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GLOBAL REGULARITY FOR THE 2D MICROPOLAR EQUATIONS WITH FRACTIONAL DISSIPATION

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Abstract. Micropolar equations, modeling micropolar fluid flows, consist of coupled equations obeyed by the evolution of the velocity u and that of the microrotation w. This paper focuses on the two-dimensional micropolar equations with the fractional dissipation $(-\Delta)^\alpha u$ and $(-\Delta)^\beta w$, where $0 < \alpha, \beta < 1$. The goal here is the global regularity of the fractional micropolar equations with minimal fractional dissipation. Recent efforts have resolved the two borderline cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$. However, the situation for the general critical case $\alpha + \beta = 1$ with $0 < \alpha < 1$ is far more complex and the global regularity appears to be out of reach. When the dissipation is split among the equations, the dissipation is no longer as efficient as in the borderline cases and different ranges of α and β require different estimates and tools. We aim at the subcritical case $\alpha + \beta > 1$ and divide $\alpha \in (0, 1)$ into five sub-intervals to seek the best estimates so that we can impose the minimal requirements on α and β. The proof of the global regularity relies on the introduction of combined quantities, sharp lower bounds for the fractional dissipation and delicate upper bounds for the nonlinearity and associated commutators.

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1. **Introduction.** Micropolar equations, derived in 1960's by Eringen [14, 15], govern the motion of micropolar fluids. Micropolar fluids are a class of fluids with microstructures such as fluids consisting of bar-like elements and liquid crystals made up of dumbbell molecules (see, e.g., [9, 13, 25, 27]). They are non-Newtonian fluids with nonsymmetric stress tensor. The micropolar equations take into account of the kinematic viscous effect, microrotational effects as well as microrotational inertia. The 3D micropolar equations are given by

$$\begin{cases} \partial_t u + u \cdot \nabla u - 2\kappa \nabla \times w + \nabla \pi = (\nu + \kappa)\Delta u, \\ \nabla \cdot u = 0, \\ \partial_t w + u \cdot \nabla w + 4\kappa w - 2\kappa \nabla \times u = \gamma \Delta w + \mu \nabla \nabla \cdot w, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the fluid velocity, $w(x,t)$ the field of microrotation representing the angular velocity of the rotation of the fluid particles, $\pi(x,t)$ the scalar pressure, and the parameter ν denotes the kinematic viscosity, κ the microrotation viscosity, and γ and μ the angular viscosities. The 3D micropolar equations reduce to the 2D micropolar equation when

$$u = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0), \quad w = (0,0,w_3(x_1,x_2,t)), \quad \pi = \pi(x_1,x_2,t).$$

More explicitly, the 2D micropolar equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u - 2\kappa \nabla \times w + \nabla \pi = (\nu + \kappa)\Delta u, \\ \nabla \cdot u = 0, \\ \partial_t w + u \cdot \nabla w + 4\kappa w - 2\kappa \nabla \times u = \gamma \Delta w, \end{cases} \tag{1.2}$$

where we have written $u = (u_1, u_2)$ and w for w_3 for notational brevity. Here and in what follows,

$$\Omega = \nabla \times u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \nabla \times w = \left(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right).$$

In addition to their applications in engineering and physics, the micropolar equations are also mathematically significant due to their special structures. The well-posedness problem on the micropolar equations and closely related equations such as the magneto-micropolar equations have attracted considerable attention recently and very interesting results have been established ([7, 10, 12, 11, 16, 23, 30, 31, 33]). Generally speaking, the global regularity problem for the micropolar equations is easier than that for the corresponding incompressible magnetohydrodynamic equations and harder than that for the corresponding incompressible Boussinesq equations.

Recent efforts are focused on the 2D micropolar equations with partial dissipation. When there is full dissipation, the global well-posedness problem on (1.2) is easy and can be solved similarly as that for the 2D Navier-Stokes equations (see, e.g., [3, 4, 6, 28]). When there is only partial dissipation, the global existence and regularity problem can be difficult. Due to recent efforts, the global regularity for several partial dissipation cases have been resolved. In [12] Dong and Zhang obtained the global regularity of (1.2) without the micro-rotation viscosity, namely $\gamma = 0$. For (1.2) with $\nu = 0$, $\gamma > 0$, $\kappa > 0$ and $\kappa \neq \gamma$, Xue obtained the global well-posedness in the frame work of Besov spaces [30]. Very recently, Dong, Li and Wu [11] proved the global well-posedness of (1.2) with only angular viscosity dissipation. [11] makes use of the maximal regularity of the heat operator and introduces a combined quantity to obtain the desired global bounds. In addition, [11] also obtains explicit decay rates of the solutions to this partially dissipated system.

This paper aims at the global existence and regularity of classical solutions to the 2D micropolar equations with fractional dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + (\nu+\kappa)\Lambda^{2\alpha} u - 2\kappa \nabla \times w + \nabla \pi = 0, \\ \nabla \cdot u = 0, \\ \partial_t w + u \cdot \nabla w + \gamma \Lambda^{2\beta} w + 4\kappa w - 2\kappa \nabla \times u = 0, \end{cases} \tag{1.3}$$

where $0 < \alpha, \beta < 1$ and $\Lambda = (-\Delta)^{1/2}$ denotes the Zygmund operator, defined via the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \, \widehat{f}(\xi).$$

Clearly, (1.3) generalizes (1.2) and reduces to (1.2) when $\alpha = \beta = 1$. Mathematically (1.3) has an advantage over (1.2) in the sense that (1.3) allows the study of a family of equations simultaneously. Our attempt is to establish the global regularity of (1.3) with the minimal amount of dissipation, namely for smallest $\alpha, \beta \in (0,1)$. As aforementioned, the two endpoint cases, $\alpha = 1$ and $\beta = 0$, and $\alpha = 0$ and $\beta = 1$ have previously been resolved in [12] and [11], respectively. The global regularity for the general critical case when $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$ appears to be extremely challenging.

When $\alpha + \beta = 1$, the dissipation is not sufficient in controlling the nonlinearity and standard energy estimates do not yield the desired global *a priori* bounds on the solutions. Due to the presence of the linear derivative terms $\nabla \times w$ and $\nabla \times u$ in (1.3), we need $\alpha + \beta > 1$ even in the proof of the global L^2-bound for the solution. It does not appear to be possible to bound the nonlinear terms when we estimate the Sobolev norms of the solutions in the critical or supercritical case $\alpha + \beta \le 1$. This paper focuses on the subcritical case $\alpha + \beta > 1$, but we intend to get as close as possible to the critical case $\alpha + \beta = 1$. We are able to prove the following global existence and regularity result for (1.3).

Theorem 1.1. *Assume $(u_0, w_0) \in H^s(\mathbb{R}^2)$ with $s > 2$ and $\nabla \cdot u_0 = 0$. If α, $\beta \in (0,\, 1)$ satisfy*

$$\beta \begin{cases} > 1 - 2\alpha^2, & 0 < \alpha \le \dfrac{1}{6}; \\ \ge 1 - \dfrac{\alpha}{3}, & \dfrac{1}{6} \le \alpha \le \dfrac{3}{4}; \\ \ge \dfrac{3}{2} - \alpha, & \dfrac{3}{4} \le \alpha \le \dfrac{7}{8}; \\ \ge 5(1-\alpha), & \dfrac{7}{8} \le \alpha \le \dfrac{39}{40}; \\ > \dfrac{1 - \alpha + \sqrt{\alpha^2 - 4\alpha + 3}}{2}, & \dfrac{39}{40} \le \alpha < 1, \end{cases} \tag{1.4}$$

then the fractional 2D micropolar equation (1.3) has a unique global regular solution (u, w) satisfying

$$u \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\alpha}(\mathbb{R}^2));$$
$$w \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\beta}(\mathbb{R}^2)).$$

Even though Theorem 1.1 requires $\alpha + \beta > 1$, we have made serious efforts towards the critical case $\alpha+\beta = 1$. We divide $\alpha \in (0, 1)$ into five different subranges to seek the best estimates so that we can impose the minimal requirements on α and β. As we can tell from (1.4), $\alpha + \beta$ is close to the critical case when either α is close to 0 or close to 1. Figure 1 below depicts the regions of α and β for which the global regularity is established in Theorem 1.1.

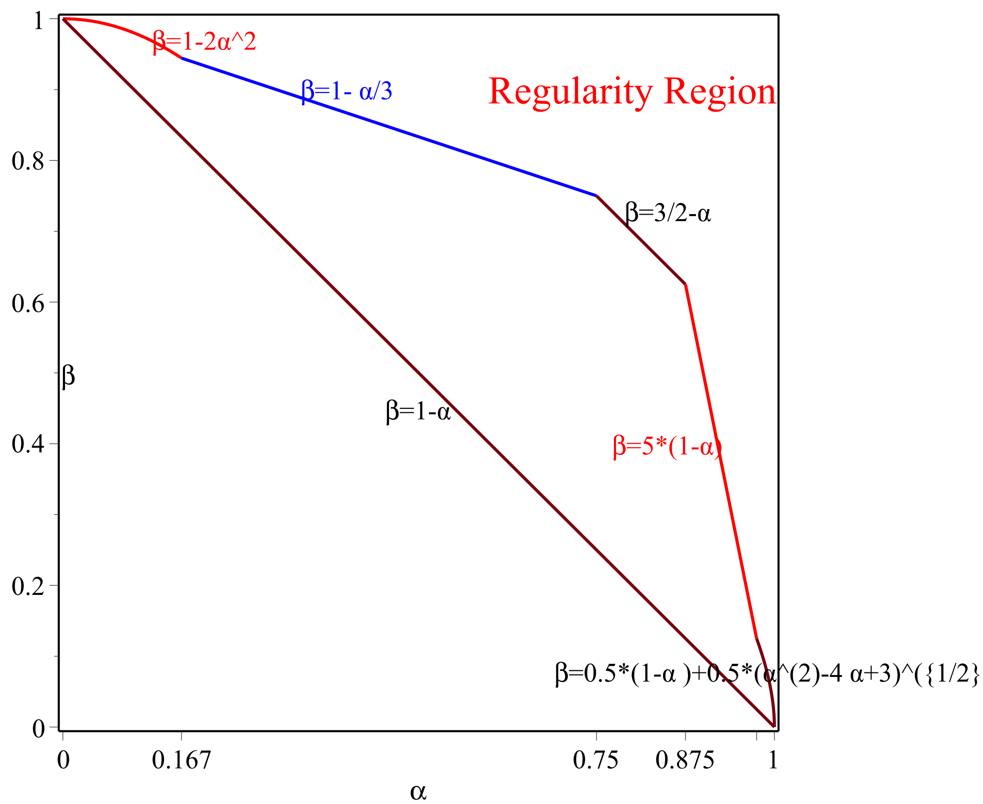

FIGURE 1. Regularity region

We briefly summarize the main challenge for each subrange and explain what we have done to achieve the global regularity. Here and in what follows, we set the viscosity coefficients $\nu = \kappa = \gamma = 1$ for simplicity. In order to prove Theorem 1.1, we need global *a priori* bounds on the solutions in sufficiently functional settings. More precisely, if we can show, for any $T > 0$,

$$\int_0^T \|(\nabla u(t), \nabla w(t))\|_{L^\infty} \, dt < \infty, \tag{1.5}$$

then Theorem 1.1 would follow from a more or less standard procedure. For any $\alpha \in (0, 1)$ and $\alpha+\beta > 1$, the L^2-norm of (u, w) is globally bounded (see Proposition 2.2). The next natural step is to obtain a global H^1-bound for (u, w). We invoke the equation of the vorticity $\Omega \equiv \nabla \times u$,

$$\partial_t \Omega + (u \cdot \nabla)\Omega + 2\Lambda^{2\alpha}\Omega + 2\Delta w = 0. \tag{1.6}$$

For $0 < \alpha < \frac{3}{4}$, we need to estimate $\|\Omega\|_{L^2}$ and $\|\Lambda^{2\beta-1}w\|_{L^2}$ simultaneously in order to bound the coupled terms. The index $2\beta - 1$ is chosen to minimize the requirement on β, which turns out to be

$$\beta \geq 1 - \frac{\alpha}{3}. \tag{1.7}$$

More regular global bound can be obtained for w,

$$\|\Lambda^{\alpha+\beta}w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\alpha+2\beta}w(s)\|_{L^2}^2 ds \leq C(t, \|(u_0, w_0)\|_{H^s}),$$

which, due to $\alpha + 2\beta > 2$, implies $\nabla w \in L_t^1 L_x^\infty$ for all $t > 0$. However,, it appears impossible to derive (1.5) from the vorticity equation (1.6) due to the presence of the term Δw. We overcome this difficulty by considering the combined quantity

$$\Gamma = \Omega + 2\Lambda^{2-2\beta}w,$$

which satisfies

$$\partial_t\Gamma + u\cdot\nabla\Gamma + 2\Lambda^{2\alpha}\Gamma = 4\Lambda^{2+2\alpha-2\beta}w - 8\Lambda^{2-2\beta}w + 4\Lambda^{2-2\beta}\Omega - 2[\Lambda^{2-2\beta}, u\cdot\nabla]w.$$

The equation of Γ eliminates the term Δw from the vorticity equation and makes it possible to estimate the L^q-norm of Γ. In fact, by making use of sharp lower bounds for the dissipative term and suitable commutator estimates, we are able to obtain the global bound for $\|\Gamma\|_{L^q}$ for q satisfying

$$2 \leq q < \frac{2\alpha}{1-\beta}.$$

Due to the regularity of w, we obtain a global bound for $\|\Omega\|_{L^q}$ as a consequence. By further assuming

$$\beta > 1 - 2\alpha^2, \tag{1.8}$$

we are able to show that

$$\begin{aligned}\|\Lambda^2 u(t)\|_{L^2}^2 + \|\Lambda^2 w(t)\|_{L^2}^2 \quad &+ \quad \int_0^t \left(\|\Lambda^{2+\alpha}u\|_{L^2}^2 + \|\Lambda^{2+\beta}w\|_{L^2}^2\right) ds \\ &\leq \quad C(t, \|(u_0, w_0)\|_{H^s}),\end{aligned}$$

which, especially, implies (1.5). (1.7) and (1.8) together yield the restriction on β in (1.4) for $0 < \alpha \leq \frac{3}{4}$.

