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Global regularity for a class of 2D generalized tropical climate models

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Abstract

This paper establishes the global existence and regularity of solutions to a two-dimensional (2D) tropical climate model (TCM) with fractional dissipation. The inviscid counterpart of this model was derived by Frierson, Majda and Pauluis [8] as a model for tropical geophysical flows. This model reflects the interaction and coupling among the barotropic mode u, the first baroclinic mode v of the velocity and the temperature θ. The systems with fractional dissipation studied here may arise in the modeling of geophysical circumstances. Mathematically these systems allow simultaneous examination of a family of systems with various levels of regularization. The aim here is the global regularity with the least dissipation. We prove two main results: first, the global regularity of the system with $(-\Delta)^{\beta} v$ and $(-\Delta)^{\gamma}\theta$ for $\beta > 1$ and $\beta + \gamma > \frac{3}{2}$; and second, the global regularity of the system with $(-\Delta)^{\beta} v$ for $\beta > \frac{3}{2}$. The proofs of these results are not trivial and the requirements on the fractional indices appear to be optimal. The key tools employed here include the maximal regularity for general fractional heat operators, the Littlewood–Paley decomposition and Besov space techniques, lower bounds involving fractional Laplacian and simultaneous estimates of several coupled quantities.

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1. Introduction

Consider the two-dimensional (2D) tropical climate model (TCM) with fractional dissipation

$$\begin{cases} \partial_t u + u\cdot\nabla u + \mu(-\Delta)^\alpha u + \nabla p + \nabla\cdot(v\otimes v) = 0, \\ \partial_t v + u\cdot\nabla v + \nu(-\Delta)^\beta v + v\cdot\nabla u + \nabla\theta = 0, \\ \partial_t \theta + u\cdot\nabla\theta + \eta(-\Delta)^\gamma\theta + \nabla\cdot v = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where the 2D vector fields $u = (u_1(x,t), u_2(x,t))$ and $v = (v_1(x,t), v_2(x,t))$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively, and the scalars θ and p denote the temperature and the pressure, respectively, and $\mu, \nu, \eta, \alpha, \beta, \gamma \geq 0$ are real parameters. Here $v\otimes v$ is the standard tensor notation and the fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi).$$

There are geophysical circumstances in which the fractional Laplacian may arise. Flows in the middle atmosphere traveling upwards undergo changes due to the changes in atmospheric properties. The effect of kinematic diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled using the space fractional Laplacian [4,9].

When $\alpha = \beta = \gamma = 1$, (1.1) reduces to the standard TCM with Laplacian dissipation. The inviscid version of (1.1), namely (1.1) with $\mu = \nu = \eta = 0$, was first derived by Frierson, Majda and Pauluis [8] as a model for tropical geophysical flows. Fundamental issues concerning (1.1) such as the global existence and regularity of solutions have attracted considerable attention. Important results have been obtained. Li and Titi proved the global well-posedness for the case when $\alpha = \beta = 1$ and $\eta = 0$ by introducing a combined quantity called pseudo baroclinic velocity [16]. The work of Li and Titi [16] inspired several subsequent studies. Dong, Wang, Wu and Zhang [7] examined (1.1) with $\alpha + \beta = 2$ and $\eta = 0$. By taking advantage of the special structure of the equations of

$$\omega = \nabla\times u, \qquad j = \nabla\times v,$$

[7] proved the global regularity for the case $\alpha + \beta = 2$ and $1 \leq \beta \leq \frac{3}{2}$. Ye [24] investigated the case when $\alpha > 0$, $\beta = 1$ and $\gamma = 1$ and proved the global existence and uniqueness of classical solutions.

This paper focuses on two cases:

$$(1) \quad \mu = 0,\ \beta > 1,\ \beta + \gamma > \frac{3}{2}; \qquad (2) \quad \frac{3}{2} < \beta \leq 2,\ \mu = \eta = 0.$$

We establish the global existence and uniqueness of classical solutions for each case. More precisely, we obtain the following theorems.

Theorem 1.1. *Consider the following TCM with fractional dissipation*

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p + \nabla\cdot(v\otimes v) = 0, \\ \partial_t v + u\cdot\nabla v + \nu(-\Delta)^{\beta} v + v\cdot\nabla u + \nabla\theta = 0, \\ \partial_t \theta + u\cdot\nabla\theta + \eta(-\Delta)^{\gamma}\theta + \nabla\cdot v = 0, \\ \nabla\cdot u = 0, \\ (u, v, \theta)(x, 0) = (u_0(x), v_0(x), \theta_0(x)) \end{cases} \tag{1.2}$$

with

$$\nu > 0, \quad \eta > 0, \quad \beta > 1, \quad \beta + \gamma > \frac{3}{2}.$$

Assume $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla\cdot u_0 = 0$. *Then (1.2) has a unique global classical solution* (u, v, θ) *satisfying, for any* $t > 0$,

$$(u, v, \theta) \in C(0, t; H^s), \quad v \in L^2(0, t; H^{s+\beta}), \quad \theta \in L^2(0, t; H^{s+\gamma}).$$

Corresponding to the second case, we have the following theorem.

Theorem 1.2. *Consider the following TCM with fractional dissipation*

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p + \nabla\cdot(v\otimes v) = 0, \\ \partial_t v + u\cdot\nabla v + \nu(-\Delta)^{\beta} v + v\cdot\nabla u + \nabla\theta = 0, \\ \partial_t \theta + u\cdot\nabla\theta + \nabla\cdot v = 0, \\ \nabla\cdot u = 0, \\ (u, v, \theta)(x, 0) = (u_0(x), v_0(x), \theta_0(x)) \end{cases} \tag{1.3}$$

with

$$\nu > 0, \quad \frac{3}{2} < \beta \le 2.$$

Assume $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla\cdot u_0 = 0$. *Then (1.3) has a unique global classical solution* (u, v, θ) *satisfying, for any* $t > 0$,

$$(u, v, \theta) \in C(0, t; H^s), \quad v \in L^2(0, t; H^{s+\beta}).$$

The aim has been to establish the global existence and regularity with the least dissipation. The proofs of these theorems are not trivial. The proofs exploit the fractional dissipation to its full capacity. The key tools employed here include the maximal regularity for general fractional heat operators, the Littlewood–Paley decomposition and Besov space techniques, lower bounds involving fractional Laplacian and simultaneous estimates of several coupled quantities. We devote one section, Section 2 to some of the tools we use. In addition, an appendix on the Littlewood–Paley decomposition and Besov spaces is also attached for reader's convenience. For notational convenience, we write $\Lambda = (-\Delta)^{\frac{1}{2}}$ and use $\|f\|_{L_t^q L_x^p}$ for $\|f\|_{L^q(0,t;L^p(\mathbb{R}^d))}$.

The core part in the proof of Theorem 1.1 consists of several key *a priori* global-in-time estimates. The desired global bounds are proven in two steps. The first step establishes global bounds for three quantities:

$$\|\Lambda^\sigma v\|_{L_t^q L_x^2}, \quad \|\theta\|_{L_t^\infty L_x^q} \quad \text{and} \quad \|(\nabla u,\, \Lambda^{2-\beta} v,\, \Lambda^\delta \theta)\|_{L_t^\infty L_x^2}$$

where $\sigma \le 2\beta - 1$, $2 \le q < \infty$ and $\delta < \gamma$. The global bound for the first quantity $\|\Lambda^\sigma v\|_{L_t^q L_x^2}$ makes use of the global L^2-bounds, the structure of the equation for v and the maximal regularity for the fractional heat operator. To obtain the global bound for $\|\theta\|_{L_t^\infty L_x^q}$, we perform L^q-estimates, make use of a lower bound for the fractional dissipation $\Lambda^\gamma \theta$ and bound the term related to $\nabla \cdot v$ suitably. The third bound is for three quantities. The quantities ∇u, $\Lambda^{2-\beta} v$ and $\Lambda^\delta \theta$ are simultaneously estimated here due to the coupling of the equations. The estimate of one of them depends on the other two. The second main step proves two key *a priori* bounds, for $2 \le p \le \infty$,

$$\|\Delta v\|_{L_t^1 L_x^p} \quad \text{and} \quad \|\omega\|_{L_t^\infty L_x^p}. \tag{1.4}$$

These global bounds are sufficient for any global bounds in more regular settings such as H^s. To prove (1.4), we exploit the nonlinear coupling structure and combine the estimates of $\|\omega\|_{L_t^\infty L_x^p}$, $\|\Delta v\|_{L_t^1 L_x^p}$ and $\|\Lambda^{\sigma_1} v\|_{L_t^1 L_x^p}$ with $\sigma_1 < 2\gamma$. The estimates of these three quantities are tangled together, with the estimate of one of them depending on the other two. The global bounds for them are obtained through suitable combination and Gronwall's inequality. In the process of the estimates, it appears that the condition $\beta + \gamma > \frac{3}{2}$ is optimal. More details of these estimates are presented in Section 3.

The proof of Theorem 1.2 boils down to prove the global *a priori* bounds

$$\int_0^t \|\Delta v\|_{L^\infty}\, d\tau, \quad \|\nabla u\|_{L^\infty} \quad \text{and} \quad \|\nabla \theta\|_{L^\infty}. \tag{1.5}$$

Clearly the H^s bound of (u, v, θ) follows easily from the bounds in (1.5). The proof of (1.5) fully exploits the dissipation in the equation of v and is split into two steps. The first step combines the estimates of $\|\omega\|_{L_t^\infty L_x^2}$ and $\|\Lambda^{\beta-1} v\|_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}^\beta}$ due to their coupling. The global bound for $\|\theta\|_{L^\infty}$ then follows as a simple consequence. The second step makes use of the maximal regularity for the fractional heat operator and show the boundedness of $\|\Delta v\|_{L_t^2 L_x^\infty}$. This regularity allows us to prove that $\|\nabla u\|_{L^\infty}$ and $\|\nabla \theta\|_{L^\infty}$. Then follows the global bound for $\|(u, v, \theta)\|_{H^s}$.

The rest of this paper is divided into three sections followed by an appendix. Section 2 contains various tools such as the maximal regularity for general fractional heat operators, commutator estimates involving fractional Laplacian operators in Besov spaces and a lower bound for fractional Laplacian operators. Section 3 proves Theorem 1.1 while Section 4 proves Theorem 1.2. Detailed *a priori* estimates are proved in these sections. An appendix on the Littlewood–Paley decomposition and Besov spaces is attached for reader's convenience.

2. Several tool lemmas

This section serves as a preparation for the proofs of our main results. Several tool lemmas and estimates are presented here.

The following lemma provides an upper bound for the fractional heat operator $\Lambda^\sigma e^{-\Lambda^\alpha t}$ as a map from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, which follows from the Young inequality combined with scaling property of the corresponding kernel. It is a natural extension of the corresponding result for the standard heat operator (see for example [18, Lemma 3.1]).

Lemma 2.1. *Let $\sigma > 0$ and $\alpha > 0$. Let $1 \le p \le q \le \infty$. Then, for any $t > 0$,*

$$\left\| \Lambda^\sigma e^{-\Lambda^\alpha t} f \right\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\frac{\sigma}{\alpha} - \frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}$$

for a constant $C = C(d, \sigma, \alpha, p, q)$.

We remark that Lemma 2.1 is also true for $\sigma = 0$ (see [18, Lemma 3.1]). More precisely, we have for $1 \le p \le q \le \infty$ and for any $t > 0$,

$$\left\| e^{-\Lambda^\alpha t} f \right\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}.$$

The maximal regularity estimate for the standard heat operator $e^{\Delta t}$ is well-known (see, e.g., [15]). The same estimate actually holds for the general heat operator with fractional Laplacian (see, e.g., [3,11]). The proof of this lemma involves fundamental tools in harmonic analysis such as the Calderon–Zygmund theory on the vector-valued singular integral operators. Actually, Lemma 2.2 below is a special case of the Theorem in part 3.1 of [11] (see pages 1654–1655 of [11]). We remark that p and q in Lemma 2.2 are not allowed to be 1 or ∞.

