < p,\ p_0,\ p_1,\ q,\ q_0,\ q_1 \le \infty$, $s,\ s_0,\ s_1 \in \mathbb{R}$ *and* $0 \le \vartheta \le 1$. *Then, the following fractional type Gagliardo–Nirenberg inequality*

$$\|v\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^n)} \le C\|v\|^{1-\vartheta}_{\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n)}\|v\|^{\vartheta}_{\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n)} \tag{2.1}$$

holds for all $v \in \dot{B}^{s_0}_{p_0,q_0} \cap \dot{B}^{s_1}_{p_1,q_1}$ *if and only if*

$$\frac{n}{p} - s = (1-\vartheta)\left(\frac{n}{p_0} - s_0\right) + \vartheta\left(\frac{n}{p_1} - s_1\right), \qquad s \le (1-\vartheta)s_0 + \vartheta s_1,$$
$$\frac{1}{q} \le \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1}, \quad \text{if } p_0 \ne p_1 \text{ and } s = (1-\vartheta)s_0 + \vartheta s_1,$$
$$s_0 \ne s_1 \text{ or } \frac{1}{q} \le \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1}, \quad \text{if } p_0 = p_1 \text{ and } s = (1-\vartheta)s_0 + \vartheta s_1,$$
$$s_0 - \frac{n}{p_0} \ne s - \frac{n}{p} \text{ or } \frac{1}{q} \le \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1}, \quad \text{if } s < (1-\vartheta)s_0 + \vartheta s_1.$$

The inequality in (2.1) *remains valid when the homogeneous Besov spaces are replaced by the nonhomogeneous Besov spaces.*

Next we recall the following refined logarithmic Gronwall inequality (Ye 2018), which generalizes Lemma 2.5 in Cao et al. (2017). We provide a proof of this inequality for the convenience of the readers.

Lemma 2.2 *Let* A *and* B *be two absolutely continuous and nonnegative functions on* $(0, T)$ *for any given* $T > 0$, *satisfying*

$$A'(t)+B(t) \le [l(t)+m(t)\ln(A+e) + n(t)\ln(A + B + e)](A + e) + f(t), \tag{2.2}$$

for any $t \in (0, T)$, *where* $l(t),\ m(t),\ n(t)$ *and* $f(t)$ *are all nonnegative and integrable functions on* $(0, T)$. *Assume further that there are three constants* $K \in [0, \infty)$, $\alpha \in [0, \infty)$ *and* $\beta \in [0, 1)$ *such that for any* $t \in (0, T)$

$$n(t) \le K\big(A(t) + e\big)^{\alpha}\big(A(t) + B(t) + e\big)^{\beta}. \tag{2.3}$$

Then, the following estimate holds

$$A(t) + \int_0^t B(s)\,ds \le \widetilde{C}(l, m, n, f, \alpha, \beta, K, t) < \infty, \tag{2.4}$$

for any $t \in (0, T)$. *Especially, for the case* $\alpha = 0$, *namely,*

$$n(t) \le K\big(A(t) + B(t) + e\big)^{\beta}, \tag{2.5}$$

(2.4) *remains valid.*

Proof of Lemma 2.2 By denoting

$$A_1 := A + e, \qquad B_1 := A + B + e,$$

we have

$$\begin{aligned} A_1' + B_1 &= A' + A + B + e \\ &\le [1 + l(t) + m(t)\ln(A+e) + n(t)\ln(A+B+e)](A+e) + f(t) \\ &= [1 + l(t) + m(t)\ln A_1 + n(t)\ln B_1]A_1 + f(t). \end{aligned} \tag{2.6}$$

Dividing both sides of (2.6) by A_1 yields

$$\frac{A_1'}{A_1} + \frac{B_1}{A_1} = [1 + l(t) + m(t)\ln A_1 + n(t)\ln B_1] + \frac{f(t)}{A_1},$$

which, due to the fact $A_1 \ge 1$, implies

$$(\ln A_1)' + \frac{B_1}{A_1} \le 1 + l(t) + m(t)\ln A_1 + n(t)\ln B_1 + f(t). \tag{2.7}$$

Since

$$n(t) \le K\big(A(t)+e\big)^{\alpha}\big(A(t)+B(t)+e\big)^{\beta} \le K A_1^{\alpha} B_1^{\beta}$$

and

$$\ln z \le \frac{z^{\eta}}{e\eta}, \quad \forall z,\ \eta \in (0,\ \infty), \tag{2.8}$$

we have by taking $\eta \in (0,\ 1-\beta)$

$$\begin{aligned} n(t)\ln B_1 &= n(t)\left[\ln\Big(\frac{B_1}{A_1^{1+\frac{\alpha+\beta}{\eta}}}\Big) + \Big(1 + \frac{\alpha+\beta}{\eta}\Big)\ln A_1\right] \\ &\le K A_1^{\alpha} B_1^{\beta}\left[\frac{1}{e\eta}\Big(\frac{B_1}{A_1^{1+\frac{\alpha+\beta}{\eta}}}\Big)^{\eta}\right] + \Big(1 + \frac{\alpha+\beta}{\eta}\Big)n(t)\ln A_1 \\ &\le \frac{K}{e\eta}\Big(\frac{B_1}{A_1}\Big)^{\beta+\eta} + \Big(1 + \frac{\alpha+\beta}{\eta}\Big)n(t)\ln A_1 \\ &\le \frac{B_1}{2A_1} + C(\alpha,\beta,\eta,K) + \Big(1 + \frac{\alpha+\beta}{\eta}\Big)n(t)\ln A_1, \end{aligned} \tag{2.9}$$

where we have used the fact $\beta + \eta < 1$ in the last line. Combining (2.7) and (2.9) yields

$$(\ln A_1)' + \frac{B_1}{2A_1} \leq l(t)+m(t)\ln A_1+C(\alpha,\beta,\eta,K)+\Big(1+\frac{\alpha+\beta}{\eta}\Big)n(t)\ln A_1+f(t). \quad (2.10)$$

Writing

$$X(t) := \ln A_1(t) + \int_0^t \frac{B_1(s)}{2A_1(s)}\,\mathrm{d}s,$$

(2.10) then implies

$$X'(t) \leq C(\alpha,\beta,\eta,K) + f(t) + l(t) + \Big(m(t) + \frac{\alpha+\beta+\eta}{\eta}n(t)\Big)X(t).$$

By the standard Gronwall inequality, we obtain

$$\begin{aligned} X(t) &\leq e^{\int_0^t \left(m(s)+\frac{\alpha+\beta+\eta}{\eta}n(s)\right)\mathrm{d}s}\Big(X(0) + \int_0^t \{C(\alpha,\beta,\eta,K) + f(s) + l(s)\}\,\mathrm{d}s\Big) \\ &:= C(l,m,n,f,\alpha,\beta,K,t). \end{aligned} \quad (2.11)$$

According to the definition of X, we infer

$$A_1(t) \leq e^{X(t)} \leq e^{C(l,m,n,f,\alpha,\beta,K,t)}.$$

Moreover,

$$\begin{aligned} \int_0^t B_1(s)\,\mathrm{d}s &= \int_0^t 2A_1(s)\frac{B_1(s)}{2A_1(s)}\,\mathrm{d}s \\ &\leq \int_0^t 2\big(\max_{0\leq\lambda\leq t} A_1(\lambda)\big)\frac{B_1(s)}{2A_1(s)}\,\mathrm{d}s \\ &\leq 2e^{C(l,m,n,f,\alpha,\beta,K,t)}\int_0^t \frac{B_1(s)}{2A_1(s)}\,\mathrm{d}s \\ &\leq 2C(l,m,n,f,\alpha,\beta,K,t)e^{C(l,m,n,f,\alpha,\beta,K,t)}. \end{aligned} \quad (2.12)$$

This concludes the proof of Lemma 2.2. □

We will also make use of the following commutator estimate (see, e.g., Jiu et al. 2014; Ye and Xu 2016).

Lemma 2.3 *Let f be a divergence-free vector field and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p \in [2,\infty)$, $p_1,\ p_2 \in [2,\infty]$, $r \in [1,\infty]$ as well as $s \in (-1, 1-\delta)$ for $\delta \in (0,2)$, then it holds*

$$\|[\Lambda^\delta, f\cdot\nabla]g\|_{B^s_{p,r}} \leq C(p,r,\delta,s)\big(\|\nabla f\|_{L^{p_1}}\|g\|_{B^{s+\delta}_{p_2,r}} + \|f\|_{L^2}\|g\|_{L^2}\big). \quad (2.13)$$

We also need a special commutator estimate obtained by Hadadifard and Stefanov (see Hadadifard and Stefanov 2017, Lemma 2.2).

Lemma 2.4 *Let s_1 and s_2 satisfy $s_1 \geq 0$ and $0 \leq s_2 - s_1 \leq 1$. If $\nabla \cdot u = 0$, then there holds*

$$\|\Lambda^{-s_1}([\Lambda^{s_2},\, u \cdot \nabla]\theta)\|_{L^p} \leq C\|\nabla u\|_{L^q}\|\Lambda^{s_2-s_1}\theta\|_{L^r}, \tag{2.14}$$

where $2 < q < \infty,\ 1 < p,\ r < \infty$ satisfy $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

3 Preliminary Bounds

We start proving Theorem 1.1. This section provides the global (in time) bounds at the L^2-level and a global bound for $\|\theta\|_{L^{\frac{2}{1-\gamma}}}$. In addition, we also write the vorticity formulation to prepare for the proof of the global H^1 bounds in subsequent sections. As aforementioned in Introduction, the vorticity formulation appears to have an advantage in proving the global H^1 bound.

The following lemma states the global L^2-bound on (u, v, θ) along with the time integrability of $\|\Lambda^\alpha u\|_{L^2}^2$, $\|\nabla v\|_{L^2}^2$ and $\|\Lambda^\gamma \theta\|_{L^2}^2$.

Lemma 3.1 *Assume that (u_0, v_0, θ_0) obeys the conditions stated in Theorem 1.1. Then, the corresponding smooth solution (u, v, θ) of (1.1) satisfies, for any $t > 0$,*

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\int_0^t (\|\Lambda^\alpha u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Lambda^\gamma \theta\|_{L^2}^2)(\tau)\, d\tau\\ &\quad = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2.\end{aligned} \tag{3.1}$$

Proof Dotting (1.1) by (u, v, θ), integrating by parts and using $\nabla \cdot u = 0$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2) + \|\Lambda^\alpha u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Lambda^\gamma \theta\|_{L^2}^2 = 0, \tag{3.2}$$

where the following cancellations have been used above

$$\int_{\mathbb{R}^2} \nabla \cdot (v \otimes v) \cdot u\, \mathrm{d}x + \int_{\mathbb{R}^2} (v \cdot \nabla) u \cdot v\, \mathrm{d}x = 0,$$
$$\int_{\mathbb{R}^2} \nabla\theta \cdot v\, \mathrm{d}x + \int_{\mathbb{R}^2} (\nabla \cdot v)\theta\, \mathrm{d}x = 0.$$

Integrating (3.2) in time over $(0, t)$ yields desired estimate (3.1). □

The following lemma establishes a global bound for $\|\theta\|_{L^{\frac{2}{1-\gamma}}}$ and for the time integrability of the corresponding dissipative part.

Lemma 3.2 *Assume that (u_0, v_0, θ_0) obeys the conditions stated in Theorem 1.1. Let (u, v, θ) be a smooth solution of (1.1). Then, for any $t > 0$,*

$$\|\theta(t)\|_{L^{\frac{2}{1-\gamma}}}^{\frac{2}{1-\gamma}} + \int_0^t \|\theta(\tau)\|_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}^{\frac{2}{1-\gamma}}\, d\tau + \int_0^t \|\theta(\tau)\|_{L^{\frac{2}{(1-\gamma)^2}}}^{\frac{2}{1-\gamma}}\, d\tau \leq C_0, \tag{3.3}$$

where C_0 is given by

$$C_0 := \widetilde{C}(\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2)\left(\|\theta_0\|^2_{L^{\frac{2}{1-\gamma}}} + \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2\right)^{\frac{\gamma}{1-\gamma}}$$

for an absolute constant $\widetilde{C} > 0$ independent of the initial data.

Proof Multiplying equation (1.1)$_3$ by $|\theta|^{\frac{2}{1-\gamma}-2}\theta$ and integrating over $\mathbb{R}^2$ lead to

$$\frac{1-\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}} + \int_{\mathbb{R}^2}\Lambda^{2\gamma}\theta(|\theta|^{\frac{2}{1-\gamma}-2}\theta)\,\mathrm{d}x = -\int_{\mathbb{R}^2}\nabla\cdot v\ (|\theta|^{\frac{2}{1-\gamma}-2}\theta)\,\mathrm{d}x.$$

We recall the following lower bounds associated with the fractional dissipation term, for any $q \in [2,\infty)$ and $s \in (0,1)$,

$$\int_{\mathbb{R}^2}|f|^{q-2}f\ \Lambda^{2s}f\ \mathrm{d}x \ge C(s,q)\|f\|^q_{\dot{B}^{\frac{2s}{q}}_{q,q}}, \tag{3.4}$$

$$\int_{\mathbb{R}^2}|f|^{q-2}f\ \Lambda^{2s}f\ \mathrm{d}x \ge C(s,q)\|f\|^q_{L^{\frac{q}{1-s}}}. \tag{3.5}$$