For $\frac{3}{4} \leq \alpha \leq \frac{7}{8}$ and $\beta \geq \frac{3}{2} - \alpha$, we estimate the L^2-norm of Ω and of $\Lambda^{\frac{1}{2}}w$ simultaneously to establish a global bound for both of them. With this global bound at our disposal, we further establish global bounds for $\|\Lambda^{\frac{3}{2}}w\|_{L^2}$ and $\|\Lambda\Omega\|_{L^2}$ and the corresponding bounds for $w \in L_t^2 H^{\frac{3}{2}+\beta}$ and $\Omega \in L_t^2 H^{1+\alpha}$, which yield the desired bound in (1.5). We remark that the estimates here actually hold for any $\alpha \in (0,1)$ and $\beta \geq \frac{3}{2} - \alpha$. We restrict α to the range $\frac{3}{4} \leq \alpha \leq \frac{7}{8}$ in order to minimize the assumption on β.

For $\frac{7}{8} \leq \alpha < 1$, it appears very difficult to obtain any global bounds beyond the L^2-norm for (u, w). The strategy here is to work with another combined quantity

$$\Gamma = \Omega - \Lambda^{2-2\alpha}w,$$

which satisfies

$$\begin{aligned}&\partial_t G + u\cdot\nabla G + 2\Lambda^{2\alpha}G + 2\Lambda^{2-2\alpha}G \\ &\qquad = \Lambda^{2+2\alpha-2\beta}w + 4\Lambda^{2-2\alpha}w - 2\Lambda^{4-4\alpha}w + [\Lambda^{2-2\alpha}, u\cdot\nabla]w.\end{aligned}$$

The advantage of the G-equation is that it removes Δw from the vorticity equation. For α and β satisfying

$$\beta \geq 5(1-\alpha), \tag{1.9}$$

we are able to establish the global L^2 bound for G, for any $t>0$,

$$\|G(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha G(s)\|_{L^2}^2\, ds \leq C(t, \|(u_0,\, w_0)\|_{H^s}).$$

This global bound serves as an adequate preparation for the following global L^q-bound for w, for any $2 \leq q < \frac{2\beta}{1-\alpha}$,

$$\|w(t)\|_{L^q}^q + \int_0^t \|w(s)\|_{L^{\frac{q}{1-\beta}}}^q\, ds + \int_0^t \|w(s)\|^q_{\dot{B}^{\frac{2\beta}{q}}_{q,q}}\, ds \leq\ C(t, \|(u_0, w_0)\|_{H^s}).$$

where $\dot{B}^{\frac{2\beta}{q}}_{q,q}$ denotes a homogenerous Besov space. More information on Besov spaces are provided in the appendix. Making use of this L^q bound and further assuming that α and β satisfy

$$2\beta^2 - 2(1-\alpha)\beta - (1-\alpha) > 0 \quad \text{or} \quad \beta > \frac{1-\alpha+\sqrt{\alpha^2-4\alpha+3}}{2}, \tag{1.10}$$

we obtain a global bound for $\|\Omega\|_{L^2}$ and $\|\nabla w\|_{L^2}$. To achieve (1.5), we further bound $\|\nabla\Omega\|_{L^2}$ and $\|\Delta w\|_{L^2}$. (1.4) for $\frac{7}{8} \leq \alpha < 1$ is a combination of (1.9) and (1.10).

As aforementioned, once the global bound in (1.5) is established, Theorem 1.1 can then be established following standard approaches. The rest of this paper is divided into four sections and one appendix. Each one of the sections is devoted to establishing the global *a priori* bounds for one of the three cases described above. Section 5 outlines the proof of Theorem 1.1. The appendix provides the definitions and related facts concerning the Besov spaces. In addition, we also supply the details on several notations and simple facts used the regular sections.

2. The case for $0 < \alpha \leq \frac{3}{4}$.

For the sake of clarity, the proof of Theorem 1.1 is split into three major cases. This section is devoted to the case when $0 < \alpha \leq \frac{3}{4}$. The aim here is to prove the global existence and regularity of solutions to (1.3) when $0 < \alpha \leq \frac{3}{4}$ and β satisfies (1.4). More precisely, we prove the following theorem.

Theorem 2.1. *Consider (1.3) with*

$$\beta \begin{cases} > 1 - 2\alpha^2, & 0 < \alpha \leq \dfrac{1}{6}; \\ \geq 1 - \dfrac{\alpha}{3}, & \dfrac{1}{6} \leq \alpha \leq \dfrac{3}{4}. \end{cases} \tag{2.1}$$

Assume (u_0, w_0) satisfies the conditions of Theorem 1.1. Then (1.3) possesses a unique global solution satisfying, for any $T>0$,

$$\begin{aligned} &u \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\alpha}(\mathbb{R}^2)); \\ &w \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\beta}(\mathbb{R}^2)). \end{aligned}$$

As aforementioned, the proof of Theorem 2.1 relies on suitable global *a priori* bounds for the solutions. This section focuses on the necessary global *a priori* bounds. These bounds are proved in Proposition 2.2, Proposition 2.6 and Proposition 2.7.

Proposition 2.2. *Consider (1.3) with α and β satisfying $\beta \geq 1 - \frac{\alpha}{3}$. Assume (u_0, w_0) satisfies the conditions of Theorem 1.1 and let (u, w) be the corresponding solution. Then (u, w) obeys the following global bounds, for any $0 < t < \infty$,*

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^\alpha u(s)\|_{L^2}^2 + \|\Lambda^\beta w(s)\|_{L^2}^2) ds \leq C, \tag{2.2}$$

$$\|\Omega(t)\|_{L^2}^2 + \|\Lambda^{2\beta-1} w(t)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^\alpha \Omega(s)\|_{L^2}^2 + \|\Lambda^{3\beta-1} w(s)\|_{L^2}^2\right) ds \leq C, \tag{2.3}$$

$$\|\Lambda^{\alpha+\beta} w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\alpha+2\beta} w(s)\|_{L^2}^2 ds \leq C, \tag{2.4}$$

where C's depend on t and $\|(u_0, w_0)\|_{H^s}$ only (the explicit dependence can be found in the proof). Especially, due to $\alpha + 2\beta > 2$ according to (2.1), (2.4) implies, for any $T > 0$,

$$\int_0^T \|\nabla w(t)\|_{L^\infty} \, dt \leq C(T, \|(u_0, w_0)\|_{H^s}). \tag{2.5}$$

To prove Proposition 2.2, we recall the following classical commutator estimate (see, e.g., [19], [20, p.334]).

Lemma 2.3. *Let $s > 0$. Let $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then,*

$$\|[\Lambda^s, f]g\|_{L^r} \leq C \left(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right),$$

where C is a constant depending on the indices s, r, p_1, q_1, p_2 and q_2.

The following lemma can be found in [21, p.614].

Lemma 2.4. *Let $0 < s < 1$ and $1 < p < \infty$. Then*

$$\|\Lambda^s (f\, g) - f\, \Lambda^s g - g\, \Lambda^s f\|_{L^p} \leq C\, \|g\|_{L^\infty}\, \|\Lambda^s f\|_{L^p}.$$

The following lemma generalizes the Kato-Ponce inequality, which requires m to be an integer (see, e.g., [21]). This lemma extends it to any real number $m \geq 2$. For the convenience of the readers, we provide a proof for this lemma.

Lemma 2.5. *Let $0 < s < \sigma < 1$, $2 \leq m < \infty$, and $p, q, r \in (1, \infty)^3$ satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, there exists $C = C(s, \sigma, m, p, q, r)$ such that*

$$\big\||f|^{m-2} f\big\|_{L^p} + \|\Lambda^s(|f|^{m-2} f)\|_{L^p} \leq C \|f\|_{B^\sigma_{q,\,p}} \|f\|^{m-2}_{L^{r(m-2)}}. \tag{2.6}$$

Proof of Lemma 2.5. It is easy to see that, for $0 < s < \sigma$ and $p, \tilde{p} \in [1, \infty]^2$, the Besov space $B^\sigma_{p,\tilde{p}}$ is embedded in the Bessel potential space $\mathcal{L}^p_s$ (see, e.g., [17, Chapter 1.3.1]), namely

$$\|g\|_{\mathcal{L}^p_s} \equiv \|g\|_{L^p} + \|\Lambda^s g\|_{L^p} \leq C\, \|g\|_{B^\sigma_{p,\tilde{p}}} \equiv C\, (\|g\|_{L^p} + \|g\|_{\dot{B}^\sigma_{p,\tilde{p}}}), \tag{2.7}$$

where $B^\sigma_{p,\tilde{p}}$ and $\dot{B}^\sigma_{p,\tilde{p}}$ denote the standard inhomogeneous and homogeneous Besov spaces, respectively. Besov spaces and their properties are provided in the appendix.

A short proof for (2.7) is also given in the appendix. Setting $\tilde{p} = p$ and invoking the equivalence definition of $\dot{B}^{\sigma}_{p,p}$ in (A.3), we have

$$\|\Lambda^s(|f|^{m-2}f)\|^p_{L^p} \le C\,\big\||f|^{m-2}f\big\|^p_{L^p} + C\int_{\mathbb{R}^2} \frac{\||f|^{m-2}f(x+\cdot) - |f|^{m-2}f(\cdot)\|^p_{L^p}}{|x|^{2+\sigma p}}\,dx.$$

By the Hölder inequality,

$$\big\||f|^{m-2}f\big\|^p_{L^p} \le \|f\|^p_{L^q}\||f|^{m-2}\|^p_{L^r} = \|f\|^p_{L^q}\|f\|^{p(m-2)}_{L^{r(m-2)}}.$$

Due to the simple inequality

$$\Big||a|^{m-2}a - |b|^{m-2}b\Big| \le C(m)|a-b|(|a|^{m-2} + |b|^{m-2})$$

and Hölder's inequality,

$$\begin{aligned} \||f|^{m-2}f(x+\cdot) - |f|^{m-2}f(\cdot)\|_{L^p} &\le C\|f(x+\cdot) - f(\cdot)\|_{L^q}\||f|^{m-2}\|_{L^r} \\ &\le C\|f(x+\cdot) - f(\cdot)\|_{L^q}\|f\|^{m-2}_{L^{r(m-2)}}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\Lambda^s(|f|^{m-2}f)\|^p_{L^p} &\le C\,\|f\|^p_{L^q}\|f\|^{p(m-2)}_{L^{r(m-2)}} \\ &\quad + C\,\|f\|^{(m-2)p}_{L^{r(m-2)}}\int_{\mathbb{R}^2} \frac{\|f(x+\cdot) - f(\cdot)\|^p_{L^q}}{|x|^{2+\sigma p}}\,dx \\ &\le C\,\|f\|^p_{L^q}\|f\|^{p(m-2)}_{L^{r(m-2)}} + C\,\|f\|^{(m-2)p}_{L^{r(m-2)}}\|f\|^p_{\dot{B}^{\sigma}_{q,\,p}} \\ &\le C\,\|f\|^{(m-2)p}_{L^{r(m-2)}}\|f\|^p_{B^{\sigma}_{q,\,p}}. \end{aligned}$$

This completes the proof of (2.6). □

We remark that, if we replace the Bessel potential space norm by the norm of the Sobolev-Slobodeckij space $\widetilde{W}^{s,p}$, the proof of Lemma 2.5 then implies

$$\big\||f|^{m-2}f\big\|_{\widetilde{W}^{s,p}} \le C\|f\|_{B^s_{q,\,p}}\|f\|^{m-2}_{L^{r(m-2)}}. \tag{2.8}$$

In fact, (2.8) follows

$$\|g\|^p_{\widetilde{W}^{s,p}} \approx \|g\|^p_{L^p} + \left(\int_{\mathbb{R}^2} \frac{\|g(x+\cdot) - g(\cdot)\|^p_{L^p}}{|x|^{2+sp}}\,dx\right)^p$$

and combined with the rest of the proof for Lemma 2.5. The definition of $\widetilde{W}^{s,p}$ and some embedding properties are given in the appendix. Here we also want to remark that unfortunately, it is not clear whether the term $\|f\|_{B^s_{q,\,p}}$ of (2.8) can be replaced by $\|f\|_{\widetilde{W}^{s,q}}$.

Proof of Proposition 2.2. Taking the L^2 inner product of (1.3) with (u, w), we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{L^2} + \|w(t)\|^2_{L^2}\right) + 2\|\Lambda^{\alpha}u(t)\|^2_{L^2} + \|\Lambda^{\beta}w(t)\|^2_{L^2} + 4\|w(t)\|^2_{L^2} \\ = \;& 2\int_{\mathbb{R}^2} \{(\nabla\times w)\cdot u + (\nabla\times u)w\}\,dx \\ \le \;& 4\|\Lambda^{\alpha}u\|_{L^2}\|\Lambda^{1-\alpha}w\|_{L^2} \\ \le \;& \|\Lambda^{\alpha}u(t)\|^2_{L^2} + \frac{1}{2}\|\Lambda^{\beta}w(t)\|^2_{L^2} + C\|w(t)\|^2_{L^2}, \end{aligned}$$

where we have used the condition $\alpha+\beta>1$ in the last inequality as well as the following facts, due to $\nabla\cdot u=0$,

$$\int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot u\;dx=0\quad\text{and}\quad\int_{\mathbb{R}^2}(u\cdot\nabla w)\;w\;dx=0.$$

Applying Gronwall inequality gives, for $0<t<\infty$,

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2+\int_0^t(\|\Lambda^\alpha u(s)\|_{L^2}^2+\|\Lambda^\beta w(s)\|_{L^2}^2)ds\\ \le\;& e^{Ct}\left(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2\right),\end{aligned}\tag{2.9}$$

which is (2.2). To prove (2.3) and (2.4), we apply $\nabla\times$ to the first equation of (1.3) to obtain the vorticity equation

$$\partial_t\Omega+(u\cdot\nabla)\Omega+2\Lambda^{2\alpha}\Omega+2\Delta w=0,\tag{2.10}$$

where we have used $-\nabla\times\nabla\times w=\Delta w$. Taking the inner product of (2.10) with Ω and the inner product of third equation of (1.3) with $\Lambda^{2(2\beta-1)}w$ leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2+\|\Lambda^{2\beta-1}w\|_{L^2}^2\right)+2\|\Lambda^\alpha\Omega\|_{L^2}^2+\|\Lambda^{3\beta-1}w\|_{L^2}^2+4\|\Lambda^{2\beta-1}w\|_{L^2}^2\\ =\;&2\int_{\mathbb{R}^2}\left(\Lambda^2w\Omega+\Omega\Lambda^{2(2\beta-1)}w\right)dx-\int_{\mathbb{R}^2}[\Lambda^{2\beta-1},u\cdot\nabla]w\Lambda^{2\beta-1}wdx\\ :=\;&I_1+I_2,\end{aligned}\tag{2.11}$$

where we have used the facts

$$\int_{\mathbb{R}^2}(u\cdot\nabla\Omega)\Omega\;dx=0\quad\text{and}\quad\int_{\mathbb{R}^2}(u\cdot\nabla\Lambda^{2\beta-1}w)\Lambda^{2\beta-1}w\;dx=0.$$