Lemma 2.2. *Let $\alpha > 0$ and $p, q \in (1, \infty)$. Then the operator*

$$G f \equiv \int_0^t (-\Delta)^\alpha e^{-(-\Delta)^\alpha (t-\tau)} f(\tau)\, d\tau$$

is bounded from $L_t^q L_x^p$ to $L_t^q L_x^p$. The case with $\alpha = 1$ represents the maximal regularity for the standard Laplacian operator.

We also state and prove the following estimate that provides explicit dependence on time t. We note that the indices p and q in the following lemma can be 1 or ∞.

Lemma 2.3. *Let $\beta > 0$ be a real parameter. Assume $f = f(x,t) \in L_t^q L_x^p$ with $1 \le p, q \le \infty$. Then the solution u of the fractional parabolic equation*

$$\begin{cases} \partial_t u + \Lambda^{2\beta} u = f, \\ u(x,0) = u_0(x) \end{cases} \tag{2.1}$$

satisfies, for any $0 < \sigma_1 < 2\beta$ and $\sigma_2 + \sigma_3 = \sigma_1$,

$$\|\Lambda^{\sigma_1} u\|_{L_t^q L_x^p} \le C t^{\frac{1}{q} - \frac{\sigma_2}{2\beta}} \|\Lambda^{\sigma_3} u_0\|_{L_x^p} + C\, t^{1 - \frac{\sigma_1}{2\beta}} \|f\|_{L_t^q L_x^p}. \tag{2.2}$$

The point of this lemma is the explicit dependence on t, even though the case when $\sigma_1 = 2\beta$ is excluded. Nevertheless, (2.2) is good enough for our purpose.

Proof. The solution u of (2.1) can be written in the form

$$u(t) = e^{-\Lambda^{2\beta} t} u_0 + \int_0^t e^{-\Lambda^{2\beta}(t-\tau)} f(\tau)\, d\tau.$$

Therefore,

$$\Lambda^{\sigma_1} u(t) = \Lambda^{\sigma_1} e^{-\Lambda^{2\beta} t} u_0 + \int_0^t \Lambda^{\sigma_1} e^{-\Lambda^{2\beta}(t-\tau)} f(\tau)\, d\tau. \tag{2.3}$$

The kernel function associated with the operator $\Lambda^{\sigma_1} e^{-\Lambda^{2\beta} t}$ is given by

$$g(x,t) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}\, |\xi|^{\sigma_1}\, e^{-|\xi|^{2\beta} t}\, d\xi = t^{-\frac{\sigma_1}{2\beta}} t^{-\frac{1}{\beta}}\, g_0\left(\frac{x}{t^{\frac{1}{2\beta}}}\right),$$

where

$$g_0(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}\, |\xi|^{\sigma_1}\, e^{-|\xi|^{2\beta}}\, d\xi.$$

Inspired by the proof of [18, Lemma 2.1], we can show that g_0 obeys the following bounds (see appendix for details)

$$\|g(\cdot,t)\|_{L^1_x} \le C\, t^{-\frac{\sigma_1}{2\beta}} \quad \text{and} \quad \|\|g(\cdot,t)\|_{L^1_x}\|_{L^1(0,t)} \le C\, t^{1-\frac{\sigma_1}{2\beta}}. \tag{2.4}$$

Applying L^p_x on (2.3), Minkowski's inequality and Lemma 2.1 yield

$$\begin{aligned}
\|\Lambda^{\sigma_1} u(t)\|_{L^p_x} &\le \|\Lambda^{\sigma_1} e^{-\Lambda^{2\beta} t} u_0\|_{L^p_x} + \left\| \int_0^t \Lambda^{\sigma_1} e^{-\Lambda^{2\beta}(t-\tau)} f(\tau)\, d\tau \right\|_{L^p_x} \\
&\le \|\Lambda^{\sigma_1} e^{-\Lambda^{2\beta} t} u_0\|_{L^p_x} + C \int_0^t \|\Lambda^{\sigma_1} e^{-\Lambda^{2\beta}(t-\tau)} f(\tau)\|_{L^p_x}\, d\tau \\
&\le \|\Lambda^{\sigma_1} e^{-\Lambda^{2\beta} t} u_0\|_{L^p_x} + C \int_0^t (t-\tau)^{-\frac{\sigma_1}{2\beta}}\, \|f(\tau)\|_{L^p_x}\, d\tau.
\end{aligned}$$

Taking the L^q_t-norm and writing the time integral above as a convolution, we have, by the convolution Young inequality,

$$\begin{aligned}\|\Lambda^{\sigma_1} u\|_{L^q_t L^p_x} &\le C\Big\| \|\Lambda^{\sigma_1-\sigma_3} e^{-\Lambda^{2\beta} t}\Lambda^{\sigma_3}u_0\|_{L^p_x}\Big\|_{L^q_t} + C\Big\|\big(\tau^{-\frac{\sigma_1}{2\beta}}\chi_{\{0\le\tau\le t\}} \star \|f(\tau)\|_{L^p_x}\big)(t)\Big\|_{L^q_t}\\
&\le C\Big\| t^{-\frac{\sigma_1-\sigma_3}{2\beta}}\|\Lambda^{\sigma_3}u_0\|_{L^p_x}\Big\|_{L^q_t} + C\,\|\tau^{-\frac{\sigma_1}{2\beta}}\chi_{\{0\le\tau\le t\}}\|_{L^1_\tau}\|f\|_{L^q_t L^p_x}\\
&\le C\|t^{-\frac{\sigma_1-\sigma_3}{2\beta}}\|_{L^q_t}\|\Lambda^{\sigma_3}u_0\|_{L^p_x} + C\,t^{1-\frac{\sigma_1}{2\beta}}\|f\|_{L^q_t L^p_x}\\
&\le C t^{\frac{1}{q}-\frac{\sigma_2}{2\beta}}\|\Lambda^{\sigma_3}u_0\|_{L^p_x} + C\,t^{1-\frac{\sigma_1}{2\beta}}\|f\|_{L^q_t L^p_x},\end{aligned}$$

where $\star$ denotes the convolution operator and $\chi_{\{0\le\tau\le t\}}$ denotes the characteristic function. This completes the proof of Lemma 2.3. □

We shall also make use of the following commutator estimate (see, e.g., [12,25]).

Lemma 2.4. *Let $p\in[2,\infty)$ and $r\in[1,\infty]$ and $\delta\in(0,1)$, $s\in(0,1)$ such that $s+\delta<1$. Then*

$$\|[\Lambda^\delta, f]g\|_{B^s_{p,r}} \le C(p,r,\delta,s)\big(\|\nabla f\|_{L^p}\|g\|_{B^{s+\delta-1}_{\infty,r}} + \|f\|_{L^2}\|g\|_{L^2}\big). \tag{2.5}$$

Additionally, if f is a divergence-free vector field and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ with $p\in[2,\infty)$, $p_1,\ p_2\in[2,\infty]$, $r\in[1,\infty]$ as well as $s\in(-1,1-\delta)$ for $\delta\in(0,1)$, then it holds

$$\|[\Lambda^\delta, f\cdot\nabla]g\|_{B^s_{p,r}} \le C(p,r,\delta,s)\big(\|\nabla f\|_{L^{p_1}}\|g\|_{B^{s+\delta}_{p_2,r}} + \|f\|_{L^2}\|g\|_{L^2}\big). \tag{2.6}$$

The following fractional type Gagliardo–Nirenberg inequality will also be used (see, e.g., [10]).

Lemma 2.5. *Let $0<p,\ p_0,\ p_1,\ q,\ q_0,\ q_1\le\infty$, $s,\ s_0,\ s_1\in\mathbb{R}$ and $0\le\vartheta\le1$. Then the following fractional type Gagliardo–Nirenberg inequality*

$$\|v\|_{\dot B^s_{p,q}(\mathbb{R}^n)} \le C\|v\|^{1-\vartheta}_{\dot B^{s_0}_{p_0,q_0}(\mathbb{R}^n)}\|v\|^{\vartheta}_{\dot B^{s_1}_{p_1,q_1}(\mathbb{R}^n)} \tag{2.7}$$

holds for all $v\in\dot B^{s_0}_{p_0,q_0}\cap\dot B^{s_1}_{p_1,q_1}$ if and only if

$$\begin{aligned}&\frac{n}{p}-s=(1-\vartheta)\Big(\frac{n}{p_0}-s_0\Big)+\vartheta\Big(\frac{n}{p_1}-s_1\Big), \qquad s\le(1-\vartheta)s_0+\vartheta s_1,\\
&\frac{1}{q}\le\frac{1-\vartheta}{q_0}+\frac{\vartheta}{q_1}, \qquad \text{if } p_0\neq p_1 \text{ and } s=(1-\vartheta)s_0+\vartheta s_1,\\
&s_0\neq s_1 \text{ or } \frac{1}{q}\le\frac{1-\vartheta}{q_0}+\frac{\vartheta}{q_1}, \qquad \text{if } p_0=p_1 \text{ and } s=(1-\vartheta)s_0+\vartheta s_1,\\
&s_0-\frac{n}{p_0}\neq s-\frac{n}{p} \text{ or } \frac{1}{q}\le\frac{1-\vartheta}{q_0}+\frac{\vartheta}{q_1}, \qquad \text{if } s<(1-\vartheta)s_0+\vartheta s_1.\end{aligned}$$

Remark 2.6. Lemma 2.5 is also true in the nonhomogeneous framework.

The following commutator and bilinear estimates involving fractional derivatives will be used (see, e.g., [13,14]).

Lemma 2.7. *Let $s > 0$. Let $p, p_1, p_3 \in (1, \infty)$ and $p_2, p_4 \in [1, \infty]$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then there exist two constants C_1 and C_2,

$$\|[\Lambda^s, f]g\|_{L^p} \le C_1 \left(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\Lambda^{s-1} g\|_{L^{p_3}} \|\nabla f\|_{L^{p_4}}\right),$$
$$\|\Lambda^s (f\, g)\|_{L^p} \le C_2 \left(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}} \|f\|_{L^{p_4}}\right).$$

We recall a lower bound involving the fractional dissipation (see, e.g., [6]).

Lemma 2.8. *For any $\gamma \in (0, 1)$ and $2 \le q < \infty$, the following lower bound holds*

$$\int_{\mathbb{R}^2} \Lambda^{2\gamma}\theta(|\theta|^{q-2}\theta)\, dx \ge \widetilde{c}\, \|\theta\|^q_{L^{\frac{q}{1-\gamma}}},$$

where $\widetilde{c}$ is a positive constant.

Finally we recall the classical Hardy–Littlewood–Sobolev inequality (see, e.g., [20]).

Lemma 2.9. *Let $0 < \varrho < d$ and $1 < q < p < \infty$ satisfy $\frac{1}{p} + \frac{\varrho}{d} = \frac{1}{q}$. Then, for any $f \in L^q(\mathbb{R}^d)$,*

$$\|\Lambda^{-\varrho} f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^q(\mathbb{R}^d)}, \tag{2.8}$$

where C is a positive constant depending only on $d, \varrho,\ p$ and q.

3. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, which boils down to establishing global *a priori* bounds for the solution (u, v, θ) in H^s. This is accomplished in two main steps. The first step contains three preliminary global *a priori* bounds. The second step proves the global bounds for $\|\Delta v\|_{L^1_t L^\infty_x}$ and $\|\omega\|_{L^\infty_t L^\infty_x}$. Once these global bounds are obtained, the global H^s bound on (u, v, θ) then follows as a special consequence.

For the sake of clarity, the rest of this section is divided into two subsections. The first subsection presents the global bounds on

$$\|\Lambda^\sigma v\|_{L^q_t L^2_x}, \quad \|\theta\|_{L^\infty_t L^q_x} \quad \text{and} \quad \|(\nabla u,\ \Lambda^{2-\beta} v,\ \Lambda^\delta \theta)\|_{L^\infty_t L^2_x}$$

where $\sigma \le 2\beta - 1$, $2 \le q < \infty$ and $\delta < \gamma$. The second subsection simultaneously estimates $\|\omega\|_{L^\infty_t L^p_x}$, $\|\Delta v\|_{L^1_t L^p_x}$ and $\|\Lambda^{\sigma_1} v\|_{L^1_t L^p_x}$ with $\sigma_1 < 2\gamma$ due to the fact that the estimate of one of them depends on the other two.