(3.4) is due to Chamorro and Lemarié-Rieusset (see Theorem 2 of Chamorro and Lemarié-Rieusset 2012) and (3.5) follows from (3.4) via the Sobolev inequality. We thus obtain

$$\int_{\mathbb{R}^2}\Lambda^{2\gamma}\theta\left(|\theta|^{\frac{2}{1-\gamma}-2}\theta\right)\mathrm{d}x \ge \widetilde{c}_1\int_{\mathbb{R}^2}\left(\Lambda^\gamma|\theta|^{\frac{1}{1-\gamma}}\right)^2 \ge \widetilde{c}_2\|\theta\|^{\frac{2}{1-\gamma}}_{\dot{B}^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}},$$

$$\int_{\mathbb{R}^2}\Lambda^{2\gamma}\theta\left(|\theta|^{\frac{2}{1-\gamma}-2}\theta\right)\mathrm{d}x \ge \widetilde{c}_1\int_{\mathbb{R}^2}\left(\Lambda^\gamma|\theta|^{\frac{1}{1-\gamma}}\right)^2 \ge \widetilde{c}_3\|\theta\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{(1-\gamma)^2}}},$$

where $\widetilde{c}_1$, $\widetilde{c}_2$ and $\widetilde{c}_3$ are absolute constants. By Hölder's inequality and the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\nabla\cdot v\ \left(|\theta|^{\frac{2}{1-\gamma}-2}\theta\right)\mathrm{d}x\right| &\le C\|\nabla v\|_{L^2}\|\theta\|^{\frac{1+\gamma}{1-\gamma}}_{L^{\frac{2(1+\gamma)}{1-\gamma}}}\\ &\le C\|\nabla v\|_{L^2}\|\theta\|^{\frac{\gamma}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}}\|\theta\|^{\frac{1}{1-\gamma}}_{L^{\frac{2}{(1-\gamma)^2}}}\\ &\le \frac{\widetilde{c}}{2}\|\theta\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{(1-\gamma)^2}}} + C\|\nabla v\|^2_{L^2}\|\theta\|^{\frac{2\gamma}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}}.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}} + \|\theta\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{(1-\gamma)^2}}} + \|\theta\|^{\frac{2}{1-\gamma}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}} \le C\|\nabla v\|^2_{L^2}\|\theta\|^{\frac{2\gamma}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}}. \tag{3.6}$$

Especially,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}} \le C\|\nabla v\|^2_{L^2}\|\theta\|^{\frac{2\gamma}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}},$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|^2_{L^{\frac{2}{1-\gamma}}} \le C\|\nabla v\|^2_{L^2}.$$

Integrating in time and using (3.1), we have

$$\|\theta(t)\|^2_{L^{\frac{2}{1-\gamma}}} \le \|\theta_0\|^2_{L^{\frac{2}{1-\gamma}}} + C(\|u_0\|^2_{L^2} + \|v_0\|^2_{L^2} + \|\theta_0\|^2_{L^2}),$$

which, together with (3.6), yields

$$\|\theta(t)\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{1-\gamma}}} + \int_0^t \|\theta(\tau)\|^{\frac{2}{1-\gamma}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}\,\mathrm{d}\tau + \int_0^t \|\theta(\tau)\|^{\frac{2}{1-\gamma}}_{L^{\frac{2}{(1-\gamma)^2}}}\,\mathrm{d}\tau \le C_0.$$

This completes the proof of (3.3). □

In order to prove the global H^1-bound for (u, v, θ), we resort to the equation of the vorticity ω. Since v is not necessarily divergence-free, we also write the equations of the curl of v and the divergence of v. It follows from system (1.1) that

$$\omega = \nabla\times u := \partial_1 u_2 - \partial_2 u_1, \qquad j = \nabla\times v := \partial_1 v_2 - \partial_2 v_1,$$
$$h = \nabla\cdot v := \partial_1 v_1 + \partial_2 v_2$$

satisfy

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega + \Lambda^{2\alpha}\omega + (v\cdot\nabla)j + 2hj - (v_1\partial_2 h - v_2\partial_1 h) = 0,\\ \partial_t j + (u\cdot\nabla)j - \Delta j + (v\cdot\nabla)\omega + h\omega = 0,\\ \partial_t h + (u\cdot\nabla)h - \Delta h + \Delta\theta + Q(\nabla u, \nabla v) = 0,\\ \nabla\cdot u = 0, \end{cases} \tag{3.7}$$

where $Q(\nabla u, \nabla v)$ is given by

$$Q(\nabla u, \nabla v) = 2\partial_1 u_1(\partial_1 v_1 - \partial_2 v_2) + 2\partial_1 u_2\partial_2 v_1 + 2\partial_2 u_1\partial_1 v_2.$$

Now by introducing the following key quantity

$$H = h - \theta \tag{3.8}$$

and combining with the third equation of (1.1), we have

$$\partial_t H + (u\cdot\nabla)H - \Delta H + Q(\nabla u, \nabla v) = \Lambda^{2\gamma}\theta + H + \theta. \tag{3.9}$$

Owing to the following identity

$$\Delta v = \nabla(\nabla \cdot v) + \nabla^{\perp}(\nabla \times v), \qquad \nabla^{\perp} = (\partial_2, -\partial_1),$$

one deduces that

$$\Delta v = \nabla h + \nabla^{\perp} j. \tag{3.10}$$

As a consequence,

$$\nabla v = \mathcal{R}_1 h + \mathcal{R}_2 j = \mathcal{R}_1 H + \mathcal{R}_1 \theta + \mathcal{R}_2 j, \tag{3.11}$$

where

$$\mathcal{R}_1 = \nabla\nabla(\Delta)^{-1}, \quad \mathcal{R}_2 = \nabla\nabla^{\perp}(\Delta)^{-1}.$$

The rest of this paper is devoted to completing the proof of Theorem 1.1. We divide the proof into two cases:

$$\textbf{Case 1} \quad \gamma \geq \frac{4+\alpha-\sqrt{\alpha^2+8\alpha+8}}{2}, \qquad 0 < \alpha < \frac{1}{2}, \tag{3.12}$$

$$\textbf{Case 2} \quad \gamma \geq 1-\alpha, \qquad \frac{1}{2} \leq \alpha \leq 1. \tag{3.13}$$

Section 4 is devoted to the first case, while Sect. 5 is on the second case.

4 Proof of Theorem 1.1: Case 1

This section proves Theorem 1.1 for Case 1:

$$\gamma \geq \frac{4+\alpha-\sqrt{\alpha^2+8\alpha+8}}{2}, \qquad 0 < \alpha < \frac{1}{2}. \tag{4.1}$$

Attention is focused on γ and α satisfying

$$\gamma = \frac{4+\alpha-\sqrt{\alpha^2+8\alpha+8}}{2}, \qquad 0 < \alpha < \frac{1}{2}$$

since the remaining parameter range can be dealt with in a similar manner and is actually easier to handle. Theorem 1.1 is proven by consecutively establishing more and more regular global (in time) bounds. The crucial step is the global H^1 bound for (u, v, θ) via (3.7). In order to close the differential inequalities, we estimate $\|(\omega, j, H)\|_{L^2}$ together with $\|\Lambda^{1-\gamma}\theta\|_{L^2}$. It is in this step that we need (4.1). These global H^1-level bounds further allow us to obtain a global bound for $\|\nabla\theta\|_{L^2}$. The next tier of bounds is time integrability of $\|\nabla v\|_{L^\infty}$, $\|\Delta v\|_{L^{\tilde{q}}}$ (for certain range of $\tilde{q}$)

and of $\|\Delta H\|_{L^2}$ shown via the maximal regularity of parabolic type equations. These bounds further lead to the global bound for $\|\nabla u\|_{L^q}$ and $\|\nabla\theta\|_{L^q}$ for any $2 \le q < \infty$, which are sufficient for all higher regularity.

The following lemma establishes the global H^1-estimate of u and v as well as a global bound for $\|\Lambda^{1-\gamma}\theta\|_{L^2}$.

Lemma 4.1 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume α and γ belong to Case 1, namely (3.12). Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\begin{aligned}
&\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\Lambda^{1-\gamma}\theta(t)\|_{L^2}^2 \\
&\quad + \int_0^t (\|\Lambda^\alpha\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)(\tau)\,d\tau \\
&\le C_0(t),
\end{aligned} \tag{4.2}$$

where the bound C_0 depends only on t and the initial data.

Proof Multiplying Eqs. $(3.7)_1$, $(3.7)_2$ and (3.9) by ω, j and H, respectively, integrating over $\mathbb{R}^2$ and summing them up lead to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2) + \|\Lambda^\alpha\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^2}\Big((v\cdot\nabla)j\omega + 2hj\omega + (v\cdot\nabla)\omega j + h\omega j\Big)\,\mathrm{d}x - \int_{\mathbb{R}^2} Q(\nabla u, \nabla v)H\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^2}(\Lambda^{2\gamma}\theta + H + \theta)H\,\mathrm{d}x + \int_{\mathbb{R}^2}(v_1\partial_2 h - v_2\partial_1 h)\omega\,\mathrm{d}x \\
&\quad := J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{4.3}$$

In what follows, we estimate the terms on the right-hand side of (4.3). Integrating by parts and using the Gagliardo–Nirenberg inequality yield

$$\begin{aligned}
J_1 &= -\int_{\mathbb{R}^2}(v\cdot\nabla)j\omega\,\mathrm{d}x - \int_{\mathbb{R}^2}\Big(2hj\omega + (v\cdot\nabla)\omega j + h\omega j\Big)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} hj\omega\,\mathrm{d}x + \int_{\mathbb{R}^2}(v\cdot\nabla)\omega j\,\mathrm{d}x - \int_{\mathbb{R}^2}\Big(2hj\omega + (v\cdot\nabla)\omega j + h\omega j\Big)\,\mathrm{d}x \\
&= -2\int_{\mathbb{R}^2} hj\omega\,\mathrm{d}x \\
&= -2\int_{\mathbb{R}^2} Hj\omega\,\mathrm{d}x - 2\int_{\mathbb{R}^2}\theta j\omega\,\mathrm{d}x \\
&\le C\|\omega\|_{L^2}\|H\|_{L^4}\|j\|_{L^4} + C\|\theta\|_{L^2}\|H\|_{L^4}\|j\|_{L^4} \\
&\le C\|\omega\|_{L^2}\|H\|_{L^2}^{\frac12}\|\nabla H\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\nabla j\|_{L^2}^{\frac12} + C\|\theta\|_{L^2}\|H\|_{L^2}^{\frac12}\|\nabla H\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\nabla j\|_{L^2}^{\frac12} \\
&\le \epsilon\|\nabla j\|_{L^2}^2 + \epsilon\|\nabla H\|_{L^2}^2 + C_\epsilon\|H\|_{L^2}\|j\|_{L^2}\|\omega\|_{L^2}^2 + C_\epsilon\|H\|_{L^2}\|j\|_{L^2}\|\theta\|_{L^2}^2 \\
&\le \epsilon\|\nabla j\|_{L^2}^2 + \epsilon\|\nabla H\|_{L^2}^2 + C_\epsilon(\|h\|_{L^2} + \|\theta\|_{L^2})\|j\|_{L^2}\|\omega\|_{L^2}^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + C_\epsilon(\|h\|_{L^2} + \|\theta\|_{L^2})\|j\|_{L^2}\|\theta\|_{L^2}^2 \\
&\le \epsilon\|\nabla j\|_{L^2}^2 + \epsilon\|\nabla H\|_{L^2}^2 + C_\epsilon(\|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2)\|\omega\|_{L^2}^2 \\
&\quad + C_\epsilon(\|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2)\|\theta\|_{L^2}^2. \qquad (4.4)
\end{aligned}
$$

By (3.11) and the Gagliardo–Nirenberg inequality, we have

$$
\begin{aligned}
J_2 &\le C\int_{\mathbb{R}^2} |\nabla u|\,|\nabla v|\,|H|\,\mathrm{d}x \\
&\le C\int_{\mathbb{R}^2} |\nabla u|\,(|\mathcal{R}_1 H| + |\mathcal{R}_1\theta| + |\mathcal{R}_2 j|)\,|H|\,\mathrm{d}x \\
&\le C\|\nabla u\|_{L^2}(\|H\|_{L^4}\|j\|_{L^4} + \|H\|_{L^4}^2) + C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\theta\|_{L^2}\|H\|_{L^{\frac{2}{\alpha}}} \\
&\le C\|\omega\|_{L^2}(\|H\|_{L^2}\|\nabla H\|_{L^2} + \|j\|_{L^2}\|\nabla j\|_{L^2}) \\
&\quad + C\|\Lambda^\alpha\omega\|_{L^2}\|\theta\|_{L^2}\|H\|_{L^2}^{\alpha}\|\nabla H\|_{L^2}^{1-\alpha} \qquad (4.5)\\
&\le \epsilon\|\nabla j\|_{L^2}^2 + \epsilon\|\nabla H\|_{L^2}^2 + \epsilon\|\Lambda^\alpha\omega\|_{L^2}^2 + C_\epsilon(\|H\|_{L^2}^2 + \|j\|_{L^2}^2)\|\omega\|_{L^2}^2 \\
&\quad + C_\epsilon\|\theta\|_{L^2}^{\frac{2}{\alpha}}\|H\|_{L^2}^2 \\
&\le \epsilon\|\nabla j\|_{L^2}^2 + \epsilon\|\nabla H\|_{L^2}^2 + \epsilon\|\Lambda^\alpha\omega\|_{L^2}^2 + C_\epsilon\|\nabla v\|_{L^2}^2\|\omega\|_{L^2}^2 + C_\epsilon\|\theta\|_{L^2}^{\frac{2}{\alpha}}\|H\|_{L^2}^2. \qquad (4.6)
\end{aligned}
$$

The term J_3 can be easily bounded as

$$
\begin{aligned}
J_3 &\le C\|\Lambda^\gamma\theta\|_{L^2}\|\Lambda^\gamma H\|_{L^2} + C\|H\|_{L^2}^2 + C\|\theta\|_{L^2}^2 \\
&\le C\|\Lambda^\gamma\theta\|_{L^2}\|H\|_{L^2}^{1-\gamma}\|\nabla H\|_{L^2}^{\gamma} + C\|H\|_{L^2}^2 + C\|\theta\|_{L^2}^2 \\
&\le \epsilon\|\nabla H\|_{L^2}^2 + C\|\Lambda^\gamma\theta\|_{L^2}^2 + C\|H\|_{L^2}^2 + C\|\theta\|_{L^2}^2.
\end{aligned}
$$

The last term J_4 is more involved. Actually, we have

$$
\begin{aligned}
J_4 &= C\int_{\mathbb{R}^2} (v_1\partial_2 H + v_1\partial_2\theta - v_2\partial_1 H - v_2\partial_1\theta)\omega\,\mathrm{d}x \\
&\le C\|v\|_{L^\infty}\|\nabla H\|_{L^2}\|\omega\|_{L^2} + C\|v\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\omega\|_{L^2} \\
&\le \epsilon\|\nabla H\|_{L^2}^2 + \epsilon\|\nabla\theta\|_{L^2}^2 + C_\epsilon\|v\|_{L^\infty}^2\|\omega\|_{L^2}^2,
\end{aligned}
$$

where $\epsilon > 0$ represents a small real number. The estimate of J_4 is not done yet, and the bound involves $\|\nabla\theta\|_{L^2}$. Plugging the estimates for $J_1 - J_4$ into (4.3) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2\right) + \|\Lambda^\alpha\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \\
&\quad \le 2\epsilon\|\nabla j\|_{L^2}^2 + 4\epsilon\|\nabla H\|_{L^2}^2 + \epsilon\|\Lambda^\alpha\omega\|_{L^2}^2 + \epsilon\|\nabla\theta\|_{L^2}^2
\end{aligned}
$$

$$+ C_\epsilon \left(\|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) \|\omega\|_{L^2}^2 + C_\epsilon \left(1 + \|\theta\|_{L^2}^{\frac{2}{\alpha}}\right) \|H\|_{L^2}^2$$
$$+ C_\epsilon \left(1 + \|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) \|\theta\|_{L^2}^2 + C_\epsilon \|v\|_{L^\infty}^2 \|\omega\|_{L^2}^2. \tag{4.7}$$

