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Calculus of Variations

Stability and exponential decay for the 2D anisotropic Boussinesq equations with horizontal dissipation

Boqing Dong[1] · Jiahong Wu[2] · Xiaojing Xu[3] · Ning Zhu[4]

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Abstract

The hydrostatic equilibrium is a prominent topic in fluid dynamics and astrophysics. Understanding the stability of perturbations near the hydrostatic equilibrium of the Boussinesq system helps gain insight into certain weather phenomena. The 2D Boussinesq system focused here is anisotropic and involves only horizontal dissipation and horizontal thermal diffusion. Due to the lack of the vertical dissipation, the stability and precise large-time behavior problem is difficult. When the spatial domain is $\mathbb{R}^2$, the stability problem in a Sobolev setting remains open. When the spatial domain is $\mathbb{T} \times \mathbb{R}$, this paper solves the stability problem and specifies the precise large-time behavior of the perturbation. By decomposing the velocity u and temperature θ into the horizontal average $(\bar{u}, \bar{\theta})$ and the corresponding oscillation $(\widetilde{u}, \widetilde{\theta})$, and deriving various anisotropic inequalities, we are able to establish the global stability in the Sobolev space H^2. In addition, we prove that the oscillation $(\widetilde{u}, \widetilde{\theta})$ decays exponentially to zero in H^1 and (u, θ) converges to $(\bar{u}, \bar{\theta})$. This result reflects the stratification phenomenon of buoyancy-driven fluids.

Keywords Boussinesq equations · Partial dissipation · Stability · Decay

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✉ Ning Zhu
mathzhu1@163.com

Boqing Dong
bqdong@szu.edu.cn

Jiahong Wu
jiahong.wu@okstate.edu

Xiaojing Xu
xjxu@bnu.edu.cn

1 College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, People's Republic of China

2 Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

3 Laboratory of Mathematics and Complex Systems, Ministry of Education, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China

4 School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

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1 Introduction

The goal of this paper is to understand the stability and large-time behavior problem on perturbations near the hydrostatic equilibrium of buoyancy-driven fluids. Being capable of capturing the key features of buoyancy-driven fluids such as stratification, the Boussinesq equations have become the most frequently used models for these circumstances (see, e.s., [35,39]). The Boussinesq system concerned here assumes the form

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu \partial_{11} u + \Theta e_2, \\ \partial_t \Theta + u \cdot \nabla \Theta = \kappa \partial_{11} \Theta, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ denotes the velocity field, P the pressure, Θ the temperature, $e_2 = (0, 1)$ (the unit vector in the vertical direction), and $\nu > 0$ and $\kappa > 0$ are the viscosity and the thermal diffusivity, respectively. (1.1) involves only horizontal dissipation and horizontal thermal diffusion, and governs the motion of anisotropic fluids when the corresponding vertical dissipation and thermal diffusion are negligible (see, e.g., [39]).

The Boussinesq systems have attracted considerable interests recently due to their broad physical applications and mathematical significance. The Boussinesq systems are the most frequently used models for buoyancy-driven fluids such as many large-scale geophysical flows and the Rayleigh–Bénard convection (see, e.g., [13,17,35,39]). The Boussinesq equations are also mathematically important. They share many similarities with the 3D Navier–Stokes and the Euler equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier–Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations can be identified as the Euler equations for the 3D axisymmetric swirling flows [36].

Many efforts have been devoted to understanding two fundamental problems concerning the Boussinesq systems. The first is the global existence and regularity problem. Substantial progress has been made on various Boussinesq systems, especially those with only partial or fractional dissipation (see, e.g., [1–4,6,8–12,14–16,20–34,37,38,41,42,47–57]). The second is the stability problem on perturbations near several physically relevant steady states. The investigations on the stability problem are relatively more recent. One particular important steady state is the hydrostatic equilibrium. Mathematically the hydrostatic equilibrium refers to the stationary solution $(u_{he}, \Theta_{he}, P_{he})$ with

$$u_{he} = 0, \quad \Theta_{he} = x_2, \quad P_{he} = \frac{1}{2} x_2^2.$$

The hydrostatic equilibrium is one of the most prominent topics in fluid dynamics, atmospherics and astrophysics. In fact, our atmosphere is mostly in the hydrostatic equilibrium with the upward pressure-gradient force balanced out by the downward gravity. The work of Doering, Wu, Zhao and Zheng [19] initiated the rigorous study on the stability problem near the hydrostatic equilibrium of the 2D Boussinesq equations with only velocity dissipation. A followup work of Tao, Wu, Zhao and Zheng establishes the large-time behavior and the eventual temperature profile [44]. The paper of Castro, Córdoba and Lear successfully established the stability and large time behavior on the 2D Boussinesq equations with velocity damping instead of dissipation [7]. There are other more recent work [40,46,52,58]. Another important steady state is the shear flow. Linear stability results on the shear flow for several partially dissipated Boussinesq systems are obtained in [43] and [58] while the nonlinear stability problem on the shear flow of the 2D Boussinesq equations with only vertical dissipation was solved by [18].

The goal of this paper is to assess the stability and the precise large-time behavior of perturbations near the hydrostatic equilibrium. To understand the stability problem, we write the equations for the perturbation (u, p, θ) where

$$p = P - \frac{1}{2}x_2^2 \quad \text{and} \quad \theta = \Theta - x_2.$$

It is easy to check that (u, p, θ) satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu\, \partial_{11} u + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \kappa\, \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.2}$$

We have observed a new phenomenon on (1.2). It appears that the type of the spatial domain plays a crucial role in the resolution of the stability problem concerned here.

When the spatial domain is the whole plane $\mathbb{R}^2$, the stability problem remains an open problem. Given any sufficiently smooth initial data $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$, (1.2) does admit a unique global solution. But the solution could potentially grow rather rapidly in time. In fact, the best upper bounds one could obtain on $\|u(t)\|_{H^1}$ and $\|\theta(t)\|_{H^1}$ grow algebraically in time. The only way to possibly establish upper bounds that are uniform in time is to combine the equation of ∇u and that of $\nabla\theta$, or equivalently the equation of ω with $\nabla\theta$, where $\omega = \nabla \times u$ is the vorticity. When we combine the equations of ω and $\nabla\theta$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu\, \partial_{11}\omega + \partial_1 \theta, \\ \partial_t \nabla\theta + u \cdot \nabla(\nabla\theta) + \nabla u_2 = \kappa\, \nabla\partial_{11}\theta - \nabla u \cdot \nabla\theta \end{cases} \tag{1.3}$$

and estimate $\|\omega\|_{L^2}$ and $\|\nabla\theta\|_{L^2}$ simultaneously, we can eliminate the term from the buoyancy, namely $\partial_1\theta$. In fact, a simple energy estimate on (1.3) yields, after suitable integration by parts,

$$\begin{aligned} &\frac{d}{dt}\left(\|\omega(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2\right) + 2\nu\, \|\partial_1\omega(t)\|_{L^2}^2 + 2\kappa\, \|\partial_1\nabla\theta(t)\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^2} \nabla\theta \cdot \nabla u \cdot \nabla\theta\, dx. \end{aligned} \tag{1.4}$$

The difficulty is how to obtain a suitable upper bound on the term on the right-hand side of (1.4). To make full use of the anisotropic dissipation, we naturally divide this term further into four component terms

$$\begin{aligned} -\int_{\mathbb{R}^2} \nabla\theta \cdot \nabla u \cdot \nabla\theta\, dx = &-\int_{\mathbb{R}^2} \partial_1 u_1 (\partial_1\theta)^2\, dx - \int_{\mathbb{R}^2} \partial_1 u_2\, \partial_1\theta\, \partial_2\theta\, dx \\ &- \int_{\mathbb{R}^2} \partial_2 u_1\, \partial_1\theta\, \partial_2\theta\, dx - \int_{\mathbb{R}^2} \partial_2 u_2 (\partial_2\theta)^2\, dx. \end{aligned} \tag{1.5}$$

Due to the lack of dissipation or thermal diffusion in the vertical direction, the last two terms on (1.5) prevents us from bounding them suitably. This is one of the difficulties that keep the stability problem on (1.2) open when the spatial domain is the whole plane $\mathbb{R}^2$.

When the spatial domain is

$$\Omega = \mathbb{T} \times \mathbb{R}$$

with $\mathbb{T} = [0, 1]$ being a 1D periodic box and $\mathbb{R}$ being the whole line, this paper is able to solve the desired stability problem on (1.2). In fact, we are able to prove the following result.