To estimate I_1, we integrate by parts and apply Hölder's inequality, the Gagliardo-Nirenberg inequality and Young's inequality to obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\Lambda^2w\Omega+\Omega\Lambda^{2(2\beta-1)}w\right)dx\\ \le\;&\|\Lambda^\alpha\Omega\|_{L^2}\|\Lambda^{2-\alpha}w\|_{L^2}+\|\Omega\|_{L^2}\|\Lambda^{2(2\beta-1)}w\|_{L^2}\\ \le\;&\|\Lambda^\alpha\Omega\|_{L^2}(\|w\|_{L^2}^{\frac{\alpha+3\beta-3}{3\beta-1}}\|\Lambda^{3\beta-1}w\|_{L^2}^{\frac{2-\alpha}{3\beta-1}})+\|\Omega\|_{L^2}(\|\Lambda^\beta w\|_{L^2}^{\frac{1-\beta}{2\beta-1}}\|\Lambda^{3\beta-1}w\|_{L^2}^{\frac{3\beta-2}{2\beta-1}})\\ \le\;&\frac{1}{2}\|\Lambda^\alpha\Omega\|_{L^2}^2+\frac{2}{3}\|\Lambda^{3\beta-1}w\|_{L^2}^2+C\|\Omega\|_{L^2}^2+C\|w\|_{L^2}^2+C\|\Lambda^\beta w\|_{L^2}^2\end{aligned}$$

where we have used (2.1),

$$\alpha+3\beta\ge3\quad\text{and}\quad\beta<2(2\beta-1)<3\beta-1.$$

To estimate I_2, we employ Hölder's inequality and Sobolev's inequality and invoke Lemma 2.3 to obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}[\Lambda^{2\beta-1},u\cdot\nabla]w\,\Lambda^{2\beta-1}wdx\\ \le\;&\left(\|\nabla u\|_{L^2}\|\Lambda^{2\beta-1}w\|_{L^q}+\|\Lambda^{2\beta-1}u\|_{L^{p_1}}\|\nabla w\|_{L^{q_1}}\right)\|\Lambda^{2\beta-1}w\|_{L^r}\\ \le\;&\|\Omega\|_{L^2}\|\Lambda^{3\beta-1}w\|_{L^2}\|\Lambda^\beta w\|_{L^2}\\ \le\;&C\|\Omega\|_{L^2}^2\|\Lambda^\beta w\|_{L^2}^2+\frac{1}{16}\|\Lambda^{3\beta-1}w\|_{L^2}^2,\end{aligned}$$

where the indices are given by

$$q=\frac{2}{1-\beta},\quad r=\frac{2}{\beta},\quad p_1=\frac{2}{2\beta-1},\quad q_1=\frac{2}{3(1-\beta)}$$

and they are so chosen to fulfill the requirements of the Sobolev inequalities,

$$\frac{1}{q}+\frac{1}{r}=\frac{1}{2}, \quad \frac{1}{q}-\frac{2\beta-1}{2}=\frac{1}{2}-\frac{3\beta-1}{2}, \quad \frac{1}{r}-\frac{2\beta-1}{2}=\frac{1}{2}-\frac{\beta}{2},$$
$$\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r}=1, \quad \frac{1}{p_1}-\frac{2\beta-1}{2}=\frac{1}{2}-\frac{1}{2}, \quad \frac{1}{q_1}-\frac{1}{2}=\frac{1}{2}-\frac{3\beta-1}{2}.$$

Inserting the bounds for I_1 and I_2 in(2.11) yields

$$\begin{aligned} &\frac{d}{dt}\left(\|\Omega\|_{L^2}^2+\|\Lambda^{2\beta-1}w\|_{L^2}^2\right)+\|\Lambda^{\alpha}\Omega\|_{L^2}^2+\|\Lambda^{3\beta-1}w\|_{L^2}^2 \\ \leq\; & C\|\Omega\|_{L^2}^2\left(\|\Lambda^{\beta}w\|_{L^2}^2+1\right)+C\|\Lambda^{\beta}w\|_{L^2}^2. \end{aligned}$$

Gronwall inequality together with (2.9) implies

$$\begin{aligned} &\|\Omega(t)\|_{L^2}^2+\|\Lambda^{2\beta-1}w(t)\|_{L^2}^2+\int_0^t\left(\|\Lambda^{\alpha}\Omega(s)\|_{L^2}^2+\|\Lambda^{3\beta-1}w(s)\|_{L^2}^2\right)ds \\ &\leq e^{\left\{Ct+e^{Ct}\left(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2\right)\right\}}\left(\|\nabla u_0\|_{L^2}^2+\|\nabla w_0\|_{L^2}^2+e^{Ct}\left(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2\right)\right), \end{aligned}$$

which is (2.3). We now prove (2.4). Taking the inner product of third equation of (1.3) with $\Lambda^{2(\alpha+\beta)}w$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^{\alpha+\beta}w\|_{L^2}^2+\|\Lambda^{\alpha+2\beta}w\|_{L^2}^2+4\|\Lambda^{\alpha+\beta}w\|_{L^2}^2 \\ =\;& 2\int_{\mathbb{R}^2}\Omega\Lambda^{2(\alpha+\beta)}w\,dx-\int_{\mathbb{R}^2}[\Lambda^{\alpha+\beta},u\cdot\nabla]w\Lambda^{\alpha+\beta}w\,dx \\ :=\;& J_1+J_2, \end{aligned} \tag{2.12}$$

J_1 is bounded by

$$J_1\leq 2\|\Lambda^{\alpha}\Omega\|_{L^2}\|\Lambda^{\alpha+2\beta}w\|_{L^2}\leq 4\|\Lambda^{\alpha}\Omega\|_{L^2}^2+\frac{1}{4}\|\Lambda^{\alpha+2\beta}w\|_{L^2}^2.$$

Similar to the estimates for I_2, we have, after applying Hölder's inequality, Sobolev's imbedding inequality and Lemma 2.3,

$$\begin{aligned} &\int_{\mathbb{R}^2}[\Lambda^{\alpha+\beta},u\cdot\nabla]w\Lambda^{\alpha+\beta}w\,dx \\ \leq\;& \left(\|\nabla u\|_{L^{\frac{2}{\beta}}}\|\Lambda^{\alpha+\beta}w\|_{L^{\frac{2}{1-\beta}}}+\|\Lambda^{\alpha+\beta}u\|_{L^2}\|\nabla w\|_{L^\infty}\right)\|\Lambda^{\alpha+\beta}w\|_{L^2} \\ \leq\;& C\left(\|\Lambda^{1-\beta}\Omega\|_{L^2}+\|\Lambda^{\alpha+\beta-1}\Omega\|_{L^2}\right)\|w\|_{H^{\alpha+2\beta}}\|\Lambda^{\alpha+\beta}w\|_{L^2} \\ \leq\;& C\left(\|\Lambda^{\alpha}\Omega\|_{L^2}^2+\|\Omega\|_{L^2}^2\right)\|\Lambda^{\alpha+\beta}w\|_{L^2}^2+C\|w\|_{L^2}^2+\frac{1}{4}\|\Lambda^{\alpha+2\beta}w\|_{L^2}^2, \end{aligned}$$

where we have used that α and β satisfy

$$0<\alpha,\beta<1, \quad \alpha+2\beta>2.$$

Inserting the bounds for J_1 and J_2 in (2.12) and applying Gronwall's inequality, we have

$$\|\Lambda^{\alpha+\beta}w(t)\|_{L^2}^2+\int_0^t\left(\|\Lambda^{\alpha+2\beta}w(s)\|_{L^2}^2\right)ds\leq C(t,u_0,w_0)$$

which is (2.4). (2.5) follows from (2.4) via Sobolev's inequality. This completes the proof of Proposition 2.2. □

The global bounds in Proposition 2.2 are not sufficient to prove Theorem 2.1. More regular global bounds are needed. In particular, if we have, for any $T > 0$,

$$\int_0^T \|\Omega(t)\|_{L^\infty}\, dt < \infty, \tag{2.13}$$

the global existence and regularity then follows. It does not appear plausible to prove (2.13) directly via (2.10) when $0 < \alpha < \frac{1}{2}$. Due to the term Δw in (2.10), we need $\Lambda^2 w \in L^1_t L^\infty$, which is unavailable at this moment. To overcome this difficulty, we work with the combined quantity

$$\Gamma = \Omega + 2\Lambda^{2-2\beta} w.$$

Applying $\Lambda^{2-2\beta}$ to the second equation in (1.3) leads to

$$\partial_t(\Lambda^{2-2\beta} w) + u\cdot\nabla\Lambda^{2-2\beta} w + \Lambda^2 w + 4\Lambda^{2-2\beta} w - 2\Lambda^{2-2\beta}\Omega = -[\Lambda^{2-2\beta}, u\cdot\nabla] w,$$

which, together with (2.10), yields the equation for Γ,

$$\begin{aligned} & \partial_t\Gamma + u\cdot\nabla\Gamma + 2\Lambda^{2\alpha}\Gamma \\ = \;& 4\Lambda^{2+2\alpha-2\beta} w - 8\Lambda^{2-2\beta} w + 4\Lambda^{2-2\beta}\Omega - 2[\Lambda^{2-2\beta}, u\cdot\nabla] w. \end{aligned} \tag{2.14}$$

Although (2.14) appears to be more complex than (2.10), it eliminates the most regularity demanding term Δw and allows us to derive the L^q bounds of Γ, which is crucial to derive the $\Omega \in L^1_t L^\infty$. More precisely, we prove the following proposition.

Proposition 2.6. *Consider (1.3) with α and β satisfying (2.1). Assume (u_0, w_0) satisfies the conditions of Theorem 1.1 and let (u, w) be the corresponding solution. Then, for q satisfying*

$$2 \le q < \frac{2\alpha}{1-\beta}, \tag{2.15}$$

(u, w) obeys the following global bounds,

$$\|\Gamma(t)\|_{L^q}^q + \int_0^t \|\Gamma(s)\|^q_{L^{\frac{q}{1-\alpha}}}\, ds \le C, \tag{2.16}$$

where $C > 0$ depends only on t and $\|(u_0, w_0)\|_{H^s}$.

Proof. To start, by the above estimates (2.2)-(2.4) and $\Gamma = \Omega + 2\Lambda^{2-2\beta} w$, one gets

$$\begin{aligned} \|\Gamma\|_{L^2} &\le \|\Omega\|_{L^2} + C\|\Lambda^{2-2\beta} w\|_{L^2} \\ &\le \|\Omega\|_{L^2} + C\|w\|_{L^2}^{\frac{\alpha+3\beta-2}{\alpha+\beta}} \|\Lambda^{\alpha+\beta} w\|_{L^2}^{\frac{2-2\beta}{\alpha+\beta}} \\ &\le C(t,\, u_0,\, w_0) \end{aligned}$$

and $\Lambda^\alpha\Gamma = \Lambda^\alpha\Omega + 2\Lambda^{\alpha+2-2\beta} w$ as well as

$$\begin{aligned} \int_0^t \|\Lambda^\alpha\Gamma\|_{L^2}^2\, ds &\le \int_0^t \|\Lambda^\alpha\Omega\|_{L^2}^2\, ds + C\int_0^t \|\Lambda^{2+\alpha-2\beta} w\|_{L^2}^2\, ds \\ &\le \int_0^t \|\Lambda^\alpha\Omega\|_{L^2}^2\, ds + C\int_0^t \left(\|\Lambda^{\alpha+2\beta} w\|_{L^2}^{\frac{2(2+\alpha-2\beta)}{\alpha+2\beta}} \|w\|_{L^2}^{\frac{2(3\beta-2)}{\alpha+2\beta}} \right) ds \\ &\le C(t,\, u_0,\, w_0), \end{aligned}$$

which imply

$$\|\Gamma(t)\|_{L^2}^2 + \int_0^t \|\Gamma(\tau)\|^2_{L^{\frac{2}{1-\alpha}}}\, d\tau \le C(t,\, u_0,\, w_0).$$