In order to close the above inequality, we need to absorb $\|\nabla\theta\|_{L^2}$ to the left-hand side. To achieve this goal, we apply $\Lambda^{1-\gamma}$ to the third equation of (1.1) and multiply by $\Lambda^{1-\gamma}\theta$ to deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{1-\gamma}\theta(t)\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\Lambda^{1-\gamma}\theta\|_{L^2}^2 = K_1 + K_2, \tag{4.8}$$

where

$$K_1 = -\int_{\mathbb{R}^2} \Lambda^{1-\gamma} H \Lambda^{1-\gamma}\theta \,\mathrm{d}x, \qquad K_2 = -\int_{\mathbb{R}^2} \Lambda^{1-\gamma}(u\cdot\nabla\theta)\Lambda^{1-\gamma}\theta\,\mathrm{d}x.$$

By an interpolation inequality,

$$\begin{aligned} K_1 &\le \|\Lambda^{1-\gamma} H\|_{L^2}\|\Lambda^{1-\gamma}\theta\|_{L^2} \\ &\le \|H\|_{L^2}^{\gamma}\|\nabla H\|_{L^2}^{1-\gamma}\|\Lambda^{1-\gamma}\theta\|_{L^2} \\ &\le \epsilon\|\nabla H\|_{L^2}^2 + C_\epsilon\|H\|_{L^2}^2 + C_\epsilon\|\Lambda^{1-\gamma}\theta\|_{L^2}^2. \end{aligned}$$

The estimate of K_2 is more delicate. We need the assumption that α and γ belong to Case 1:

$$\gamma \ge \frac{4+\alpha-\sqrt{\alpha^2+8\alpha+8}}{2} \quad \text{or} \quad \gamma^2 - 4\gamma + 2 \le \alpha\gamma. \tag{4.9}$$

We set

$$p_0 = \frac{2}{-\gamma^2+4\gamma-1} \quad \text{and} \quad \mu_0 = \frac{p_0-2}{\alpha p_0} = \frac{1}{\alpha}(\gamma^2-4\gamma+2) \le \gamma < 1.$$

Making use of commutator estimate (2.14) and the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned} K_2 &= -\int_{\mathbb{R}^2} [\Lambda^{1-\gamma},\ u\cdot\nabla]\theta\ \Lambda^{1-\gamma}\theta\,\mathrm{d}x \\ &\le C\|\Lambda^{-(1-\gamma)}[\Lambda^{1-\gamma},\ u\cdot\nabla]\theta\|_{L^{\frac{1}{\gamma}}}\|\Lambda^{2(1-\gamma)}\theta\|_{L^{\frac{1}{1-\gamma}}} \\ &\le C\|\nabla u\|_{L^{p_0}}\|\theta\|_{L^{\frac{p_0}{\gamma p_0-1}}}\|\nabla\theta\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}^{1-\mu_0}\|\Lambda^\alpha\nabla u\|_{L^2}^{\mu_0}\|\theta\|_{L^{\frac{2}{(1-\gamma)^2}}}\|\nabla\theta\|_{L^2} \\ &\le \epsilon\|\nabla\theta\|_{L^2}^2 + \epsilon\|\Lambda^\alpha\omega\|_{L^2}^2 + C_\epsilon\|\theta\|_{L^{\frac{2}{(1-\gamma)^2}}}^{\frac{2}{1-\mu_0}}\|\omega\|_{L^2}^2 \\ &\le \epsilon\|\nabla\theta\|_{L^2}^2 + \epsilon\|\Lambda^\alpha\omega\|_{L^2}^2 + C_\epsilon\|\theta\|_{L^{\frac{2}{(1-\gamma)^2}}}^{\frac{2}{1-\gamma}}\|\omega\|_{L^2}^2, \end{aligned} \tag{4.10}$$

where we have used the fact that α and γ are in Case 1, namely (4.9) or

$$\mu_0 \leq \gamma \quad \text{or} \quad \frac{2}{1-\mu_0} \leq \frac{2}{1-\gamma}$$

in the last step of (4.10). Inserting the bounds for K_1 and K_2 in (4.8) yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{1-\gamma}\theta(t)\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\Lambda^{1-\gamma}\theta\|_{L^2}^2 \\
&\quad \leq \epsilon\|\nabla H\|_{L^2}^2 + \epsilon\|\nabla\theta\|_{L^2}^2 + \epsilon\|\Lambda^{\alpha}\omega\|_{L^2}^2 + C_\epsilon\|H\|_{L^2}^2 + C_\epsilon\|\Lambda^{1-\gamma}\theta\|_{L^2}^2 \\
&\quad\quad + C_\epsilon\|\theta\|_{L^{\frac{2}{(1-\gamma)^2}}}^{\frac{2}{1-\gamma}}\|\omega\|_{L^2}^2.
\end{aligned} \tag{4.11}$$

Summing up (4.7) and (4.11) and taking ϵ small enough, we find that

$$\begin{aligned}
X(t) &:= \|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 + \|\Lambda^{1-\gamma}\theta(t)\|_{L^2}^2, \\
Y(t) &:= \|\Lambda^{\alpha}\omega(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2 + \|\nabla H(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2, \\
G(t) &:= 1 + \|\nabla v(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^{\frac{2}{\alpha}} + \|\theta(t)\|_{L^{\frac{2}{(1-\gamma)^2}}}^{\frac{2}{1-\gamma}}, \\
H(t) &:= (1 + \|\nabla v(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2)\|\theta(t)\|_{L^2}^2
\end{aligned}$$

satisfy the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) + Y(t) \leq C(\|v\|_{L^\infty}^2 + G(t))X(t) + CH(t).$$

To bound $\|v\|_{L^\infty}$, we invoke the following logarithmic Sobolev embedding inequality (see, e.g., Brezis and Gallouet 1980), for $\varrho > 1$,

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\Big(1 + \|f\|_{L^2(\mathbb{R}^2)} + \|\nabla f\|_{L^2(\mathbb{R}^2)}\sqrt{\ln(e + \|\Lambda^{\varrho} f\|_{L^2(\mathbb{R}^2)})}\Big). \tag{4.12}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) + Y(t) \leq C(1 + \|\nabla v\|_{L^2}^2 \ln(e + \|\Delta v\|_{L^2})X(t) + CG(t)X(t) + CH(t).$$

Thanks to (3.8) and (3.10),

$$\|\Delta v(t)\|_{L^2} \leq C(\|\nabla j(t)\|_{L^2}^2 + \|\nabla H(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) \leq CY(t)$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t)+Y(t) \leq C(1+\|\nabla v\|_{L^2}^2 \ln\big(e+X(t)+Y(t)\big)X(t)+CG(t)X(t)+CH(t). \tag{4.13}$$

Due to the simple inequality

$$\|\nabla v\|_{L^2}^2 \le C\|v\|_{L^2}\|\Delta v\|_{L^2} \le C\|\Delta v\|_{L^2} \le CY^{\frac{1}{2}}(t),$$

we apply Lemma 2.2 to (4.13) to obtain

$$X(t) + \int_0^t Y(s)\,\mathrm{d}s \le C,$$

which is (4.2). This finishes the proof of Lemma 4.1. □

Next, we prove the global H^1-estimate of θ.

Lemma 4.2 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume α and γ belong to Case 1, namely (3.12). Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\gamma \nabla\theta(\tau)\|_{L^2}^2\, d\tau \le C(t). \tag{4.14}$$

Proof of Lemma 4.2 Applying ∇ to the third equation of (1.1) and dotting it by $\nabla\theta$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t)\|_{L^2}^2 + \|\Lambda^\gamma\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\\
&\quad\le \int_{\mathbb{R}^2}\nabla H\nabla\theta\,\mathrm{d}x - \int_{\mathbb{R}^2}\nabla u\cdot\nabla\theta\nabla\theta\,\mathrm{d}x\\
&\quad\le C\|\nabla H\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^4}^2\\
&\quad\le C\|\nabla H\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^{2-\frac{1}{\gamma}}\|\Lambda^\gamma\nabla\theta\|_{L^2}^{\frac{1}{\gamma}}\\
&\quad\le \frac{1}{2}\|\Lambda^\gamma\nabla\theta\|_{L^2}^2 + C\|\nabla H\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\omega\|_{L^2}^{\frac{2\gamma}{2\gamma-1}}\|\nabla\theta\|_{L^2}^2.
\end{aligned}$$

The global bound in (4.2) and Gronwall's inequality imply

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\gamma\nabla\theta(\tau)\|_{L^2}^2\,\mathrm{d}\tau \le C.$$

This concludes the proof of Lemma 4.2. □

Our next step is to obtain higher regularity by making use of the maximal regularity estimate for parabolic equations (see, e.g., Lemarié-Rieusset 2002).

Lemma 4.3 *The operator A defined by*

$$Af(x,t) := \int_0^t e^{(t-s)\Delta}\Delta f(s,x)\,ds$$

is bounded from $L^p(0,T; L^q(\mathbb{R}^n))$ to $L^p(0,T; L^q(\mathbb{R}^n))$ for every $(p,q)\in(1,\infty)\times(1,\infty)$ and $T\in(0,\infty]$, namely

$$\Big\| \int_0^t e^{(t-s)\Delta} \Delta f(s,x)\, ds \Big\|_{L^p(0,T;L^q(\mathbb{R}^n))} \le C \|f\|_{L^p(0,T;L^q(\mathbb{R}^n))},$$

where the constant C is independent of T.

Lemmas 4.1 through 4.3 allow us to prove the following estimates.

Lemma 4.4 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume α and γ belong to Case 1, namely (3.12). Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\|v(t)\|_{L^\infty(0,\,T;L^\infty)} + \|\nabla v(t)\|_{L^2(0,\,T;L^\infty)} \le C(t). \tag{4.15}$$

Proof of Lemma 4.4 We recall the following property of the heat equation (see Ye 2017), which states that any solution v of

$$\partial_t v - \Delta v = \sum_{l=1}^{k} f_l, \quad \|f_l(t)\|_{L^{p_l}(0,\,T;L^{q_l}(\mathbb{R}^n))} \le C, \quad \frac{2}{p_l} + \frac{n}{q_l} < 2, \;\; l = 1, 2, \cdots, k$$

satisfies

$$\|v(t)\|_{L^\infty(0,\,T;L^\infty)} \le C. \tag{4.16}$$

Combining (4.14), (4.2) and (3.1), we can show that

$$\|\nabla\theta(t)\|_{L^2\left(0,\,T;L^{\frac{2}{1-\gamma}}\right)} \le C\|\Lambda^\gamma \nabla\theta\|_{L^2(0,\,T;L^2)} \le C;$$

$$\|u \cdot \nabla v(t)\|_{L^2\left(0,\,T;L^{\frac{2-\gamma}{1-\gamma}}\right)} \le C; \tag{4.17}$$

$$\|v \cdot \nabla u(t)\|_{L^2\left(0,\,T;L^{\frac{2-\alpha}{1-\alpha}}\right)} \le C. \tag{4.18}$$

In fact, by Hölder's inequality,

$$\|u \cdot \nabla v\|_{L^{\frac{2-\gamma}{1-\gamma}}} \le C\|u\|_{L^{\frac{2(2-\gamma)}{\gamma(1-\gamma)}}} \|\nabla v\|_{L^{\frac{2}{1-\gamma}}},$$

which, together with (4.2) and (4.14), implies (4.17). The proof of (4.18) is the same. Recall the third equation of (1.1)

$$\partial_t v - \Delta v = -\nabla\theta - (u \cdot \nabla)v - (v \cdot \nabla)u. \tag{4.19}$$

According to (4.16),

$$\|v(t)\|_{L^\infty(0,\,T;L^\infty)} \le C. \tag{4.20}$$

Applying operator Δ to (4.19) and making use of the Duhamel Principle yield

$$\Delta v(x,t) = e^{t\Delta}\Delta v_0(x) - \int_0^t e^{(t-s)\Delta}\Delta\Big((u\cdot\nabla)v + \nabla\theta + (v\cdot\nabla)u\Big)\,\mathrm{d}s. \quad (4.21)$$

By Lemma 4.3,

$$\int_0^t \|\Delta v(s)\|^2_{L^{\widetilde{q}}}\,\mathrm{d}s \le C\int_0^t \Big(\|u\cdot\nabla v(s)\|^2_{L^{\widetilde{q}}} + \|\nabla\theta(s)\|^2_{L^{\widetilde{q}}} + \|v\cdot\nabla u(s)\|^2_{L^{\widetilde{q}}}\Big)\,\mathrm{d}s < \infty,$$

for some $2 < \widetilde{q} \le \min\{\frac{2-\alpha}{1-\alpha}, \ \frac{2-\gamma}{1-\gamma}\}$. Desired bound (4.15) follows as a sequence. This completes the proof. □

The following lemma proves a global bound for ΔH.