3.1. Three global a priori bounds

This subsection proves three preliminary global *a priori* bounds, which serve as a foundation for bounds in higher regularity spaces. For simplicity, we set $\nu = \eta = 1$ throughout the rest of this section.

The first one is on the $\Lambda^\sigma v$ with $\sigma \le 2\beta - 1$ in $L^q_t L^2_x$.

Proposition 3.1. *Consider (1.2) with the initial data satisfying the assumptions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then, for any $\sigma \le 2\beta - 1$ and any $2 \le q < \infty$,*

$$\int_0^t \|\Lambda^\sigma v(\tau)\|_{L^2}^q \, d\tau \le C(t, u_0, v_0, \theta_0). \tag{3.1}$$

To prove (3.1), we first state the following global L^2 bound, which follows easily from simple energy estimates.

Lemma 3.2. *Consider (1.2) with the initial data satisfying the assumptions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then (u, v, θ) admits the following global L^2 bound*

$$\|(u, v, \theta)(t)\|_{L^2}^2 + 2\int_0^t \left(\|\Lambda^\beta v\|_{L^2}^2 + \|\Lambda^\gamma \theta\|_{L^2}^2 \right) d\tau = \|(u_0, v_0, \theta_0)\|_{L^2}^2.$$

Proof of Proposition 3.1. The proof makes use of Lemma 2.2. To start, we write the second equation of (1.2) as

$$\partial_t v + (-\Delta)^\beta v = -u \cdot \nabla v - v \cdot \nabla u - \nabla \theta.$$

By the Duhamel principle, the k-th component of v can be written in the following integral form

$$v_k(t) = e^{-\Lambda^{2\beta} t} v_{0k} - \int_0^t e^{-\Lambda^{2\beta}(t-\tau)} \left(\nabla \cdot (u v_k + v u_k) - u_k \nabla \cdot v + \partial_k \theta \right) d\tau.$$

Applying $\Lambda^{2\beta-1}$ yields

$$\partial_t \Lambda^{2\beta-1} v_k + \Lambda^{2\beta} \Lambda^{2\beta-1} v_k = -\Lambda^{2\beta-1} \left(\nabla \cdot (u v_k + v u_k) - u_k \nabla \cdot v + \partial_k \theta \right)$$

or

$$\begin{aligned} &\Lambda^{2\beta-1} v_k(t) \\ &= e^{-\Lambda^{2\beta} t} \Lambda^{2\beta-1} v_{0k} - \int_0^t e^{-\Lambda^{2\beta}(t-\tau)} \Lambda^{2\beta-1} \left(\nabla \cdot (u v_k + v u_k) - u_k \nabla \cdot v + \partial_k \theta \right) d\tau \end{aligned}$$

$$
\begin{aligned}
&= e^{-\Lambda^{2\beta} t}\Lambda^{2\beta-1}v_{0k} - \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta-1}\left(\nabla\cdot(uv_k+vu_k)\right)d\tau \\
&\quad + \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta-1}\left(u_k\nabla\cdot v\right)d\tau - \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta-1}\left(\partial_k\theta\right)d\tau \\
&= e^{-\Lambda^{2\beta} t}\Lambda^{2\beta-1}v_{0k} - \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta}\Lambda^{-1}\nabla\cdot\left((uv_k+vu_k)\right)d\tau \\
&\quad + \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta}\Lambda^{-1}\left(u_k\nabla\cdot v\right)d\tau - \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta}\Lambda^{-1}\partial_k\theta\, d\tau \\
&:= N_1+N_2+N_3+N_4.
\end{aligned}
$$

Applying $L_t^qL_x^2$ to the equation above leads to

$$
\int_0^t \|\Lambda^{2\beta-1}v\|_{L^2}^q d\tau \le \int_0^t \|N_1\|_{L^2}^q d\tau + \int_0^t \|N_2\|_{L^2}^q d\tau + \int_0^t \|N_3\|_{L^2}^q d\tau + \int_0^t \|N_4\|_{L^2}^q d\tau.
$$

By Lemma 2.1 with $f=\Lambda^{2\beta-1}v_{0k}$, $\sigma=0$ and $q=p=2$,

$$
\begin{aligned}
\int_0^t \|N_1\|_{L^2}^q d\tau &= \int_0^t \|e^{-\Lambda^{2\beta} t}\Lambda^{2\beta-1}v_{0k}\|_{L^2}^q\, d\tau \\
&\le C\int_0^t \|\Lambda^{2\beta-1}v_{0k}\|_{L^2}^q\, d\tau \\
&= C\|\Lambda^{2\beta-1}v_{0k}\|_{L^2}^q \int_0^t 1\, d\tau \\
&= Ct\|\Lambda^{2\beta-1}v_0\|_{L^2}^q.
\end{aligned}
$$

Applying Lemma 2.1, Minkowski's inequality, the Hardy–Littlewood–Sobolev inequality (2.8) and the convolution Young inequality, we obtain

$$
\begin{aligned}
\int_0^t \|N_3\|_{L^2}^q d\tau &= \left\| \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta}\Lambda^{-1}\left(u_k\nabla\cdot v\right)d\tau \right\|_{L_t^qL_x^2}^q \\
&= \left\| \int_0^t e^{-\Lambda^{2\beta}(t-\tau)}\Lambda^{2\beta+\delta-1}\Lambda^{-\delta}\left(u_k\nabla\cdot v\right)d\tau \right\|_{L_t^qL_x^2}^q
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \left\| \int_0^t e^{-\Lambda^{2\beta}(t-\tau)} \Lambda^{2\beta+\delta-1} \Lambda^{-\delta} (u_k \nabla \cdot v) \, d\tau \right\|_{L_x^2} \right\|_{L_t^q}^q \\
&\le \left\| \int_0^t \left\| e^{-\Lambda^{2\beta}(t-\tau)} \Lambda^{2\beta+\delta-1} \Lambda^{-\delta} (u_k \nabla \cdot v) \right\|_{L_x^2} d\tau \right\|_{L_t^q}^q \\
&\le C \left\| \int_0^t (t-\tau)^{-\frac{2\beta+\delta-1}{2\beta}} \| \Lambda^{-\delta} (u_k \nabla \cdot v) \|_{L_x^2} \, d\tau \right\|_{L_t^q}^q \\
&\le C \left\| \int_0^t (t-\tau)^{-\frac{2\beta+\delta-1}{2\beta}} \| (u_k \nabla \cdot v) \|_{L_x^{\frac{2}{1+\delta}}} \, d\tau \right\|_{L_t^q}^q \\
&\le C \left\| \int_0^t \tau^{-\frac{2\beta+\delta-1}{2\beta}} \, d\tau \right\|_{L_\tau^1}^q \int_0^t \| (u_k \nabla \cdot v) \|_{L_x^{\frac{2}{1+\delta}}}^q \, d\tau \\
&\le C t^{\frac{(1-\delta)q}{2\beta}} \int_0^t \|u\|_{L_x^2}^q \|\nabla v\|_{L_x^{\frac{2}{\delta}}}^q \, d\tau \\
&\le C t^{\frac{(1-\delta)q}{2\beta}} \int_0^t \|u\|_{L_x^2}^q \|v\|_{L_x^2}^{\frac{q(2\beta+\delta-3)}{2\beta-1}} \|\Lambda^{2\beta-1} v\|_{L_x^2}^{\frac{q(2-\delta)}{2\beta-1}} \, d\tau \\
&\le \frac{1}{4} \int_0^t \|\Lambda^{2\beta-1} v\|_{L^2}^q \, d\tau + C t^{\frac{(1-\delta)(2\beta-1)q}{2\beta(2\beta+\delta-3)}} \int_0^t (\|u\|_{L^2}^{\frac{q(2\beta-1)}{2\beta+\delta-3}} \|v\|_{L^2}^q) \, d\tau,
\end{aligned}
$$

where $\delta > 0$ satisfies $3 - 2\beta < \delta < 1$. Such δ exists since $\beta > 1$. By Lemma 2.2,

$$
\begin{aligned}
\int_0^t \|N_2\|_{L^2}^q d\tau &= \left\| \int_0^t e^{-\Lambda^{2\beta}(t-\tau)} \Lambda^{2\beta} \Lambda^{-1} \nabla \cdot ((u v_k + v u_k)) \, d\tau \right\|_{L_t^q L_x^2}^q \\
&\le C \int_0^t \|\Lambda^{-1} \nabla \cdot (u v_k + v u_k)\|_{L^2}^q \, d\tau \\
&\le C \int_0^t \|u v_k + v u_k\|_{L^2}^q \, d\tau \\
&\le C \int_0^t \|u\|_{L^2}^q \|v\|_{L^\infty}^q \, d\tau
\end{aligned}
$$

$$\leq C \int_0^t (\|u\|_{L^2}^q \|v\|_{L^2}^{(1-\frac{1}{2\beta-1})q} \|\Lambda^{2\beta-1} v\|_{L^2}^{\frac{q}{2\beta-1}})\, d\tau$$

$$\leq \frac{1}{4} \int_0^t \|\Lambda^{2\beta-1} v\|_{L^2}^q \, d\tau + C \int_0^t (\|u\|_{L^2}^{\frac{q(2\beta-1)}{2\beta-2}} \|v\|_{L^2}^q)\, d\tau$$

and

$$\int_0^t \|N_4\|_{L^2}^q d\tau = \left\| \int_0^t e^{-\Lambda^{2\beta}(t-\tau)} \Lambda^{2\beta} \Lambda^{-1} \partial_k \theta \, d\tau \right\|_{L_t^q L_x^2}^q$$

$$\leq C \int_0^t \|\Lambda^{-1} \nabla\theta\|_{L^2}^q \, d\tau$$

$$\leq C \int_0^t \|\theta\|_{L^2}^q \, d\tau.$$

Putting these estimates together leads to

$$\int_0^t \|\Lambda^{2\beta-1} v\|_{L^2}^q d\tau \leq \frac{1}{2} \int_0^t \|\Lambda^{2\beta-1} v\|_{L^2}^q dt + Ct \|\Lambda^{2\beta-1} v_0\|_{L^2}^q$$

$$+ Ct^{\frac{(1-\delta)(2\beta-1)q}{2\beta(2\beta+\delta-3)}} \int_0^t (\|u\|_{L^2}^{\frac{q(2\beta-1)}{2\beta+\delta-3}} \|v\|_{L^2}^q)\, d\tau$$

$$+ C \int_0^t (\|u\|_{L^2}^{\frac{q(2\beta-1)}{2\beta-2}} \|v\|_{L^2}^q)\, d\tau + C \int_0^t \|\theta\|_{L^2}^q \, d\tau.$$

Since $\|(u, v, \theta)\|_{L^2}$ is bounded in terms of the initial data (u_0, v_0, θ_0), we obtain

$$\int_0^t \|\Lambda^{2\beta-1} v(\tau)\|_{L^2}^q \, d\tau \leq C(t, u_0, v_0, \theta_0) < \infty.$$

By interpolation, for any $\sigma \leq 2\beta - 1$,

$$\int_0^t \|\Lambda^\sigma v(\tau)\|_{L^2}^q \, d\tau \leq \int_0^t \left(\|v(\tau)\|_{L^2}^{\frac{2\beta-1-\sigma}{2\beta-1}} \|\Lambda^{2\beta-1} v(\tau)\|_{L^2}^{\frac{\sigma}{2\beta-1}}\right)^q \, d\tau \leq C(t, u_0, v_0, \theta_0).$$

This completes the proof of Proposition 3.1. □

The next proposition assesses a global bound for $\|\theta\|_{L_t^\infty L_x^q}$ for any $2 \le q < \infty$.