Theorem 1 *Let $\mathbb{T} = [0, 1]$ be a 1D periodic box and let $\Omega = \mathbb{T} \times \mathbb{R}$. Assume $u_0, \theta_0 \in H^2(\Omega)$ and $\nabla \cdot u_0 = 0$. Then there exists $\varepsilon > 0$ such that, if*

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \leq \varepsilon, \tag{1.6}$$

then (1.2) has a unique global solution that remains uniformly bounded for all time,

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \nu \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 \, d\tau + \kappa \int_0^t \|\partial_1 \theta(\tau)\|_{H^2}^2 \, d\tau \leq C_0^2 \varepsilon^2$$

for some pure constant $C_0 > 0$ and for all $t > 0$.

How does this domain makes a difference? The key point is that Ω allows us to separate the horizontal average (or the zeroth horizontal Fourier mode) from the corresponding oscillation part. These two different parts have different physical behavior. In fact, this decomposition is partially motivated by the stratification phenomenon observed in numerical results [19]. The numerical simulations performed in [19] show that the temperature becomes horizontally homogeneous and stratify in the vertical direction as time evolves. Mathematically we do not expect the horizontal average to decay in time since it is associated with the zeroth horizontal Fourier mode and the dissipative effect at this mode vanishes. The oscillation part could decay exponentially. In addition, this decomposition and the oscillation part possess several desirable mathematical properties such as a strong Poincaré type inequality. The two difficult terms in (1.5) are now handled by decomposing both u and θ into the aforementioned two parts, and different terms induced by the decomposition are estimated differently. This is the main reason why the impossible stability problem in the $\mathbb{R}^2$ case becomes solvable when the domain is $\Omega = \mathbb{T} \times \mathbb{R}$.

To make the idea described above more precise, we introduce a few notations. Since the functional setting for our solution (u, θ) is $H^2(\Omega)$, it is meaningful to define the horizontal average,

$$\bar{u}(x_2, t) = \int_{\mathbb{T}} u(x_1, x_2, t) \, dx_1.$$

We set $\widetilde{u}$ to be the corresponding oscillation part

$$\widetilde{u} = u - \bar{u} \quad \text{or} \quad u = \bar{u} + \widetilde{u}.$$

$\bar{\theta}$ and $\widetilde{\theta}$ are similarly defined. This decomposition is orthogonal,

$$(\bar{u}, \widetilde{u}) := \int_{\Omega} \bar{u} \, \widetilde{u} \, dx = 0, \qquad \|u\|_{L^2(\Omega)}^2 = \|\bar{u}\|_{L^2(\Omega)}^2 + \|\widetilde{u}\|_{L^2(\Omega)}^2.$$

A crucial property of $\widetilde{u}$ is that it obeys a strong version of the Poincaré type inequality,

$$\|\widetilde{u}\|_{L^2(\Omega)} \leq C \, \|\partial_1 \widetilde{u}\|_{L^2(\Omega)}, \tag{1.7}$$

where the full gradient in the standard Poincaré type inequality is replaced by ∂_1. With this decomposition at our disposal, we are ready to handle the two difficult terms in (1.5). For the sake of conciseness, we focus on the first term. By invoking the decomposition, we can further split it into four terms,

$$
\begin{aligned}
\int_\Omega \partial_2 u_1\, \partial_1\omega\, \partial_2\omega\, dx &= \int_\Omega \partial_2(\bar{u}_1+\widetilde{u}_1)\, \partial_1(\bar{\omega}+\widetilde{\omega})\, \partial_2(\bar{\omega}+\widetilde{\omega})\, dx \\
&= \int_\Omega \partial_2\bar{u}_1\, \partial_1\widetilde{\omega}\, \partial_2\bar{\omega}\, dx + \int_\Omega \partial_2\bar{u}_1\, \partial_1\widetilde{\omega}\, \partial_2\widetilde{\omega}\, dx \\
&\quad + \int_\Omega \partial_2\widetilde{u}_1\, \partial_1\widetilde{\omega}\, \partial_2\bar{\omega}\, dx + \int_\Omega \partial_2\widetilde{u}_1\, \partial_1\widetilde{\omega}\, \partial_2\widetilde{\omega}\, dx,
\end{aligned}
\tag{1.8}
$$

where we have used $\partial_1\bar{\omega} = 0$. The first term in (1.8) is clearly zero,

$$\int_\Omega \partial_2\bar{u}_1\, \partial_1\widetilde{\omega}\, \partial_2\bar{\omega}\, dx = 0.$$

The other three terms in (1.8) can all be bounded suitably by applying (1.7) and several other anisotropic inequalities, as stated in the lemmas in Sect. 2. We leave more technical details to the proof of Theorem 1 in Sect. 3.

Our second main result states that the oscillation part $(\widetilde{u}, \widetilde{\theta})$ decays to zero exponentially in time in the H^1-norm. This result reflects the stratification phenomenon of buoyancy driven fluids. It also rigorously confirms the observation of the numerical simulations in [19], the temperature eventually stratifies and converges to the horizontal average. As we have explained before, the horizontal average $(\bar{u}, \bar{\theta})$ is not expected to decay in time.

Theorem 2 *Let $u_0, \theta_0 \in H^2(\Omega)$ with $\nabla\cdot u_0 = 0$. Assume that (u_0, θ_0) satisfies (1.6) for sufficiently small $\varepsilon > 0$. Let (u, θ) be the corresponding solution of (1.2). Then the H^1 norm of the oscillation part $(\widetilde{u}, \widetilde{\theta})$ decays exponentially in time,*

$$\|\widetilde{u}(t)\|_{H^1} + \|\widetilde{\theta}(t)\|_{H^1} \le (\|u_0\|_{H^1} + \|\theta_0\|_{H^1})e^{-C_1 t},$$

for some pure constant $C_1 > 0$ and for all $t > 0$.

As a special consequence of this decay result, the solution (u, θ) approaches the horizontal average $(\bar{u}, \bar{\theta})$ asymptotically, and the Boussinesq system (1.2) evolves to the following 1D system

$$
\begin{cases}
\partial_t\bar{u} + \overline{u\cdot\nabla\widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2\bar{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{\theta} \end{pmatrix}, \\
\partial_t\bar{\theta} + \overline{u\cdot\nabla\widetilde{\theta}} = 0,
\end{cases}
$$

as given in (4.2). The proof of Theorem 2 starts with the system governing the oscillation $(\widetilde{u}, \widetilde{\theta})$,

$$
\begin{cases}
\partial_t\widetilde{u} + \widetilde{u\cdot\nabla\widetilde{u}} + u_2\partial_2\bar{u} - \nu\partial_1^2\widetilde{u} + \nabla\widetilde{p} = \widetilde{\theta}e_2, \\
\partial_t\widetilde{\theta} + \widetilde{u\cdot\nabla\widetilde{\theta}} + u_2\partial_2\bar{\theta} - \kappa\partial_1^2\widetilde{\theta} + \widetilde{u}_2 = 0.
\end{cases}
$$

By performing separate energy estimates for $\|(\widetilde{u}, \widetilde{\theta})\|_{L^2}^2$ and $\|(\nabla\widetilde{u}, \nabla\widetilde{\theta})\|_{L^2}^2$ and carefully evaluating the nonlinear terms with the strong Poincaré type inequality and other anisotropic tools, we are able to establish the inequality

$$
\begin{aligned}
&\frac{d}{dt}\|(\widetilde{u}, \widetilde{\theta})\|_{H^1}^2 + (2\nu - C_1\,\|(u,\theta)\|_{H^2})\,\|\partial_1\widetilde{u}\|_{H^1}^2 \\
&\quad + (2\kappa - C_1\,\|(u,\theta)\|_{H^2})\,\|\partial_1\widetilde{\theta}\|_{H^1}^2 \le 0.
\end{aligned}
\tag{1.9}
$$

When the initial data (u_0, θ_0) is taken to be sufficiently small in H^2, say

$$\|(u_0, \theta_0)\|_{H^2} \le \varepsilon$$

for sufficiently small $\varepsilon > 0$, then $\|(u, \theta)\|_{H^2} \le C_0\, \varepsilon$ and

$$2\nu - C_1\, \|(u, \theta)\|_{H^2} \ge \nu, \qquad 2\eta - C_1\, \|(u, \theta)\|_{H^2} \ge \eta.$$

Applying the strong Poincaré inequality to (1.9) yields the desired exponential decay.

The rest of this paper is divided into three sections. Section 2 serves as a preparation. It presents several anisotropic inequalities and some fine properties related to the orthogonal decomposition. Section 3 proves Theorem 1 while Sect. 4 is devoted to verifying Theorem 2.