Lemma 4.5 *Assume* (u_0, v_0, θ_0) *satisfies the conditions stated in Theorem 1.1. Assume* α *and* γ *belong to Case 1, namely (3.12). Then, the corresponding smooth solution* (u, v, θ) *of (1.1) obeys the global bound, for any* $t > 0$,

$$\|\Delta H(t)\|_{L^{\frac{2\alpha+2}{\alpha+2}}(0,T;L^2)} \le C. \quad (4.22)$$

Proof It is not difficult to see that the global bounds in (4.14), (4.15), (4.2) and (3.1) imply

$$\|\Lambda^{2\gamma}\theta(t)\|_{L^{\frac{2\gamma}{2\gamma-1}}(0,T;L^2)} \le C; \quad (4.23)$$

$$\|H(t)\|_{L^\infty(0,T;L^2)} + \|\theta(t)\|_{L^\infty(0,T;L^2)} \le C; \quad (4.24)$$

$$\|Q(\nabla u, \nabla v)(t)\|_{L^2(0,T;L^2)} \le C; \quad (4.25)$$

$$\|(u\cdot\nabla)H(t)\|_{L^{\frac{2\alpha+2}{\alpha+2}}(0,T;L^2)} \le C. \quad (4.26)$$

Now recalling equation (3.9)

$$\partial_t H - \Delta H = \Lambda^{2\gamma}\theta + H + \theta - Q(\nabla u, \nabla v) - (u\cdot\nabla)H, \quad (4.27)$$

and applying Δ to (4.27) and making use of the Duhamel Principle yield

$$\Delta H(x,t) = e^{t\Delta}\Delta H_0(x) + \int_0^t e^{(t-s)\Delta}\Delta\Big(\Lambda^{2\gamma}\theta + H + \theta - Q(\nabla u, \nabla v) - (u\cdot\nabla)H\Big)\,\mathrm{d}s.$$

By (4.23)–(4.26) and Lemma 4.3, we have

$$\|\Delta H(t)\|_{L^{\frac{2\alpha+2}{\alpha+2}}(0,T;L^2)} \le C.$$

This ends the proof. □

Now we establish the following key estimates.

Lemma 4.6 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume α and γ belong to Case 1, namely (3.12). Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $2 \le q < \infty$ and for any $t > 0$,*

$$\|\nabla\theta(t)\|_{L^q} + \|\nabla u(t)\|_{L^q} \le C(t). \tag{4.28}$$

Proof of Lemma 4.6 We apply ∇ to the third equation of (1.1) and multiply it by $|\nabla\theta|^{q-2}\nabla\theta$ to deduce

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t)\|_{L^q}^q + \int_{\mathbb{R}^2} \Lambda^{2\gamma}\nabla\theta(|\nabla\theta|^{q-2}\nabla\theta)\,\mathrm{d}x + \|\nabla\theta\|_{L^q}^q = M_1 + M_2, \tag{4.29}$$

where

$$M_1 = -\int_{\mathbb{R}^2} \nabla H(|\nabla\theta|^{q-2}\nabla\theta)\,\mathrm{d}x, \qquad M_2 = -\int_{\mathbb{R}^2} \nabla u\cdot\nabla\theta(|\nabla\theta|^{q-2}\nabla\theta)\,\mathrm{d}x.$$

By means of the pointwise inequality (see, e.g., Córdoba and Córdoba 2004) and Sobolev embedding inequality, it entails

$$\int_{\mathbb{R}^2} \Lambda^{2\gamma}\nabla\theta(|\nabla\theta|^{q-2}\nabla\theta)\,\mathrm{d}x \ge C_1\int_{\mathbb{R}^2} (\Lambda^{\gamma}|\nabla\theta|^{\frac{q}{2}})^2\,\mathrm{d}x \ge C_2\,\|\nabla\theta\|_{L^{\frac{q}{1-\gamma}}}^q. \tag{4.30}$$

According to the Hölder inequality, we have

$$\begin{aligned} M_1 &\le C\|\nabla H\|_{L^q}\|\nabla\theta\|_{L^q}^{q-1} \\ &\le C\|\nabla H\|_{L^2}^{\frac{2}{q}}\|\Delta H\|_{L^2}^{1-\frac{2}{q}}\|\nabla\theta\|_{L^q}^{q-1}, \end{aligned} \tag{4.31}$$

$$\begin{aligned} M_2 &\le C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^{2q}}^q \\ &\le C\|\omega\|_{L^2}\|\nabla\theta\|_{L^q}^{\frac{(2\gamma-1)q}{2\gamma}}\|\nabla\theta\|_{L^{\frac{q}{1-\gamma}}}^{\frac{q}{2\gamma}}. \\ &\le \frac{C_2}{8}\|\nabla\theta\|_{L^{\frac{q}{1-\gamma}}}^q + C\|\omega\|_{L^2}^{\frac{2\gamma}{2\gamma-1}}\|\nabla\theta\|_{L^q}^q \end{aligned} \tag{4.32}$$

Putting the above estimates (4.29)–(4.32) together yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t)\|_{L^q}^q \le C\|\nabla H\|_{L^2}^{\frac{2}{q}}\|\Delta H\|_{L^2}^{1-\frac{2}{q}}\|\nabla\theta\|_{L^q}^{q-1} + C\|\omega\|_{L^2}^{\frac{2\gamma}{2\gamma-1}}\|\nabla\theta\|_{L^q}^q$$

or

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t)\|_{L^q} &\le C\|\nabla H\|_{L^2}^{\frac{2}{q}}\|\Delta H\|_{L^2}^{1-\frac{2}{q}} + C\|\omega\|_{L^2}^{\frac{2\gamma}{2\gamma-1}}\|\nabla\theta\|_{L^q} \\ &\le C\|\nabla H\|_{L^2} + C\|\Delta H\|_{L^2} + C\|\omega\|_{L^2}^{\frac{2\gamma}{2\gamma-1}}\|\nabla\theta\|_{L^q}. \end{aligned}$$

By Gronwall's inequality along with (4.2) and (4.22), one gets

$$\|\nabla\theta(t)\|_{L^q} \le C. \tag{4.33}$$

Recalling the vorticity equation

$$\partial_t\omega + (u\cdot\nabla)\omega + \Lambda^{2\alpha}\omega = -\nabla\times\nabla\cdot(v\otimes v), \tag{4.34}$$

multiplying (4.34) by $|\omega|^{q-2}\omega$ and integrating by parts, we have, for any $2 \le q < \infty$.

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^q}^q &\le -\int_{\mathbb{R}^2}\nabla\times\nabla\cdot(v\otimes v)\omega|\omega|^{q-2}\,\mathrm{d}x\\ &\le C\|\nabla\times\nabla\cdot(v\otimes v)\|_{L^q}\|\omega\|_{L^q}^{q-1}\\ &\le C\|\Delta(vv)\|_{L^q}\|\omega\|_{L^q}^{q-1}\\ &\le C\|v\|_{L^\infty}\|\Delta v\|_{L^q}\|\omega\|_{L^q}^{q-1}\\ &\le C(\|\Delta v\|_{L^q}^2 + \|\omega\|_{L^q}^2)\|\omega\|_{L^q}^{q-2}.\end{aligned}$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^q}^2 \le C(\|\Delta v\|_{L^q}^2 + \|\omega\|_{L^q}^2). \tag{4.35}$$

Integrating (4.35) in time and using Lemma 4.3 lead to

$$\begin{aligned}\|\omega\|_{L^q}^2 &\le \|\omega_0\|_{L^q}^2 + C\int_0^t (\|\Delta v(s)\|_{L^q}^2 + \|\omega(s)\|_{L^q}^2)\,\mathrm{d}s\\ &\le \|\omega_0\|_{L^q}^2 + C\int_0^t (\|u\cdot\nabla v(s)\|_{L^q}^2 + \|\nabla\theta(s)\|_{L^q}^2\\ &\quad + \|v\cdot\nabla u(s)\|_{L^q}^2 + \|\omega(s)\|_{L^q}^2)\,\mathrm{d}s\\ &\le \|\omega_0\|_{L^q}^2 + C\int_0^t (\|u\|_{L^{2q}}^2\|\nabla v\|_{L^{2q}}^2 + \|\nabla\theta\|_{L^q}^2\\ &\quad + \|v\|_{L^\infty}^2\|\nabla u\|_{L^q}^2 + \|\omega\|_{L^q}^2)(s)\,\mathrm{d}s\\ &\le \|\omega_0\|_{L^q}^2 + C\int_0^t (\|u\|_{H^1}^2\|v\|_{H^2}^2 + \|\nabla\theta\|_{L^q}^2 + \|\omega\|_{L^q}^2)(s)\,\mathrm{d}s.\end{aligned}$$

By the Gronwall inequality and (4.33),

$$\|\omega(t)\|_{L^q} \le C, \quad 2 \le q < \infty.$$

By the boundedness of the Calderon–Zygmund operator on the L^q space,

$$\|\nabla u(t)\|_{L^q} \le C, \quad 2 \le q < \infty. \tag{4.36}$$

This completes the proof of Lemma 4.6. □

With the estimates above at our disposal, we are now ready to give the proof of Theorem 1.1 for Case 1. The following homogeneous commutator estimate will be used several times. For any $s > 0$, we have

$$\|\Lambda^s(f\,g) - f\,\Lambda^s g\|_{L^p} \le C\,(\|\nabla f\|_{L^q}\,\|\Lambda^{s-1}g\|_{L^r} + \|g\|_{L^{q_1}}\,\|\Lambda^s f\|_{L^{r_1}}),$$

where $p, r, r_1 \in (1, \infty)$ and $q, q_1 \in [1, \infty]$ satisfy

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r} = \frac{1}{q_1} + \frac{1}{r_1}.$$

This type of commutator estimates can be found in several references (see, e.g., Li in press; Kenig et al. 1993).

Proof of Theorem 1.1 (Case 1) Applying Λ^s with $s > 2$ to (1.1) and taking the L^2 inner product with $(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2) + \|\Lambda^{s+\alpha}u\|_{L^2}^2 + \|\Lambda^{s+1}v\|_{L^2}^2 \\
&\quad + \|\Lambda^{s+\gamma}\theta\|_{L^2}^2 = -\int_{\mathbb{R}^2}\Big(\Lambda^s\nabla\cdot(v\otimes v)\cdot\Lambda^s u + \Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\Big)\,\mathrm{d}x \\
&\quad -\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\theta)\cdot\Lambda^s\theta\,\mathrm{d}x - \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\,\mathrm{d}x - \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,\mathrm{d}x \\
&\quad := H_1 + H_2 + H_3 + H_4.
\end{aligned} \tag{4.37}$$

By the commutator and bilinear estimates (see, e.g., Li in press; Kato and Ponce 1988; Kenig et al. 1991)

$$\begin{aligned}
\|[\Lambda^s, f]g\|_{L^p} &\le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}), \\
\|\Lambda^s(fg)\|_{L^p} &\le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}})
\end{aligned}$$

with $s > 0,\ p_2, p_3 \in (1, \infty)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, we have

$$\begin{aligned}
H_1 &\le C\|\Lambda^s\nabla\cdot(v\otimes v)\|_{L^2}\|\Lambda^s u\|_{L^2} + C\|\Lambda^{s-1}(v\cdot\nabla u)\|_{L^2}\|\Lambda^{s+1}v\|_{L^2} \\
&\le C\|v\|_{L^\infty}\|\Lambda^{s+1}v\|_{L^2}\|\Lambda^s u\|_{L^2} + C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^{s-1}v\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s+1}v\|_{L^2} \\
&\le C\|v\|_{L^\infty}\|\Lambda^{s+1}v\|_{L^2}\|\Lambda^s u\|_{L^2} + C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\big(\|v\|_{L^2} + \|\Lambda^s v\|_{L^2}\big)\|\Lambda^{s+1}v\|_{L^2} \\
&\le \frac{1}{8}\|\Lambda^{s+1}v\|_{L^2}^2 + C\|v\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2 + C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^2\Big(\|v\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2\Big), \\
H_2 &\le C\|[\Lambda^s, u\cdot\nabla]\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2} \\
&\le C\left(\|\nabla u\|_{L^{\frac{2}{\gamma}}}\|\Lambda^s\theta\|_{L^{\frac{2}{1-\gamma}}} + \|\nabla\theta\|_{L^{\frac{2}{\alpha}}}\|\Lambda^s u\|_{L^{\frac{2}{1-\alpha}}}\right)\|\Lambda^s\theta\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\|\nabla u\|_{L^{\frac{2}{\gamma}}}\|\Lambda^{s+\gamma}\theta\|_{L^2}+\|\nabla\theta\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s+\alpha}u\|_{L^2}\right)\|\Lambda^s\theta\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+\gamma}\theta\|_{L^2}^2+\frac{1}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2+C\left(\|\nabla u\|_{L^{\frac{2}{\gamma}}}^2+\|\nabla\theta\|_{L^{\frac{2}{\alpha}}}^2\right)\|\Lambda^s\theta\|_{L^2}^2,\\
H_3 &\le C\|\Lambda^s(u\cdot\nabla v)\|_{L^2}\|\Lambda^s v\|_{L^2}\\
&\le C\left(\|u\|_{L^\infty}\|\Lambda^{s+1}v\|_{L^2}+\|\nabla v\|_{L^\infty}\|\Lambda^s u\|_{L^2}\right)\|\Lambda^s v\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+1}v\|_{L^2}^2+C\|\nabla v\|_{L^\infty}\|\Lambda^s u\|_{L^2}\|\Lambda^s v\|_{L^2}+C\|u\|_{L^\infty}^2\|\Lambda^s v\|_{L^2}^2,\\
H_4 &\le C\int_{\mathbb{R}^2}|[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u|\,\mathrm{d}x\\
&\le C\|\Lambda^s u\|_{L^{2(\alpha+1)}}\|[\Lambda^s,u\cdot\nabla]u\|_{L^{\frac{2(\alpha+1)}{2\alpha+1}}}\\
&\le C\|\Lambda^s u\|_{L^{2(\alpha+1)}}\|[\Lambda^s,u_i]\partial_i u\|_{L^{\frac{2(\alpha+1)}{2\alpha+1}}}\\
&\le C\|\Lambda^s u\|_{L^{2(\alpha+1)}}\left(\|\nabla u\|_{L^{\frac{\alpha+1}{\alpha}}}\|\Lambda^{s-1}\partial_i u\|_{L^{2(\alpha+1)}}+\|\partial_i u\|_{L^{\frac{\alpha+1}{\alpha}}}\|\Lambda^s u\|_{L^{2(\alpha+1)}}\right)\\
&\le C\|\Lambda^s u\|_{L^{2(\alpha+1)}}\|\nabla u\|_{L^{\frac{\alpha+1}{\alpha}}}\|\Lambda^s u\|_{L^{2(\alpha+1)}}\\
&\le C\|\Lambda^s u\|_{L^2}^{\frac{2\alpha}{\alpha+1}}\|\Lambda^{s+\alpha}u\|_{L^2}^{\frac{2}{\alpha+1}}\|\nabla u\|_{L^{\frac{\alpha+1}{\alpha}}}\\
&\le \frac{1}{2}\|\Lambda^{s+\alpha}u\|_{L^2}^2+C\|\nabla u\|_{L^{\frac{\alpha+1}{\alpha}}}^{\frac{\alpha+1}{\alpha}}\|\Lambda^s u\|_{L^2}^2.
\end{aligned}
$$