Multiplying (2.14) by $|\Gamma|^{q-2}\Gamma$ and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned} & \frac{d}{dt}\|\Gamma\|_{L^q}^q + C_0\|\Gamma\|^q_{\dot{B}^{\frac{2\alpha}{q}}_{q,q}} + C_1\|\Gamma\|^q_{L^{\frac{q}{1-\alpha}}} + C_2\|\Lambda^\alpha\left(|\Gamma|^{\frac{q}{2}}\right)\|_{L^2}^2 \\ \le\ & \frac{q}{2}\int_{\mathbb{R}^2}\left(4\Lambda^{2+2\alpha-2\beta}w - 8\Lambda^{2-2\beta}w\right)|\Gamma|^{q-2}\Gamma dx \\ & + \frac{q}{2}\int_{\mathbb{R}^2}4\Lambda^{2-2\beta}\Omega\,|\Gamma|^{q-2}\Gamma\,dx - \frac{q}{2}\int_{\mathbb{R}^2}2[\Lambda^{2-2\beta}, u\cdot\nabla]w|\Gamma|^{q-2}\Gamma\,dx \\ :=\ & K_1 + K_2 + K_3, \end{aligned} \tag{2.17}$$

where we have invoked the following lower bounds associated with the fractional dissipation term, for any $q \in [2,\infty)$ and $s \in (0,1)$,

$$\int_{\mathbb{R}^2}|f|^{q-2}f\,\Lambda^{2s}f\,dx \ge C(s,q)\|\Lambda^s\left(|f|^{\frac{q}{2}}\right)\|_{L^2}^2, \tag{2.18}$$

$$\int_{\mathbb{R}^2}|f|^{q-2}f\,\Lambda^{2s}f\ dx \ge C(s,q)\|f\|^q_{L^{\frac{q}{1-s}}}, \tag{2.19}$$

$$\int_{\mathbb{R}^2}|f|^{q-2}f\,\Lambda^{2s}f\ dx \ge C(s,q)\|f\|^q_{\dot{B}^{\frac{2s}{q}}_{q,q}}, \tag{2.20}$$

where $\dot{B}^s_{p,q}$ denotes the standard homogeneous Besov space (see the appendix for more details). (2.18) can be found in ([8]), (2.19) follows from (2.18) via the Sobolev inequality and (2.20) is due to [5, Theorem 2]. By the Hölder inequality and the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned} K_1 \ \le\ & C\left\|\Lambda^\alpha\big(|\Gamma|^{\frac{q}{2}}\big)\right\|_{L^2}\left\|\Lambda^{-\alpha}\big\{(4\Lambda^{2+2\alpha-2\beta}w - 8\Lambda^{2-2\beta}w)\ |\Gamma|^{\frac{q}{2}-2}\Gamma\big\}\right\|_{L^2} \\ \le\ & C\left\|\Lambda^\alpha\big(|\Gamma|^{\frac{q}{2}}\big)\right\|_{L^2}\left\|(4\Lambda^{2+2\alpha-2\beta}w - 8\Lambda^{2-2\beta}w)\ |\Gamma|^{\frac{q}{2}-2}\Gamma\right\|_{L^{\frac{2}{1+\alpha}}} \\ \le\ & C\left\|\Lambda^\alpha\big(|\Gamma|^{\frac{q}{2}}\big)\right\|_{L^2}\left\|(4\Lambda^{2+2\alpha-2\beta}w - 8\Lambda^{2-2\beta}w)\right\|_{L^{\frac{2}{1+\alpha-\beta}}}\left\||\Gamma|^{\frac{q}{2}-2}\Gamma\right\|_{L^{\frac{2}{\beta}}} \\ \le\ & C\left\|\Lambda^\alpha\big(|\Gamma|^{\frac{q}{2}}\big)\right\|_{L^2}\Big(\|\Lambda^{2+2\alpha-2\beta}w\|_{L^{\frac{2}{1+\alpha-\beta}}} + \|\Lambda^{2-2\beta}w\|_{L^{\frac{2}{1+\alpha-\beta}}}\Big)\|\Gamma\|^{\frac{q}{2}-1}_{L^{\frac{q-2}{\beta}}} \\ \le\ & C\left\|\Lambda^\alpha\big(|\Gamma|^{\frac{q}{2}}\big)\right\|_{L^2}(\|w\|_{L^2} + \|\Lambda^{\alpha+2\beta}w\|_{L^2})\|\Gamma\|^{\frac{q}{2}-1}_{L^{\frac{q-2}{\beta}}} \\ \le\ & \frac{C_2}{8}\left\|\Lambda^\alpha\big(|\Gamma|^{\frac{q}{2}}\big)\right\|^2_{L^2} + C(\|w\|_{L^2} + \|\Lambda^{\alpha+2\beta}w\|_{L^2})^2\,(\|\Gamma\|^{q-2}_{L^2} + \|\Gamma\|^{q-2}_{L^q}), \end{aligned}$$

where we have used the following inequality

$$\|\Lambda^{2+2\alpha-2\beta}w\|_{L^{\frac{2}{1+\alpha-\beta}}} + \|\Lambda^{2-2\beta}w\|_{L^{\frac{2}{1+\alpha-\beta}}} \le C(\|w\|_{L^2} + \|\Lambda^{\alpha+2\beta}w\|_{L^2}),$$

following from the Gagliardo-Nirenberg inequality, for any $\alpha - \beta \le s \le 2\alpha + \beta$,

$$\|\Lambda^s w\|_{L^{\frac{2}{1+\alpha-\beta}}} \le C\|w\|^{1-\vartheta}_{L^2}\|\Lambda^{\alpha+2\beta}w\|^{\vartheta}_{L^2}, \quad \vartheta = \frac{s+\beta-\alpha}{\alpha+2\beta}, \quad \beta \ge \alpha.$$

Noting that $\Gamma = \Omega + 2\Lambda^{2-2\beta}w$, we split K_2 as

$$\begin{aligned} K_2 \ =\ & C\int_{\mathbb{R}^2}\Lambda^{2-2\beta}\Omega\ |\Gamma|^{q-2}\Gamma\,dx \\ =\ & -C\int_{\mathbb{R}^2}\Lambda^{4-4\beta}w\ |\Gamma|^{q-2}\Gamma\,dx + C\int_{\mathbb{R}^2}\Lambda^{2-2\beta}\Gamma\ |\Gamma|^{q-2}\Gamma\,dx \\ :=\ & K_{21} + K_{22}. \end{aligned}$$

Clearly, due to $4-4\beta \le 2+2\alpha-2\beta$, K_{21} can be estimated similarly as K_1 and

$$K_{21} \le \frac{C_2}{16}\left\|\Lambda^\alpha\big(|\Gamma|^{\frac{q}{2}}\big)\right\|_{L^2}^2 + C(\|w\|_{L^2}+\|\Lambda^{\alpha+2\beta}w\|_{L^2})^2\,(\|\Gamma\|_{L^2}^{q-2}+\|\Gamma\|_{L^q}^{q-2}).$$

To estimate K_{22}, we assume that q satisfies (2.15), namely

$$2 \le q < \frac{2\alpha}{1-\beta},$$

which implies $2-2\beta-\frac{2\alpha}{q} < \frac{2\alpha}{q}$. We then choose $0<s<\sigma<1$ satisfying

$$2-2\beta-\frac{2\alpha}{q} < s < \sigma < \frac{2\alpha}{q}. \tag{2.21}$$

By Lemma 2.5,

$$\begin{aligned}
K_{22} &\le C\|\Lambda^{2-2\beta-s}\Gamma\|_{L^q}\left\||\Lambda^s(|\Gamma|^{q-2}\Gamma)\right\|_{L^{\frac{q}{q-1}}}\\
&\le C\left(\|\Gamma\|_{\dot B^{\frac{2\alpha}{q}}_{q,q}}+\|\Gamma\|_{L^q}\right)\|\Gamma\|_{B^\sigma_{q,\frac{q}{q-1}}}\||\Gamma|^{q-2}\|_{L^{\frac{q}{q-2}}}\\
&\le C\left(\|\Gamma\|_{\dot B^{\frac{2\alpha}{q}}_{q,q}}+\|\Gamma\|_{L^q}\right)\|\Gamma\|_{B^\sigma_{q,\frac{q}{q-1}}}\|\Gamma\|^{q-2}_{L^q}\\
&\le C\left(\|\Gamma\|_{\dot B^{\frac{2\alpha}{q}}_{q,q}}+\|\Gamma\|_{L^q}\right)\|\Gamma\|_{B^{\frac{2\alpha}{q}}_{q,q}}\|\Gamma\|^{q-2}_{L^q}\\
&\le \frac{C_0}{4}\|\Gamma\|^q_{\dot B^{\frac{2\alpha}{q}}_{q,q}}+C\|\Gamma\|^q_{L^q}.
\end{aligned}$$

where we have used the embeddings, due to (2.21),

$$B^{\frac{2\alpha}{q}}_{q,q}\hookrightarrow \dot W^{2-2\beta-s,q},\qquad B^{\frac{2\alpha}{q}}_{q,q}\hookrightarrow B^\sigma_{q,\frac{q}{q-1}}.$$

We now turn to the estimate of K_3. By Lemma 2.4,

$$\begin{aligned}
K_3 &= -q\int_{\mathbb{R}^2}[\Lambda^{2-2\beta},u\cdot\nabla]w|\Gamma|^{q-2}\Gamma dx\\
&\le C\|\Lambda^{2-2\beta}(u\cdot\nabla w)-u\cdot\nabla\Lambda^{2-2\beta}w\|_{L^q}\left\||\Gamma|^{q-2}\Gamma\right\|_{L^{\frac{q}{q-1}}}\\
&\le C\left(\|\nabla w\,\Lambda^{2-2\beta}u\|_{L^q}+\|\Lambda^{2-2\beta}u\|_{L^q}\|\nabla w\|_{L^\infty}\right)\|\Gamma\|^{q-1}_{L^q}\\
&\le C\,\|\Lambda^{2-2\beta}u\|_{L^q}\,\|\nabla w\|_{L^\infty}\,\|\Gamma\|^{q-1}_{L^q}.
\end{aligned}$$

Due to $\alpha+2\beta>2$, Sobolev's inequality implies

$$\|\Lambda^{2-2\beta}u\|_{L^q}\le C\,(\|u\|_{L^2}+\|\Lambda^\alpha\Omega\|_{L^2}),\qquad \|\nabla w\|_{L^\infty}\le C\,(\|w\|_{L^2}+\|\Lambda^{\alpha+2\beta}w\|_{L^2}).$$

Therefore,

$$K_3\le C\left(\|u\|^2_{L^2}+\|w\|^2_{L^2}++\|\Lambda^\alpha\Omega\|^2_{L^2}+\|\Lambda^{\alpha+2\beta}w\|^2_{L^2}\right)(1+\|\Gamma\|^q_{L^q}).$$

Inserting the estimates for K_1, K_2 and K_3 in (2.17), we have

$$\begin{aligned}
&\frac{d}{dt}\|\Gamma\|^q_{L^q}+\frac{3}{4}C_0\|\Gamma\|^q_{\dot B^{\frac{2\alpha}{q}}_{q,q}}+C_1\|\Gamma\|^q_{L^{\frac{q}{1-\alpha}}}+\frac{3}{4}C_2\|\Lambda^\alpha\left(|\Gamma|^{\frac{q}{2}}\right)\|^2_{L^2}\\
\le\;& C(1+\|u\|^2_{L^2}+\|w\|^2_{L^2}+\|\Lambda^{\alpha+2\beta}w\|^2_{L^2}+\|\Lambda^\alpha\Omega\|^2_{L^2})(1+\|\Gamma\|^q_{L^q}).
\end{aligned}$$

Gronwall's inequality then implies

$$\|\Gamma(t)\|^q_{L^q}+\int_0^t\|\Gamma(s)\|^q_{L^{\frac{q}{1-\alpha}}}\,ds\le C.$$

This completes the proof of Proposition 2.6. $\square$

With the global *a priori* bounds in the two previous propositions at our disposal, we are ready to show that ∇u is in $L^1_t L^\infty_x$, which especially implies (2.13).

Proposition 2.7. *Consider (1.3) with α and β satisfying (2.1). Assume (u_0, w_0) satisfies the conditions of Theorem 1.1 and let (u, w) be the corresponding solution. Then (u, w) satisfies, for any $0 < t < \infty$,*

$$\begin{aligned} \|\Lambda^2 u(t)\|_{L^2}^2 + \|\Lambda^2 w(t)\|_{L^2}^2 \quad &+ \quad \int_0^t \left(\|\Lambda^{2+\alpha} u\|_{L^2}^2 + \|\Lambda^{2+\beta} w\|_{L^2}^2\right)(s) ds \\ &\le \quad C(t, u_0, w_0). \end{aligned} \tag{2.22}$$

As a special consequence, for any $t > 0$,

$$\int_0^t \|\nabla u(s)\|_{L^\infty}\, ds \le C(t, \|(u_0, w_0)\|_{H^s}) < \infty. \tag{2.23}$$

Proof. Taking the L^2 inner product of (1.3) with $(\Lambda^4 u, \Lambda^4 w)$, we find

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\|\Lambda^2 u\|_{L^2}^2 + \|\Lambda^2 w\|_{L^2}^2\right) + 2\|\Lambda^{2+\alpha} u\|_{L^2}^2 + \|\Lambda^{2+\beta} w\|_{L^2}^2 + 4\|\Lambda^2 w\|_{L^2}^2 \\ = \quad & 2\int_{\mathbb{R}^2} \left((\nabla \times u)\Lambda^4 w + (\nabla \times w)\cdot \Lambda^4 u\right) dx \\ & -\int_{\mathbb{R}^2} [\Lambda^2, u\cdot\nabla]u\ \Lambda^2 u\, dx - \int_{\mathbb{R}^2} [\Lambda^2, u\cdot\nabla]w\ \Lambda^2 w\, dx \\ := \quad & L_1 + L_2 + L_3. \end{aligned} \tag{2.24}$$

Noting that $2 < 3 - \alpha < 2 + \beta$, we obtain by applying Hölder's inequality

$$\begin{aligned} L_1 \quad &\le \quad C\|\Lambda^{2+\alpha} u\|_{L^2}\|\Lambda^{3-\alpha} w\|_{L^2} \\ &\le \quad \frac{1}{2}\|\Lambda^{2+\alpha} u\|_{L^2}^2 + C\|\Lambda^{3-\alpha} w\|_{L^2}^2 \\ &\le \quad \frac{1}{2}\|\Lambda^{2+\alpha} u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{2+\beta} w\|_{L^2}^2 + C\|\Lambda^2 w\|_{L^2}^2. \end{aligned}$$

By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned} L_2 \quad &\le \quad C\Big|\int_{\mathbb{R}^2} \nabla u\, \nabla^2 u\, \nabla^2 u\, dx\Big| \\ &\le \quad C\|\nabla u\|_{L^{\frac{q}{1-\alpha}}}\|\Lambda^2 u\|^2_{L^{\frac{2q}{q+\alpha-1}}} \\ &\le \quad C\|\Omega\|_{L^{\frac{q}{1-\alpha}}}\|\Lambda^2 u\|_{L^2}^{2(1-\frac{1-\alpha}{\alpha q})}\|\Lambda^\alpha \Lambda^2 u\|_{L^2}^{\frac{2(1-\alpha)}{\alpha q}} \\ &\le \quad \frac{1}{8}\|\Lambda^{2+\alpha} u\|_{L^2}^2 + C\|\Omega\|_{L^{\frac{q}{1-\alpha}}}^{\frac{\alpha q}{\alpha q+\alpha-1}}\|\Lambda^2 u\|_{L^2}^2. \end{aligned}$$

where $\nabla^2 u$ denotes all second-order partial derivatives of u. We will need the global bound, for any $T > 0$,

$$\int_0^T \|\Omega(s)\|_{L^{\frac{q}{1-\alpha}}}^{\frac{\alpha q}{\alpha q+\alpha-1}}\, ds < \infty. \tag{2.25}$$

By Propositions 2.2 and 2.6, we have, for q satisfying (2.15),

$$\int_0^t \|\Omega(s)\|_{L^{\frac{q}{1-\alpha}}}^q\, ds \quad \le \quad \int_0^t \|\Gamma(s)\|_{L^{\frac{q}{1-\alpha}}}^q\, ds + C\int_0^t \|\Lambda^{2-2\beta} w(s)\|_{L^{\frac{q}{1-\alpha}}}^q\, ds$$

$$\begin{aligned} &\le \int_0^t \|\Gamma(s)\|^q_{L^{\frac{q}{1-\alpha}}}\, ds + C\int_0^t (\|w\|_{L^2} + \|\Lambda^{\alpha+\beta} w\|_{L^2})^q\, ds \\ &\le C, \end{aligned}$$