2 Anisotropic inequalities

This section presents several anisotropic inequalities to be used extensively in the proofs of Theorem 1 and Theorem 2. In addition, several key properties of the horizontal average and the corresponding oscillation are also listed for the convenience of later applications.

We first recall the horizontal average and the corresponding orthogonal decomposition. For any function $f = f(x_1, x_2)$ that is integrable in x_1 over the 1D periodic box $\mathbb{T} = [0, 1]$, its horizontal average $\bar{f}$ is given by

$$\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1. \tag{2.1}$$

We decompose f into $\bar{f}$ and the corresponding oscillation portion $\widetilde{f}$,

$$f = \bar{f} + \widetilde{f}. \tag{2.2}$$

The following lemma collects a few properties of $\bar{f}$ and $\widetilde{f}$ to be used in the subsequent sections. These properties can be easily verified via (2.1) and (2.2).

Lemma 1 *Assume that the 2D function f defined on $\Omega = \mathbb{T} \times \mathbb{R}$ is sufficiently regular, say $f \in H^2(\Omega)$. Let $\bar{f}$ and $\widetilde{f}$ be defined as in (2.1) and (2.2).*

(a) $\bar{f}$ and $\widetilde{f}$ obey the following basic properties,

$$\overline{\partial_1 f} = \partial_1 \bar{f} = 0, \quad \overline{\partial_2 f} = \partial_2 \bar{f}, \quad \overline{\widetilde{f}} = 0, \quad \widetilde{\partial_2 f} = \partial_2 \widetilde{f}.$$

(b) If f is a divergence-free vector field, namely $\nabla \cdot f = 0$, then $\bar{f}$ and $\widetilde{f}$ are also divergence-free,

$$\nabla \cdot \bar{f} = 0 \quad \textit{and} \quad \nabla \cdot \widetilde{f} = 0.$$

(c) $\bar{f}$ and $\widetilde{f}$ are orthogonal in L^2, namely

$$(\bar{f}, \widetilde{f}) := \int_\Omega \bar{f}\, \widetilde{f}\, dx = 0, \quad \|f\|^2_{L^2(\Omega)} = \|\bar{f}\|^2_{L^2(\Omega)} + \|\widetilde{f}\|^2_{L^2(\Omega)}.$$

In particular, $\|\bar{f}\|_{L^2} \le \|f\|_{L^2}$ and $\|\widetilde{f}\|_{L^2} \le \|f\|_{L^2}$.

Proof (a) follows from the definitions of $\bar{f}$ and $\widetilde{f}$ directly. If $\nabla \cdot f = 0$, then

$$0 = \overline{\partial_1 f + \partial_2 f} = \partial_1 \bar{f} + \partial_2 \bar{f} = \nabla \cdot \bar{f} = \nabla \cdot f - \nabla \cdot \widetilde{f} = -\nabla \cdot \widetilde{f},$$

which gives (b). For (c), according to the definitions of $\bar{f}$ and $\widetilde{f}$,

$$(\bar{f}, \widetilde{f}) = \int_{\Omega} \bar{f}\,\widetilde{f}\,dx = \int_{\mathbb{R}} \bar{f}\left(\int_{\mathbb{T}} f(x_1, x_2)\,dx_1\right)dx_2 - \int_{\mathbb{R}} |\bar{f}|^2\,dx_2 = 0.$$

This completes the proof of Lemma 1. □

We now present several anisotropic inequalities. Basic 1D inequalities play a role in these anisotropic inequalities. We emphasize that 1D inequalities on the whole line $\mathbb{R}$ are not always the same as the corresponding ones on bounded domains including periodic domains. For any 1D function $f \in H^1(\mathbb{R})$,

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\,\|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{2.3}$$

For a bounded domain such as $\mathbb{T}$ and $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty(\mathbb{T})} \le \sqrt{2}\,\|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\,\|f'\|_{L^2(\mathbb{T})}^{\frac{1}{2}} + \|f\|_{L^2(\mathbb{T})}. \tag{2.4}$$

However, if a function has mean zero such as the oscillation part $\widetilde{f}$, the 1D inequality for $\widetilde{f}$ is the same as the whole line case, that is, for $\widetilde{f} \in H^1(\mathbb{T})$,

$$\|\widetilde{f}\|_{L^\infty(\mathbb{T})} \le C\,\|\widetilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\,\|(\widetilde{f})'\|_{L^2(\mathbb{T})}^{\frac{1}{2}}. \tag{2.5}$$

These basic inequalities are incorporated into the anisotropic inequalities stated in the following lemmas.

Lemma 2 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\left|\int_{\Omega} f\,g\,h\,dx\right| \le C\,\|f\|_{L^2}^{\frac{1}{2}}\left(\|f\|_{L^2} + \|\partial_1 f\|_{L^2}\right)^{\frac{1}{2}}\,\|g\|_{L^2}^{\frac{1}{2}}\,\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\,\|h\|_{L^2}. \tag{2.6}$$

For any $f \in H^2(\Omega)$, we have

$$\begin{aligned}\|f\|_{L^\infty(\Omega)} \le C\,&\|f\|_{L^2(\Omega)}^{\frac{1}{4}}\left(\|f\|_{L^2(\Omega)} + \|\partial_1 f\|_{L^2(\Omega)}\right)^{\frac{1}{4}}\,\|\partial_2 f\|_{L^2(\Omega)}^{\frac{1}{4}}\\ &\times\left(\|\partial_2 f\|_{L^2(\Omega)} + \|\partial_1\partial_2 f\|_{L^2(\Omega)}\right)^{\frac{1}{4}}.\end{aligned} \tag{2.7}$$

Proof The upper bound for the triple product in (2.6) on $\mathbb{R}^2$ was stated and proven in [5], but (2.6) for the domain Ω includes an extra lower-order term. For the convenience of the readers, we provide the proofs of (2.6) and (2.7). Applying Hölder's inequality in each direction, Minkowski's inequality, and (2.3) and (2.4), we have

$$\begin{aligned}\left|\int_{\Omega} f\,g\,h\,dx\right| &\le \|f\|_{L^2_{x_2}L^\infty_{x_1}}\,\|g\|_{L^\infty_{x_2}L^2_{x_1}}\,\|h\|_{L^2}\\ &\le \|f\|_{L^2_{x_2}L^\infty_{x_1}}\,\|g\|_{L^2_{x_1}L^\infty_{x_2}}\,\|h\|_{L^2}\\ &\le C\left\|\|f\|_{L^2_{x_1}}^{\frac{1}{2}}\,\|\partial_1 f\|_{L^2_{x_1}}^{\frac{1}{2}} + \|f\|_{L^2_{x_1}}\right\|_{L^2_{x_2}}\\ &\quad\times\left\|\|g\|_{L^2_{x_2}}^{\frac{1}{2}}\,\|\partial_2 g\|_{L^2_{x_2}}^{\frac{1}{2}}\right\|_{L^2_{x_1}}\,\|h\|_{L^2}\\ &\le C\,\|f\|_{L^2}^{\frac{1}{2}}\left(\|f\|_{L^2} + \|\partial_1 f\|_{L^2}\right)^{\frac{1}{2}}\,\|g\|_{L^2}^{\frac{1}{2}}\,\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\,\|h\|_{L^2}.\end{aligned}$$

Here $\|f\|_{L^2_{x_2}L^\infty_{x_1}}$ represents the L^∞-norm in the x_1-variable, followed by the L^2-norm in the x_2-variable. To prove (2.7), we again use Hölder's inequality, Minkowski's inequality, and (2.3) and (2.4),

$$\begin{aligned}
\|f\|_{L^\infty_{x_1}L^\infty_{x_2}} &\le C\,\left\|\|f\|^{\frac{1}{2}}_{L^2_{x_2}}\,\|\partial_2 f\|^{\frac{1}{2}}_{L^2_{x_2}}\right\|_{L^\infty_{x_1}} \\
&\le C\,\left\|\|f\|_{L^\infty_{x_1}}\right\|^{\frac{1}{2}}_{L^2_{x_2}}\,\left\|\|\partial_2 f\|_{L^\infty_{x_1}}\right\|^{\frac{1}{2}}_{L^2_{x_2}} \\
&\le C\,\left\|\|f\|^{\frac{1}{2}}_{L^2_{x_1}}\,\|\partial_1 f\|^{\frac{1}{2}}_{L^2_{x_1}} + \|f\|_{L^2_{x_1}}\right\|^{\frac{1}{2}}_{L^2_{x_2}} \\
&\quad \times \left\|\|\partial_2 f\|^{\frac{1}{2}}_{L^2_{x_1}}\,\|\partial_1\partial_2 f\|^{\frac{1}{2}}_{L^2_{x_1}} + \|\partial_2 f\|_{L^2_{x_1}}\right\|^{\frac{1}{2}}_{L^2_{x_2}} \\
&\le C\,\|f\|^{\frac{1}{4}}_{L^2}\left(\|f\|_{L^2} + \|\partial_1 f\|_{L^2}\right)^{\frac{1}{4}}\|\partial_2 f\|^{\frac{1}{4}}_{L^2} \\
&\quad \times \left(\|\partial_2 f\|_{L^2} + \|\partial_1\partial_2 f\|_{L^2}\right)^{\frac{1}{4}}.
\end{aligned}$$

This completes the proof of Lemma 2. □

If we replace f by the oscillation part $\widetilde{f}$, some of the lower-order parts in (2.6) and (2.7) can be dropped, as the following lemma states.