Substituting all the preceding estimates into (4.37), one gets

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2)+\|\Lambda^{s+\alpha}u\|_{L^2}^2+\|\Lambda^{s+1}v\|_{L^2}^2+\|\Lambda^{s+\gamma}\theta\|_{L^2}^2\\
&\quad\le C\mathcal{B}(t)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{B}(t)=\Bigg(&1+\|v(t)\|_{L^\infty}^2+\|\nabla u(t)\|_{L^{\frac{2}{1-\alpha}}}^2+\|\nabla u(t)\|_{L^{\frac{2}{\gamma}}}^2+\|\nabla\theta(t)\|_{L^{\frac{2}{\alpha}}}\\
&+\|u(t)\|_{L^\infty}^2+\|\nabla v(t)\|_{L^\infty}^2+\|\nabla u(t)\|_{L^{\frac{\alpha+1}{\alpha}}}^{\frac{\alpha+1}{\alpha}}\Bigg).
\end{aligned}
$$

Thanks to (3.1), (4.2), (4.15), (4.28), one has

$$
\int_0^T\mathcal{B}(t)\,\mathrm{d}t<\infty.
$$

The Gronwall inequality allows us to conclude that

$$
\begin{aligned}
&\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2+\int_0^t(\|\Lambda^{s+\alpha}u(\tau)\|_{L^2}^2+\|\Lambda^{s+1}v(\tau)\|_{L^2}^2\\
&+\|\Lambda^{s+\gamma}\theta(\tau)\|_{L^2}^2)\,\mathrm{d}\tau<\infty,
\end{aligned}
$$

which is the desired global bounds in Theorem 1.1. This completes the proof of Theorem 1.1 for Case 1. □

5 Proof of Theorem 1.1: Case 2

This section provides the proof for Theorem 1.1: Case 2. The focus again is on the range

$$\gamma = 1-\alpha, \qquad \frac{1}{2} \le \alpha \le 1$$

since the case $\gamma > 1-\alpha$ can be dealt with in a similar manner and is actually easier. We further divide into two subcases: $\gamma = 1-\alpha$, $\frac{1}{2} \le \alpha < 1$ and $\alpha = 1, \gamma = 0$. The rest of this section is correspondingly split into two subsections with each devoted to one of the cases. Each subcase starts with the estimate of the H^1-norm of (u, v, θ).

5.1 The Subcase: $\gamma = 1-\alpha$, $\frac{1}{2} \le \alpha < 1$

Lemma 5.1 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume α and γ belong to Case 2, namely (3.13). Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\begin{aligned} &\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\Lambda^{1-\alpha}\theta(t)\|_{L^2}^2 \\ &\quad + \int_0^t (\|\Lambda^\alpha \omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\Lambda^{1-\alpha+\gamma}\nabla v\|_{L^2}^2 + \|\Lambda^{1-\alpha+\gamma}\theta\|_{L^2}^2)(\tau)\, d\tau \\ &\quad \le C_0(t), \end{aligned} \tag{5.1}$$

where C_0 depends only on t and the initial data.

Proof of Lemma 5.1 It follows from (4.3) that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2) + \|\Lambda^\alpha \omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^2} \Big((v\cdot\nabla) j\omega + 2hj\omega + (v\cdot\nabla)\omega j + h\omega j\Big)\,\mathrm{d}x - \int_{\mathbb{R}^2} Q(\nabla u, \nabla v) H\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^2} (\Lambda^{2\gamma}\theta + H + \theta) H\,\mathrm{d}x + \int_{\mathbb{R}^2} (v_1\partial_2 h - v_2\partial_1 h)\omega\,\mathrm{d}x \\ &\quad := J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{5.2}$$

The terms J_1, J_2, J_3 can be similarly estimated as those of H_1, H_2, H_3. However, the last term J_4 is handled differently. It is bounded by

$$J_4 = C\int_{\mathbb{R}^2} (v_1\partial_2 H + v_1\partial_2\theta - v_2\partial_1 H - v_2\partial_1\theta)\,\omega\,\mathrm{d}x$$

$$
\begin{aligned}
&\le C\|v\|_{L^\infty}\|\nabla H\|_{L^2}\|\omega\|_{L^2} + C\|\Lambda^{1-\alpha}\theta\|_{L^2}\|\Lambda^\alpha (v\omega)\|_{L^2}\\
&\le C\|v\|_{L^\infty}\|\nabla H\|_{L^2}\|\omega\|_{L^2} + C\|\Lambda^{1-\alpha}\theta\|_{L^2}\left(\|v\|_{L^\infty}\|\Lambda^\alpha\omega\|_{L^2} + \|\Lambda^\alpha v\|_{L^{\frac{2}{\alpha}}}\|\omega\|_{L^{\frac{2}{1-\alpha}}}\right)\\
&\le C\|v\|_{L^\infty}\|\nabla H\|_{L^2}\|\omega\|_{L^2} + C\|\Lambda^{1-\alpha}\theta\|_{L^2}\left(\|v\|_{L^\infty}\|\Lambda^\alpha\omega\|_{L^2} + \|\nabla v\|_{L^2}\|\Lambda^\alpha\omega\|_{L^2}\right)\\
&\le \epsilon\|\nabla H\|_{L^2}^2 + \epsilon\|\Lambda^\alpha\omega\|_{L^2}^2 + C_\epsilon\|v\|_{L^\infty}^2\|\omega\|_{L^2}^2 + C_\epsilon\left(\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^2}^2\right)\|\Lambda^{1-\alpha}\theta\|_{L^2}^2.
\end{aligned}
$$

Putting the estimations of $J_1 - J_4$ into (5.2), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2) + \|\Lambda^\alpha\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\\
&\quad\le 2\epsilon\|\nabla j\|_{L^2}^2 + 4\epsilon\|\nabla H\|_{L^2}^2 + 2\epsilon\|\Lambda^\alpha\omega\|_{L^2}^2 + \epsilon\|\nabla\theta\|_{L^2}^2\\
&\qquad + C_\epsilon(\|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2)\|\omega\|_{L^2}^2 + C_\epsilon(1 + \|\theta\|_{L^2}^{\frac{2}{\alpha}})\|H\|_{L^2}^2\\
&\qquad + C_\epsilon(1 + \|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2)\|\theta\|_{L^2}^2 + C_\epsilon(\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^2}^2)\|\Lambda^{1-\alpha}\theta\|_{L^2}^2.
\end{aligned}
\tag{5.3}
$$

The right-hand side of (5.3) involves $\|\Lambda^{1-\alpha}\theta\|_{L^2}$, and we need to estimate $\|\Lambda^{1-\alpha}\theta\|_{L^2}$ simultaneously. To this end, applying $\Lambda^{1-\alpha}$ to the third equation of (1.1) and multiply by $\Lambda^{1-\alpha}\theta$ to deduce

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{1-\alpha}\theta(t)\|_{L^2}^2 + \|\Lambda^{1-\alpha+\gamma}\theta\|_{L^2}^2 + \|\Lambda^{1-\alpha}\theta\|_{L^2}^2 = \widetilde{K}_1 + \widetilde{K}_2, \tag{5.4}
$$

where

$$
\widetilde{K}_1 = -\int_{\mathbb{R}^2}\Lambda^{1-\alpha}H\,\Lambda^{1-\alpha}\theta\,\mathrm{d}x, \qquad \widetilde{K}_2 = -\int_{\mathbb{R}^2}\Lambda^{1-\alpha}(u\cdot\nabla\theta)\Lambda^{1-\alpha}\theta\,\mathrm{d}x.
$$

By Hölder's inequality and an interpolation inequality,

$$
\begin{aligned}
\widetilde{K}_1 &\le \|\Lambda^{1-\alpha}H\|_{L^2}\|\Lambda^{1-\alpha}\theta\|_{L^2}\\
&\le \|H\|_{L^2}^{\alpha}\|\nabla H\|_{L^2}^{1-\alpha}\|\Lambda^{1-\alpha}\theta\|_{L^2}\\
&\le \epsilon\|\nabla H\|_{L^2}^2 + C_\epsilon\|H\|_{L^2}^2 + C_\epsilon\|\Lambda^{1-\alpha}\theta\|_{L^2}^2.
\end{aligned}
$$

Thanks to the commutator estimate (2.13) and the Gagliardo–Nirenberg inequality (2.1), it yields for $\gamma^2 < \sigma < \gamma$ that

$$
\begin{aligned}
\widetilde{K}_2 &= -\int_{\mathbb{R}^2}[\Lambda^{1-\alpha},\ u\cdot\nabla]\theta\ \Lambda^{1-\alpha}\theta\,\mathrm{d}x\\
&\le \|[\Lambda^{1-\alpha},\ u\cdot\nabla]\theta\|_{H^{-\sigma}}\|\Lambda^{1-\alpha}\theta\|_{H^\sigma}\\
&\le \|[\Lambda^{1-\alpha},\ u\cdot\nabla]\theta\|_{B_{2,2}^{-\sigma}}\|\Lambda^{1-\alpha}\theta\|_{L^2}^{1-\frac{\sigma}{\gamma}}\|\Lambda^{1-\alpha+\gamma}\theta\|_{L^2}^{\frac{\sigma}{\gamma}}\\
&\le C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\theta\|_{B^{\gamma-\sigma}_{\frac{2}{1-\gamma},2}}\|\Lambda^{1-\alpha}\theta\|_{L^2}^{1-\frac{\sigma}{\gamma}}\|\Lambda^{1-\alpha+\gamma}\theta\|_{L^2}^{\frac{\sigma}{\gamma}}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\Lambda^\alpha\omega\|_{L^2}\|\theta\|^{1-\frac{\gamma-\sigma}{\gamma(1-\gamma)}}_{B^0_{\frac{2}{1-\gamma},\infty}}\|\theta\|^{\frac{\gamma-\sigma}{\gamma(1-\gamma)}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}\|\Lambda^{1-\alpha}\theta\|^{1-\frac{\sigma}{\gamma}}_{L^2}\|\Lambda^{1-\alpha+\gamma}\theta\|^{\frac{\sigma}{\gamma}}_{L^2}\\
&\le C\|\Lambda^\alpha\omega\|_{L^2}\|\theta\|^{1-\frac{\gamma-\sigma}{\gamma(1-\gamma)}}_{L^{\frac{2}{1-\gamma}}}\|\theta\|^{\frac{\gamma-\sigma}{\gamma(1-\gamma)}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}\|\Lambda^{1-\alpha}\theta\|^{1-\frac{\sigma}{\gamma}}_{L^2}\|\Lambda^{1-\alpha+\gamma}\theta\|^{\frac{\sigma}{\gamma}}_{L^2}\\
&\le \epsilon\|\Lambda^\alpha\omega\|^2_{L^2}+\epsilon\|\Lambda^{1-\alpha+\gamma}\theta\|^2_{L^2}+C_\epsilon\|\theta\|^{\frac{\sigma-\gamma^2}{(1-\gamma)(\gamma-\sigma)}}_{L^{\frac{2}{1-\gamma}}}\|\theta\|^{\frac{2}{1-\gamma}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}\|\Lambda^{1-\alpha}\theta\|^2_{L^2},
\end{aligned}
$$

where we have used $\gamma = 1-\alpha$ in the commutator estimate and invoked the following Gagliardo–Nirenberg inequality (see (2.1) or (Bahouri et al. 2011, Proposition 2.22)

$$
\|\theta\|_{B^{\gamma-\sigma}_{\frac{2}{1-\gamma},2}} \le C\|\theta\|^{1-\frac{\gamma-\sigma}{\gamma(1-\gamma)}}_{B^0_{\frac{2}{1-\gamma},\infty}}\|\theta\|^{\frac{\gamma-\sigma}{\gamma(1-\gamma)}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}, \qquad \gamma^2<\sigma<\gamma.
$$

Consequently, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{1-\alpha}\theta(t)\|^2_{L^2}+\|\Lambda^{1-\alpha+\gamma}\theta\|^2_{L^2}+\|\Lambda^{1-\alpha}\theta\|^2_{L^2}\\
&\quad\le \epsilon\|\nabla H\|^2_{L^2}+\epsilon\|\Lambda^{1-\alpha+\gamma}\theta\|^2_{L^2}+\epsilon\|\Lambda^\alpha\omega\|^2_{L^2}+C_\epsilon\|H\|^2_{L^2}+C_\epsilon\|\Lambda^{1-\alpha}\theta\|^2_{L^2}\\
&\qquad+C_\epsilon\|\theta\|^{\frac{\sigma-\gamma^2}{(1-\gamma)(\gamma-\sigma)}}_{L^{\frac{2}{1-\gamma}}}\|\theta\|^{\frac{2}{1-\gamma}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}}\|\Lambda^{1-\alpha}\theta\|^2_{L^2}+C_\epsilon(1+\|u\|^2_{L^2})\|\theta\|^2_{L^2},
\end{aligned}\tag{5.5}
$$

Summing up (5.3) and (5.5) and taking ϵ small enough, we find that

$$
\begin{aligned}
\widetilde{X}(t) &:= \|\omega(t)\|^2_{L^2}+\|j(t)\|^2_{L^2}+\|H(t)\|^2_{L^2}+\|\Lambda^{1-\alpha}\theta(t)\|^2_{L^2},\\
\widetilde{Y}(t) &:= \|\Lambda^\alpha\omega(t)\|^2_{L^2}+\|\nabla j(t)\|^2_{L^2}+\|\nabla H(t)\|^2_{L^2}+\|\Lambda^{1-\alpha+\gamma}\theta(t)\|^2_{L^2},\\
\widetilde{G}(t) &:= 1+\|\nabla v(t)\|^2_{L^2}+\|\theta(t)\|^{\frac{2}{\alpha}}_{L^2}+\|\theta\|^{\frac{\sigma-\gamma^2}{(1-\gamma)(\gamma-\sigma)}}_{L^{\frac{2}{1-\gamma}}}\|\theta\|^{\frac{2}{1-\gamma}}_{B^{\gamma(1-\gamma)}_{\frac{2}{1-\gamma},\frac{2}{1-\gamma}}},\\
\widetilde{H}(t) &:= (1+\|\nabla v(t)\|^2_{L^2}+\|\theta(t)\|^2_{L^2}+\|u(t)\|^2_{L^2})\|\theta(t)\|^2_{L^2}
\end{aligned}
$$

satisfy

$$
\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{X}(t)+\widetilde{Y}(t)\le C(\|v\|^2_{L^\infty}+\widetilde{G}(t))\widetilde{X}(t)+C\widetilde{H}(t). \tag{5.6}
$$