Proposition 3.3. *Consider (1.2) with $\beta > \frac{3}{2} - \gamma$. Assume that (u_0, v_0, θ_0) satisfy the conditions stated in Theorem 1.1. Then the corresponding smooth solution (u, v, θ) of (1.2) satisfies, for any $t > 0$ and for any $2 \le q < \infty$,*

$$\|\theta(t)\|_{L^q} \le C(t, u_0, v_0, \theta_0). \tag{3.2}$$

Proof of Proposition 3.3. Multiplying $(1.2)_3$ by $|\theta|^{q-2}\theta$ and integrating the resulting equality with respect to x, we have

$$\frac{1}{q}\frac{d}{dt}\|\theta(t)\|_{L^q}^q + \int_{\mathbb{R}^2} \Lambda^{2\gamma}\theta(|\theta|^{q-2}\theta)\,dx = -\int_{\mathbb{R}^2} \nabla \cdot v\ (|\theta|^{q-2}\theta)\,dx.$$

We remark that we may assume $\beta < \frac{3}{2}$. Actually, if $\beta \ge \frac{3}{2}$, one easily obtains from (3.1) that, for any $2 \le p,\ q < \infty$

$$\int_0^t \|\nabla v(\tau)\|_{L^p}^q\,d\tau \le C(t, u_0, v_0, \theta_0).$$

This immediately implies the desired estimate (3.2). According to the lower bound in Lemma 2.8, we have, for a constant $\widetilde{c}$,

$$\int_{\mathbb{R}^2} \Lambda^{2\gamma}\theta(|\theta|^{q-2}\theta)\,dx \ge \widetilde{c}\,\|\theta\|_{L^{\frac{q}{1-\gamma}}}^q.$$

By Hölder's inequality,

$$\left|\int_{\mathbb{R}^2} \nabla \cdot v\ (|\theta|^{q-2}\theta)\,dx\right| \le C\|\nabla v\|_{L^{\frac{2}{2-\sigma}}}\|\theta\|_{L^{\frac{2(q-1)}{\sigma}}}^{q-1}. \tag{3.3}$$

For any $1 < \sigma < 2\beta - 1$, by Sobolev's inequality and an interpolation inequality,

$$\|\nabla v\|_{L^{\frac{2}{2-\sigma}}} \le C\,\|\Lambda^\sigma v\|_{L^2}, \qquad \|\theta\|_{L^{\frac{2(q-1)}{\sigma}}} \le \|\theta\|_{L^q}^{1-\lambda}\|\theta\|_{L^{\frac{q}{1-\gamma}}}^{\lambda},$$

where

$$\lambda = \frac{(2-\sigma)q - 2}{2\gamma(q-1)}.$$

Inserting these inequalities in (3.3) and applying Young's inequality, we have

$$\Big|\int_{\mathbb{R}^2} \nabla\cdot v\,(|\theta|^{q-2}\theta)\,dx\Big| \le \frac{\widetilde{c}}{2}\|\theta\|^q_{L^{\frac{q}{1-\gamma}}} + C\|\Lambda^\sigma v\|_{L^2}^{\frac{q}{q-(q-1)\lambda}}\|\theta\|_{L^q}^{\frac{q(q-1)(1-\lambda)}{q-(q-1)\lambda}}$$

$$\le \frac{\widetilde{c}}{2}\|\theta\|^q_{L^{\frac{q}{1-\gamma}}} + C\|\Lambda^{2\beta-1} v\|_{L^2}^{\frac{q}{q-(q-1)\lambda}}(1+\|\theta\|^q_{L^q}).$$

In order for

$$\lambda = \frac{(2-\sigma)q-2}{2\gamma(q-1)} \in (0,\ 1),$$

we need, noticing that $\sigma < 2\beta - 1$ but close to $2\beta - 1$,

$$(3-2\beta-2\gamma)q < 2(1-\gamma). \tag{3.4}$$

The condition $\beta > \frac{3}{2} - \gamma$ is imposed to ensure (3.4) holds. Therefore, for any $2 \le q < \infty$,

$$\frac{d}{dt}\|\theta(t)\|^q_{L^q} \le C\|\Lambda^\sigma v\|_{L^2}^{\frac{q}{q-(q-1)\lambda}}(1+\|\theta\|^q_{L^q}). \tag{3.5}$$

By Gronwall's inequality and (3.1), we obtain the desired global bound for $\|\theta\|_{L^q}$. □

Next we establish the following global bound of (u, v, θ).

Proposition 3.4. *Suppose that (u_0, v_0, θ_0) satisfies the assumptions stated in Theorem 1.1, $\beta > 1$ and $\beta > \frac{3}{2} - \gamma$. Let (u, v, θ) be the corresponding solution, then (u, v, θ) obeys the following global bound, for any $3 - 2\beta - \gamma < \delta < \gamma$ and for any $t > 0$,*

$$\|(\nabla u, \Lambda^{2-\beta} v, \Lambda^\delta\theta)(t)\|^2_{L^2} + \int_0^t \|(\Delta v, \Lambda^{\delta+\gamma}\theta)\|^2_{L^2}d\tau \le C(t, u_0, v_0, \theta_0). \tag{3.6}$$

Proof of Proposition 3.4. Multiplying both sides of the first three equations of (1.2) with $(-\Delta u, \Lambda^{2(2-\beta)} v, \Lambda^{2\delta}\theta)$, we find

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u, \Lambda^{2-\beta} v, \Lambda^\delta\theta)\|^2_{L^2} + \|(\Delta v, \Lambda^{\delta+\gamma}\theta)\|^2_{L^2}$$
$$= \int_{\mathbb{R}^2} \nabla\cdot(v\otimes v)\cdot\Delta u\,dx + \int_{\mathbb{R}^2} \Lambda^{2-\beta}(v\cdot\nabla u)\cdot\Lambda^{2-\beta} v\,dx$$
$$+ \int_{\mathbb{R}^2} \Lambda^{2-\beta}(u\cdot\nabla v)\cdot\Lambda^{2-\beta} v\,dx + \int_{\mathbb{R}^2} \Lambda^{2-\beta}\nabla\theta\cdot\Lambda^{2-\beta} v\,dx$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^2} \Lambda^\delta \nabla \cdot v \Lambda^\delta \theta + \int_{\mathbb{R}^2} \Lambda^\delta (u \cdot \nabla\theta) \Lambda^\delta \theta \, dx \\
=:& \, I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned} \tag{3.7}
$$

where we have used the following identity, due to $\nabla \cdot u = 0$ (see [22, (3.2)])

$$
\int_{\mathbb{R}^2} (u \cdot \nabla) u \cdot \Delta u \, dx = 0.
$$

By Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} \partial_i (v_i v_j) \partial_k^2 u_j \, dx \\
&= - \int_{\mathbb{R}^2} \partial_k^2 (v_i v_j) \partial_i u_j \, dx \\
&= - \int_{\mathbb{R}^2} (\partial_k^2 v_i v_j + 2 \partial_k v_i \partial_k v_j) \partial_i u_j \, dx \\
&\le C (\|\Delta v\|_{L^2} \|v\|_{L^\infty} + \|\nabla v\|_{L^4}^2) \|\nabla u\|_{L^2} \\
&\le C (\|\Delta v\|_{L^2} \|v\|_{L^\infty} + \|\nabla v\|_{L^2} \|\Delta v\|_{L^2}) \|\nabla u\|_{L^2} \\
&\le C \, (\|v\|_{L^2} + \|\Lambda^\beta v\|_{L^2}) \, \|\Delta v\|_{L^2} \, \|\nabla u\|_{L^2} \\
&\le \frac{1}{6} \|\Delta v\|_{L^2}^2 + C (1 + \|\Lambda^\beta v\|_{L^2}^2) \|\nabla u\|_{L^2}^2 .
\end{aligned}
$$

Since the case when $\beta > \frac{3}{2}$ is dealt with in Theorem 1.2, we restrict to $\beta \in (1, \frac{3}{2}]$ here. By the Hardy–Littlewood–Sobolev inequality (2.8), we have, for $\beta < \frac{3}{2}$,

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} \Lambda^{2-2\beta} (v \cdot \nabla u) \cdot \Lambda^2 v \, dx \\
&\le \|\Lambda^{2-2\beta} (v \cdot \nabla u)\|_{L^2} \|\Delta v\|_{L^2} \\
&= \|\Lambda^{-(2\beta-2)} (v \cdot \nabla u)\|_{L^2} \|\Delta v\|_{L^2} \\
&\le C \|v \cdot \nabla u\|_{L^{\frac{2}{2\beta-1}}} \|\Delta v\|_{L^2} \\
&\le C \|v\|_{L^{\frac{1}{\beta-1}}} \|\nabla u\|_{L^2} \|\Delta v\|_{L^2} \\
&\le \frac{1}{6} \|\Delta v\|_{L^2}^2 + C (1 + \|\Lambda^\beta v\|_{L^2}^2) \|\nabla u\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_{\mathbb{R}^2} \Lambda^{2-2\beta}(u\cdot\nabla v)\cdot\Lambda^2 v\,dx \\
&\le \|\Lambda^{2-2\beta}(u\cdot\nabla v)\|_{L^2}\|\Delta v\|_{L^2} \\
&\le C\|u\cdot\nabla v\|_{L^{\frac{2}{2\beta-1}}}\|\Delta v\|_{L^2} \\
&\le C\|u\|_{L^{\frac{2}{3\beta-3}}}\|\nabla v\|_{L^{\frac{2}{2-\beta}}}\|\Delta v\|_{L^2} \\
&\le C(\|u\|_{L^2}+\|\nabla u\|_{L^2})\|\Lambda^\beta v\|_{L^2}\|\Delta v\|_{L^2} \\
&\le \frac{1}{6}\|\Delta v\|_{L^2}^2 + C(1+\|\Lambda^\beta v\|_{L^2}^2)\|\nabla u\|_{L^2}^2.
\end{aligned}
$$

In terms of the case $\beta=\frac{3}{2}$, one has, for some $p\in(1,\,2)$,

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} \Lambda^{-1}(v\cdot\nabla u)\cdot\Lambda^2 v\,dx \\
&\le \|v\cdot\nabla u\|_{L^p}\|\nabla v\|_{L^{\frac{p}{p-1}}} \\
&\le C\|v\|_{L^{\frac{2p}{2-p}}}\|\nabla u\|_{L^2}\|v\|_{L^2}^{\frac{p-1}{p}}\|\Delta v\|_{L^2}^{\frac{1}{p}} \\
&\le C(\|v\|_{L^2}+\|\Lambda^\beta v\|_{L^2})\|\nabla u\|_{L^2}\|v\|_{L^2}^{\frac{p-1}{p}}\|\Delta v\|_{L^2}^{\frac{1}{p}} \\
&\le \frac{1}{6}\|\Delta v\|_{L^2}^2 + C(1+\|\Lambda^\beta v\|_{L^2}^2)(1+\|\nabla u\|_{L^2}^2)
\end{aligned}
$$

and, for some $\widetilde{p}\in(\frac{4}{3},\,2)$,

$$
\begin{aligned}
I_3 &= \int_{\mathbb{R}^2} \Lambda^{-1}(u\cdot\nabla v)\cdot\Lambda^2 v\,dx \\
&\le \|u\cdot\nabla v\|_{L^{\widetilde{p}}}\|\nabla v\|_{L^{\frac{\widetilde{p}}{\widetilde{p}-1}}} \\
&\le C\|u\|_{L^{\frac{2\widetilde{p}}{2-(2-\beta)\widetilde{p}}}}\|\nabla v\|_{L^{\frac{2}{2-\beta}}}\|v\|_{L^2}^{\frac{\widetilde{p}-1}{\widetilde{p}}}\|\Delta v\|_{L^2}^{\frac{1}{\widetilde{p}}} \\
&\le C(\|u\|_{L^2}+\|\nabla u\|_{L^2})\|\Lambda^\beta v\|_{L^2}\|v\|_{L^2}^{\frac{\widetilde{p}-1}{\widetilde{p}}}\|\Delta v\|_{L^2}^{\frac{1}{\widetilde{p}}} \\
&\le \frac{1}{6}\|\Delta v\|_{L^2}^2 + C(1+\|\Lambda^\beta v\|_{L^2}^2)(1+\|\nabla u\|_{L^2}^2).
\end{aligned}
$$

Thanks to $3 - 2\beta - \gamma < \delta$, by Young's inequality

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^2} \Lambda^{2-\beta} \nabla\theta \cdot \Lambda^{2-\beta} v \, dx \\
&\le \|\Lambda^{3-2\beta}\theta\|_{L^2} \|\Delta v\|_{L^2} \\
&\le \frac{1}{6}\|\Delta v\|_{L^2}^2 + \frac{1}{6}\|\Lambda^{\delta+\gamma}\theta\|_{L^2}^2 + C\|\theta\|_{L^2}^2.
\end{aligned}
$$