Lemma 3 *Let $\Omega = \mathbb{T}\times\mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\left|\int_\Omega \widetilde{f}\, g\, h\, dx\right| \le C\,\|\widetilde{f}\|^{\frac{1}{2}}_{L^2}\,\|\partial_1\widetilde{f}\|^{\frac{1}{2}}_{L^2}\,\|g\|^{\frac{1}{2}}_{L^2}\,\|\partial_2 g\|^{\frac{1}{2}}_{L^2}\,\|h\|_{L^2}. \tag{2.8}$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C\,\|\widetilde{f}\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_1\widetilde{f}\|^{\frac{1}{4}}_{L^2(\Omega)}\,\|\partial_2\widetilde{f}\|^{\frac{1}{4}}_{L^2(\Omega)}\,\|\partial_1\partial_2\widetilde{f}\|^{\frac{1}{4}}_{L^2(\Omega)}.$$

Proof The two inequalities in this lemma can be shown similarly as those in Lemma 2. The only modification here is to use (2.5) instead of (2.4). Since (2.5) does not contain the lower-order part, the inequalities in this lemma do not have the lower-order terms. □

The next lemma assesses that the oscillation part $\widetilde{f}$ obeys a strong Poincaré type inequality with the upper bound in terms of $\partial_1\widetilde{f}$ instead of $\nabla\widetilde{f}$.

Lemma 4 *Let f be a smooth function, $\bar{f}$ and $\widetilde{f}$ be defined as in (2.1) and (2.2). If $\|\partial_1\widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C\|\partial_1\widetilde{f}\|_{L^2(\Omega)}, \tag{2.9}$$

where C is a pure constant. In addition, if $\|\partial_1\widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C\|\partial_1\widetilde{f}\|_{H^1(\Omega)}. \tag{2.10}$$

Proof For fixed $x_2 \in \mathbb{R}$,

$$\int_0^1 \widetilde{f}(x_1, x_2)\, dx_1 = 0.$$

According to the mean-value theorem, there exists $\eta \in [0, 1]$ such that $\widetilde{f}(\eta, x_2) = 0$. Therefore, by Hölder inequality,

$$|\widetilde{f}(x_1, x_2)| = \left| \int_\eta^{x_1} \partial_{y_1} \widetilde{f}(y_1, x_2)\, dy_1 \right| \le \left(\int_0^1 (\partial_{y_1} \widetilde{f}(y_1, x_2))^2\, dy_1 \right)^{\frac{1}{2}}. \tag{2.11}$$

Taking the L^2-norm of (2.11) over Ω yields

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}.$$

To prove (2.10), we use (2.4), namely, for any fixed $x_2 \in \mathbb{R}$,

$$\|\widetilde{f}(x_1, x_2)\|_{L^\infty_{x_1}(\mathbb{T})} \le C \|\widetilde{f}\|^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{T})} \|\partial_1 \widetilde{f}\|^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{T})}.$$

Taking the L^∞-norm in x_2, and using (2.3) and (2.9), we find

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

This completes the proof of Lemma 4. □

As an application of Lemma 4, the inequality in (2.8) can be converted to

$$\left| \int_\Omega \widetilde{f}\, g\, h\, dx \right| \le C\, \|\partial_1 \widetilde{f}\|_{L^2}\, \|g\|^{\frac{1}{2}}_{L^2}\, \|\partial_2 g\|^{\frac{1}{2}}_{L^2}\, \|h\|_{L^2}.$$

3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Since the local well-posedness can be shown via standard methods such as Friedrichs' Fourier cutoff, our focus is on the global *a priori* bound on the solution in $H^2(\Omega)$.

The framework for proving the global H^2-bound is the bootstrapping argument (see, e.g., [45, p.21]). By selecting a suitable energy functional at the H^2-level, we devote our main efforts to showing that this energy functional obeys a desirable energy inequality. This process is lengthy and involves establishing suitable upper bounds for several nonlinear terms such as the one in (1.8). As described in the introduction, we invoke the orthogonal decomposition $u = \bar{u} + \widetilde{u}$ and $\theta = \bar{\theta} + \widetilde{\theta}$, apply various anisotropic inequalities in the previous section and make use of the fine properties of $\widetilde{u}$ and $\widetilde{\theta}$. More details are given in the proof of Theorem 1.

Proof of Theorem 1 We define the natural energy functional,

$$E(t) := \sup_{0 \le \tau \le t} \left(\|u(\tau)\|^2_{H^2} + \|\theta(\tau)\|^2_{H^2} \right) + \nu \int_0^t \|\partial_1 u(\tau)\|^2_{H^2}\, d\tau + \kappa \int_0^t \|\partial_1 \theta(\tau)\|^2_{H^2}\, d\tau.$$

Our main efforts are devoted to proving that, for a constant C uniform for all $t > 0$,

$$E(t) \le E(0) + C\, E(t)^2 + C\, E(t)^3. \tag{3.1}$$

Once (3.1) is established, the bootstrapping argument implies that, if

$$E(0) = \|u_0\|^2_{H^2} + \|\theta_0\|^2_{H^2} \le \varepsilon^2, \quad \text{for } \varepsilon^2 < \min\{\frac{1}{16C}, \frac{1}{8\sqrt{C}}\},$$

then $E(t)$ admits the desired uniform global bound $E(t) \le C\,\varepsilon^2$. To initiate the bootstrapping argument, we make the ansatz

$$E(t) \le \min\{\frac{1}{4C}, \frac{1}{2\sqrt{C}}\}. \tag{3.2}$$

(3.1) will allow us to conclude that $E(t)$ actually admits an even smaller bound. In fact, if (3.2) holds, then (3.1) implies

$$E(t) \le E(0) + \frac{1}{4}E(t) + \frac{1}{4}E(t),$$

or

$$E(t) \le 2E(0) \le 2\varepsilon^2 \le \frac{1}{2}\,\min\{\frac{1}{4C}, \frac{1}{2\sqrt{C}}\}$$

with the bound being half of the one in (3.2). The bootstrapping argument then asserts that $E(t)$ is bounded uniformly for all time,

$$E(t) \le C\,\varepsilon^2. \tag{3.3}$$

Then we can deduce the global existence as well as the stability result from the global bound in (3.3). Now we show (3.1). A L^2-estimate yields

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_1 u(\tau)\|_{L^2}^2\,d\tau + 2\kappa\int_0^t \|\partial_1\theta(\tau)\|_{L^2}^2\,d\tau \\ &= \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \end{aligned} \tag{3.4}$$

To estimate the H^1-norm, we make use of the vorticity equation associated with the velocity equation in (1.2),

$$\partial_t\omega + u\cdot\nabla\omega = \partial_1\theta,$$

where $\omega = \nabla\times u$. Taking the inner product of $(\omega, \Delta\theta)$ with the equations of vorticity and temperature, we have, due to $\nabla\cdot u = 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\omega(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \nu\|\partial_1\omega\|_{L^2}^2 + \kappa\|\partial_1\nabla\theta\|_{L^2}^2 \\ &\qquad = -\int \nabla(u\cdot\nabla\theta)\cdot\nabla\theta\,dx := M, \end{aligned} \tag{3.5}$$

where we have used

$$\int \partial_1\theta\,\omega\,dx = -\int \theta\,\partial_1\omega\,dx = -\int \theta\,\Delta u_2\,dx = -\int \Delta\theta\,u_2\,dx.$$