Bounding $\|v\|_{L^\infty}$ via (4.12), we obtain, for any $\delta>0$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{X}(t)+\widetilde{Y}(t)\le{}& C\|\nabla v(t)\|^2_{L^2}\ln(e+\|\Lambda^\delta\nabla v(t)\|_{L^2})\widetilde{X}(t)\\
&+C\widetilde{G}(t)\widetilde{X}(t)+C\widetilde{H}(t).
\end{aligned}\tag{5.7}
$$

According to (3.8) and (3.10), we have by taking $0 < \delta < \min\{1-\alpha+\gamma,\ 1\}$

$$\begin{aligned}\|\Lambda^\delta \nabla v(t)\|_{L^2} &\le C(\|\Lambda^\delta H(t)\|_{L^2} + \|\Lambda^\delta j(t)\|_{L^2} + \|\Lambda^\delta \theta(t)\|_{L^2})\\ &\le C(\|H(t)\|_{L^2}^{1-\delta}\|\nabla H(t)\|_{L^2}^{\delta} + \|j(t)\|_{L^2}^{1-\delta}\|\nabla j(t)\|_{L^2}^{\delta}\\ &\quad + \|\theta(t)\|_{L^2}^{1-\frac{\delta}{1-\alpha+\gamma}}\|\Lambda^{1-\alpha+\gamma}\theta(t)\|_{L^2}^{\frac{\delta}{1-\alpha+\gamma}})\\ &\le C\big(e + \widetilde{X}^{\frac{1}{2}}(t) + \widetilde{Y}^{\frac{1}{2}}(t)\big). \end{aligned} \tag{5.8}$$

Furthermore,

$$\|\nabla v\|_{L^2}^2 \le C\|v\|_{L^2}^{\frac{2\delta}{\delta+1}}\|\Lambda^\delta \nabla v\|_{L^2}^{\frac{2}{\delta+1}} \le C\|\Lambda^\delta\nabla v\|_{L^2}^{\frac{2}{\delta+1}} \le C\big(e + \widetilde{X}(t) + \widetilde{Y}(t)\big)^{\frac{1}{\delta+1}}. \tag{5.9}$$

Combining (5.8), (5.9) and applying Lemma 2.2 lead to

$$\widetilde{X}(t) + \int_0^t \widetilde{Y}(s)\,\mathrm{d}s \le C,$$

which is (5.1). This completes the proof of Lemma 5.1. □

Next, we prove the global H^1-estimate of θ.

Lemma 5.2 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume α and γ belong to Case 2, namely (3.13). Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\gamma \nabla\theta(\tau)\|_{L^2}^2\, d\tau \le C(t). \tag{5.10}$$

Proof of Lemma 5.2 Applying ∇ to the third equation of (1.1) and multiplying it by $\nabla\theta$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t)\|_{L^2}^2 + \|\Lambda^\gamma\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\\ &\quad\le \int_{\mathbb{R}^2}\nabla H\nabla\theta\,\mathrm{d}x - \int_{\mathbb{R}^2}\nabla u\cdot\nabla\theta\nabla\theta\,\mathrm{d}x\\ &\quad\le C\|\nabla H\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla\theta\|_{L^{\frac{4}{1+\alpha}}}^2\\ &\quad\le C\|\nabla H\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\Lambda^\alpha\omega\|_{L^2}\|\nabla\theta\|_{L^2}^{2-\frac{1-\alpha}{\gamma}}\|\Lambda^\gamma\nabla\theta\|_{L^2}^{\frac{1-\alpha}{\gamma}}\\ &\quad\le \frac{1}{2}\|\Lambda^\gamma\nabla\theta\|_{L^2}^2 + C\|\nabla H\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\Lambda^\alpha\omega\|_{L^2}^{\frac{2\gamma}{2\gamma+\alpha-1}}\|\nabla\theta\|_{L^2}^2\\ &\quad\le \frac{1}{2}\|\Lambda^\gamma\nabla\theta\|_{L^2}^2 + C\|\nabla H\|_{L^2}\|\nabla\theta\|_{L^2} + C(1+\|\Lambda^\alpha\omega\|_{L^2}^2)\|\nabla\theta\|_{L^2}^2,\end{aligned}$$

where we have used $\gamma \geq 1-\alpha$ in the last line. Gronwall's inequality along with estimate (5.1) leads to

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\gamma \nabla\theta(\tau)\|_{L^2}^2 \, d\tau \leq C.$$

This concludes the proof of Lemma 5.2. □

Lemma 5.3 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume α and γ belong to Case 2, namely (3.13). Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\|v(t)\|_{L^\infty(0,T;L^\infty)} \leq C(t). \tag{5.11}$$

Proof of Lemma 5.3 By (5.1) and (5.10), we have, for any $2 \leq p < \infty$,

$$\begin{aligned}
&\|\nabla\theta(t)\|_{L^2(0,T;L^{\frac{2}{1-\gamma}})} \leq C;\\
&\|u\cdot\nabla v(t)\|_{L^\infty(0,T;L^{\frac{2p}{p+2}})} \leq C;\\
&\|v\cdot\nabla u(t)\|_{L^\infty(0,T;L^{\frac{2p}{p+2}})} \leq C.
\end{aligned}$$

It then follows from (4.16) that

$$\|v(t)\|_{L^\infty(0,T;L^\infty)} \leq C(t).$$

This proves Lemma 5.3. □

With the estimates above at our disposal, we are now ready to give the proof of the second case of Theorem 1.1.

Proof of Theorem 1.1 (Subcase 1) Applying Λ^s with $s > 2$ to (1.1) and then taking the L^2 inner product with $(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s\theta(t)\|_{L^2}^2\Big) + \|\Lambda^{s+\alpha}u\|_{L^2}^2 + \|\Lambda^{s+1}v\|_{L^2}^2 + \|\Lambda^{s+\gamma}\theta\|_{L^2}^2\\
&= -\int_{\mathbb{R}^2}\Big(\Lambda^s\nabla\cdot(v\otimes v)\cdot\Lambda^s u + \Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\Big)\,dx - \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\theta)\cdot\Lambda^s\theta\,dx\\
&\quad - \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx\\
&:= H_1 + H_2 + H_3 + H_4.
\end{aligned} \tag{5.12}$$

By Hölder's inequality, Sobolev's inequality and commutator estimates, we have

$$\begin{aligned}
H_1 &\leq \frac{1}{8}\|\Lambda^{s+1}v\|_{L^2}^2 + C\|v\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2 + C\|\nabla u\|_{H^\alpha}^2(\|v\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2),\\
H_2 &\leq C\|[\Lambda^s, u\cdot\nabla]\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\|\nabla u\|_{L^{\frac{2}{\gamma}}}\|\Lambda^s\theta\|_{L^{\frac{2}{1-\gamma}}}+\|\nabla\theta\|_{L^{\frac{2}{\alpha}}}\|\Lambda^s u\|_{L^{\frac{2}{1-\alpha}}}\right)\|\Lambda^s\theta\|_{L^2}\\
&\le C\left(\|\nabla u\|_{L^{\frac{2}{\gamma}}}\|\Lambda^{s+\gamma}\theta\|_{L^2}+\|\nabla\theta\|_{L^{\frac{2}{\alpha}}}\|\Lambda^{s+\alpha}u\|_{L^2}\right)\|\Lambda^s\theta\|_{L^2}\\
&\le C\left(\|\nabla u\|_{L^2}^{\frac{\alpha+\gamma-1}{\alpha}}\|\Lambda^\alpha\nabla u\|_{L^2}^{\frac{1-\gamma}{\alpha}}\|\Lambda^{s+\gamma}\theta\|_{L^2}\right.\\
&\quad\left.+\|\nabla\theta\|_{L^2}^{\frac{\alpha+\gamma-1}{\gamma}}\|\Lambda^\gamma\nabla\theta\|_{L^2}^{\frac{1-\alpha}{\gamma}}\|\Lambda^{s+\alpha}u\|_{L^2}\right)\|\Lambda^s\theta\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+\gamma}\theta\|_{L^2}^2+\frac{1}{8}\|\Lambda^{s+\alpha}u\|_{L^2}^2+C(\|\nabla u\|_{H^\alpha}^2+\|\nabla\theta\|_{H^\gamma}^2)\|\Lambda^s\theta\|_{L^2}^2,\\
H_3&\le C\|\Lambda^s(uv)\|_{L^2}\|\Lambda^{s+1}v\|_{L^2}\\
&\le C(\|u\|_{L^\infty}\|\Lambda^s v\|_{L^2}+\|v\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^{s+1}v\|_{L^2}\\
&\le \frac{1}{8}\|\Lambda^{s+1}v\|_{L^2}^2+C(\|u\|_{L^\infty}^2+\|v\|_{L^\infty}^2)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2)\\
&\le \frac{1}{8}\|\Lambda^{s+1}v\|_{L^2}^2+C(\|u\|_{H^{1+\alpha}}^2+\|v\|_{L^\infty}^2)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2)
\end{aligned}
\tag{5.13}
$$

and

$$
\begin{aligned}
H_4&\le C\int_{\mathbb{R}^2}|[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u|\,\mathrm{d}x\\
&\le C\|\Lambda^s u\|_{L^{\frac{4}{1+\alpha}}}\|[\Lambda^s,u\cdot\nabla]u\|_{L^{\frac{4}{3-\alpha}}}\\
&\le C\|\Lambda^s u\|_{L^{\frac{4}{1+\alpha}}}\|[\Lambda^s,u_i]\partial_i u\|_{L^{\frac{4}{3-\alpha}}}\\
&\le C\|\Lambda^s u\|_{L^{\frac{4}{1+\alpha}}}\left(\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^{s-1}\partial_i u\|_{L^{\frac{4}{1+\alpha}}}+\|\partial_i u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^s u\|_{L^{\frac{4}{1+\alpha}}}\right)\\
&\le C\|\Lambda^s u\|_{L^{\frac{4}{1+\alpha}}}\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^s u\|_{L^{\frac{4}{1+\alpha}}}\\
&\le C\|\nabla u\|_{H^\alpha}\|\Lambda^s u\|_{L^2}^{\frac{3\alpha-1}{\alpha}}\|\Lambda^{s+\alpha}u\|_{L^2}^{\frac{1-\alpha}{\alpha}}\\
&\le \frac{1}{2}\|\Lambda^{s+\alpha}u\|_{L^2}^2+C\|\nabla u\|_{H^\alpha}^{\frac{2\alpha}{3\alpha-1}}\|\Lambda^s u\|_{L^2}^2\\
&\le \frac{1}{2}\|\Lambda^{s+\alpha}u\|_{L^2}^2+C\left(1+\|\nabla u\|_{H^\alpha}^2\right)\|\Lambda^s u\|_{L^2}^2,
\end{aligned}
$$

where the condition $\alpha\ge\frac{1}{2}$ has been used in the last line. Inserting the preceding estimates in (5.12) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2)+\|\Lambda^{s+\alpha}u\|_{L^2}^2+\|\Lambda^{s+1}v\|_{L^2}^2+\|\Lambda^{s+\gamma}\theta\|_{L^2}^2\\
&\le C\mathcal{D}(t)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2+\|\Lambda^s\theta\|_{L^2}^2),
\end{aligned}
\tag{5.14}
$$

where

$$\mathcal{D}(t)=1+\|v(t)\|_{L^\infty}^2+\|\nabla u(t)\|_{H^\alpha}^2+\|\nabla\theta(t)\|_{H^\gamma}^2.$$

By (3.1), (5.1), (5.5) and (5.11), we have

$$\int_0^T \mathcal{D}(t)\,\mathrm{d}t < \infty.$$

Gronwall's inequality then implies

$$\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{s+\alpha} u(\tau)\|_{L^2}^2 + \|\Lambda^{s+1} v(\tau)\|_{L^2}^2 + \|\Lambda^{s+\gamma}\theta(\tau)\|_{L^2}^2)\,\mathrm{d}\tau < \infty,$$

which is the desired global bound in Theorem 1.1. This proves Theorem 1.1 for the subcase $\gamma = 1-\alpha,\ \ \frac{1}{2} \le \alpha < 1.$ □

5.2 The Subcase : $\alpha = 1, \gamma = 0$

Lemma 5.4 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume $\alpha = 1$ and $\gamma = 0$. Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|v(t)\|_{L^\infty}^2 + \|\theta(t)\|_{L^p}^2 + \int_0^t (\|\nabla\omega(\tau)\|_{L^2}^2 + \|\nabla j(\tau)\|_{L^2}^2 + \|\nabla H(\tau)\|_{L^2}^2)\,d\tau \le C_0(t), \tag{5.15}$$

where C_0 depends only on t and the initial data.