According to a simple interpolation inequality and Young's inequality,

$$
\begin{aligned}
I_5 &\le \|\Lambda^{\delta+1-\gamma} v\|_{L^2} \|\Lambda^{\delta+\gamma}\theta\|_{L^2} \\
&\le \frac{1}{6}\|\Delta v\|_{L^2}^2 + \frac{1}{6}\|\Lambda^{\delta+\gamma}\theta\|_{L^2}^2 + C\|v\|_{L^2}^2.
\end{aligned}
$$

By the commutator estimate (2.5),

$$
\begin{aligned}
I_6 &= -\int_{\mathbb{R}^2} [\Lambda^\delta, \ u\cdot\nabla]\theta \ \Lambda^\delta\theta \, dx \\
&\le \|[\Lambda^\delta, \ u]\theta\|_{H^{1-\gamma}} \|\Lambda^{\delta+\gamma}\theta\|_{L^2} \\
&\le C\big(\|\nabla u\|_{L^2}\|\theta\|_{B^{\delta-\gamma}_{\infty,2}} + \|u\|_{L^2}\|\theta\|_{L^2}\big)\|\Lambda^{\delta+\gamma}\theta\|_{L^2} \\
&\le C\big(\|\nabla u\|_{L^2}\|\theta\|_{L^q} + \|u\|_{L^2}\|\theta\|_{L^2}\big)\|\Lambda^{\delta+\gamma}\theta\|_{L^2} \\
&\le \frac{1}{6}\|\Lambda^{\delta+\gamma}\theta\|_{L^2}^2 + C\|\theta\|_{L^q}^2\|\nabla u\|_{L^2}^2 + C\|u\|_{L^2}^2\|\theta\|_{L^2}^2,
\end{aligned}
$$

where we have used the estimate, for $\delta < \gamma$,

$$
\|\theta\|_{B^{\delta-\gamma}_{\infty,2}} \le C\|\theta\|_{L^q}, \qquad q > \frac{2}{\gamma-\delta}.
$$

Combining all the estimates above and using (3.2), we reach

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla u, \Lambda^{2-\beta}v, \Lambda^\delta\theta)(t)\|_{L^2}^2 + \|(\Delta v, \Lambda^{\delta+\gamma}\theta)\|_{L^2}^2 \\
&\le C\,\|\Lambda^\gamma\theta\|_{L^2}^2 + C(1 + \|\Lambda^\beta v\|_{L^2}^2 + \|\theta\|_{L^q}^2)(1 + \|\nabla u\|_{L^2}^2).
\end{aligned}
$$

Gronwall's inequality and Proposition 4.2 imply (3.6). □

3.2. Global bounds for $\|\Delta v\|_{L^1_t L^\infty_x}$ and $\|\omega\|_{L^\infty_t L^\infty_x}$

The goal of this subsection is to show that the following proposition.

Proposition 3.5. *Consider (1.2) with $\beta > 1$ and $\beta > \frac{3}{2} - \gamma$. Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then, for any $2 \le p \le \infty$,*

$$\|\Delta v\|_{L^1_t L^p_x} \le C < \infty, \qquad \|\omega\|_{L^\infty_t L^p_x} \le C < \infty,$$

where the upper bounds $C = C(t, u_0, v_0, \theta_0)$.

To prove Proposition 3.5, we need a global bound for v in a more regular setting than provided in the previous subsection.

Proposition 3.6. *Consider (1.2) with $\beta > 1$ and $\beta > \frac{3}{2} - \gamma$. Assume that (u_0, v_0, θ_0) satisfy the conditions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then for any $t > 0$,*

$$\|\Lambda^\rho v(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\rho+\beta} v(\tau)\|_{L^2}^2 d\tau \le C(t, u_0, v_0, \theta_0), \tag{3.8}$$

where $\rho = \delta + \gamma + \beta - 1$ with $\delta < 1 - \gamma$. Especially, we have, by taking ρ close to β,

$$\|v(t)\|_{L^\infty} \le C(t, u_0, v_0, \theta_0)$$

and for any $2 \le q \le \infty$,

$$\int_0^t \|\nabla v\|_{L^q}^2 \, d\tau \le C \int_0^t \|\Lambda^{\rho+\beta} v\|_{L^2}^2 \, d\tau \le C(t, u_0, v_0, \theta_0),$$

which, by the proof of Proposition 3.3, implies

$$\|\theta(t)\|_{L^\infty} \le C(t, u_0, v_0, \theta_0).$$

Proof of Proposition 3.6. Taking the inner product of the second equation in (1.2) with $\Lambda^{2\rho} v$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^\rho v\|_{L^2}^2 + \|\Lambda^{\rho+\beta} v\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^2} (v \cdot \nabla u) \cdot \Lambda^{2\rho} v \, dx - \int_{\mathbb{R}^2} (u \cdot \nabla v) \cdot \Lambda^{2\rho} v \, dx - \int_{\mathbb{R}^2} \nabla\theta \cdot \Lambda^{2\rho} v \, dx \\
&=: \widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3.
\end{aligned} \tag{3.9}$$

By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
\widetilde{I}_1 &\le C\|v\|_{L^\infty}\|\nabla u\|_{L^2}\|\Lambda^{2\rho}v\|_{L^2}\\
&\le C(\|v\|_{L^2}+\|\Lambda^{\rho+\beta}v\|_{L^2})\|\Lambda^{\beta}v\|_{L^2}\,\|\nabla u\|_{L^2}\\
&\le \frac{1}{6}\|\Lambda^{\rho+\beta}v\|_{L^2}^2+C\|\Lambda^{\beta}v\|_{L^2}^2\|\nabla u\|_{L^2}^2+C\|v\|_{L^2}\|\Lambda^{\beta}v\|_{L^2}\|\nabla u\|_{L^2}\\
&\le \frac{1}{6}\|\Lambda^{\rho+\beta}v\|_{L^2}^2+C\|\Lambda^{\beta}v\|_{L^2}^2\|\nabla u\|_{L^2}^2+C\|v\|_{L^2}^2.
\end{aligned}$$

Similarly, for suitable small $\epsilon > 0$,

$$\begin{aligned}
\widetilde{I}_2 &\le C\|u\|_{L^{\frac{2}{\beta-\rho+\epsilon}}}\|\nabla v\|_{L^{\frac{2}{1-\epsilon}}}\|\Lambda^{2\rho}v\|_{L^{\frac{2}{1+\rho-\beta}}}\\
&\le C\|u\|_{H^1}^2\|\Lambda^{\rho}v\|_{L^2}^{1-\lambda(\epsilon)}\|\Lambda^{\rho+\beta}v\|_{L^2}^{\lambda(\epsilon)}\|\Lambda^{\rho+\beta}v\|_{L^2}\\
&\le \frac{1}{6}\|\Lambda^{\rho+\beta}v\|_{L^2}^2+C\|u\|_{H^1}^{\frac{2}{1-\lambda(\epsilon)}}\|\Lambda^{\rho}v\|_{L^2}^2,
\end{aligned}$$

where $\lambda(\epsilon)=\frac{1-\rho+\epsilon}{\beta}\in(0,\ 1)$. Thanks to $\rho=\delta+\gamma+\beta-1$, by Young's inequality

$$\begin{aligned}
\widetilde{I}_3 &= -\int_{\mathbb{R}^2}\nabla\theta\cdot\Lambda^{2\rho}v\,dx \le \|\Lambda^{\rho-\beta+1}\theta\|_{L^2}\|\Lambda^{\rho+\beta}v\|_{L^2}\\
&\le \frac{1}{6}\|\Lambda^{\rho+\beta}v\|_{L^2}^2+C\|\Lambda^{\delta+\gamma}\theta\|_{L^2}^2.
\end{aligned}$$

Summing up all the estimates above, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\Lambda^{\rho}v\|_{L^2}^2+\|\Lambda^{\rho+\beta}v\|_{L^2}^2\\
&\le C\|\Lambda^{\beta}v\|_{L^2}^2\|\nabla u\|_{L^2}^2+C\|v\|_{L^2}^2+C\|\Lambda^{\delta+\gamma}\theta\|_{L^2}^2+C\|u\|_{H^1}^{\frac{2}{1-\lambda(\epsilon)}}\|\Lambda^{\rho}v\|_{L^2}^2.
\end{aligned}$$

Gronwall's inequality then yields the desired bound in (3.8). This completes the proof of Proposition 3.6. □

The proof of Proposition 3.5 is divided into three closely related steps. The result of the first step is stated in the following Lemma.

Lemma 3.7. *Consider (1.2) with $\beta > 1$ and $\beta > \frac{3}{2}-\gamma$. Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then, for any $2 \le p \le \infty$, the vorticity $\omega = \nabla \times u$ obeys, for any $t > 0$,*

$$\|\omega(t)\|_{L^p}\le\|\omega_0\|_{L^p}+C\int_0^t\|\Delta v(\tau)\|_{L^p}\,d\tau, \tag{3.10}$$

where $C = C(t, u_0, v_0, \theta_0)$.

Proof. It follows from the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega + \nabla \times \nabla \cdot (v \otimes v) = 0 \tag{3.11}$$

that

$$\begin{aligned}\|\omega(t)\|_{L^p} &\le \|\omega_0\|_{L^p} + \int_0^t \|\nabla \times \nabla \cdot (v \otimes v)\|_{L^p}\, d\tau \\ &\le \|\omega_0\|_{L^p} + C \int_0^t \|v\|_{L^\infty} \|\Delta v\|_{L^p}\, d\tau \\ &\le \|\omega_0\|_{L^p} + C \int_0^t \|\Delta v\|_{L^p}\, d\tau,\end{aligned}$$

where we have used the global bound on $\|v\|_{L^\infty}$ from Proposition 3.6. This proves Lemma 3.7. □

The second step controls $\|\Lambda^\sigma v\|_{L^1_t L^p_x}$ for any $0 < \sigma < 2\beta$ in terms of ω and θ, as provided by the following lemma.

Lemma 3.8. *Consider (1.2) with $\beta > 1$ and $\beta > \frac{3}{2} - \gamma$. Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then, for any $2 \le p \le \infty$ and any $\epsilon > 0$,*

$$\begin{aligned}\|\Delta v\|_{L^1_t L^p_x} \le\, & t^{1-\frac{1}{2\beta}} \|\Lambda u_0\|_{L^p} + C\, t^{1-\frac{1}{\beta}} \\ & + C\, t^{1-\frac{1}{\beta}} \|\omega\|_{L^1_t L^p_x} + C\, t^{\frac{\epsilon}{2\beta}} \|\Lambda^{3-2\beta+\epsilon}\theta\|_{L^1_t L^p_x}.\end{aligned} \tag{3.12}$$

In addition, for any $2 < \tilde{\mu} < 2\beta$ but close to 2, say

$$\tilde{\mu} = 2 + \frac{1}{10}(2\beta + 2\gamma - 3),$$

we have

$$\begin{aligned}\|\Lambda^{\tilde{\mu}} v\|_{L^1_t L^p_x} \le\, & t^{1-\frac{\tilde{\mu}}{2\beta}} \|\Lambda u_0\|_{L^p} + C\, t^{1-\frac{\tilde{\mu}}{2\beta}} \\ & + C\, t^{1-\frac{\tilde{\mu}}{2\beta}} \|\omega\|_{L^1_t L^p_x} + C\, t^{\frac{\tilde{\mu}-2}{2\beta}} \|\Lambda^{3-2\beta+(\tilde{\mu}-2)}\theta\|_{L^1_t L^p_x}.\end{aligned} \tag{3.13}$$

Proof. We proceed as in the proof of Lemma 2.3. First, we write the equation of v in (1.2) in the integral form

$$v(t) = e^{-\nu\Lambda^{2\beta}t} v_0 - \int_0^t e^{-\nu\Lambda^{2\beta}(t-\tau)} (u \cdot \nabla v + v \cdot \nabla u + \nabla\theta)\, d\tau.$$

Applying Λ^2 to the integral form above, we obtain, as the proof of Lemma 2.3,

$$\begin{aligned}\|\Lambda^2 v\|_{L^p_x} \le \|\Lambda^2 e^{-\nu\Lambda^{2\beta}t} v_0\|_{L^p_x} &+ C \int_0^t (t-\tau)^{-\frac{1}{\beta}} \|u(\tau)\|_{L^{2p}} \|\nabla v(\tau)\|_{L^{2p}}\, d\tau \\ &+ C \int_0^t (t-\tau)^{-\frac{1}{\beta}} \|v(\tau)\|_{L^\infty} \|\nabla u(\tau)\|_{L^p}\, d\tau \\ &+ C \int_0^t (t-\tau)^{-1+\frac{\epsilon}{2\beta}} \|\Lambda^{3-2\beta+\epsilon}\theta(\tau)\|_{L^p}\, d\tau\end{aligned}$$

for any $\epsilon > 0$ small. Invoking the basic inequalities

$$\|u\|_{L^{2p}} \le C\|u\|_{H^1}, \qquad \|\nabla u\|_{L^p} \le C\,\|\omega\|_{L^p}$$

and the global bound for $\|\nabla v(\tau)\|_{L^1_t L^{2p}_x}$ in Proposition 3.6, we obtain, after integrating in time,

$$\begin{aligned}\|\Delta v\|_{L^1_t L^p_x} \le & t^{1-\frac{1}{2\beta}} \|\Lambda u_0\|_{L^p} + C\, t^{1-\frac{1}{\beta}} \\ & + C\, t^{1-\frac{1}{\beta}} \|\omega\|_{L^1_t L^p_x} + C\, t^{\frac{\epsilon}{2\beta}} \|\Lambda^{3-2\beta+\epsilon}\theta\|_{L^1_t L^p_x}.\end{aligned}$$

The proof of (3.13) is very similar. The difference is that one applies $\Lambda^{\tilde{\mu}}$ instead of Λ^2. This proves Lemma 3.8. □

The third step makes use of the equation of θ and proves the following lemma.