We further write M in (3.5) as

$$\begin{aligned} M &= -\int \nabla(u\cdot\nabla\theta)\cdot\nabla\theta\,dx \\ &= -\sum_{i,j=1}^{2}\int \partial_j(u_i\partial_i\theta)\,\partial_j\theta\,dx \\ &= -\sum_{i,j=1}^{2}\int \partial_j u_i\partial_i\theta\,\partial_j\theta\,dx - \sum_{i,j=1}^{2}\int u_i\partial_i\partial_j\theta\,\partial_j\theta\,dx \end{aligned}$$

$$
\begin{aligned}
&= -\int \partial_1 u_1\, \partial_1\theta\, \partial_1\theta\, dx - \int \partial_1 u_2\, \partial_2\theta\, \partial_1\theta\, dx \\
&\quad - \int \partial_2 u_1\, \partial_1\theta\, \partial_2\theta\, dx - \int \partial_2 u_2\, \partial_2\theta\, \partial_2\theta\, dx \\
&:= M_1 + M_2 + M_3 + M_4.
\end{aligned}
$$

Here, due to $\nabla \cdot u = 0$, we have used

$$
\sum_{i,j=1}^{2} \int u_i \partial_i \partial_j \theta\, \partial_j \theta\, dx = 0.
$$

By Lemma 1, Lemma 3, Lemma 4 and Young's inequality,

$$
\begin{aligned}
M_1 &= -\int \partial_1 \widetilde{u}_1\, \partial_1 \widetilde{\theta}\, \partial_1 \theta\, dx \\
&\le C\|\partial_1\theta\|_{L^2} \|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_1\theta\|_{L^2}^2 (\|\partial_1 u\|_{L^2}^2 + \|\partial_1\theta\|_{L^2}^2) + \delta(\|\partial_1\nabla u\|_{L^2}^2 + \|\partial_1\nabla\theta\|_{L^2}^2),
\end{aligned}
$$

where $\delta > 0$ is a small fixed constant to be specified later. Similarly,

$$
\begin{aligned}
M_2 &= -\int \partial_1 \widetilde{u}_2\, \partial_2 \theta\, \partial_1 \widetilde{\theta}\, dx \\
&\le C\|\partial_2\theta\|_{L^2} \|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1\widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_2\theta\|_{L^2}^2 (\|\partial_1 u\|_{L^2}^2 + \|\partial_1\theta\|_{L^2}^2) + \delta(\|\partial_1\nabla u\|_{L^2}^2 + \|\partial_1\nabla\theta\|_{L^2}^2).
\end{aligned}
$$

To deal with M_3, we invoke the decompositions $u = \bar{u} + \widetilde{u}$ and $\theta = \bar{\theta} + \widetilde{\theta}$ to write it into four terms,

$$
\begin{aligned}
M_3 &= -\int \partial_2 u_1\, \partial_1\theta\, \partial_2\theta\, dx \\
&= -\int \partial_2 \bar{u}_1\, \partial_1\widetilde{\theta}\, \partial_2\bar{\theta}\, dx - \int \partial_2 \bar{u}_1\, \partial_1\widetilde{\theta}\, \partial_2\widetilde{\theta}\, dx \\
&\quad - \int \partial_2 \widetilde{u}_1\, \partial_1\widetilde{\theta}\, \partial_2\bar{\theta}\, dx - \int \partial_2 \widetilde{u}_1\, \partial_1\widetilde{\theta}\, \partial_2\widetilde{\theta}\, dx \\
&:= M_{31} + M_{32} + M_{33} + M_{34}.
\end{aligned}
$$

According to Lemma 1, it is easy to see $M_{31} = 0$. To bound M_{32} and M_{33}, we use Lemma 1, Lemma 3, Lemma 4 and Young's inequality to obtain

$$
\begin{aligned}
M_{32} &= -\int \partial_2 \bar{u}_1\, \partial_1\widetilde{\theta}\, \partial_2\widetilde{\theta}\, dx \\
&\le C\|\partial_2\bar{u}_1\|_{L^2} \|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \\
&\le C\|\partial_2 u\|_{L^2} \|\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{3}{2}} \\
&\le C\|\partial_2 u\|_{L^2}^4 \|\partial_1\theta\|_{L^2}^2 + \delta\|\partial_1\nabla\theta\|_{L^2}^2
\end{aligned}
$$

and

$$
M_{33} = -\int \partial_2 \widetilde{u}_1\, \partial_1\widetilde{\theta}\, \partial_2\bar{\theta}\, dx
$$

$$
\begin{aligned}
&\le C\|\partial_2\bar\theta\|_{L^2}\|\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le C\|\partial_2\theta\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}_1\|_{L^2}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le C\|\partial_2\theta\|_{L^2}^4\|\partial_1\theta\|_{L^2}^2+\delta(\|\partial_1\nabla u\|_{L^2}^2+\|\partial_1\nabla\theta\|_{L^2}^2).
\end{aligned}
$$

M_{34} can be similarly bounded as M_{32}. For M_4, we also write it as

$$
\begin{aligned}
M_4 &= \int \partial_1 u_1\, \partial_2\theta\, \partial_2\theta\, dx\\
&= 2\int \partial_1\widetilde{u}_1\, \partial_2\widetilde{\theta}\, \partial_2\bar\theta\, dx + \int \partial_1\widetilde{u}_1\, \partial_2\widetilde{\theta}\, \partial_2\widetilde{\theta}\, dx\\
&:= M_{41} + M_{42}.
\end{aligned}
$$

Similar as M_{33}, we can bound M_{41} and M_{42} by

$$
M_{41},\ M_{42} \le C\|\partial_2\theta\|_{L^2}^4\|\partial_1 u\|_{L^2}^2+\delta(\|\partial_1\nabla u\|_{L^2}^2+\|\partial_1\nabla\theta\|_{L^2}^2).
$$

Collecting the estimates for M and taking $\delta > 0$ to be small, say

$$
\delta \le \frac{1}{16}\min\{\nu,\eta\},
$$

we obtain

$$
\begin{aligned}
&\|\omega(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^2}^2+\nu\int_0^t\|\partial_1\omega(\tau)\|_{L^2}^2\tau+\kappa\int_0^t\|\partial_1\nabla\theta(\tau)\|_{L^2}^2\,d\tau\\
&\le C\int_0^t(\|\nabla\theta\|_{L^2}^2+\|\nabla\theta\|_{L^2}^4+\|\nabla u\|_{L^2}^4)\times(\|\partial_1 u\|_{L^2}^2+\|\partial_1\theta\|_{L^2}^2)\,d\tau. \qquad (3.6)
\end{aligned}
$$

Next we estimate the H^2-norm of (u,θ). We take the inner product of $\Delta\omega$ and $\Delta^2\theta$ with the equations of vorticity and temperature, respectively. Due to $\nabla\cdot u = 0$ and after integrating by parts, we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}(\|\nabla\omega(t)\|_{L^2}^2+\|\Delta\theta(t)\|_{L^2}^2)+\nu\|\partial_1\nabla\omega\|_{L^2}^2+\kappa\|\partial_1\Delta\theta\|_{L^2}^2\\
&= -\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx-\int\Delta(u\cdot\nabla\theta)\Delta\theta\,dx. \qquad (3.7)
\end{aligned}
$$

For the first term on the right hand side, we can decompose it as

$$
\begin{aligned}
N &:= -\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx\\
&= -\int\partial_1 u_1\,(\partial_1\omega)^2\,dx-\int\partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx\\
&\quad -\int\partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx-\int\partial_2 u_2\,(\partial_2\omega)^2\,dx\\
&:= N_1+N_2+N_3+N_4.
\end{aligned}
$$

N_1 and N_2 can be bounded directly. According to Lemma 1, $\partial_1\bar u = 0$ and $\partial_1 u = \partial_1\widetilde{u}$. By Lemma 3,

$$
\begin{aligned}
N_1 &= -\int\partial_1 u_1\,\partial_1\omega\,\partial_1\widetilde{\omega}\,dx\\
&\le C\,\|\partial_1 u_1\|_{L^2}^{\frac12}\,\|\partial_1\partial_1 u_1\|_{L^2}^{\frac12}\,\|\partial_1\omega\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{\omega}\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\,\|\partial_1 u\|_{H^1}\,\|\partial_1 u\|_{H^2}\|\partial_1\nabla\omega\|_{L^2}\\
&\le C\,\|\partial_1 u\|_{H^1}^2\,\|\partial_1 u\|_{H^2}^2+\delta'\|\partial_1\nabla\omega\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
N_2 &\le C\,\|\partial_1 u_2\|_{L^2}^{\frac12}\,\|\partial_1\partial_1 u_2\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\omega\|_{L^2}\\
&\le C\,\|u\|_{H^2}^2\,\|\partial_1 u\|_{H^2}^2+\delta'\|\partial_1\nabla\omega\|_{L^2}^2,
\end{aligned}
$$

where $\delta' > 0$ is a small but fixed parameter. The estimate of N_3 is slightly more delicate.