Proof of Lemma 5.4 It follows from (4.3) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2) + \|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\\ &\quad= -\int_{\mathbb{R}^2}\Big((v\cdot\nabla)j\omega + 2hj\omega + (v\cdot\nabla)\omega j + h\omega j\Big)\,\mathrm{d}x - \int_{\mathbb{R}^2} Q(\nabla u, \nabla v)H\,\mathrm{d}x\\ &\qquad+ \int_{\mathbb{R}^2}(H+\theta)H\,\mathrm{d}x + \int_{\mathbb{R}^2}(v_1\partial_2 h - v_2\partial_1 h)\omega\,\mathrm{d}x\\ &\quad:= J_1 + J_2 + J_3 + J_4.\end{aligned} \tag{5.16}$$

The terms J_1, J_3 can be easily estimated as before

$$J_1 \le \epsilon\|\nabla j\|_{L^2}^2 + \epsilon\|\nabla H\|_{L^2}^2 + C_\epsilon(\|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2)\|\omega\|_{L^2}^2 \tag{5.17}$$

$$\quad + C_\epsilon\,(\|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^2)\|\theta\|_{L^2}^2, \tag{5.18}$$

$$J_3 \le C\|H\|_{L^2}^2 + C\|\theta\|_{L^2}^2. \tag{5.19}$$

The Gagliardo–Nirenberg inequality implies

$$
\begin{aligned}
J_2 &\le C\int_{\mathbb{R}^2} |\nabla u|\,|\nabla v|\,|H|\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^2} |\nabla u|\,(|\mathcal{R}_1 H| + |\mathcal{R}_1\theta| + |\mathcal{R}_2 j|)\,|H|\,\mathrm{d}x\\
&\le C\|\nabla u\|_{L^2}(\|H\|_{L^4}\|j\|_{L^4} + \|H\|_{L^4}^2) + C\|\nabla u\|_{L^4}\|\theta\|_{L^2}\|H\|_{L^4}\\
&\le C\|\omega\|_{L^2}(\|H\|_{L^2}\|\nabla H\|_{L^2} + \|j\|_{L^2}\|\nabla j\|_{L^2})\\
&\quad + C\|\omega\|_{L^2}^{\frac12}\|\nabla\omega\|_{L^2}^{\frac12}\|\theta\|_{L^2}\|H\|_{L^2}^{\frac12}\|\nabla H\|_{L^2}^{\frac12}\\
&\le \epsilon\|\nabla j\|_{L^2}^2 + \epsilon\|\nabla H\|_{L^2}^2 + \epsilon\|\nabla\omega\|_{L^2}^2 + C_\epsilon(\|\omega\|_{L^2}^2 + \|\theta\|_{L^2}^2)(\|H\|_{L^2}^2 + \|j\|_{L^2}^2).
\end{aligned}
\tag{5.20}
$$

We further split J_4 into two terms,

$$
J_4 = \int_{\mathbb{R}^2} hj\omega\,\mathrm{d}x - \int_{\mathbb{R}^2} hv\cdot\nabla^\perp\omega\,\mathrm{d}x := J_{41} + J_{42}.
$$

J_{41} admits the same bound as J_1, while J_{42} can be bounded by

$$
\begin{aligned}
J_{42} &\le \|v\|_{L^\infty}\|h\|_{L^2}\|\nabla\omega\|_{L^2}\\
&\le \|v\|_{L^\infty}(\|H\|_{L^2} + \|\theta\|_{L^2})\|\nabla\omega\|_{L^2}\\
&\le \epsilon\|\nabla\omega\|_{L^2}^2 + C_\epsilon\|v\|_{L^\infty}^2(\|H\|_{L^2}^2 + \|\theta\|_{L^2}^2).
\end{aligned}
$$

Putting the estimates of $J_1 - J_4$ into (5.16) and taking ϵ small enough, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|H(t)\|_{L^2}^2) + \|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla H\|_{L^2}^2\\
&\quad\le C(1 + \|\nabla v\|_{L^2}^2 + \|\theta\|_{L^2}^{\frac{2}{\alpha}} + \|\omega\|_{L^2}^2 + \|v\|_{L^\infty}^2)(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \|H\|_{L^2}^2).
\end{aligned}
\tag{5.21}
$$

We still need to bound $\|v\|_{L^\infty}$. As we shall see later in the proof, we need a bound for $\|v\|_{L^p}$ with $2 < p < \infty$, which depends on $\|\theta\|_{L^p}$. As a consequence, we need to include the estimate of $\|\theta(t)\|_{L^p}$. Using (3.8) and (3.10), we have, for any $2 < p < \infty$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{L^p}^2 &\le C\|\nabla v\|_{L^p}\|\theta\|_{L^p}\\
&\le C\|\nabla v\|_{L^p}^2 + C\|\theta\|_{L^p}^2\\
&\le C(\|H\|_{L^p}^2 + \|j\|_{L^p}^2 + \|\theta\|_{L^p}^2)\\
&\le C(\|H\|_{L^2}^{\frac4p}\|\nabla H\|_{L^2}^{\frac{2p-4}{p}} + \|j\|_{L^2}^{\frac4p}\|\nabla j\|_{L^2}^{\frac{2p-4}{p}} + \|\theta\|_{L^p}^2)\\
&\le \frac12\|\nabla j\|_{L^2}^2 + \frac12\|\nabla H\|_{L^2}^2 + C(\|j\|_{L^2}^2 + \|H\|_{L^2}^2 + \|\theta\|_{L^p}^2).
\end{aligned}
\tag{5.22}
$$

Summing up (5.21) and (5.22) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+\|H(t)\|_{L^2}^2+\|\theta(t)\|_{L^p}^2\right)+\|\nabla\omega\|_{L^2}^2+\|\nabla j\|_{L^2}^2+\|\nabla H\|_{L^2}^2\\&\quad\le C\left(1+\|\nabla v\|_{L^2}^2+\|\theta\|_{L^2}^{\frac{2}{\alpha}}+\|\omega\|_{L^2}^2+\|v\|_{L^\infty}^2\right)\left(\|\omega\|_{L^2}^2+\|j\|_{L^2}^2+\|H\|_{L^2}^2+\|\theta\|_{L^p}^2\right).\end{aligned}$$

Writing

$$\begin{aligned}X_1(t)&:=\|\omega(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2+\|H(t)\|_{L^2}^2+\|\theta(t)\|_{L^p}^2,\\Y_1(t)&:=\|\nabla\omega(t)\|_{L^2}^2+\|\nabla j(t)\|_{L^2}^2+\|\nabla H(t)\|_{L^2}^2,\\G_1(t)&:=\|\nabla v(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^{\frac{2}{\alpha}}+\|\omega(t)\|_{L^2}^2,\end{aligned}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}X_1(t)+Y_1(t)\le CG_1(t)X_1(t)+C\|v\|_{L^\infty}^2X_1(t).$$

Invoking the following logarithmic Sobolev embedding inequality (Kozono et al. 2002), for $p>2$,

$$\|f\|_{L^\infty(\mathbb{R}^2)}\le C\left(1+\|f\|_{L^2(\mathbb{R}^2)}+\|f\|_{\dot{B}^0_{\infty,2}(\mathbb{R}^2)}\sqrt{\ln(e+\|\nabla f\|_{L^p(\mathbb{R}^2)})}\right),\tag{5.23}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}X_1(t)+Y_1(t)\le CG_1(t)X_1(t)+C(1+\|v(t)\|_{\dot{B}^0_{\infty,2}}^2\ln(e+\|\nabla v(t)\|_{L^p})X_1(t).$$

Furthermore,

$$\begin{aligned}\|\nabla v(t)\|_{L^p}&\le C(\|H\|_{L^p}+\|j\|_{L^p}+\|\theta\|_{L^p})&(5.24)\\&\le C\left(\|H\|_{L^2}^{\frac{2}{p}}\|\nabla H\|_{L^2}^{\frac{p-2}{p}}+\|j\|_{L^2}^{\frac{2}{p}}\|\nabla j\|_{L^2}^{\frac{p-2}{p}}+\|\theta\|_{L^p}\right)&(5.25)\\&\le C(e+X_1(t)+Y_1(t))^{\frac{1}{2}}&(5.26)\end{aligned}$$

and

$$\begin{aligned}\|v(t)\|_{\dot{B}^0_{\infty,2}}^2&\le C\|v(t)\|_{L^2}^{\frac{p-2}{p-1}}\|\nabla v(t)\|_{L^p}^{\frac{p}{p-1}}&(5.27)\\&\le C\|\nabla v(t)\|_{L^p}^{\frac{p}{p-1}}\le C(e+X_1(t)+Y_1(t))^{\frac{p}{2(p-1)}}.&(5.28)\end{aligned}$$

Lemma 2.2 then implies, any $2<p<\infty$,

$$X_1(t)+\int_0^t Y_1(s)\,\mathrm{d}s\le C,$$

which is (5.15). This completes the proof of Lemma 5.4. □

The following lemma provides a bound for $\|\theta\|_{L^q}$.

Lemma 5.5 *Assume (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Assume $\alpha = 1$ and $\gamma = 0$. Then, the corresponding smooth solution (u, v, θ) of (1.1) obeys the global bound, for any $t > 0$,*

$$\sup_{q\geq 2} \frac{\|\theta(t)\|_{L^q}}{\sqrt{q}} \leq C_0(t), \tag{5.29}$$

where $C_0(t)$ depends only on t and the initial data.

Proof of Lemma 5.5 Multiplying the equation of θ,

$$\partial_t\theta + (u \cdot \nabla)\theta + \theta = -H$$

by $\theta|\theta|^{q-2}$ yields

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{L^q}^q + \|\theta\|_{L^q}^q \leq \|H\|_{L^q}\|\theta\|_{L^q}^{q-1}.$$

Especially,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{L^q} \leq \|H\|_{L^q}.$$

Recall the fact that

$$\|H\|_{L^q} \leq C\sqrt{q}\|H\|_{H^1}$$

with a constant C independent of q. We thus have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{L^q} \leq C\sqrt{q}\|H\|_{H^1}.$$

Integrating in time and using bound (5.15), we obtain (5.29) immediately. □

Proof of Theorem 1.1 for $\alpha = 1$ and $\gamma = 0$. We prove the global H^s-estimate. To do so, we first recall the following logarithmic Sobolev interpolation inequalities

$$\|f\|_{L^\infty} \leq C + C\Big(\sup_{q\geq 2}\frac{\|f\|_{L^q}}{\sqrt{q}}\Big)\sqrt{\ln\Big(e + \|f\|_{H^\sigma}\Big)}, \qquad \forall\sigma > 1, \tag{5.30}$$

$$\|\nabla u\|_{L^\infty} \leq C + C\|\nabla\omega\|_{L^2}\sqrt{\ln(e + \|u\|_{H^s})}, \qquad \forall s > 2. \tag{5.31}$$

We apply Λ^s with $s > 2$ to (1.1) and take the L^2 inner product with $(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s \theta(t)\|_{L^2}^2)+\|\Lambda^{s+1}u\|_{L^2}^2+\|\Lambda^{s+1}v\|_{L^2}^2\\
&\quad=-\int_{\mathbb{R}^2}\Big(\Lambda^s\nabla\cdot(v\otimes v)\cdot\Lambda^s u+\Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\Big)\,\mathrm{d}x-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla\theta)\cdot\Lambda^s\theta\,\mathrm{d}x\\
&\qquad-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\,\mathrm{d}x-\int_{\mathbb{R}^2}\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,\mathrm{d}x\\
&\quad:=H_1+H_2+H_3+H_4.
\end{aligned}$$

The terms H_1 through H_4 can be bounded as follows.

$$\begin{aligned}
H_1&\le\frac{1}{8}\|\Lambda^{s+1}v\|_{L^2}^2+C\|v\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2+C\|\nabla u\|_{L^\infty}^2(\|v\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2),\\
H_2&\le C\|[\Lambda^s\nabla\cdot,u]\theta\|_{L^2}\|\Lambda^s\theta\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}+\|\theta\|_{L^\infty}\|\Lambda^{s+1}u\|_{L^2})\|\Lambda^s\theta\|_{L^2}\\
&\le\frac{1}{8}\|\Lambda^{s+1}u\|_{L^2}^2+C(\|\nabla u\|_{L^\infty}+\|\theta\|_{L^\infty}^2)\|\Lambda^s\theta\|_{L^2}^2,\\
H_3&\le C\|\Lambda^s(uv)\|_{L^2}\|\Lambda^{s+1}v\|_{L^2}\\
&\le C(\|u\|_{L^\infty}\|\Lambda^s v\|_{L^2}+\|v\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^{s+1}v\|_{L^2}\\
&\le\frac{1}{8}\|\Lambda^{s+1}v\|_{L^2}^2+C(\|u\|_{L^\infty}^2+\|v\|_{L^\infty}^2)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s v\|_{L^2}^2)
\end{aligned}$$

and

$$\begin{aligned}
H_4&\le C\int_{\mathbb{R}^2}|[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u|\,\mathrm{d}x\\
&\le C\|[\Lambda^s,u\cdot\nabla]u\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\le C\|[\Lambda^s,u_i]\partial_i u\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\le C(\|\nabla u_i\|_{L^\infty}\|\Lambda^{s-1}\partial_i u\|_{L^2}+\|\partial_i u\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2.
\end{aligned}$$

Collecting all the estimates above implies that

$$Z(t):=\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s v(t)\|_{L^2}^2+\|\Lambda^s\theta(t)\|_{L^2}^2$$

satisfies the inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}Z(t)+\|\Lambda^{s+1}u\|_{L^2}^2+\|\Lambda^{s+1}v\|_{L^2}^2\\
&\quad\le C(1+\|\nabla u\|_{L^\infty}^2+\|\theta\|_{L^\infty}^2)Z(t)\\
&\quad\le C\left(1+\|\nabla\omega\|_{L^2}^2+\left(\sup_{q\ge2}\frac{\|\theta\|_{L^q}}{\sqrt{q}}\right)^2\right)\ln\big(e+Z(t)\big)\,Z(t).
\end{aligned}$$

Gronwall's inequality ensures that

$$\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{s+1} u(\tau)\|_{L^2}^2 + \|\Lambda^{s+1} v(\tau)\|_{L^2}^2)\, \mathrm{d}\tau < \infty.$$

This completes the proof of (1.1) for the case $\alpha = 1$ and $\gamma = 0$. □

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Appendix A: Local Well-Posedness Theory on (1.1)

For the sake of completeness, this Appendix presents the local existence and uniqueness result for (1.1) with initial data $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$. More precisely, in this Appendix, we prove the following local well-posedness result.