Lemma 3.9. *Consider (1.2) with $\beta > 1$ and $\beta > \frac{3}{2} - \gamma$. Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.1. Let (u, v, θ) be the corresponding solution. Then, for any $2 \le p < \infty$ and for any $\sigma < 2\gamma$,*

$$\|\Lambda^\sigma \theta\|_{L^1_t L^p_x} \le \|\theta_0\|_{L^p} + C\,\|\omega\|_{L^1_t L^p_x} + C, \tag{3.14}$$

where $C = C(t, u_0, v_0, \theta_0)$.

Proof. Applying the Fourier localization operator Δ_j with $j \in \mathbb{Z}$ and $j \ge -1$ to the equation of θ and then dotting the resulting equation with $\Delta_j\theta|\Delta_j\theta|^{p-2}$ yields

$$\frac{1}{p}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}^p + C\,2^{2\gamma j}\,\|\Delta_j\theta\|_{L^p}^p = -\int \Delta_j\theta\,|\Delta_j\theta|^{p-2}\,[\Delta_j, u\cdot\nabla\theta]\,dx + \int \Delta_j\theta\,|\Delta_j\theta|^{p-2}\,\Delta_j\nabla\cdot v\,dx,$$

where we have invoked the lower bound (see [5] or [17])

$$\int \Delta_j\theta\,|\Delta_j\theta|^{p-2}\,\Lambda^{2\gamma}\,\Delta_j\theta\,dx \ge C\,2^{2\gamma j}\,\|\Delta_j\theta\|_{L^p}^p.$$

More details on the Littlewood–Paley decomposition, Besov spaces, Bernstein's inequalities and other related materials can be found in the appendix. Recall a standard commutator estimate,

$$\|[\Delta_j, u\cdot\nabla]\theta\|_{L^p} \le C\,\|\theta\|_{B^0_{\infty,\infty}}\,\|\nabla u\|_{L^p}.$$

Applying Hölder's inequality then yields

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^p} + C\,2^{2\gamma j}\,\|\Delta_j\theta\|_{L^p} \le C\,\|\theta\|_{L^\infty}\,\|\omega\|_{L^p} + C\,\|\nabla v\|_{L^p}.$$

Integrating in time yields

$$\|\Delta_j\theta(t)\|_{L^p} \le C\,e^{-2^{2\gamma j}t}\,\|\Delta_j\theta_0\|_{L^p} + C\int_0^t e^{-2^{2\gamma j}(t-\tau)}\,(\|\omega(\tau)\|_{L^p} + \|\nabla v(\tau)\|_{L^p})\,d\tau.$$

Taking the L^1-norm in time and applying Young's inequality for convolution, we have

$$\int_0^t \|\Delta_j\theta(\tau)\|_{L^p}\,d\tau \le C\,2^{-2\gamma j}\,\|\Delta_j\theta_0\|_{L^p} + C\,2^{-2\gamma j}\,\Big(\|\omega\|_{L^1_tL^p_x} + \|\nabla v\|_{L^1_tL^p_x}\Big).$$

Multiplying both sides by $2^{2\gamma j}$ yields, for any $j\ge 0$,

$$2^{2\gamma j}\int_0^t \|\Delta_j\theta(\tau)\|_{L^p}\,d\tau \le C\,\|\Delta_j\theta_0\|_{L^p} + C\,\|\omega\|_{L^1_tL^p_x} + C.$$

As a special consequence, for any $\sigma < 2\gamma$,

$$\begin{aligned}\int_0^t \|\Lambda^\sigma\theta(\tau)\|_{L^p}\,d\tau &\le \int_0^t \|\Lambda^\sigma\theta(\tau)\|_{B^0_{p,1}}\,d\tau\\ &= \sum_{j\ge -1}\int_0^t \|\Delta_j\Lambda^\sigma\theta(\tau)\|_{L^p}\,d\tau\end{aligned}$$

$$\begin{aligned}
&\le \sum_{j\ge -1} \int_0^t 2^{\sigma j} \|\Delta_j \theta(\tau)\|_{L^p}\, d\tau \\
&= \sum_{j\ge -1} 2^{(\sigma-2\gamma)j}\, 2^{2\gamma j} \int_0^t \|\Delta_j \theta(\tau)\|_{L^p}\, d\tau \\
&\le \|\theta_0\|_{L^p} + C\, \|\omega\|_{L^1_t L^p_x} + C,
\end{aligned}$$

which is (3.14). This completes the proof of Lemma 3.9. □

We are now ready to prove Proposition 3.5.

Proof of Proposition 3.5. Since

$$\beta + \gamma > \frac{3}{2},$$

we can choose $\epsilon > 0$ and $\sigma > 0$ such that

$$3 - 2\beta + \epsilon < \sigma < 2\gamma. \tag{3.15}$$

It follows from Lemma 3.7, (3.12) in Lemma 3.8 and Lemma 3.9 that

$$\begin{aligned}
\|\omega(t)\|_{L^p} &\le \|\omega_0\|_{L^p} + C \int_0^t \|\Delta v(\tau)\|_{L^p}\, d\tau \\
&\le \|\omega_0\|_{L^p} + C \int_0^t \|\omega(\tau)\|_{L^p}\, d\tau + C \int_0^t \|\Lambda^{3-2\beta+\epsilon}\theta\|_{L^p}\, d\tau + C \\
&\le C + C \int_0^t \|\omega(\tau)\|_{L^p}\, d\tau + C \int_0^t \|\Lambda^{\sigma}\theta\|_{L^p}\, d\tau \\
&\le C + C \int_0^t \|\omega(\tau)\|_{L^p}\, d\tau.
\end{aligned}$$

Gronwall's inequality then implies that, for any $\epsilon > 0$ and $\sigma > 0$ satisfying (3.15)

$$\|\omega(t)\|_{L^p} \le C, \qquad \int_0^t \|\Lambda^{\sigma}\theta\|_{L^p}\, d\tau \le C.$$

Then we apply (3.13) in Lemma 3.8 with $2 < \tilde{\mu} < 2\beta$ but close to 2 to obtain, for $\sigma < 2\gamma$ close to 2γ,

$$\int_0^t \|\Lambda^{\tilde{\mu}} v\|_{L^p}\, d\tau \le C + C\,\|\omega\|_{L_t^1 L_x^p} + C\,\|\Lambda^\gamma \theta\||_{L_t^1 L_x^p} < \infty.$$

Due to the Sobolev embedding, for $\tilde{\mu} > 2$,

$$\|\Delta v\|_{L_x^\infty} \le C\,(\|v\|_{L^2} + \|\Lambda^{\tilde{\mu}} v\|_{L^p}),$$

we obtain, for any $t > 0$,

$$\int_0^t \|\Delta v\|_{L^\infty}\, dt < \infty.$$

The vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega + \nabla \times \nabla \cdot (v \otimes v) = 0$$

then implies

$$\|\omega(t)\|_{L^\infty} < \infty.$$

These global bounds are then sufficient for any higher regularity. This completes the proof of Proposition 3.5. □

4. The proof of Theorem 1.2

This section proves Theorem 1.2. The main efforts are devoted to showing the global *a priori* bound for the solution (u, v, θ) in H^s. The key component is to obtain the global L^∞ for ∇u. This is accomplished via several propositions.

Proposition 4.1 *(Uniform H^1 bound for the velocity). Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.2. Let (u, v, θ) be the corresponding solution of (1.3). Then the following uniform H^1 bound holds*

$$\|\omega\|_{L^2}^2 + \|\Lambda^{\beta-1} v\|_{L^2}^2 + \int_0^t \|\Lambda^{2\beta-1} v\|_{L^2}^2\, d\tau \le C(\|(u_0, v_0, \theta_0)\|_{H^1}). \tag{4.1}$$

As a consequence,

$$\int_0^\infty \|\nabla v\|_{L^\infty}\, dt < \infty.$$

We start with the global L^2 bound.

Lemma 4.2 *(Global L^2 bound). Consider (1.3). Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.2. Let (u, v, θ) be the corresponding solution of (1.3). Then (u, v, θ) obeys the following global L^2-bound*

$$\|(u, v, \theta)(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^\beta v\|_{L^2}^2 \, d\tau = \|(u_0, v_0, \theta_0)\|_{L^2}^2, \tag{4.2}$$

for any $t > 0$, which implies $\int_0^t \|v\|_{L^\infty}^2 d\tau < +\infty$ due to $\beta > 1$.

Proof. Taking the L^2-inner product of (1.3) with (u, v, θ), integrating by parts and using $\nabla \cdot u = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(u, v, \theta)\|_{L^2}^2 + \|\Lambda^\beta v\|_{L^2}^2 = 0, \tag{4.3}$$

where we have used the following facts

$$\int_{\mathbb{R}^2} (u \cdot \nabla)u \cdot u \, dx = \int_{\mathbb{R}^2} (u \cdot \nabla)v \cdot v \, dx = \int_{\mathbb{R}^2} (u \cdot \nabla\theta)\theta \, dx = 0,$$

$$\int_{\mathbb{R}^2} \nabla \cdot (v \otimes v) \cdot u \, dx + \int_{\mathbb{R}^2} (v \cdot \nabla)u \cdot v \, dx = 0,$$

$$\int_{\mathbb{R}^2} \nabla\theta \cdot v \, dx + \int_{\mathbb{R}^2} (\nabla \cdot v)\theta \, dx = 0.$$

Integrating (4.3) in time from 0 to t implies (4.2). □

We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. The vorticity $\omega = \nabla \times u$ satisfies

$$\partial_t \omega + u \cdot \nabla\omega + \nabla \times \nabla \cdot (v \otimes v) = 0. \tag{4.4}$$

Taking the L^2 inner product of (4.4) with ω and integrating by parts yield

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \nabla \times \nabla \cdot (v \otimes v)\omega dx \\
&\le C\|\Delta v\|_{L^2}\|v\|_{L^\infty}\|\omega\|_{L^2} \\
&\le \frac{1}{4}\|\Lambda^{2\beta-1} v\|_{L^2}^2 + C\|v\|_{L^\infty}^2\|\omega\|_{L^2}^2
\end{aligned}$$