$$
\begin{aligned}
N_3 &= -\int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx\\
&= -\int \partial_2(\bar{u}_1+\widetilde{u}_1)\,\partial_1\widetilde{\omega}\,\partial_2(\bar{\omega}+\widetilde{\omega})\,dx\\
&= -\int \partial_2\bar{u}_1\ \partial_1\widetilde{\omega}\,\partial_2\bar{\omega}\,dx-\int \partial_2\bar{u}_1\ \partial_1\widetilde{\omega}\,\partial_2\widetilde{\omega}\,dx\\
&\quad -\int \partial_2\widetilde{u}_1\ \partial_1\widetilde{\omega}\,\partial_2\bar{\omega}\,dx-\int \partial_2\widetilde{u}_1\ \partial_1\widetilde{\omega}\,\partial_2\widetilde{\omega}\,dx\\
&:= N_{31}+N_{32}+N_{33}+N_{34}.
\end{aligned}
$$

The first term N_{31} is clearly zero,

$$
N_{31} = -\int_{\mathbb{R}} \partial_2\bar{u}_1\,\partial_2\bar{\omega}\int_{\mathbb{T}}\partial_1\widetilde{\omega}\,dx_1\,dx_2 = 0.
$$

To bound N_{32} and N_{33}, we first use (2.8) of Lemma 3 and then Lemma 4 to obtain

$$
\begin{aligned}
N_{32} &\le C\,\|\partial_2\bar{u}_1\|_{L^2}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\,\|\partial_2\bar{u}_1\|_{L^2}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac32}\\
&\le C\,\|u\|_{H^1}^4\,\|\partial_1 u\|_{H^2}^2+\delta'\|\partial_1\nabla\omega\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
N_{33} &\le C\,\|\partial_2\bar{\omega}\|_{L^2}\,\|\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\,\|u\|_{H^2}^2\,\|\partial_1 u\|_{H^2}^2+\delta'\|\partial_1\nabla\omega\|_{L^2}^2,
\end{aligned}
$$

N_{34} can be similarly bounded as N_{32}. N_4 can also be bounded similarly.

$$
\begin{aligned}
N_4 &= -\int \partial_1\widetilde{u}_1\,(\partial_2\bar{\omega}+\partial_2\widetilde{\omega})^2\,dx\\
&= -2\int \partial_1\widetilde{u}_1\,\partial_2\bar{\omega}\,\partial_2\widetilde{\omega}\,dx-\int \partial_1\widetilde{u}_1\,(\partial_2\widetilde{\omega})^2\,dx\\
&\le C\,(\|\partial_2\bar{\omega}\|_{L^2}+\|\partial_2\widetilde{\omega}\|_{L^2})\|\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\,\|u\|_{H^2}^2\,\|\partial_1 u\|_{H^2}^2+\delta'\|\partial_1\nabla\omega\|_{L^2}^2.
\end{aligned}
$$

Thus we have shown that

$$
|N| \le C\,(\|u\|_{H^2}^2+\|u\|_{H^2}^4)\,\|\partial_1 u\|_{H^2}^2+6\delta'\|\partial_1\nabla\omega\|_{L^2}^2. \tag{3.8}
$$

For the last term of (3.7), we can write it as

$$
\begin{aligned}
-\int \Delta(u\cdot\nabla\theta)\,\Delta\theta\,dx &= -\int \Delta u\cdot\nabla\theta\,\Delta\theta\,dx - 2\int \nabla u\cdot\nabla^2\theta\,\Delta\theta\,dx\\
&= -\int \Delta u_1\,\partial_1\theta\,\Delta\theta\,dx - \int \Delta u_2\,\partial_2\theta\,\Delta\theta\,dx\\
&\quad -2\int \partial_1 u_1\,\partial_1\partial_1\theta\,\Delta\theta\,dx - 2\int \partial_1 u_2\,\partial_2\partial_1\theta\,\Delta\theta\,dx\\
&\quad -2\int \partial_2 u_1\,\partial_1\partial_2\theta\,\Delta\theta\,dx - 2\int \partial_2 u_2\,\partial_2\partial_2\theta\,\Delta\theta\,dx\\
&:= P_1+P_2+P_3+P_4+P_5+P_6.
\end{aligned}
$$

According to Lemma 1, we can divide P_1 into four terms,

$$
\begin{aligned}
P_1 &= -\int \Delta\bar{u}_1\,\partial_1\widetilde{\theta}\,\Delta\bar{\theta}\,dx - \int \Delta\bar{u}_1\,\partial_1\widetilde{\theta}\,\Delta\widetilde{\theta}\,dx\\
&\quad -\int \Delta\widetilde{u}_1\,\partial_1\widetilde{\theta}\,\Delta\bar{\theta}\,dx - \int \Delta\widetilde{u}_1\,\partial_1\widetilde{\theta}\,\Delta\widetilde{\theta}\,dx\\
&:= P_{11}+P_{12}+P_{13}+P_{14}.
\end{aligned}
$$

It is clear that $P_{11}=0$. For P_{12}, we can bound it using Lemma 1, Lemma 3 and Lemma 4,

$$
\begin{aligned}
P_{12} &\le C\|\Delta\bar{u}_1\|_{L^2}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_1\Delta\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le C\|\Delta u\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1\nabla\theta\|_{L^2}^{\frac12}\|\partial_1\Delta\widetilde{\theta}\|_{L^2}\\
&\le C\|\Delta u\|_{L^2}^2(\|\partial_1\theta\|_{L^2}^2+\|\partial_1\nabla\theta\|_{L^2}^2)+\delta'\|\partial_1\Delta\theta\|_{L^2}^2.
\end{aligned}
$$

Similarly, P_{14} shares the same bounded with P_{12}. For P_{13}, we can bound it by

$$
\begin{aligned}
P_{13} &\le C\|\Delta\bar{\theta}\|_{L^2}\|\Delta\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_1\Delta\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le C\|\Delta\theta\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac12}\|\partial_1\Delta\widetilde{u}\|_{L^2}\|\partial_1\Delta\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\le C\|\Delta\theta\|_{L^2}^4\|\partial_1\theta\|_{L^2}^2+\delta'(\|\partial_1\Delta u\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2).
\end{aligned}
$$

Therefore,

$$
P_1 \le C(\|\Delta u\|_{L^2}^2+\|\Delta\theta\|_{L^2}^4)\times\|\partial_1\theta\|_{H^1}^2+\delta'(\|\partial_1\Delta u\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2).
$$

According to the relation $\Delta u_2=\partial_1\omega$, we can decompose P_2 as follows,

$$
\begin{aligned}
P_2 &= -\int \partial_1\omega\,\partial_2\theta\,\Delta\theta\,dx\\
&= -\int \partial_1\widetilde{\omega}\,\partial_2\bar{\theta}\,\Delta\bar{\theta}\,dx - \int \partial_1\widetilde{\omega}\,\partial_2\bar{\theta}\,\Delta\widetilde{\theta}\,dx\\
&\quad -\int \partial_1\widetilde{\omega}\,\partial_2\widetilde{\theta}\,\Delta\bar{\theta}\,dx - \int \partial_1\widetilde{\omega}\,\partial_2\widetilde{\theta}\,\Delta\widetilde{\theta}\,dx\\
&:= P_{21}+P_{22}+P_{23}+P_{24}.
\end{aligned}
$$

Clearly, $P_{21}=0$. Making use of Lemma 1, Lemma 3 and Lemma 4, we obtain

$$
P_{22},\,P_{24} \le C\|\partial_2\bar{\theta}\|_{L^2}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\|\Delta\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\Delta\widetilde{\theta}\|_{L^2}^{\frac12}
$$

$$\leq C\|\nabla\theta\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{\theta}\|_{L^2}$$
$$\leq C\|\nabla\theta\|_{L^2}^4\|\partial_1\omega\|_{L^2}^2+\delta'(\|\partial_1\nabla\omega\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2)$$

and

$$\begin{aligned}P_{23} &\leq C\|\Delta\bar{\theta}\|_{L^2}\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\\&\leq C\|\Delta\theta\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\widetilde{\omega}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{\theta}\|_{L^2}\\&\leq C\|\Delta\theta\|_{L^2}^4\|\partial_1\omega\|_{L^2}^2+\delta'(\|\partial_1\nabla\omega\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2).\end{aligned}$$

Thus,

$$P_2\leq C(\|\nabla u\|_{L^2}^4+\|\Delta\theta\|_{L^2}^4)\times\|\partial_1\omega\|_{L^2}^2+\delta'(\|\partial_1\Delta u\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2).$$