Therefore, if

$$\frac{\alpha q}{\alpha q + \alpha - 1} \le q \qquad \text{or} \qquad q \ge \frac{1}{\alpha},$$

then (2.25) holds. This is where we need $\beta > 1 - 2\alpha^2$ in (2.1) on α and β. When α and β satisfy (2.1), we can then choose q satisfying (2.15) such that

$$\frac{2\alpha}{1-\beta} > q > \frac{1}{\alpha}.$$

Noting that $2 < \frac{5-\alpha}{2} < 2 + \beta$, we have

$$\begin{aligned} L_3 &\le C\Big|\int_{\mathbb{R}^2} \nabla u\, \nabla^2 w\, \nabla^2 w\, dx\Big| + C\Big|\int_{\mathbb{R}^2} \nabla^2 u\, \nabla w\, \nabla^2 w\, dx\Big| \\ &\le C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}} \|\Lambda^2 w\|^2_{L^{\frac{4}{1+\alpha}}} + C\|\nabla w\|_{L^\infty} \left(\|\Lambda^2 u\|^2_{L^2} + \|\Lambda^2 w\|^2_{L^2}\right) \\ &\le C\|\Lambda^{1+\alpha} u\|_{L^2} \|\Lambda^{\frac{5-\alpha}{2}} w\|^2_{L^2} + C\|\nabla w\|_{L^\infty} \left(\|\Lambda^2 u\|^2_{L^2} + \|\Lambda^2 w\|^2_{L^2}\right) \\ &\le C\|\Lambda^{1+\alpha} u\|_{L^2} \|\Lambda^2 w\|^{2\sigma}_{L^2} \|\Lambda^{2+\beta} w\|^{2-2\sigma}_{L^2} + C\|\nabla w\|_{L^\infty} \left(\|\Lambda^2 u\|^2_{L^2} + \|\Lambda^2 w\|^2_{L^2}\right) \\ &\le \frac{1}{4}\|\nabla^{2+\beta} w\|^2_{L^2} + C\|\Lambda^{1+\alpha} u\|^{\frac{1}{\sigma}}_{L^2} \|\Lambda^2 w\|^2_{L^2} \\ &\quad + C\|\nabla w\|_{L^\infty} \left(\|\Lambda^2 u\|^2_{L^2} + \|\Lambda^2 w\|^2_{L^2}\right), \end{aligned}$$

where, due to $\alpha + \beta > 1$,

$$\sigma = \frac{\alpha + 2\beta - 1}{2\beta} > \frac{1}{2}.$$

Inserting the estimates of L_1, L_2 and L_3 in (2.24), it follows that

$$\begin{aligned} &\frac{d}{dt}\left(\|\Lambda^2 u\|^2_{L^2} + \|\Lambda^2 w\|^2_{L^2}\right) + \|\Lambda^{2+\alpha} u\|^2_{L^2} + \|\Lambda^{2+\beta} w\|^2_{L^2} \\ \le\ & C\left(1 + \|\Lambda^{1+\alpha} u\|^2_{L^2} + \|\nabla w\|_{L^\infty} + \|\Omega\|^{\frac{\alpha q}{\alpha q + \alpha - 1}}_{L^{\frac{q}{1-\alpha}}}\right)\left(\|\Lambda^2 u\|^2_{L^2} + \|\Lambda^2 w\|^2_{L^2}\right). \end{aligned}$$

As explained previously, when α and β satisfy (2.1), (2.25) holds and Gronwall's inequality then implies (2.22). Sobolev's inequality with (2.22) then implies (2.23). This completes the proof of Proposition 2.7. □

3. **The case for $\frac{3}{4} \le \alpha \le \frac{7}{8}$.** This section proves that the 2D micropolar equation (1.3) with any $0 < \alpha < 1$ and $\beta = \frac{3}{2} - \alpha$ always possesses a unique global solution when the initial data is sufficiently smooth. More precisely, the following global regularity result holds.

Theorem 3.1. *Consider (1.3) with α and β satisfying*

$$0 < \alpha < 1, \quad \beta \ge \frac{3}{2} - \alpha. \tag{3.1}$$

Assume (u_0, w_0) satisfies the conditions of Theorem 1.1. Then (1.3) possesses a unique global solution satisfying, for any $T > 0$,

$$\begin{aligned} &u \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\alpha}(\mathbb{R}^2)); \\ &w \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\beta}(\mathbb{R}^2)). \end{aligned}$$

When $\alpha < \frac{3}{4}$, the requirement on β in (3.1) is more than those in (2.1) and thus Theorem 2.1 is sharper for $\alpha < \frac{3}{4}$. Similarly, as we shall see in the coming section, for $\alpha > \frac{7}{8}$, Theorem 4.1 in the subsequent section is stronger, Theorem 3.1 is significant only for α in the range between $\frac{3}{4}$ and $\frac{7}{8}$.

As explained in the previous section, it suffices to provide the necessary global *a priori* bounds. The following proposition establishes the global bound for $\|(\nabla u, \nabla w)\|_{L^1_t L^\infty_x}$, which is sufficient for the proof of Theorem 3.1.

Proposition 3.2. *Consider (1.3) with α and β satisfying (3.1). Assume (u_0, w_0) satisfies the conditions of Theorem 1.1. Then the corresponding solution (u, w) of (1.3) obeys, for any $0 < t < \infty$,*

$$\|\Omega(t)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} w(t)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^\alpha \Omega(s)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\beta} w(s)\|_{L^2}^2\right) ds \le C, \tag{3.2}$$

$$\|\Lambda^{\frac{3}{2}} w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}+\beta} w(s)\|_{L^2}^2 ds \le C, \tag{3.3}$$

$$\|\Lambda\Omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{1+\alpha}\Omega(s)\|_{L^2}^2 ds \le C, \tag{3.4}$$

where C's depend on t, u_0, w_0 only. Especially, (3.3) and (3.4) imply that

$$\int_0^t \|(\nabla u(s), \nabla w(s))\|_{L^\infty_x} \, ds < \infty. \tag{3.5}$$

Proof. We first remark that the global L^2-bound in (2.2) remains valid since it only requires $\alpha+\beta > 1$. To show the global bound in (3.2), we take the L^2 inner product of (2.10) with Ω and L^2 inner product of third equation of (1.3) with Λw to obtain

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} w\|_{L^2}^2\right) + 2\|\Lambda^\alpha\Omega\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\beta} w\|_{L^2}^2 + 4\|\Lambda^{\frac{1}{2}} w\|_{L^2}^2 \\
= \; & 2\int_{\mathbb{R}^2} \left(\Lambda^2 w\, \Omega + \Omega\, \Lambda w\right) dx - \int_{\mathbb{R}^2} [\Lambda^{\frac{1}{2}}, u\cdot\nabla] w\, \Lambda^{\frac{1}{2}} w\, dx \\
:= \; & M_1 + M_2.
\end{aligned} \tag{3.6}$$

For the conciseness of our presentation, attention is focused on the case $\beta = \frac{3}{2} - \alpha$. The case $\beta > \frac{3}{2} - \alpha$ is even simpler. Noting that $\alpha + \beta = \frac{3}{2}$ and $0 < 1 - \alpha < \beta$, we have, by the interpolation inequality,

$$\begin{aligned}
M_1 &\le 2\|\Lambda^\alpha\Omega\|_{L^2}\|\Lambda^{\frac{1}{2}+\beta} w\|_{L^2} + 2\|\Lambda^\alpha\Omega\|_{L^2}\|\Lambda^{1-\alpha} w\|_{L^2} \\
&\le \frac{3}{2}\|\Lambda^\alpha\Omega\|_{L^2}^2 + \frac{3}{4}\|\Lambda^{\frac{1}{2}+\beta} w\|_{L^2}^2 + C\|\Lambda^\beta w\|_{L^2}^2 + C\|w\|_{L^2}^2,
\end{aligned}$$

where we have used the following facts

$$\|\Lambda^\alpha\Omega\|_{L^2}\|\Lambda^{\frac{1}{2}+\beta} w\|_{L^2} \le \frac{2}{3}\|\Lambda^\alpha\Omega\|_{L^2}^2 + \frac{3}{8}\|\Lambda^{\frac{1}{2}+\beta} w\|_{L^2}^2,$$

$$\|\Lambda^\alpha\Omega\|_{L^2}\|\Lambda^{1-\alpha} w\|_{L^2} \le \frac{1}{12}\|\Lambda^\alpha\Omega\|_{L^2}^2 + 3\|\Lambda^{1-\alpha} w\|_{L^2}^2.$$

By the divergence-free condition of u, we will show

$$[\Lambda^{\frac{1}{2}}, u\cdot\nabla] w = [\Lambda^{\frac{1}{2}}\partial_{x_1}, u_1] w + [\Lambda^{\frac{1}{2}}\partial_{x_2}, u_2] w.$$

Thanks to the following variant version of Lemma 2.3 (its proof is the same one as for Lemma 2.3)

$$\|[\Lambda^{s-1}\partial_{x_i}, f] g\|_{L^r} \le C\left(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right), \quad i = 1, 2$$

and Sobolev's inequality, it ensures that

$$\begin{aligned}
M_2 &\le \left(\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^{\frac12}w\|_{L^{\frac{2}{\alpha}}}+\|\Lambda^{\frac32}u\|_{L^{\frac{2}{\beta}}}\|w\|_{L^{\frac{2}{1-\beta}}}\right)\|\Lambda^{\frac12}w\|_{L^2}\\
&\le C\left(\|\Lambda^{\alpha}\Omega\|_{L^2}\|\Lambda^{\frac32-\alpha}w\|_{L^2}+\|\Lambda^{\frac32-\beta}\Omega\|_{L^2}\|\Lambda^{\beta}w\|_{L^2}\right)\|\Lambda^{\frac12}w\|_{L^2}\\
&\le C\left(\|\Lambda^{\alpha}\Omega\|_{L^2}\|\Lambda^{\beta}w\|_{L^2}+\|\Lambda^{\alpha}\Omega\|_{L^2}\|\Lambda^{\beta}w\|_{L^2}\right)\|\Lambda^{\frac12}w\|_{L^2}\\
&\le \frac14\|\Lambda^{\alpha}\Omega\|_{L^2}^2+C\|\Lambda^{\beta}w\|_{L^2}^2\|\Lambda^{\frac12}w\|_{L^2}^2.
\end{aligned}$$

Inserting the estimates for M_1 and M_2 in (3.6) and applying Gronwall's inequality, we obtain (3.2). To prove (3.3), we take the L^2 inner product of third equation of (1.3) with $\Lambda^3 w$ to obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\Lambda^{\frac32}w\|_{L^2}^2+\|\Lambda^{\frac32+\beta}w\|_{L^2}^2+4\|\Lambda^{\frac32}w\|_{L^2}^2\\
=\;&2\int_{\mathbb{R}^2}\Omega\,\Lambda^3w\,dx-\int_{\mathbb{R}^2}[\Lambda^{\frac32},u\cdot\nabla]w\,\Lambda^{\frac32}w\,dx\\
:=\;&N_1+N_2,
\end{aligned}\tag{3.7}$$

Again, due to $\alpha+\beta=\frac32$,

$$N_1\le 2\|\Lambda^{\alpha}\Omega\|_{L^2}\|\Lambda^{\frac32+\beta}w\|_{L^2}\le 4\|\Lambda^{\alpha}\Omega\|_{L^2}^2+\frac14\|\Lambda^{\frac32+\beta}w\|_{L^2}^2.$$

By Lemma 2.3 and Sobolev's inequality,

$$\begin{aligned}
N_2 &\le \left(\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\Lambda^{\frac32}w\|_{L^{\frac{2}{\alpha}}}+\|\Lambda^{\frac32}u\|_{L^{\frac{4}{3-2\alpha}}}\|\nabla w\|_{L^{\frac{4}{2\alpha-1}}}\right)\|\Lambda^{\frac32}w\|_{L^2}\\
&\le C\,\|\Lambda^{\alpha}\Omega\|_{L^2}\|\Lambda^{\frac52-\alpha}w\|_{L^2}\|\Lambda^{\frac32}w\|_{L^2}\\
&\le C\|\Lambda^{\alpha}\Omega\|_{L^2}^2\|\Lambda^{\frac12}w\|_{L^2}^2+\frac14\|\Lambda^{\frac32+\beta}w\|_{L^2}^2.
\end{aligned}$$

Inserting the estimates of N_1, N_2 in (3.7) and applying Gronwall's inequality yield

$$\|\Lambda^{\frac32}w(t)\|_{L^2}^2+\int_0^t\left(\|\Lambda^{\frac32+\beta}w(s)\|_{L^2}^2\right)ds\le C(t,u_0,w_0),\tag{3.8}$$

which is (3.3). We now prove (3.4). Taking the L^2 inner product of (2.10) with $\Delta\Omega$, we have, noting that $1>\alpha>\frac12$,

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\Lambda\Omega\|_{L^2}^2+2\|\Lambda^{1+\alpha}\Omega\|_{L^2}^2\\
\le\;&2\left|\int_{\mathbb{R}^2}\Lambda^2w\,\Lambda^2\Omega\;dx\right|+\left|\int_{\mathbb{R}^2}(\nabla u\cdot\nabla\Omega)\,\nabla\Omega\,dx\right|\\
\le\;&2\|\Lambda^{1+\alpha}\Omega\|_{L^2}\|\Lambda^{\frac32+\beta}w\|_{L^2}+\|\Omega\|_{L^2}\|\nabla\Omega\|_{L^4}^2\\
\le\;&\frac12\|\Lambda^{1+\alpha}\Omega\|_{L^2}^2+C\|\Lambda^{\frac32+\beta}w\|_{L^2}+\|\Omega\|_{L^2}\|\Lambda^{\frac32}\Omega\|_{L^2}^2\\
\le\;&\frac12\|\Lambda^{1+\alpha}\Omega\|_{L^2}^2+C\|\Lambda^{\frac32+\beta}w\|_{L^2}+\|\Omega\|_{L^2}(\|\Omega\|_{L^2}^{\frac{2\alpha-1}{\alpha+1}}\|\Lambda^{1+\alpha}\Omega\|_{L^2}^{\frac{3}{\alpha+1}})\\
\le\;&\frac32\|\Lambda^{1+\alpha}\Omega\|_{L^2}^2+C\|\Lambda^{\frac32+\beta}w\|_{L^2}+C\,\|\Omega\|_{L^2}^{\frac{6\alpha}{2\alpha-1}}.
\end{aligned}$$

Integrating in time and using (3.2), we obtain (3.4). Finally, (3.5) follows from (3.3) and 3.4 via Sobolev's inequality. This completes the proof of Proposition 3.2. □

4. **The Case when $\frac{7}{8} \leq \alpha < 1$.** This section focuses on the case when $\frac{7}{8} \leq \alpha < 1$. We prove theorem 1.1 for this range of α. More precisely, the following theorem holds.