Dotting both sides of the v-equation in (1.3) with $\Lambda^{2\beta-2}v$ and integrating on $\mathbb{R}^2$, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{\beta-1}v\|_{L^2}^2 + \|\Lambda^{2\beta-1}v\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^2} (u\cdot\nabla v)\Lambda^{2\beta-2}v dx - \int_{\mathbb{R}^2} (v\cdot\nabla u)\Lambda^{2\beta-2}v dx - \int_{\mathbb{R}^2} \nabla\theta\Lambda^{2\beta-2}v dx \\
&\le C\|u\|_{L^2}\|v\|_{L^\infty}\|\Lambda^{2\beta-1}v\|_{L^2} + C\|\nabla u\|_{L^2}\|v\|_{L^\infty}\|\Lambda^{2\beta-2}v\|_{L^2} + C\|\theta\|_{L^2}\|\Lambda^{2\beta-1}v\|_{L^2} \\
&\le C\|v\|_{L^\infty}\|\Lambda^{2\beta-1}v\|_{L^2} + C\|v\|_{L^\infty}^2\|\omega\|_{L^2}^2 + \frac{1}{8}\|\Lambda^{2\beta-1}v\|_{L^2}^2 + C\|\theta\|_{L^2}\|\Lambda^{2\beta-1}v\|_{L^2} \\
&\le C\|v\|_{L^\infty}^2\|\omega\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{2\beta-1}v\|_{L^2}^2
\end{aligned}$$

Combining the inequalities above, we have

$$\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|\Lambda^{\beta-1}v\|_{L^2}^2) + \|\Lambda^{2\beta-1}v\|_{L^2}^2 \le C\|v\|_{L^\infty}^2\|\omega\|_{L^2}^2 \tag{4.5}$$

Integrating (4.5) in time from 0 to t implies (4.1). Since $2\beta-1>2$, (4.1), together with Sobolev embedding inequality, implies

$$\int_0^t \|\nabla v\|_{L^\infty} d\tau \le t^{\frac{1}{2}}\left(\int_0^t \|\nabla v\|_{L^\infty}^2 d\tau\right)^{\frac{1}{2}} \le t^{\frac{1}{2}}\left(\int_0^t (\|v\|_{L^2}^2 + \|\Lambda^{2\beta-1}v\|_{L^2}^2)d\tau\right)^{\frac{1}{2}} < \infty.$$

This completes the proof of Proposition 4.1. □

The following global L^p bound for θ holds.

Proposition 4.3. *Consider (1.3). Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.2. Let (u, v, θ) be the corresponding solution of (1.3). Then, for any $2 \le p \le \infty$,*

$$\|\theta(t)\|_{L^p} \le C(t, u_0, v_0, \theta_0). \tag{4.6}$$

Proof. For any $2 \le p < \infty$, by multiplying both sides of the θ-equation in (1.3) by $|\theta|^{p-2}\theta$ and integrating on $\mathbb{R}^2$, we have

$$\frac{1}{p}\frac{d}{dt}\|\theta\|_{L^p}^p \le -\int_{\mathbb{R}^2} \nabla\cdot v|\theta|^{p-2}\theta dx \le C\|\nabla v\|_{L^p}\|\theta\|_{L^p}^{p-1}.$$

Integrating in time leads to

$$\|\theta\|_{L^p} \le \|\theta_0\|_{L^p} + C\int_0^t \|\nabla v\|_{L^p} ds \le C\int_0^t \|v\|_{L^2}^{1-\frac{2(p-1)}{(2\beta-1)p}}\|\Lambda^{2\beta-1}v\|_{L^2}^{\frac{2(p-1)}{(2\beta-1)p}} ds,$$

which yields the bound for $p < \infty$. Now taking the limit as $p \to \infty$, we have

$$\|\theta(t)\|_{L^\infty} \le C \int_0^t (\|v\|_{L^2}^2 + \|\Lambda^{2\beta-1} v\|_{L^2}^2) ds < +\infty.$$

This completes the proof. □

Next we establish a bound for $\int_0^t \|\Delta v\|_{L^\infty}^2$.

Proposition 4.4. *Consider (1.3). Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.2. Let (u, v, θ) be the corresponding solution of (1.3). Then, for any $t > 0$,*

$$\int_0^t \|\Delta v\|_{L^\infty}^2 \, d\tau < +\infty.$$

Proof. Applying the argument of Lemma 2.3, we have, for any $\delta < 2\beta - 1$ and for any $2 \le p < \infty$,

$$\begin{aligned} \int_0^t \|\Lambda^\delta v\|_{L^p}^2 \, d\tau &\le t^{\frac{2}{\beta}-1} \|\nabla v_0\|_{L^p}^2 + C \int_0^t (\|uv\|_{L^p}^2 + \|\theta\|_{L^p}^2) \, d\tau + C \int_0^t \|u \nabla \cdot v\|_{L^p}^2 \, d\tau \\ &\le C(t, \|v_0\|_{H^s}) + C \int_0^t (\|u\|_{H^1}^2 \|v\|_{L^\infty}^2 + \|u\|_{H^1}^2 \|\nabla v\|_{L^\infty}^2 + \|\theta\|_{L^p}^2) \, d\tau \\ &\le C(t, u_0, v_0, \theta_0) < \infty. \end{aligned}$$

Since $2\beta - 1 > 2$, a simple embedding inequality then implies

$$\int_0^t \|\Delta v\|_{L^\infty}^2 \, d\tau < \infty.$$

This completes the proof of Proposition 4.4. □

The next proposition proves a key component in the proof of Theorem 1.2, a global bound for $\|\nabla u\|_{L^\infty}$.

Proposition 4.5 *(L^∞ estimate of the ω). Consider (1.3). Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.2. Let (u, v, θ) be the corresponding solution of (1.3). Then, for any $t > 0$,*

$$\|\nabla u(t)\|_{L^\infty} < \infty.$$

Proof. We start by recalling the vorticity equation in (4.4),

$$\partial_t \omega + u \cdot \nabla \omega = -\nabla \times \nabla \cdot (v \otimes v). \tag{4.7}$$

By a bound on solutions to the transport equation in Besov spaces with zero regularity index, namely $B^0_{\infty,1}$ (see, e.g., [1,17,23]),

$$\|\omega\|_{B^0_{\infty,1}} \le \Big(\|\omega_0\|_{B^0_{\infty,1}} + \int_0^t \|\nabla \times \nabla \cdot (v \otimes v)\|_{B^0_{\infty,1}} \, d\tau \Big) \Big(1 + \int_0^t \|\nabla u\|_{L^\infty} \, d\tau \Big).$$

For $p > \frac{2}{2\beta - 3}$,

$$\begin{aligned} \|\nabla \times \nabla \cdot (v \otimes v)\|_{B^0_{\infty,1}} &\le C \|vv\|_{B^2_{\infty,1}} \\ &\le C \|v\|_{L^\infty} \|v\|_{B^2_{\infty,1}} \\ &\le C \|v\|^2_{B^2_{\infty,1}} \\ &\le C \|v\|^2_{L^2} + C \|\Lambda^{2\beta-1} v\|^2_{L^p}. \end{aligned} \tag{4.8}$$

In addition,

$$\|\nabla u\|_{L^\infty} \le C \|u\|_{L^2} + C \|\omega\|_{B^0_{\infty,1}}.$$

Therefore,

$$\|\omega\|_{B^0_{\infty,1}} \le C \Big(1 + \int_0^t \|\Lambda^{2\beta-1} v\|^2_{L^p} \, d\tau \Big) \Big(1 + \int_0^t \|\omega\|_{B^0_{\infty,1}} \, d\tau \Big).$$

Thanks to Gronwall's inequality, we have

$$\|\omega\|_{B^0_{\infty,1}} \le C(t),$$

which further implies

$$\|\nabla u(t)\|_{B^0_{\infty,1}} \le C(t).$$

This completes the proof of Proposition 4.5. □

A simple consequence of (4.8) is a global bound for $\|\nabla \theta\|_{L^p}$.

Proposition 4.6 *(L^p estimate of $\nabla\theta$). Consider (1.3). Assume that (u_0, v_0, θ_0) satisfies the conditions stated in Theorem 1.2. Let (u, v, θ) be the corresponding solution of (1.3). Then, for any $2 \le p \le \infty$ and any $t > 0$,*

$$\|\nabla\theta(t)\|_{L^p} \le C(t, u_0, v_0, \theta_0) < \infty. \tag{4.9}$$

Proof. Applying ∇ to the θ-equation in (1.3)

$$\partial_t\theta + u \cdot \nabla\theta + \nabla \cdot v = 0,$$

then multiplying the resultant by $|\nabla\theta|^{p-2}\nabla\theta$, and integrating over $\mathbb{R}^2$, we have, after integration by parts,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\nabla\theta\|_{L^p}^p &\le \int_{\mathbb{R}^2} \nabla\theta \cdot \nabla u \cdot |\nabla\theta|^{p-2}\nabla\theta\, dx + \int_{\mathbb{R}^2} \nabla\nabla \cdot v \cdot |\nabla\theta|^{p-2}\nabla\theta\, dx \\ &\le \|\nabla\theta\|_{L^p}^p \|\nabla u\|_{L^\infty} + \|\Delta v\|_{L^p} \|\nabla\theta\|_{L^p}^{p-1}\end{aligned}$$

or

$$\frac{d}{dt}\|\nabla\theta\|_{L^p} \le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p} + C\|\Delta v\|_{L^p},$$

where C's are constants independent of p. Gronwall's inequality then implies

$$\|\nabla\theta\|_{L^p}^2(t) \le C\left(1 + \int_0^t \|\Delta v\|_{L^p}\, d\tau\right) \exp\left(\int_0^t \|\nabla u\|_{L^\infty}\, d\tau\right) < \infty.$$

Taking the limit as $p \to \infty$ yields (4.9). This completes the proof of Proposition 4.6. □

We are now ready to prove a global bound for (u, v, θ) in H^s.

Proof of Theorem 1.2. Applying Λ^s to (1.3) and then dotting with $(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)$, we obtain, after integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\Lambda^s u, \Lambda^s v, \Lambda^s\theta)\|_{L^2}^2 + \|\Lambda^{s+\beta} v\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^2} \Lambda^s(u \cdot \nabla u) \cdot \Lambda^s u\, dx - \int_{\mathbb{R}^2} \Lambda^s\nabla \cdot (v \otimes v) \cdot \Lambda^s u\, dx \\ &\quad - \int_{\mathbb{R}^2} \Lambda^s(u \cdot \nabla v) \cdot \Lambda^s v\, dx - \int_{\mathbb{R}^2} \Lambda^s(v \cdot \nabla u) \cdot \Lambda^s v\, dx\end{aligned}$$

$$-\int_{\mathbb{R}^2} \Lambda^s(u\cdot\nabla\theta)\cdot\Lambda^s\theta\,dx$$
$$=: H_1+H_2+H_3+H_4+H_5.$$

Now we estimate H_1, H_2, H_3, H_4 and H_5 one by one. By Lemma 2.7,

$$\begin{aligned} H_1 &= -\int_{\mathbb{R}^2} (\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u)\cdot\Lambda^s u\,dx \\ &= -\int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u\,dx \\ &\le C\|\Lambda^s u\|_{L^2}\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2} \\ &\le C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2. \end{aligned}$$