For P_3, we can bound it by

$$\begin{aligned}P_3 &= -2\int\partial_1\widetilde{u}_1\,\partial_1\partial_1\widetilde{\theta}\,\Delta\theta\,dx\\&\leq C\|\Delta\theta\|_{L^2}\|\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\partial_1\widetilde{\theta}\|_{L^2}^{\frac{1}{2}}\\&\leq C\|\Delta\theta\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta\widetilde{\theta}\|_{L^2}\\&\leq C\|\Delta\theta\|_{L^2}^4\|\partial_1 u\|_{L^2}^2+\delta'(\|\partial_1\Delta u\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2).\end{aligned}$$

P_4 can be bounded in the same way. Using Lemma 1, we can write P_5 as

$$\begin{aligned}P_5 &= -2\int\partial_2\bar{u}_1\,\partial_1\partial_2\widetilde{\theta}\,\Delta\bar{\theta}\,dx-2\int\partial_2\bar{u}_1\,\partial_1\partial_2\widetilde{\theta}\,\Delta\widetilde{\theta}\,dx\\&\quad-2\int\partial_2\widetilde{u}_1\,\partial_1\partial_2\widetilde{\theta}\,\Delta\bar{\theta}\,dx-2\int\partial_2\widetilde{u}_1\,\partial_1\partial_2\widetilde{\theta}\,\Delta\widetilde{\theta}\,dx\\&:= P_{51}+P_{52}+P_{53}+P_{54}.\end{aligned}$$

It is easy to check that $P_{51}=0$. P_{52} and P_{54} can be bounded by

$$P_{52},\,P_{54}\leq C\|\nabla u\|_{L^2}^4\|\partial_1\nabla\theta\|_{L^2}^2+\delta'\|\partial_1\Delta\theta\|_{L^2}^2,$$

and P_{53} have the bound

$$P_{53}\leq C\|\Delta\theta\|_{L^2}^4\|\partial_1\nabla u\|_{L^2}^2+\delta'(\|\partial_1\Delta u\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2).$$

Finally we estimate P_6, which can be written as

$$\begin{aligned}P_6 &= 2\int\partial_1 u_1\,\partial_2\partial_2\theta\,\Delta\theta\,dx\\&= 2\int\partial_1\widetilde{u}_1\,\partial_2\partial_2\bar{\theta}\,\Delta\bar{\theta}\,dx+2\int\partial_1\widetilde{u}_1\,\partial_2\partial_2\bar{\theta}\,\Delta\widetilde{\theta}\,dx\\&\quad+2\int\partial_1\widetilde{u}_1\,\partial_2\partial_2\widetilde{\theta}\,\Delta\bar{\theta}\,dx+2\int\partial_1\widetilde{u}_1\,\partial_2\partial_2\widetilde{\theta}\,\Delta\widetilde{\theta}\,dx\\&:= P_{61}+P_{62}+P_{63}+P_{64}.\end{aligned}$$

As in the estimate of P_1, we have $P_{61}=0$ and

$$P_{62},\,P_{63},\,P_{64}\leq C\|\Delta\theta\|_{L^2}^4\|\partial_1 u\|_{L^2}^2+\delta'(\|\partial_1\Delta u\|_{L^2}^2+\|\partial_1\Delta\theta\|_{L^2}^2).$$

Inserting (3.8) and the estimates for P_1 through P_6 in (3.7), and choosing $\delta' > 0$ sufficiently small, we obtain

$$\begin{aligned}&\|\nabla\omega(t)\|_{L^2}^2+\|\Delta\theta(t)\|_{L^2}^2+\nu\int_0^t\|\partial_1\nabla\omega(\tau)\|_{L^2}^2\,d\tau+\kappa\int_0^t\|\partial_1\Delta\theta(\tau)\|_{L^2}^2\,d\tau\\&\leq C\int_0^t(\|\partial_1 u\|_{H^2}^2+\|\partial_1\theta\|_{H^2}^2)\times(\|u\|_{H^2}^2+\|u\|_{H^2}^4+\|\theta\|_{H^2}^2+\|\theta\|_{H^2}^4)\,d\tau.\qquad(3.9)\end{aligned}$$

Combining (3.9), (3.4) and (3.6) leads to the desired inequality in (3.1). This completes the proof of Theorem 1. □

4 Proof of Theorem 2

This section proves Theorem 2. We work with the equations of $(\widetilde{u},\widetilde{\theta})$ and make use of the properties of the orthogonal decomposition and various anisotropic inequalities.

Proof of Theorem 2 We first write the equation of $(\bar{u},\bar{\theta})$. Making use of Lemma 1, we have $\partial_1\bar{u}=0$ and

$$\overline{u\cdot\nabla\bar{u}}=\overline{u_1\partial_1\bar{u}}+\overline{u_2\partial_2\bar{u}}=\bar{u}_2\partial_2\bar{u}.$$

Since $\nabla\cdot u=0$ in Ω, there exists a stream function ψ such that

$$u=\nabla^{\perp}\psi:=(-\partial_2\psi,\partial_1\psi).$$

Then

$$\bar{u}_2=\overline{\partial_1\psi}=0$$

and

$$\overline{u\cdot\nabla\bar{u}}=0. \qquad (4.1)$$

Taking the average of (1.2) and making use of (4.1) yield

$$\begin{cases}\partial_t\bar{u}+\overline{u\cdot\nabla\widetilde{u}}+\begin{pmatrix}0\\ \partial_2\bar{p}\end{pmatrix}=\begin{pmatrix}0\\ \bar{\theta}\end{pmatrix},\\ \partial_t\bar{\theta}+\overline{u\cdot\nabla\widetilde{\theta}}=0.\end{cases} \qquad (4.2)$$

Taking the difference of (1.2) and (4.2), we find

$$\begin{cases}\partial_t\widetilde{u}+\widetilde{u\cdot\nabla\widetilde{u}}+u_2\partial_2\bar{u}-\nu\partial_1^2\widetilde{u}+\nabla\widetilde{p}=\widetilde{\theta}e_2,\\ \partial_t\widetilde{\theta}+\widetilde{u\cdot\nabla\widetilde{\theta}}+u_2\partial_2\bar{\theta}-\kappa\partial_1^2\widetilde{\theta}+\widetilde{u}_2=0.\end{cases} \qquad (4.3)$$

The L^2-estimate gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}(t)\|_{L^2}^2+\|\widetilde{\theta}(t)\|_{L^2}^2)+\nu\|\partial_1\widetilde{u}\|_{L^2}^2+\kappa\|\partial_1\widetilde{\theta}\|_{L^2}^2\\&=-\int\widetilde{u\cdot\nabla\widetilde{u}}\cdot\widetilde{u}\,dx-\int u_2\,\partial_2\bar{u}\cdot\widetilde{u}\,dx-\int\widetilde{u\cdot\nabla\widetilde{\theta}}\,\widetilde{\theta}\,dx-\int u_2\,\partial_2\bar{\theta}\,\widetilde{\theta}\,dx\\&:=A_1+A_2+A_3+A_4.\end{aligned}$$

For A_1, according to the divergence-free condition of u and Lemma 1, we have

$$A_1 = -\int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u}\, dx = -\int u \cdot \nabla \widetilde{u} \cdot \widetilde{u}\, dx + \int \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u}\, dx = 0.$$

Similarly, $A_3 = 0$. Then we estimate A_2. By Lemma 1, Lemma 3 and Lemma 4,

$$\begin{aligned} A_2 &= -\int \widetilde{u}_2\, \partial_2 \bar{u} \cdot \widetilde{u}\, dx \\ &\le C\|\partial_2 \bar{u}\|_{L^2} \|\widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le C\|\partial_2 \bar{u}\|_{L^2} \|\partial_1 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le C\|u\|_{H^1} \|\partial_1 \widetilde{u}\|_{L^2}^2. \end{aligned}$$

Similarly,

$$A_4 = -\int \widetilde{u}_2\, \partial_2 \bar{\theta}\, \widetilde{\theta}\, dx \le C\|\theta\|_{H^1} (\|\partial_1 \widetilde{u}\|_{L^2}^2 + \|\partial_1 \widetilde{\theta}\|_{L^2}^2).$$

Collecting the estimates for A_1 through A_4, we obtain

$$\begin{aligned} &\frac{d}{dt}(\|\widetilde{u}(t)\|_{L^2}^2 + \|\widetilde{\theta}(t)\|_{L^2}^2) + (2\nu - C\|(u,\theta)\|_{H^1})\|\partial_1 \widetilde{u}\|_{L^2}^2 \\ &\qquad + (2\kappa - C\|(u,\theta)\|_{H^1})\|\widetilde{\theta}\|_{L^2}^2 \le 0. \end{aligned}$$