Theorem 4.1. *Consider (1.3) with α and β satisfying*

$$\beta \begin{cases} \geq 5(1-\alpha), & \frac{7}{8} \leq \alpha \leq \frac{39}{40}; \\ > \frac{1-\alpha+\sqrt{\alpha^2-4\alpha+3}}{2}, & \frac{39}{40} \leq \alpha < 1. \end{cases} \tag{4.1}$$

Assume (u_0, w_0) satisfies the conditions of Theorem 1.1. Then (1.3) possesses a unique global solution satisfying, for any $T > 0$,

$$\begin{aligned} &u \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\alpha}(\mathbb{R}^2)); \\ &w \in C([0,\infty); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\beta}(\mathbb{R}^2)). \end{aligned}$$

One of the main difficulties to prove Theorem 4.1 is that direct energy estimates on (1.3) do not yield the desired global bounds on the derivatives of u and w. To overcome this difficulty, we consider the combined quantity

$$G = \Omega - \Lambda^{2-2\alpha} w,$$

which satisfies

$$\begin{aligned} &\partial_t G + u \cdot \nabla G + 2\Lambda^{2\alpha} G + 2\Lambda^{2-2\alpha} G \\ = \;&\Lambda^{2+2\beta-2\alpha} w + 4\Lambda^{2-2\alpha} w - 2\Lambda^{4-4\alpha} w + [\Lambda^{2-2\alpha}, u \cdot \nabla] w. \end{aligned} \tag{4.2}$$

The following proposition establishes that $\|G\|_{L^2}$ admits a global bound.

Proposition 4.2. *Consider (1.3) with α and β satisfying (4.1). Assume (u_0, w_0) satisfies the conditions of Theorem 1.1. Let (u, w) denote the corresponding solution of (1.3). Then, for any $0 < t < \infty$,*

$$\|G(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha G(s)\|_{L^2}^2 \, ds \leq C, \tag{4.3}$$

where $C > 0$ depends on t, u_0, w_0.

In order to prove this proposition, we need the following commutator type estimates involving the fractional Laplacian operator. The following lemma is taken from [32]. Similar commutator estimates have been used previously (see, e.g., [18]).

Lemma 4.3. *Assume $p \in [2,\infty)$, $r \in [1,\infty]$, $\delta \in (0,1)$ and $s \in (0,1)$ such that $s + \delta < 1$. Then*

$$\|[\Lambda^\delta, f] g\|_{B^s_{p,r}} \leq C(p,r,\delta,s)\big(\|\nabla f\|_{L^p} \|g\|_{B^{s+\delta-1}_{\infty,r}} + \|f\|_{L^2}\|g\|_{L^2}\big). \tag{4.4}$$

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Taking the inner product of (4.2) with G, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + 2\|\Lambda^\alpha G\|_{L^2}^2 + 2\|\Lambda^{1-\alpha} G\|_{L^2}^2 \\ = \;&\int_{\mathbb{R}^2} \big(\Lambda^{2+2\beta-2\alpha} w + 4\Lambda^{2-2\alpha} w - 2\Lambda^{4-4\alpha} w\big)\, G \, dx + \int_{\mathbb{R}^2} [\Lambda^{2-2\alpha}, u \cdot \nabla] w \, G \, dx \\ := \;&H_1 + H_2. \end{aligned} \tag{4.5}$$

For the sake of conciseness, we focus on the case $\beta = 5(1-\alpha)$ when $\alpha \in [\frac{7}{8}, \frac{39}{40}]$ since the case $\beta > 5(1-\alpha)$ is even easier. Noting that $0 < 2+2\beta-3\alpha, 2-2\alpha, 4-4\alpha < \beta$, we obtain, by applying Hölder's inequality and Young's inequality,

$$\begin{aligned} H_1 &\leq \|\Lambda^{2+2\beta-3\alpha} w\|_{L^2} \|\Lambda^\alpha G\|_{L^2} + 4\|\Lambda^{2-2\alpha} w\|_{L^2} \|G\|_{L^2} + 2\|\Lambda^{4-4\alpha} w\|_{L^2} \|G\|_{L^2} \\ &\leq C\left(\|w\|_{L^2}^2 + \|\Lambda^\beta w\|_{L^2}^2\right) + \frac{1}{4}\|\Lambda^\alpha G\|_{L^2}^2 + \|G\|_{L^2}^2. \end{aligned}$$

Identifying H^s with the Besov space $B^s_{2,2}$ and applying Lemma 4.3, Sobolev's inequality and Young's inequality, we obtain

$$\begin{aligned} H_2 &\leq \|[\Lambda^{2-2\alpha}, u]w\|_{H^{1-\alpha}} \|\Lambda^\alpha G\|_{L^2} \\ &\leq C(\|\nabla u\|_{L^2} \|w\|_{B^{2-3\alpha}_{\infty,2}} + \|u\|_{L^2}\|w\|_{L^2}) \|\Lambda^\alpha G\|_{L^2} \\ &\leq C\|\Omega\|_{L^2} \|w\|_{B^{2-3\alpha}_{\infty,2}} \|\Lambda^\alpha G\|_{L^2} + C\|u\|_{L^2}\|w\|_{L^2}\|\Lambda^\alpha G\|_{L^2} \\ &\leq C(\|G\|_{L^2} + \|\Lambda^{2-2\alpha} w\|_{L^2}) \|w\|_{B^{3-3\alpha}_{2,2}} \|\Lambda^\alpha G\|_{L^2} + C\|u\|_{L^2}\|w\|_{L^2}\|\Lambda^\alpha G\|_{L^2} \\ &\leq \frac{1}{4}\|\Lambda^\alpha G\|_{L^2}^2 + C\|G\|_{L^2}^2 \|w\|_{H^{3-3\alpha}}^2 + C\|\Lambda^{2-2\alpha} w\|_{L^2}^2 \|w\|_{H^{3-3\alpha}}^2 \\ &\quad + C\|u\|_{L^2}^2 \|w\|_{L^2}^2. \end{aligned}$$

By a simple interpolation inequality,

$$\|\Lambda^{2-2\alpha} w\|_{L^2}^2 \|w\|_{H^{3-3\alpha}}^2 = \|\Lambda^{\frac{2\beta}{5}} w\|_{L^2}^2 \|w\|_{H^{\frac{3}{5}\beta}}^2 \leq C\|w\|_{L^2}^2 \|w\|_{H^\beta}^2.$$

Therefore, H_2 is bounded by

$$H_2 \leq \frac{1}{4}\|\Lambda^\alpha G\|_{L^2}^2 + C\|G\|_{L^2}^2 \|w\|_{H^\beta}^2 + C\,\|w\|_{L^2}^2 \|w\|_{H^\beta}^2 + C\|u\|_{L^2}^2 \|w\|_{L^2}^2.$$

Inserting the estimates of H_1, H_2 in (4.5) yields

$$\begin{aligned} &\frac{d}{dt}\|G\|_{L^2}^2 + \|\Lambda^\alpha G\|_{L^2}^2 \\ &\qquad \leq C\left(1 + \|w\|_{H^\beta}^2\right)\|G\|_{L^2}^2 + C(1 + \|w\|_{L^2}^2)\|w\|_{H^\beta}^2 + C\|u\|_{L^2}^2 \|w\|_{L^2}^2. \end{aligned}$$

Gronwall's inequality then implies

$$\|G(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha G(s)\|_{L^2}^2\, ds \leq C.$$

This completes the proof of Proposition 4.2. $\square$

The global bound for G in the previous proposition serves as a bridge to the global bounds on w and Ω. The following lemma controls the L^q-norm of w.

Proposition 4.4. *Under the same condition as in Proposition 4.2 and for any q satisfying*

$$2 \leq q < \frac{2\beta}{1-\alpha}, \tag{4.6}$$

we have, for any $0 < t < \infty$,

$$\|w(t)\|_{L^q}^q + \int_0^t \|w(s)\|_{L^{\frac{q}{1-\beta}}}^q\, ds + \int_0^t \|w(s)\|_{\dot{B}^{\frac{2\beta}{q}}_{q,q}}^q\, ds \leq C(t, u_0, w_0). \tag{4.7}$$

Proof. Multiplying the third equation of (1.3) by $|w|^{q-2}w$, integrating over $\mathbb{R}^2$ and using the divergence-free condition, we obtain

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|w(t)\|_{L^q}^q + 4\|w\|_{L^q}^q + \int_{\mathbb{R}^2} (\Lambda^{2\beta} w)\; w|w|^{q-2}\,dx \\
=\;& \int_{\mathbb{R}^2} \Omega\; w|w|^{q-2}\,dx \\
=\;& \int_{\mathbb{R}^2} G\; w|w|^{q-2}\,dx + \int_{\mathbb{R}^2} \Lambda^{2-2\alpha} w\; w|w|^{q-2}\,dx, \qquad (4.8)
\end{aligned}$$

where, in the last line above, we have used $\Omega = G + \Lambda^{2-2\alpha} w$. As in (2.17), the following lower bound holds

$$\begin{aligned}
\int_{\mathbb{R}^2} (\Lambda^{2\beta} w)|w|^{q-2} w\,dx \;&\geq\; C\int_{\mathbb{R}^2} \left(\Lambda^{\beta}|w|^{\frac{q}{2}}\right)^2 dx \\
&\geq\; C_0\|w\|_{L^{\frac{q}{1-\beta}}}^q + C_0\left\|\Lambda^{\beta}\left(|w|^{\frac{q}{2}}\right)\right\|_{L^2}^2 + C_0\|\Gamma\|^q_{\dot{B}^{\frac{2\beta}{q}}_{q,q}}, \qquad (4.9)
\end{aligned}$$

where $C_0 = C_0(\beta, q) > 0$. On one hand, for $q \leq \frac{2}{1-\alpha}$, we obtain by the Hölder inequality

$$\begin{aligned}
\int_{\mathbb{R}^2} G\; w|w|^{q-2}\,dx \;&\leq\; \|G\|_{L^q}\|w\|_{L^q}^{q-1} \\
&\leq\; C\|G\|_{H^\alpha}\|w\|_{L^q}^{q-1} \\
&\leq\; C\|G\|_{H^\alpha}(1+\|w\|_{L^q}^q).
\end{aligned}$$

On the other hand, for $q > \frac{2}{1-\alpha}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} G\; w|w|^{q-2}\,dx \;&\leq\; \|G\|_{L^{\frac{2}{1-\alpha}}}\|w\|^{q-1}_{L^{\frac{2(q-1)}{1+\alpha}}} \\
&\leq\; C\|G\|_{H^\alpha}\|w\|_{L^q}^{(q-1)(1-\frac{(1-\alpha)q-2}{2\beta(q-1)})}\|w\|_{L^{\frac{q}{1-\beta}}}^{(q-1)\frac{(1-\alpha)q-2}{2\beta(q-1)}} \\
&\leq\; \frac{C_0}{8}\|w\|^q_{L^{\frac{q}{1-\beta}}} + C\|G\|_{H^\alpha}^{\frac{2\beta q}{(\alpha+2\beta-1)q+2}}\|w\|_{L^q}^{(1-\frac{2\beta}{(\alpha+2\beta-1)q+2})q} \\
&\leq\; \frac{C_0}{8}\|w\|^q_{L^{\frac{q}{1-\beta}}} + C(1+\|G\|_{H^\alpha}^2)(1+\|w\|_{L^q}^q),
\end{aligned}$$

where we have used the simple fact that $\frac{2\beta q}{(\alpha+2\beta-1)q+2} < 2$. Therefore, for any $q \geq 2$, the first term in (4.8) can be bounded by

$$\int_{\mathbb{R}^2} G\; w|w|^{q-2}\,dx \leq \frac{C_0}{8}\|w\|^q_{L^{\frac{q}{1-\beta}}} + C(1+\|G\|_{H^\alpha}^2)(1+\|w\|_{L^q}^q). \qquad (4.10)$$

To bound the second term in (4.8), we use Lemma 2.5. For q satisfying (4.6), we choose $0 < \widetilde{s} < \widetilde{\sigma} < 1$ satisfying

$$2 - 2\alpha - \frac{2\beta}{q} < \widetilde{s} < \widetilde{\sigma} < \frac{2\beta}{q}. \qquad (4.11)$$

By Hölder's inequality and Lemma 2.5,

$$\begin{aligned}
\int_{\mathbb{R}^2} \Lambda^{2-2\alpha} w\; w|w|^{q-2}\,dx \;&\leq\; C\|\Lambda^{2-2\alpha-\widetilde{s}} w\|_{L^q}\left\|\Lambda^{\widetilde{s}}(|w|^{q-2}w)\right\|_{L^{\frac{q}{q-1}}} \\
&\leq\; C\|w\|_{B^{\frac{2\beta}{q}}_{q,q}}\|w\|_{B^{\widetilde{\sigma}}_{q,\frac{q}{q-1}}}\|w\|_{L^q}^{q-2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|w\|^2_{B^{\frac{2\beta}{q}}_{q,q}} \|w\|^{q-2}_{L^q} \\
&\leq \frac{C_0}{4}\|w\|^q_{\dot B^{\frac{2\beta}{q}}_{q,q}} + C\|w\|^q_{L^q},
\end{aligned}
\tag{4.12}
$$

where we have used the embeddings, due to (4.11),

$$
B^{\frac{2\beta}{q}}_{q,q} \hookrightarrow W^{2-2\alpha-\widetilde{s},q}, \qquad B^{\frac{2\beta}{q}}_{q,q} \hookrightarrow B^{\widetilde{\sigma}}_{q,\frac{q}{q-1}}.
$$

This explains why we need to restrict q to the range in (4.6). Combining (4.10) and (4.12), we obtain

$$
\frac{d}{dt}\|w\|^q_{L^q} + \|w\|^q_{L^{\frac{q}{1-\beta}}} \leq C(1+\|G\|^2_{H^\alpha} + \|\Lambda^\beta w\|^2_{L^2})(1+\|w\|^q_{L^q}).
$$

Gronwall's inequality, together with Proposition 4.2, yields

$$
\|w(t)\|^q_{L^q} + \int_0^t \|w(s)\|^q_{L^{\frac{q}{1-\beta}}}\,ds + \int_0^t \|w(s)\|^q_{\dot B^{\frac{2\beta}{q}}_{q,q}}\,ds \leq C(t,\,u_0,\,w_0),
$$

which is (4.7). □

We now show that, for any $0<t<\infty$, $\nabla u, \nabla w \in L^1_t L^\infty$, which allows us to establish the desired global regularity.