By Lemma 2.7 and Sobolev's inequality,

$$\begin{aligned} H_2 &\le C\|\Lambda^{s+1}v\|_{L^2}\|v\|_{L^\infty}\|\Lambda^s u\|_{L^2} \\ &\le C(\|v\|_{L^2}+\|\Lambda^{s+\beta}v\|_{L^2})\|\Lambda^\beta v\|_{L^2}\|\Lambda^s u\|_{L^2} \\ &\le \frac{1}{6}\|\Lambda^{s+\beta}v\|_{L^2}^2 + C + C\|\Lambda^\beta v\|_{L^2}^2\|\Lambda^s u\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} H_3 &\le \|\Lambda^s(u\cdot\nabla v)\|_{L^2}\|\Lambda^s v\|_{L^2} \\ &\le C(\|\Lambda^s u\|_{L^2}\|\nabla v\|_{L^\infty} + \|\Lambda^{s+1}v\|_{L^2}\|u\|_{L^\infty})\|\Lambda^s v\|_{L^2} \\ &\le C\big(\|\Lambda^s u\|_{L^2}\|\nabla v\|_{L^\infty} + (\|v\|_{L^2}+\|\Lambda^{s+\beta}v\|_{L^2})\|u\|_{L^\infty}\big)\|\Lambda^s v\|_{L^2} \\ &\le \frac{1}{6}\|\Lambda^{s+\beta}v\|_{L^2}^2 + C(1+\|\nabla v\|_{L^\infty}^2)\|(\Lambda^s u,\Lambda^s v)\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} H_4 &\le \|\Lambda^{s-1}(v\cdot\nabla u)\|_{L^2}\|\Lambda^{s+1}v\|_{L^2} \\ &\le C(\|\Lambda^s u\|_{L^2}\|v\|_{L^\infty} + \|\Lambda^{s-1}v\|_{L^2}\|\nabla u\|_{L^\infty})\|\Lambda^{s+1}v\|_{L^2} \\ &\le C\big(\|\Lambda^s u\|_{L^2}\|\Lambda^\beta v\|_{L^2} + (\|v\|_{L^2}+\|\Lambda^s v\|_{L^2})\|\omega\|_{L^\infty}\big)(\|v\|_{L^2}+\|\Lambda^{s+\beta}v\|_{L^2}) \\ &\le \frac{1}{6}\|\Lambda^{s+\beta}v\|_{L^2}^2 + C(1+\|\Lambda^\beta v\|_{L^2}^2)\|(\Lambda^s u,\Lambda^s v)\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} H_5 &= -\int_{\mathbb{R}^2} (\Lambda^s(u\cdot\nabla\theta) - u\cdot\nabla\Lambda^s\theta)\cdot\Lambda^s\theta\,dx \\ &= -\int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla]\theta\cdot\Lambda^s\theta\,dx \\ &\le C(\|\Lambda^s u\|_{L^2}\|\nabla\theta\|_{L^\infty} + \|\Lambda^s\theta\|_{L^2}\|\nabla u\|_{L^\infty})\|\Lambda^s\theta\|_{L^2} \\ &\le C\|(\Lambda^s u,\Lambda^s\theta)\|_{L^2}^2. \end{aligned}$$

Combining all estimates above, we get

$$\begin{aligned}&\frac{d}{dt}\|(\Lambda^s u, \Lambda^s v, \Lambda^s \theta)\|_{L^2}^2 + \|\Lambda^{s+\beta} v\|_{L^2}^2 \\ &\le C + C(1 + \|\Lambda^\beta v\|_{L^2}^2 + \|\nabla v\|_{L^\infty}^2)\|(\Lambda^s u, \Lambda^s v, \Lambda^s \theta)\|_{L^2}^2,\end{aligned}$$

which implies the desired global bound in Theorem 1.2. □

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Appendix A. Frequency localization and Besov spaces

This appendix provides the definition of the Littlewood–Paley decomposition and the definition of Besov spaces. Some related facts used in the previous sections are also included. The material presented in this appendix can be found in several books and many papers (see, e.g., [1,2,17,19,21]).

We start with several notational conventions. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. To introduce the Littlewood–Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{\xi \in \mathbb{R}^d : \, 2^{j-1} \le |\xi| < 2^{j+1}\right\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1\,, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0\,, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{A.1}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leq -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A.2}$$

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k.$$

For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. It is clear from (A.1) that $S_j \to Id$ as $j \to \infty$ in the distributional sense. In addition, the notation $\widetilde{\Delta}_k$, defined by

$$\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1},$$

is also useful and has been used in the previous sections.

Definition A.1. The inhomogeneous Besov space $B^s_{p,q}$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty,$$

where $\Delta_j f$ is as defined in (A.2).

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.2. *For any $s \in \mathbb{R}$,*

$$H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$B^s_{q,\min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,\max\{q,2\}}.$$

For any non-integer $s > 0$, the Hölder space C^s is equivalent to $B^s_{\infty,\infty}$.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Proposition A.3. *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

1) *If f satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \leq K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If f satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

Appendix B. Proof of (2.4)

This appendix provides a detailed proof of (2.4).

Proof of (2.4). We start with the definition of $g_0(x)$,

$$g_0(x) = \int\limits_{\mathbb{R}^2} e^{i x\cdot\xi}\, |\xi|^{\sigma_1}\, e^{-|\xi|^{2\beta}}\, d\xi.$$

Clearly, for any $\sigma_1 \geq 0$ and $\beta > 0$,

$$\|g_0\|_{L^\infty} \leq \int_{\mathbb{R}^2} |\xi|^{\sigma_1} e^{-|\xi|^{2\beta}} d\xi := C. \tag{B.1}$$

Denoting

$$N(x, \nabla) = \frac{x \cdot \nabla_\xi}{i|x|^2},$$

we have

$$N(x, \nabla) e^{ix\cdot\xi} = e^{ix\cdot\xi}. \tag{B.2}$$

We write $N^*(x, \nabla)$ as the dual operator of $N(x, \nabla)$, namely

$$N^*(x, \nabla) = -\frac{x \cdot \nabla_\xi}{i|x|^2}.$$

Let $\chi(\xi) \in C_c^\infty(\mathbb{R}^2)$ be the standard smooth cutoff function satisfying

$$\chi(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| > 2. \end{cases}$$

For $N_0 > 0$ to be fixed letter, we split $g_0(x)$ into two parts,

$$\begin{aligned} g_0(x) &= \int_{\mathbb{R}^2} e^{ix\cdot\xi} \chi\Big(\frac{\xi}{N_0}\Big) |\xi|^{\sigma_1} e^{-|\xi|^{2\beta}} d\xi + \int_{\mathbb{R}^2} e^{ix\cdot\xi} \left(1 - \chi\Big(\frac{\xi}{N_0}\Big)\right) |\xi|^{\sigma_1} e^{-|\xi|^{2\beta}} d\xi \\ &:= LF + HF. \end{aligned}$$

The low frequency part LF is bounded by

$$|LF| \leq \int_{|\xi| \leq 2N_0} |\xi|^{\sigma_1} d\xi \leq C\, N_0^{\sigma_1 + 2}. \tag{B.3}$$

To bound the high frequency part HF, we fix $k > 0$ to be a positive integer, invoke (B.2) and integrate by parts to obtain

$$\begin{aligned} HF &= \int_{\mathbb{R}^2} N^k(x, \nabla)(e^{ix\cdot\xi}) \left(1 - \chi\Big(\frac{\xi}{N_0}\Big)\right) |\xi|^{\sigma_1} e^{-|\xi|^{2\beta}} d\xi \\ &= \int_{\mathbb{R}^2} e^{ix\cdot\xi} \big(N^*(x, \nabla)\big)^k \left\{ \left(1 - \chi\Big(\frac{\xi}{N_0}\Big)\right) |\xi|^{\sigma_1} e^{-|\xi|^{2\beta}} \right\} d\xi \end{aligned}$$

$$
\begin{aligned}
&\leq C|x|^{-k} \int\limits_{|\xi|\geq N_0} \left|\nabla^k\big(|\xi|^{\sigma_1} e^{-|\xi|^{2\beta}}\big)\right| d\xi \\
&\quad + C|x|^{-k} \int\limits_{N_0\leq|\xi|\leq 2N_0} \sum_{l=1}^{k} \left|\nabla^l \left(1-\chi\Big(\frac{\xi}{N_0}\Big)\right)\right| \left|\nabla^{k-l}\big(|\xi|^{\sigma_1} e^{-|\xi|^{2\beta}}\big)\right| d\xi \\
&\leq C|x|^{-k} \int\limits_{|\xi|\geq N_0} \left|\nabla^k\big(|\xi|^{\sigma_1} e^{-|\xi|^{2\beta}}\big)\right| d\xi \\
&\quad + C|x|^{-k} \int\limits_{N_0\leq|\xi|\leq 2N_0} \sum_{l=1}^{k} N_0^{-l} \left|\nabla^{k-l}\big(|\xi|^{\sigma_1} e^{-|\xi|^{2\beta}}\big)\right| d\xi .
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
\left|\nabla^l |\xi|^{\sigma_1}\right| &\leq C|\xi|^{\sigma_1-l}, \\
\left|\nabla^{k-l} e^{-|\xi|^{2\beta}}\right| &\leq C \sum_{m=1}^{k-l} |\xi|^{2\beta m-(k-l)} e^{-|\xi|^{2\beta}} \\
&\leq C(|\xi|^{2\beta+l-k} + |\xi|^{(2\beta-1)(k-l)}) e^{-|\xi|^{2\beta}} .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\left|\nabla^k\big(|\xi|^{\sigma_1} e^{-|\xi|^{2\beta}}\big)\right| &\leq \sum_{l=0}^{k} \left|\nabla^l |\xi|^{\sigma_1}\right| \left|\nabla^{k-l} e^{-|\xi|^{2\beta}}\right| \\
&\leq C \sum_{l=0}^{k} (|\xi|^{\sigma_1+2\beta-k} + |\xi|^{\sigma_1-l+(2\beta-1)(k-l)}) e^{-|\xi|^{2\beta}} \\
&\leq C(|\xi|^{\sigma_1+2\beta-k} + |\xi|^{\sigma_1+(2\beta-1)k} + |\xi|^{\sigma_1-k}) e^{-|\xi|^{2\beta}} \\
&\leq C(|\xi|^{\sigma_1-k}|\xi|^{2\beta} e^{-|\xi|^{2\beta}} + |\xi|^{\sigma_1-k}|\xi|^{2\beta k} e^{-|\xi|^{2\beta}} + |\xi|^{\sigma_1-k}) \\
&\leq C|\xi|^{\sigma_1-k},
\end{aligned}
$$

where we have used the simple facts

$$
|\xi|^{2\beta} e^{-|\xi|^{2\beta}} \leq C, \qquad \xi|^{2\beta k} e^{-|\xi|^{2\beta}} \leq C.
$$

The same argument yields

$$
\left|\nabla^{k-l}\big(|\xi|^{\sigma_1} e^{-|\xi|^{2\beta}}\big)\right| \leq C|\xi|^{\sigma_1-(k-l)}.
$$

Therefore, for $k > 2 + \sigma_1$, the high-frequency part HF is bounded by

$$
\begin{aligned}
|HF| &\le C|x|^{-k} \int\limits_{|\xi|\ge N_0} |\xi|^{\sigma_1-k}\, d\xi \\
&\quad + C|x|^{-k} \int\limits_{N_0\le|\xi|\le 2N_0} \sum_{l=1}^{k} N_0^{-l} |\xi|^{\sigma_1-(k-l)}\, d\xi \\
&\le C|x|^{-k} N_0^{\sigma_1-k+2} \\
&\quad + C|x|^{-k} \int\limits_{N_0\le|\xi|\le 2N_0} \sum_{l=1}^{k} N_0^{\sigma_1-k}\, d\xi \\
&\le C|x|^{-k} N_0^{\sigma_1-k+2}.
\end{aligned}
$$

Putting these estimates together yields

$$
|g_0(x)| \le C(N_0^{\sigma_1+2} + |x|^{-k} N_0^{\sigma_1-k+2}) \le C|x|^{-\sigma_1-2} \tag{B.4}
$$

by choosing $N_0 \approx |x|^{-1}$. (B.1) and (B.4) together imply, for any $x \in \mathbb{R}^2$,

$$
|g_0(x)| \le C(1+|x|)^{-\sigma_1-2}.
$$

Therefore,

$$
\|g_0(x)\|_{L^1} \le C.
$$

Consequently,

$$
\begin{aligned}
\|g(\cdot,t)\|_{L^1_x} &= \int\limits_{\mathbb{R}^2} t^{-\frac{\sigma_1}{2\beta}} t^{-\frac{1}{\beta}} \left| g_0\left(\frac{x}{t^{\frac{1}{2\beta}}}\right)\right| dx \\
&= C t^{-\frac{\sigma_1}{2\beta}} t^{-\frac{1}{\beta}} \int\limits_{\mathbb{R}^2} |g_0(\eta)| \big(t^{\frac{1}{2\beta}}\big)^2\, d\eta \\
&= C t^{-\frac{\sigma_1}{2\beta}} \|g_0(x)\|_{L^1} \\
&\le C t^{-\frac{\sigma_1}{2\beta}},
\end{aligned}
$$

which leads to

$$
\| \|g(\cdot,t)\|_{L^1_x} \|_{L^1(0,t)} \le C t^{1-\frac{\sigma_1}{2\beta}}.
$$

This completes the proof of (2.4). □

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