By Theorem 1, if $\varepsilon > 0$ is sufficiently small and $\|u_0\|_{H_2} + \|\theta_0\|_{H_2} \le \varepsilon$, then $\|(u(t), \theta(t))\|_{H^2} \le C\varepsilon$ and

$$2\nu - C\|(u(t), \theta(t))\|_{H^2} \ge \nu, \quad 2\kappa - C\|(u(t), \theta(t))\|_{H^2} \ge \kappa.$$

Invoking the Poincaré type inequality in Lemma 4 leads to the desired exponential decay for $\|(\widetilde{u}, \widetilde{\theta})\|_{L^2}$,

$$\|\widetilde{u}(t)\|_{L^2} + \|\widetilde{\theta}(t)\|_{L^2} \le (\|u_0\|_{L^2} + \|\theta_0\|_{L^2}) e^{-C_1 t}, \tag{4.4}$$

where $C_1 = C_1(\nu, \eta) > 0$. We now turn to the exponential decay of $\|(\nabla \widetilde{u}(t), \nabla \widetilde{\theta}(t))\|_{L^2}$. Applying ∇ to (4.3) yields

$$\begin{cases} \partial_t \nabla \widetilde{u} + \nabla(\widetilde{u \cdot \nabla \widetilde{u}}) + \nabla(u_2 \partial_2 \bar{u}) - \nu \partial_1^2 \nabla \widetilde{u} + \nabla\nabla \widetilde{p} = \nabla(\widetilde{\theta} e_2), \\ \partial_t \nabla \widetilde{\theta} + \nabla(\widetilde{u \cdot \nabla \widetilde{\theta}}) + \nabla(u_2 \partial_2 \bar{\theta}) - \kappa \partial_1^2 \nabla \widetilde{\theta} + \nabla \widetilde{u}_2 = 0. \end{cases} \tag{4.5}$$

Taking the L^2 inner product of system (4.5) with $(\nabla \widetilde{u},\ \nabla \widetilde{\theta})$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla \widetilde{u}(t)\|_{L^2}^2 + \|\nabla \widetilde{\theta}(t)\|_{L^2}^2) + \nu\|\partial_1 \nabla \widetilde{u}\|_{L^2}^2 + \kappa\|\partial_1 \nabla \widetilde{\theta}\|_{L^2}^2 \\ &= -\int \nabla(\widetilde{u \cdot \nabla \widetilde{u}}) \cdot \nabla \widetilde{u}\, dx - \int \nabla(u_2 \partial_2 \bar{u}) \cdot \nabla \widetilde{u}\, dx \\ &- \int \nabla(\widetilde{u \cdot \nabla \widetilde{\theta}}) \cdot \nabla \widetilde{\theta}\, dx - \int \nabla(u_2 \partial_2 \bar{\theta}) \cdot \nabla \widetilde{\theta}\, dx \\ &:= B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{4.6}$$

By Lemma 1, B_1 can be further written as

$$B_1 = -\int \nabla(u \cdot \nabla \widetilde{u}) \cdot \nabla \widetilde{u}\, dx + \int \nabla(\overline{u \cdot \nabla \widetilde{u}}) \cdot \nabla \widetilde{u}\, dx$$

$$
\begin{aligned}
&= -\int \partial_1 u_1\, \partial_1\widetilde{u}\cdot\partial_1\widetilde{u}\,dx + \int \partial_1 u_2\, \partial_2\widetilde{u}\cdot\partial_1\widetilde{u}\,dx\\
&\quad -\int \partial_2 u_1\, \partial_1\widetilde{u}\cdot\partial_2\widetilde{u}\,dx + \int \partial_2 u_2\, \partial_2\widetilde{u}\cdot\partial_2\widetilde{u}\,dx\\
&:= B_{11}+B_{12}+B_{13}+B_{14}.
\end{aligned}
$$

Using Lemma 3 and Lemma 4, B_{11} can be bounded by

$$
\begin{aligned}
B_{11} &\le C\|\partial_1 u_1\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\
&\le C\|\partial_1 u_1\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\
&\le C\|u\|_{H^1}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

B_{12} and B_{13} can be bounded similarly and

$$
B_{12},\ B_{13} \le C\|u\|_{H^1}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.
$$

For B_{14}, according to the divergence-free condition of u and similar as B_{11}, we have

$$
\begin{aligned}
B_{14} &= -\int \partial_1 u_1\, \partial_2\widetilde{u}\cdot\partial_2\widetilde{u}\,dx \ = \int \partial_1 \widetilde{u}_1\, \partial_2\widetilde{u}\cdot\partial_2\widetilde{u}\,dx\\
&\le C\|\partial_2\widetilde{u}\|_{L^2}\|\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac12}\\
&\le C\|u\|_{H^1}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

Therefore, B_1 is bounded by

$$
|B_1| \le C\|u\|_{H^1}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.
$$

Similarly, we can bound B_3 by

$$
|B_3| \le C(\|u\|_{H^1}+\|\theta\|_{H^1})\times(\|\partial_1\nabla\widetilde{u}\|_{L^2}^2+\|\partial_1\nabla\widetilde{\theta}\|_{L^2}^2).
$$

To bound B_2, we first write it explicitly as

$$
\begin{aligned}
B_2 &= -\int \nabla(u_2\partial_2\bar{u})\cdot\nabla\widetilde{u}\,dx\\
&= -\int \partial_1 u_2\, \partial_2\bar{u}\cdot\partial_1\widetilde{u}\,dx - \int u_2\, \partial_2\partial_1\bar{u}\cdot\partial_1\widetilde{u}\,dx\\
&\quad -\int \partial_2 u_2\, \partial_2\bar{u}\cdot\partial_2\widetilde{u}\,dx - \int u_2\, \partial_2\partial_2\bar{u}\cdot\partial_2\widetilde{u}\,dx\\
&:= B_{21}+B_{22}+B_{23}+B_{24}.
\end{aligned}
$$

By Lemma 1, Lemma 3 and Lemma 4,

$$
\begin{aligned}
B_{21} &= -\int \partial_1\widetilde{u}_2\, \partial_2\bar{u}\cdot\partial_1\widetilde{u}\,dx\\
&\le C\|\partial_2\bar{u}\|_{L^2}\|\partial_1\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\
&\le C\|u\|_{H^1}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}
$$

According to the definition of $\bar{u}$,

$$
B_{22} = -\int u_2\, \partial_2\partial_1\bar{u}\cdot\partial_1\widetilde{u}\,dx = 0.
$$

Similarly, B_{23} and B_{24} can be bounded by

$$\begin{aligned} B_{23} &= \int \partial_1 \widetilde{u}_1\, \partial_2 \bar{u} \cdot \partial_2 \widetilde{u}\, dx \\ &\le C \|\partial_2 \bar{u}\|_{L^2} \|\partial_1 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le C \|u\|_{H^1} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} B_{24} &= -\int \widetilde{u}_2\, \partial_2 \partial_2 \bar{u} \cdot \partial_2 \widetilde{u}\, dx \\ &\le C \|\partial_2 \partial_2 \bar{u}\|_{L^2} \|\widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le C \|u\|_{H^2} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2. \end{aligned}$$

Thus we obtain the bound for B_2,

$$|B_2| \le C \|u\|_{H^2} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2.$$

Similarly, B_4 is bounded by

$$|B_4| \le C \|\theta\|_{H^2} \times (\|\partial_1 \nabla \widetilde{u}\|_{L^2}^2 + \|\partial_1 \nabla \widetilde{\theta}\|_{L^2}^2).$$

Inserting the estimates for B_1 through B_4 in (4.6), we obtain

$$\begin{aligned} &\frac{d}{dt}(\|\nabla \widetilde{u}(t)\|_{L^2}^2 + \|\nabla \widetilde{\theta}(t)\|_{L^2}^2) + (2\nu - C\|(u,\theta)\|_{H^2}) \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2 \\ &+ (2\kappa - C\|(u,\theta)\|_{H^2}) \|\partial_1 \nabla \widetilde{\theta}\|_{L^2}^2 \le 0. \end{aligned}$$

Choosing ε sufficiently small and invoking the Poincaré type inequality in Lemma 4, we obtain the exponential decay result for $\|(\nabla \widetilde{u}, \nabla \widetilde{\theta})\|_{L^2}$,

$$\|\nabla \widetilde{u}(t)\|_{L^2} + \|\nabla \widetilde{\theta}(t)\|_{L^2} \le (\|\nabla u_0\|_{L^2} + \|\nabla \theta_0\|_{L^2}) e^{-C_1 t}. \tag{4.7}$$

Combining the estimates (4.4) and (4.7), we obtain the desired decay result. This completes the proof of Theorem 2. □

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