Proposition 4.5. *Assume (u_0, w_0) satisfies the conditions of Theorem 1.1 and let (u, w) be the corresponding solution of (1.3) with α and β satisfying (4.1). Then, for any $0<t<\infty$,*

$$
\|\Omega(t)\|^2_{L^2} + \|\Lambda w(t)\|^2_{L^2} + \int_0^t \left(\|\Lambda^\alpha \Omega(s)\|^2_{L^2} + \|\Lambda^{1+\beta} w(s)\|^2_{L^2}\right) ds \leq C, \tag{4.13}
$$

$$
\|\Lambda\Omega(t)\|^2_{L^2} + \|\Lambda^2 w(t)\|^2_{L^2} + \int_0^t \left(\|\Lambda^{1+\alpha} \Omega(s)\|^2_{L^2} + \|\Lambda^{2+\beta} w(s)\|^2_{L^2}\right) ds \leq C. \tag{4.14}
$$

In particular, (4.14) implies $\nabla u, \nabla w \in L^1_t L^\infty$.

Proof. Taking the L^2 inner product of (2.10) with Ω and the L^2 inner product of third equation of (1.3) with $\Lambda^2 w$, we have

$$
\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|^2_{L^2} + \|\Lambda w\|^2_{L^2}\right) + 2\|\Lambda^\alpha\Omega\|^2_{L^2} + \|\Lambda^{1+\beta} w\|^2_{L^2} + 4\|\Lambda w\|^2_{L^2} = J_1 + J_2,
$$

where

$$
J_1 = 2\int_{\mathbb{R}^2} \left(\Lambda^2 w\, \Omega + \Omega\, \Lambda^2 w\right) dx, \qquad J_2 = -\int_{\mathbb{R}^2} [\Lambda, u\cdot\nabla] w\, \Lambda w\, dx.
$$

Due to $\beta \leq 1$, by Sobolev's inequality and Young's inequality,

$$
\begin{aligned}
J_1 &\leq C\|\Lambda^{1-\beta}\Omega\|_{L^2}\|\Lambda^{1+\beta} w\|_{L^2} \\
&\leq C\|\Omega\|^{1-\frac{1-\beta}{\alpha}}_{L^2} \|\Lambda^\alpha\Omega\|^{\frac{1-\beta}{\alpha}}_{L^2} \|\Lambda^{1+\beta} w\|_{L^2} \\
&\leq \frac{1}{4}\left(\|\Lambda^\alpha\Omega\|^2_{L^2} + \|\Lambda^{1+\beta} w\|^2_{L^2}\right) + C\|\Omega\|^2_{L^2}.
\end{aligned}
$$

Due to $G = \Omega - \Lambda^{2-2\alpha} w$ and the Biot-Savart law $\nabla u = \nabla\nabla^\perp\Delta^{-1}\Omega$, we write

$$
\nabla u = \nabla\nabla^\perp\Delta^{-1} G + \nabla\nabla^\perp\Delta^{-1}\Lambda^{2-2\alpha} w.
$$

Correspondingly, J_2 can be written into two parts,

$$\begin{aligned} J_2 &= \int_{\mathbb{R}^2} u\cdot\nabla w\Delta w dx \\ &\le -\int_{\mathbb{R}^2} \nabla u\,\nabla w\,\nabla w\,dx \\ &= J_{21}+J_{22}, \end{aligned}$$

where

$$J_{21}=-\int_{\mathbb{R}^2}\nabla\nabla^\perp\Delta^{-1}G\,\nabla w\,\nabla w\,dx,\quad J_{22}=-\int_{\mathbb{R}^2}\nabla\nabla^\perp\Delta^{-1}\Lambda^{2-2\alpha}w\,\nabla w\,\nabla w\,dx.$$

J_{21} and J_{22} can be bounded as follows. By Hölder's inequality, the boundedness of Riesz transforms and Sobolev's inequality,

$$\begin{aligned} J_{21} &\le C\,\|G\|_{L^{\frac{2}{1-\alpha}}}\,\|\nabla w\|_{L^2}\,\|\nabla w\|_{L^{\frac{2}{\alpha}}} \\ &\le \frac18\|\Lambda^{1+\beta}w\|_{L^2}^2+C\,\|G\|_{H^\alpha}^2\,\|\nabla w\|_{L^2}^2, \end{aligned}$$

where we have used the embedding inequalities, due to $\alpha+\beta>1$,

$$\|G\|_{L^{\frac{2}{1-\alpha}}}\le C\,\|G\|_{H^\alpha},\qquad \|\nabla w\|_{L^{\frac{2}{\alpha}}}\le C\,\|w\|_{H^{1+\beta}}.$$

We set $\frac{\beta}{1-\alpha}<q_0<\frac{2\beta}{1-\alpha}$ to be specified later and let $q_0'=\frac{q_0}{q_0-1}$ be its dual index. By the duality of the Besov spaces and Sobolev's inequality, we have

$$\begin{aligned} J_{22} &\le C\|w\|_{\dot B^{\frac{2\beta}{q_0}}_{q_0,q_0}}\,\|\nabla w\nabla w\|_{\dot B^{2-2\alpha-\frac{2\beta}{q_0}}_{q_0',q_0'}} \\ &\le C\|w\|_{\dot B^{\frac{2\beta}{q_0}}_{q_0,q_0}}\,\|\nabla w\nabla w\|_{B^{2-2\alpha-\frac{2\beta}{q_0}}_{q_0',q_0'}} \\ &\le C\|w\|_{\dot B^{\frac{2\beta}{q_0}}_{q_0,q_0}}\,\|\nabla w\nabla w\|_{\widetilde{W}^{2-2\alpha-\frac{2\beta}{q_0},q_0'}} \\ &\le C\|w\|_{\dot B^{\frac{2\beta}{q_0}}_{q_0,q_0}}\,\|\nabla w\|_{B^{2-2\alpha-\frac{2\beta}{q_0}}_{l,q_0'}}\,\|\nabla w\|_{L^\lambda} \\ &\le C\|w\|_{\dot B^{\frac{2\beta}{q_0}}_{q_0,q_0}}\,(\|\nabla w\|_{L^2}^{1-\theta}\|\nabla w\|_{H^\beta}^{\theta})(\|\nabla w\|_{L^2}^{1-\theta}\|\nabla w\|_{H^\beta}^{\theta}) \\ &\le \frac18\|\Lambda w\|_{H^\beta}^2+C\|w\|_{\dot B^{\frac{2\beta}{q_0}}_{q_0,q_0}}^{\frac{\beta q_0}{(\alpha+\beta-1)q_0+\beta-1}}\|\Lambda w\|_{L^2}^2 \\ &\le \frac18\|\Lambda^{1+\beta}w\|_{L^2}^2+C\Big(1+\|w\|_{\dot B^{\frac{2\beta}{q_0}}_{q_0,q_0}}^{\frac{\beta q_0}{(\alpha+\beta-1)q_0+\beta-1}}\Big)\|\Lambda w\|_{L^2}^2, \end{aligned}$$

where $q_0\in\Big(\frac{1-\beta}{\alpha+\beta-1},\,\frac{2\beta}{1-\alpha}\Big)$ and l,λ,θ are given by

$$l=\frac{2q_0}{(2-\alpha)q_0-(1+\beta)},\quad \lambda=\frac{2q_0}{\alpha q_0+\beta-1},\quad \theta=\frac{(1-\alpha)q_0+1-\beta}{\beta q_0}.$$

We note that, in the third inequality above, we used the norm equivalence $B^s_{p,p}\approx\widetilde{W}^{s,p}$, as explained in the appendix. In addition, we invoked (2.8) in the fourth inequality above. Combining all the estimates above, we have

$$\frac{d}{dt}\left(\|\Omega\|_{L^2}^2+\|\Lambda w\|_{L^2}^2\right)+2\|\Lambda^\alpha\Omega\|_{L^2}^2+\|\Lambda^{1+\beta}w\|_{L^2}^2+4\|\Lambda w\|_{L^2}^2$$

$$\leq \quad C\left(\|G\|_{H^\alpha}^2+\|w\|_{\dot{B}^{\frac{2\beta}{q_0}}_{q_0,q_0}}^{\frac{\beta q_0}{(\alpha+\beta-1)q_0+\beta-1}}\right)\|\nabla w\|_{L^2}^2+C\,\|\Omega\|_{L^2}^2.$$

We explain that, when α and β satisfy (4.1), we have

$$\int_0^T \|w\|_{\dot{B}^{\frac{2\beta}{q_0}}_{q_0,q_0}}^{\frac{\beta q_0}{(\alpha+\beta-1)q_0+\beta-1}}\,dt<\infty. \tag{4.15}$$

In fact, (4.1) implies

$$2\beta^2-2(1-\alpha)\beta-(1-\alpha)>0 \quad \text{or} \quad \frac{2\beta}{1-\alpha}>\frac{1}{\alpha+\beta-1}.$$

Therefore, we can choose $q_0<\frac{2\beta}{1-\alpha}$ such that

$$q_0\geq \frac{1}{\alpha+\beta-1} \quad \text{or} \quad \frac{\beta q_0}{(\alpha+\beta-1)q_0+\beta-1}\leq q_0.$$

Then (4.15) follows from Proposition 4.4. Gronwall's inequality then implies the desired bound (4.13).

In order to prove (4.14), we take the L^2 inner product of (2.10) with $\Lambda^2\Omega$ and the L^2 inner product of the third equation of (1.3) with $\Lambda^4 w$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda\Omega\|_{L^2}^2+\|\Lambda^2 w\|_{L^2}^2\right)+2\|\Lambda^{1+\alpha}\Omega\|_{L^2}^2+\|\Lambda^{2+\beta}w\|_{L^2}^2+4\|\Lambda^2 w\|_{L^2}^2\\
=\;& 2\int_{\mathbb{R}^2}\left(\Lambda^2 w\,\Lambda^2\Omega+\Omega\,\Lambda^4 w\right)dx-\int_{\mathbb{R}^2}\left([\Lambda,u\cdot\nabla]\Omega\;\Lambda\Omega+[\Lambda^2,u\cdot\nabla]w\,\Lambda^2 w\right)dx.
\end{aligned}$$

Due to $\alpha+\beta>1$ and $\beta\leq 1$,

$$\begin{aligned}
2\int_{\mathbb{R}^2}\left(\Lambda^2 w\,\Lambda^2\Omega+\Omega\,\Lambda^4 w\right)dx \;&\leq\; C\,\|\Lambda^{2-\beta}\Omega\|_{L^2}\,\|\Lambda^{2+\beta}w\|_{L^2}\\
&\leq\; C\,\|\Lambda\Omega\|_{L^2}^{1-\frac{1-\beta}{\alpha}}\,\|\Lambda^{1+\alpha}\Omega\|_{L^2}^{\frac{1-\beta}{\alpha}}\,\|\Lambda^{2+\beta}w\|_{L^2}\\
&\leq\; \frac{1}{4}\left(\|\Lambda^{1+\alpha}\Omega\|_{L^2}^2+\|\Lambda^{2+\beta}w\|_{L^2}^2\right)+C\|\Lambda\Omega\|_{L^2}^2.
\end{aligned}$$

Noting the following fact due to $\nabla\cdot u=0$

$$\begin{aligned}
&-\int_{\mathbb{R}^2}[\Lambda^2,u\cdot\nabla]w\,\Lambda^2 w dx\\
=\;& -\int_{\mathbb{R}^2}\Delta(u\cdot\nabla w)\,\Delta w dx+\int_{\mathbb{R}^2}(u\cdot\nabla\Delta w)\,\Delta w dx\\
=\;& -\sum_{k,i=1}^2\int_{\mathbb{R}^2}\partial_k^2 u_i\partial_i w\,\Delta w dx-2\sum_{k,i=1}^2\int_{\mathbb{R}^2}\partial_k u_i\partial_k\partial_i w\,\Delta w dx\\
&-\sum_{k,i=1}^2\int_{\mathbb{R}^2}u_i\partial_i\partial_k^2 w\,\Delta w dx+\int_{\mathbb{R}^2}(u\cdot\nabla\Delta w)\,\Delta w dx\\
=\;& -\sum_{k,i=1}^2\int_{\mathbb{R}^2}\partial_k^2 u_i\partial_i w\,\Delta w dx-2\sum_{k,i=1}^2\int_{\mathbb{R}^2}\partial_k u_i\partial_k\partial_i w\,\Delta w dx,
\end{aligned}$$

we have by using Lemma 2.3, the Gagliardo-Nirenberg inequality and Young's inequality