Proposition A.1 *Let $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$ and $\nabla \cdot u_0 = 0$. Then, there exists a positive time T depending on $\|u_0\|_{H^s}$, $\|v_0\|_{H^s}$ and $\|\theta_0\|_{H^s}$ such that (1.1) admits a unique solution $(u, v, \theta) \in C([0, T]; H^s(\mathbb{R}^2))$.*

We remark that we only consider the case $s > 2$. Actually, $s > 2$ can be weakened to $s > f(\alpha, \gamma)$ with the function $f(\alpha, \gamma) \leq 2$. However, to make the idea clear, we assume this condition throughout this Appendix. The proof of Proposition A.1 can be performed by the method similar to Chapter 3 in Majda and Bertozzi (2002). To prove Proposition A.1, the main step is to approximate (1.1) in order to easily produce a family of global smooth solutions. In order to do this, we may, for instance, make use of the Friedrichs method. Now we define the spectral cutoff as follows

$$\widehat{\mathcal{J}_N f}(\xi) = \chi_{B(0,N)}(\xi)\widehat{f}(\xi),$$

where $N > 0$, $B(0, N) = \{\xi \in \mathbb{R}^2 | \, |\xi| \leq N\}$ and $\chi_{B(0,N)}$ is the characteristic function on $B(0, N)$. Also we define

$$L_N^2 \triangleq \{f \in L^2(\mathbb{R}^2) | \ \mathrm{supp}\ \widehat{f} \subset B(0, N)\}.$$

Proof of Proposition A.1 The first step is to consider the following approximate system of (1.1),

$$\begin{cases} \partial_t u^N + \mathcal{P}\mathcal{J}_N((\mathcal{J}_N u^N \cdot \nabla)\mathcal{J}_N u^N) + \nu\Lambda^{2\alpha}\mathcal{J}_N u^N + \mathcal{P}\mathcal{J}_N \nabla \cdot (\mathcal{J}_N v^N \otimes \mathcal{J}_N v^N) = 0, \\ \partial_t v^N + \mathcal{J}_N(\mathcal{J}_N u^N \cdot \nabla)\mathcal{J}_N v^N - \mu\Delta\mathcal{J}_N v^N + \nabla\mathcal{J}_N\theta^N + \mathcal{J}_N(\mathcal{J}_N v^N \cdot \nabla)\mathcal{J}_N u^N = 0, \\ \partial_t\theta^N + \mathcal{J}_N(\mathcal{J}_N u^N \cdot \nabla)\mathcal{J}_N\theta^N + \eta\Lambda^{2\gamma}\mathcal{J}_N\theta^N + \nabla \cdot \mathcal{J}_N v^N = 0, \\ \nabla \cdot u^N = 0, \\ u^N(x,0) = \mathcal{J}_N u_0(x), \quad v^N(x,0) = \mathcal{J}_N v_0(x), \quad \theta^N(x,0) = \mathcal{J}_N\theta_0(x), \end{cases} \tag{A.1}$$

where $\mathcal{P}$ denotes the standard projection onto divergence-free vector fields. Taking advantage of the Cauchy–Lipschitz theorem (Picard's Theorem, see Majda and Bertozzi 2002), we can find that for any fixed N, there exists a unique local solution (u^N, v^N, θ^N) on $[0, T_N)$ in the functional setting L^2_N with $T_N = T(N, u_0, v_0, \theta_0)$. Due to $\mathcal{J}_N^2 = \mathcal{J}_N$, $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P}\mathcal{J}_N = \mathcal{J}_N\mathcal{P}$, we find that $(\mathcal{J}_N u^N, \ \mathcal{J}_N v^N, \ \mathcal{J}_N\theta^N)$ is also a solution to (A.1) with the same initial datum. Thanks to the uniqueness, we thus find

$$\mathcal{J}_N u^N = u^N, \quad \mathcal{J}_N v^N = v^N, \quad \mathcal{J}_N\theta^N = \theta^N.$$

Consequently, approximate system (A.1) reduces to

$$\begin{cases} \partial_t u^N + \mathcal{P}\mathcal{J}_N((u^N \cdot \nabla)u^N) + \nu\Lambda^{2\alpha}u^N + \mathcal{P}\mathcal{J}_N \nabla \cdot (v^N \otimes v^N) = 0, \\ \partial_t v^N + \mathcal{J}_N(u^N \cdot \nabla)v^N - \mu\Delta v^N + \nabla\theta^N + \mathcal{J}_N(v^N \cdot \nabla)u^N = 0, \\ \partial_t\theta^N + \mathcal{J}_N(u^N \cdot \nabla)\theta^N + \eta\Lambda^{2\gamma}\theta^N + \nabla \cdot v^N = 0, \\ \nabla \cdot u^N = 0, \\ u^N(x,0) = \mathcal{J}_N u_0(x), \quad v^N(x,0) = \mathcal{J}_N v_0(x), \quad \theta^N(x,0) = \mathcal{J}_N\theta_0(x). \end{cases} \tag{A.2}$$

A basic energy estimate implies (u^N, v^N, θ^N) of (A.2) satisfies

$$\begin{aligned} &\|u^N(t)\|_{L^2}^2 + \|v^N(t)\|_{L^2}^2 + \|\theta^N(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^\alpha u^N\|_{L^2}^2 + \|\nabla v^N\|_{L^2}^2 + \|\Lambda^\gamma\theta^N\|_{L^2}^2)(\tau)\,\mathrm{d}\tau \\ &\quad \le \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \end{aligned}$$

Therefore, the local solution can be extended into a global one, by the standard Picard extension theorem (see, e.g., Majda and Bertozzi 2002). Moreover, by direct H^s-estimates, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u^N(t)\|_{H^s}^2 + \|v^N(t)\|_{H^s}^2 + \|\theta^N(t)\|_{H^s}^2\right) + \|\Lambda^\alpha u^N\|_{H^s}^2 + \|\nabla v^N\|_{H^s}^2 + \|\Lambda^\gamma\theta^N\|_{H^s}^2 \\ &\quad \le C(\|\nabla u^N\|_{L^\infty} + \|\nabla v^N\|_{L^\infty} + \|\nabla\theta^N\|_{L^\infty} + \|v^N\|_{L^\infty}^2)(\|u^N\|_{H^s}^2 + \|v^N\|_{H^s}^2 + \|\theta^N\|_{H^s}^2) \\ &\quad \le C(\|u^N\|_{H^s} + \|v^N\|_{H^s} + \|\theta^N\|_{H^s} + \|v^N\|_{H^s}^2)(\|u^N\|_{H^s}^2 + \|v^N\|_{H^s}^2 + \|\theta^N\|_{H^s}^2) \\ &\quad \le C(\|u^N\|_{H^s}^2 + \|v^N\|_{H^s}^2 + \|\theta^N\|_{H^s}^2)^2, \end{aligned} \tag{A.3}$$

where here and in what follows we use the fact that, in 2D case

$$\|\nabla f\|_{L^\infty} \le C\|f\|_{H^s}, \quad s > 2.$$

Notice that in (A.3) we assume that $\|u^N\|_{H^s} + \|v^N\|_{H^s} + \|\theta^N\|_{H^s} \geq 1$, otherwise we replace $\|u^N\|_{H^s} + \|v^N\|_{H^s} + \|\theta^N\|_{H^s}$ by $1 + \|u^N\|_{H^s} + \|v^N\|_{H^s} + \|\theta^N\|_{H^s}$. For the convenience of notation, we denote

$$X(t) \triangleq \|u^N(t)\|_{H^s}^2 + \|v^N(t)\|_{H^s}^2 + \|\theta^N(t)\|_{H^s}^2.$$

Consequently, (A.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) \leq \widetilde{C} X(t)^2,$$

where $\widetilde{C} > 0$ is an absolute constant. Standard calculations show that for all N

$$\sup_{0\leq t\leq T} (\|u^N(t)\|_{H^s}^2 + \|v^N(t)\|_{H^s}^2 + \|\theta^N(t)\|_{H^s}^2) \leq \frac{\|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2}{1 - \widetilde{C}T(\|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2)}.$$

Therefore, the family (u^N, v^N, θ^N) is uniformly bounded in $C([0, T]; H^s)$ with $s > 2$, provided that

$$T < \frac{1}{\widetilde{C}(\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2)}.$$

Thus, it is not hard to see that

$$\partial_t u^N,\ \partial_t v^N,\ \partial_t \theta^N \in L_t^\infty([0, T]);\ H_x^{-\sigma}(\mathbb{R}^2) \quad \text{for some } \sigma \geq 2.$$

Since the embedding $L^2 \hookrightarrow H^{-\sigma}$ is locally compact, the well-known Aubin–Lions argument allows us to conclude that, up to extraction, subsequence $(u^N, v^N, \theta^N)_{N\in\mathbb{N}}$ satisfies

$$\|u^N - u^{N'}\|_{L^2}, \quad \|v^N - v^{N'}\|_{L^2}, \quad \|\theta^N - \theta^{N'}\|_{L^2} \to 0, \quad as \quad N,\ N' \to \infty.$$

Thanks to the interpolation ($\|f\|_{H^{s'}} \leq C\|f\|_{L^2}^{1-\frac{s'}{s}} \|f\|_{H^s}^{\frac{s'}{s}}$ for any $s' < s$), we deduce that

$$\|u^N - u^{N'}\|_{H^{s'}}, \quad \|v^N - v^{N'}\|_{H^{s'}}, \quad \|\theta^N - \theta^{N'}\|_{H^{s'}} \to 0, \quad as \quad N,\ N' \to \infty,$$

which imply that we have strong convergence limit $(u, v, \theta) \in C([0, T]; H^{s'})$ for any $s' < s$. Therefore, this is enough for us to show that up to extraction, sequence $(u^N, v^N, \theta^N)_{N\in\mathbb{N}}$ has a limit $(u,\ v,\ \theta)$ satisfying

$$\begin{cases} \partial_t u + \mathcal{P}((u\cdot\nabla)u) + \nu\Lambda^{2\alpha}u + \mathcal{P}\nabla\cdot(v\otimes v) = 0, \\ \partial_t v + (u\cdot\nabla)v - \mu\Delta v + \nabla\theta + (v\cdot\nabla)u = 0, \\ \partial_t\theta + (u\cdot\nabla)\theta + \eta\Lambda^{2\gamma}\theta + \nabla\cdot v = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{A.4}$$

Moreover, one may show that $(u, v, \theta) \in L^\infty([0,T]; H^s(\mathbb{R}^2))$. Finally, we claim that $(u, v, \theta) \in C([0,T]; H^s(\mathbb{R}^2))$; namely, (u, v, θ) is strongly continuous in $H^s(\mathbb{R}^2)$ in time. It suffices to consider $u \in C([0,T]; H^s(\mathbb{R}^2)$ as the same fashion can be applied to v and θ to obtain the desired result. From the above argument, we first have

$$\sup_{0\le t\le T}(\|u\|_{H^s} + \|v\|_{H^s} + \|\theta\|_{H^s}) < \infty.$$

By the equivalent norm, it yields

$$\|u(t_1) - u(t_2)\|_{H^s} = \left\{\left(\sum_{k<N} + \sum_{k\ge N}\right)(2^{ks}\|\Delta_k u(t_1) - \Delta_k u(t_2)\|_{L^2})^2\right\}^{\frac{1}{2}}, \tag{A.5}$$

where the Fourier localization operator Δ_k is defined through the Littlewood–Paley decomposition (see Chapter 2 in Bahouri et al. 2011 for details). Let $\varepsilon > 0$ be arbitrarily small. Due to $u \in L^\infty([0,T]; H^s(\mathbb{R}^2))$, there exists an integer $M = M(\varepsilon) > 0$ such that

$$\left\{\sum_{k\ge M}(2^{ks}\|\Delta_k u(t_1) - \Delta_k u(t_2)\|_{L^2})^2\right\}^{\frac{1}{2}} < \frac{\varepsilon}{2}. \tag{A.6}$$

Recalling system (A.4)$_1$, we obtain

$$\begin{aligned} \Delta_k u(t_1) - \Delta_k u(t_2) &= \int_{t_1}^{t_2} \frac{d}{d\tau}\Delta_k u(\tau)\,d\tau \\ &= -\int_{t_1}^{t_2} \Delta_k\mathcal{P}[\nabla\cdot(v\otimes v) + (u\cdot\nabla)u + \nu\Lambda^{2\alpha}u](\tau)\,d\tau. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\sum_{k<M} 2^{2ks}\|\Delta_k u(t_1) - \Delta_k u(t_2)\|_{L^2}^2 \\ &\quad = \sum_{k<M} 2^{2ks}\left(\left\|\int_{t_1}^{t_2}\Delta_k\mathcal{P}[\nabla\cdot(v\otimes v) + (u\cdot\nabla)u + \nu\Lambda^{2\alpha}u](\tau)\,d\tau\right\|_{L^2}\right)^2 \\ &\quad \le \sum_{k<M} 2^{2ks}\left(\int_{t_1}^{t_2}\|\Delta_k[\nabla\cdot(v\otimes v) + (u\cdot\nabla)u + \nu\Lambda^{2\alpha}u]\|_{L^2}(\tau)\,d\tau\right)^2 \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k<M} 2^{2ks}\Big(\int_{t_1}^{t_2}[\|\Delta_k(\nabla\cdot(v\otimes v))\|_{L^2}+\|\|\Delta_k(u\cdot\nabla u)\|_{L^2}+\nu\|\|\Delta_k\Lambda^{2\alpha}u\|_{L^2}](\tau)\,\mathrm{d}\tau\Big)^2\\
&=\sum_{k<M} 2^{4k}\Big(\int_{t_1}^{t_2}[2^{k(s-2)}\|\Delta_k\nabla\cdot(v\otimes v)\|_{L^2}+2^{k(s-2)}\|\Delta_k\nabla\cdot(u\otimes u)\|_{L^2}](\tau)\,\mathrm{d}\tau\Big)^2\\
&\quad+\sum_{k<M} 2^{4k}\Big(\int_{t_1}^{t_2}\nu 2^{k(s-2+2\alpha)}\|\Delta_k u(\tau)\|_{L^2}\,\mathrm{d}\tau\Big)^2\\
&\le C\sum_{k<M} 2^{4k}\Big(\|vv\|^2_{L^\infty_t H^s}|t_1-t_2|^2+\|uu\|^2_{L^\infty_t H^s}|t_1-t_2|^2+\|u\|^2_{L^\infty_t H^s}|t_1-t_2|^2\Big)\\
&\le C\sum_{k<M} 2^{4k}|t_1-t_2|^2\Big(\|v\|^2_{L^\infty_t L^\infty}\|v\|^2_{L^\infty_t H^s}+\|u\|^2_{L^\infty_t L^\infty}\|u\|^2_{L^\infty_t H^s}+\nu\|u\|^2_{L^\infty_t H^s}\Big)\\
&\le C2^{4M}|t_1-t_2|^2\Big(\|v\|^4_{L^\infty_t H^s}+\|u\|^4_{L^\infty_t H^s}+\nu\|u\|^2_{L^\infty_t H^s}\Big).
\end{aligned}
$$

Thus, the following holds true

$$
\Big\{\sum_{k<M}(2^{ks}\|\Delta_k u(t_1)-\Delta_k u(t_2)\|_{L^2})^2\Big\}^{\frac{1}{2}}<\frac{\varepsilon}{2} \tag{A.7}
$$

provided $|t_1-t_2|$ small enough. Combining (A.5), (A.6) with (A.7) implies $u\in C([0,T];H^s(\mathbb{R}^2))$. The uniqueness can be easily obtained since (u,v,θ) are all in Lipschitz space. Therefore, the proof of Proposition A.1 is completed. □

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