$$
\begin{aligned}
& -\int_{\mathbb{R}^2} \left([\Lambda, u\cdot\nabla]\Omega\ \Lambda\Omega + [\Lambda^2, u\cdot\nabla]w\,\Lambda^2 w\right) dx \\
= & -\int_{\mathbb{R}^2} [\Lambda, u\cdot\nabla]\Omega\ \Lambda\Omega dx - \int_{\mathbb{R}^2} [\Lambda^2, u\cdot\nabla]w\,\Lambda^2 w dx \\
\leq & \ C\|[\Lambda, u\cdot\nabla]\Omega\|_{L^{\frac{4}{2+\alpha}}}\|\Lambda\Omega\|_{L^{\frac{4}{2-\alpha}}} + C\|\Lambda^2 u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla w\|_{L^{\frac{4}{1+\alpha}}}\|\Lambda^2 w\|_{L^{\frac{4}{1+\alpha}}} \\
& + C\|\nabla u\|_{L^{\frac{2}{\beta}}}\|\Lambda^2 w\|^2_{L^{\frac{4}{2-\beta}}} \\
\leq & \ C(\|\nabla u\|_{L^{\frac{2}{\alpha}}}\|\Lambda\Omega\|_{L^{\frac{4}{2-\alpha}}} + \|\Omega\|_{L^{\frac{2}{\alpha}}}\|\nabla\Omega\|_{L^{\frac{4}{2-\alpha}}})\|\Lambda\Omega\|_{L^{\frac{4}{2-\alpha}}} \\
& + C\|\Lambda^2 u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla w\|_{L^{\frac{4}{1+\alpha}}}\|\Lambda^2 w\|_{L^{\frac{4}{1+\alpha}}} + C\|\nabla u\|_{L^{\frac{2}{\beta}}}\|\Lambda^2 w\|^2_{L^{\frac{4}{2-\beta}}} \\
\leq & \ C\|\nabla u\|_{L^{\frac{2}{\alpha}}}\|\Lambda\Omega\|^2_{L^{\frac{4}{2-\alpha}}} + C\|\Lambda^2 u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla w\|_{L^{\frac{4}{1+\alpha}}}\|\Lambda^2 w\|_{L^{\frac{4}{1+\alpha}}} \\
& + C\|\nabla u\|_{L^{\frac{2}{\beta}}}\|\Lambda^2 w\|^2_{L^{\frac{4}{2-\beta}}} \\
\leq & \ C\|\Lambda^{1-\alpha}\Omega\|_{L^2}\|\Lambda\Omega\|_{L^2}\|\Lambda^{1+\alpha}\Omega\|_{L^2} + C\|\Lambda^{1-\beta}\Omega\|_{L^2}\|\Lambda^2 w\|_{L^2}\|\Lambda^{2+\beta} w\|_{L^2} \\
& \quad + C\|\Lambda^{1+\alpha}\Omega\|_{L^2}\|\Lambda w\|_{L^2}^{1-\frac{1-\alpha}{2\beta}}\|\Lambda^{1+\beta} w\|_{L^2}^{\frac{1-\alpha}{2\beta}}\|\Lambda^2 w\|_{L^2}^{1-\frac{1-\alpha}{2\beta}}\|\Lambda^{2+\beta} w\|_{L^2}^{\frac{1-\alpha}{2\beta}} \\
\leq & \ \frac{1}{2}\left(\|\Lambda^{1+\alpha}\Omega\|^2_{L^2} + \|\Lambda^{2+\beta} w\|^2_{L^2}\right) \\
& \quad + C\left(\|\Omega\|^2_{L^2} + \|\Lambda^\alpha\Omega\|^2_{L^2} + \|\Lambda^{1+\beta} w\|_{L^2}^{\frac{2(1-\alpha)}{2\beta+\alpha-1}}\|\Lambda w\|^2_{L^2}\right) \\
& \times\left(\|\Lambda^2 w\|^2_{L^2} + \|\Lambda\Omega\|^2_{L^2}\right) \\
\leq & \ \frac{1}{2}\left(\|\Lambda^{1+\alpha}\Omega\|^2_{L^2} + \|\Lambda^{2+\beta} w\|^2_{L^2}\right) \\
& \quad + C\left(\|\Omega\|^2_{L^2} + \|\Lambda^\alpha\Omega\|^2_{L^2} + (\|\Lambda^{1+\beta} w\|^2_{L^2} + 1)\|\Lambda w\|^2_{L^2}\right) \\
& \times\left(\|\Lambda^2 w\|^2_{L^2} + \|\Lambda\Omega\|^2_{L^2}\right),
\end{aligned}
$$

where we have used the fact $\frac{2(1-\alpha)}{2\beta+\alpha-1} < 2$. These estimates combined with Gronwall's inequality then allow us to obtain (4.14). This completes the proof of Proposition 4.5. □

5. **Proof of Theorem 1.1.** The global *a priori* bounds obtained in the previous three sections, especially

$$\int_0^T \|(\nabla u, \nabla w)(t)\|_{L^\infty(\mathbb{R}^2)}\, dt \leq C(T, \|(u_0, w_0)\|_{H^s}) < \infty$$

is sufficient for the proofs of Theorems 2.1, 3.1 and 4.1. Since Theorem 1.1 combines all three of them, it suffices to provide the proof for Theorem 1.1.

Proof of Theorem 1.1. The existence of desired solutions to (1.3) can be obtained by standard approaches such as the Friedrichs method. For $n \in \mathbb{N}$, define the operator J_n by

$$J_n\varphi = \mathcal{F}^{-1}(\chi_{B(0,n)}(\xi)\mathcal{F}(\varphi)(\xi)),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms, respectively, and $\chi_{B(0,n)}$ denotes the characteristic function on the ball $B(0, n)$. Consider the approximate equations of (1.3)

$$\begin{cases} \partial_t u_n + 2J_n\Lambda^{2\alpha}u_n = 2J_nP\nabla\times w_n - J_nP(J_nu_n\cdot\nabla J_nu_n), \\ \nabla\cdot u_n = 0, \\ \partial_t w_n + J_n\Lambda^{2\beta}w_n + 4J_nw_n = 2J_n\nabla\times u_n - J_n(J_nu_n\cdot\nabla J_nw_n), \\ u_n(x,0) = J_nu_0, \quad w_n(x,0) = J_nw_0, \end{cases} \tag{5.1}$$

where P denotes the standard projection onto divergence-free vector fields. The standard Picard type theorem ensures that, for some $T_n > 0$, there exists a unique local solution (u_n, ω_n) on $[0, T_n)$ in the functional setting $\{f \in L^2(\mathbb{R}^2) : \operatorname{supp}\mathcal{F}(f) \subset B(0,n)\}$. Due to $J_n^2 = J_n$ and $PJ_n = J_nP$, it is easy to see that (J_nu_n, J_nw_n) is also a solution. The uniqueness of such local solutions implies

$$u_n = J_nu_n, \quad w_n = J_nw_n.$$

Therefore, (5.1) becomes

$$\begin{cases} \partial_t u_n + 2\Lambda^{2\alpha}u_n = 2P\nabla\times w_n - J_nP(u_n\cdot\nabla u_n), \\ \nabla\cdot u_n = 0, \\ \partial_t w_n + \Lambda^{2\beta}w_n + 4w_n = 2\nabla\times u_n - J_n(u_n\cdot\nabla w_n), \\ u_n(x,0) = J_nu_0, \quad w_n(x,0) = J_nw_0. \end{cases} \tag{5.2}$$

A basic L^2 energy estimate implies (u_n, w_n) of (5.2) satisfies

$$\|u_n(t)\|_{L^2}^2 + \|w_n(t)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^\alpha u_n(\tau)\|_{L^2}^2 + \|\Lambda^\beta w_n(\tau)\|_{L^2}^2\right)\tau \le C(t, u_0, w_0),$$

where C is independent of n. Therefore, the local solution can be extended into a global one, by the standard Picard Extension Theorem (see, e.g., [6]). Next we show that (u_n, w_n) admits a uniform global bound in $H^s(\mathbb{R}^2)$ with $(s > 2)$. Following the proofs of Propositions 2.7, 3.2 and 4.5, we obtain, for any $t > 0$,

$$\int_0^t \|(\nabla u_n, \nabla w_n)(s)\|_{L^\infty}\, ds \le C(t, \|(u_0, w_0)\|_{H^s}) < \infty,$$

where we have used the fact that $\|(u_n(x,0), w_n(x,0))\|_{H^s} \le \|(u_0, w_0)\|_{H^s}$. By a standard energy estimate involving (5.2), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_n\|_{H^s}^2 + \|w_n\|_{H^s}^2\right) + 2\|\Lambda^\alpha u_n\|_{H^s}^2 + \|\Lambda^\beta w_n\|_{H^s}^2 + \|\nabla w_n\|_{H^s}^2 \\ = \;& 2\int_{\mathbb{R}^2} \{(\nabla\times w_n)\cdot\Lambda^{2s}u_n + (\nabla\times u_n)\Lambda^{2s}w_n\}\, dx \\ &+ \int_{\mathbb{R}^2} [\Lambda^s, u_n\cdot\nabla]u_n\cdot\Lambda^s u_n dx + \int_{\mathbb{R}^2} [\Lambda^s, u_n\cdot\nabla]w_n\,\Lambda^s w_n dx \\ \le \;& \frac{1}{2}\left(\|\Lambda^\alpha u_n\|_{H^s}^2 + \|\Lambda^\beta w_n\|_{H^s}^2\right) + C\left(\|\nabla u_n\|_{L^\infty} + \|\nabla w_n\|_{L^\infty} + 1\right) \\ &\times\left(\|u_n\|_{H^s}^2 + \|w_n\|_{H^s}^2\right). \end{aligned}$$

Gronwall's inequality then allows us to conclude that, for any $t > 0$,

$$\begin{aligned} &\|u_n\|_{H^s}^2 + \|w_n\|_{H^s}^2 + \int_0^t (\|\Lambda^\alpha u_n\|_{H^s}^2 + \|\Lambda^\beta w\|_{H^s}^2)ds \\ \le\;& (\|u_0\|_{H^s}^2 + \|w_0\|_{H^s}^2)e^{C\int_0^t(\|\nabla u_n\|_{L^\infty} + \|\nabla w_n\|_{L^\infty} + 1)ds} \\ \le\;& C(t, \|(u_0, w_0)\|_{H^s}). \end{aligned}$$

Once this uniform global bound is at our disposal, a standard compactness argument allows us to obtain the global existence of the desired solution (u, w) to (1.3). The uniqueness part for solutions at this regularity level is standard and is thus omitted. This completes the proof of Theorem 1.1. □

Appendix A. **Besov spaces.** This appendix provides the definition of the Littlewood-Paley decomposition and the definition of Besov spaces. Some related facts used in the previous sections are also included. The material presented in this appendix can be found in several books and many papers (see, e.g., [1, 2, 22, 24, 26, 29]).

We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma \, dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ denotes the space of multinomials. We also recall the standard Fourier transform and the inverse Fourier transform,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi} dx, \qquad g^\vee(x) = \int_{\mathbb{R}^d} g(\xi)\, e^{2\pi i x\cdot \xi} d\xi.$$

To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{ \xi \in \mathbb{R}^d :\ 2^{j-1} \le |\xi| < 2^{j+1} \right\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 & , \quad \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 & , \quad \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\dot{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}. \tag{A.1}$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. We set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A.2}$$

For notational convenience, we write Δ_j for $\dot{\Delta}_j$ when there is no confusion. They are different for $j \le -1$. As provided below, the homogeneous Besov spaces are defined in terms of $\dot{\Delta}_j$ while the inhomogeneous Besov spaces are defined in Δ_j. Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where Δ_k is given by (A.2). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j and

$$S_j f \rightharpoonup f \qquad \text{in } \mathcal{S}'.$$

In addition, for two tempered distributions u and v, we also recall the notion of paraproducts

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \qquad R(u, v) = \sum_{|i-j| \le 2} \Delta_i u \Delta_j v$$

and Bony's decomposition, see e.g. [1],

$$u\,v = T_u v + T_v u + R(u, v).$$

In addition, the notation $\widetilde{\Delta}_k$, defined by

$$\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1},$$

is also useful.

Definition A.1. *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying*

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \|2^{js} \|\dot{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

An equivalent norm of the the homogeneous Besov space $\dot{B}^s_{p,q}$ with $s \in (0,1)$ is given by

$$\|f\|_{\dot{B}^s_{p,q}} = \left[\int_{\mathbb{R}^d} \frac{\|f(x+\cdot)-f(\cdot)\|^q_{L^p(\mathbb{R}^d)}}{|x|^{d+sq}}\,dx\right]^{\frac{1}{q}}. \tag{A.3}$$

Definition A.2. *The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.3. *For any $s \in \mathbb{R}$,*

$$H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$B^s_{q,\min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,\max\{q,2\}}.$$

For any non-integer $s > 0$, the Hölder space C^s is equivalent to $B^s_{\infty,\infty}$.

Bernstein's inequalities are useful tools in dealing with Fourier localized function-s. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Proposition A.4. *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

1) If f satisfies

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \le K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \le |\xi| \le K_2 2^j\}$$

for some integer j and constants $0 < K_1 \le K_2$, then

$$C_1\, 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

We now provide the proof of (2.7). By Proposition A.4,

$$\begin{aligned}
\|\Lambda^s g\|_{L^p} &\le \sum_{k\ge -1}\|\Lambda^s\Delta_k g\|_{L^p} = \|\Lambda^s\Delta_{-1}g\|_{L^p} + \sum_{k\ge 0}\|\Lambda^s\Delta_k g\|_{L^p} \\
&\le C\,\|g\|_{L^p} + \sum_{k\ge 0} 2^{-(\sigma-s)k}\|\Lambda^\sigma\Delta_k g\|_{L^p} \\
&\le C\,\|g\|_{L^p} + C\,\|g\|_{\dot{B}^\sigma_{p,\widetilde{p}}}.
\end{aligned}$$

Finally we provide the definition of Sobolev-Slobodeckij space $\widetilde{W}^{s,p}$. Let us assume $s \geq 0$ and $1 \leq p \leq \infty$. When $s \geq 0$ is an integer, the Sobolev norm is standard, namely

$$\|f\|_{W^{s,p}} = \Big(\sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^p}^p \Big)^{\frac{1}{p}}.$$

When $s > 0$ is a fraction, the norm in $\widetilde{W}^{s,p}$ is given by

$$\|f\|_{\widetilde{W}^{s,p}} = \|f\|_{W^{[s],p}} + \Big(\sum_{|\alpha|=[s]} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{d+(s-[s])p}} \, dxdy \Big)^{\frac{1}{p}}.$$

We remark that, except for $p = 2$, $\widetilde{W}^{s,p}$ with this norm is different from the most frequently used definition of Sobolev spaces of fractional order, or the Bessel potential space $\mathcal{L}_s^p$ (or sometimes denoted by $W^{s,p}$ or H_p^s) (see, e.g., [17, Chapter 1.3.1], [26, p.13]). The norm in $\mathcal{L}_s^p$ is given by

$$\|f\|_{\mathcal{L}_s^p} = \|f\|_{L^p} + \|\Lambda^s f\|_{L^p}.$$

$\widetilde{W}^{s,p}$ is closely related to Besov spaces (see, e.g., [26, 29]). In fact,

$$\widetilde{W}^{s,p} \approx B_{p,p}^s \hookrightarrow \mathcal{L}_s^p, \quad 1 < p \leq 2; \quad \mathcal{L}_s^p \hookrightarrow B_{p,p}^s \approx \widetilde{W}^{s,p}, \quad 2 \leq p < \infty.$$

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