

Global Regularity for the 2D MHD Equations with Partial Hyper-resistivity

Bo-Qing Dong¹, Jingna Li^{2,*}, and Jiahong Wu³

¹College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China, ²Department of Mathematics, Jinan University, Guangzhou 510632, China, and ³Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

*Correspondence to be sent to: e-mail: jingna8005@hotmail.com

This article establishes the global existence and regularity for a system of the two-dimensional (2D) magnetohydrodynamic (MHD) equations with only directional hyper-resistivity. More precisely, the equation of b_1 (the horizontal component of the magnetic field) involves only vertical hyperdiffusion (given by $\Lambda_2^{2\beta} b_1$) while the equation of b_2 (the vertical component) has only horizontal hyperdiffusion (given by $\Lambda_1^{2\beta} b_2$), where Λ_1 and Λ_2 are directional Fourier multiplier operators with the symbols being $|\xi_1|$ and $|\xi_2|$, respectively. We prove that, for $\beta > 1$, this system always possesses a unique global-in-time classical solution when the initial data is sufficiently smooth. The model concerned here is rooted in the MHD equations with only magnetic diffusion, which play a significant role in the study of magnetic reconnection and magnetic turbulence. In certain physical regimes and under suitable scaling, the magnetic diffusion becomes partial (given by part of the Laplacian operator). There have been considerable recent developments on the fundamental issue of whether classical solutions of these equations remain smooth for all time. The papers of Cao–Wu–Yuan [8] and of Jiu–Zhao [26] obtained the global regularity when the magnetic diffusion is given by the full fractional Laplacian $(-\Delta)^\beta$ with $\beta > 1$. The main result presented in this article requires only directional fractional diffusion and yet we prove the regularization in all directions. The proof makes use of

Received 25 May 2017; Revised 3 September 2017; Accepted 18 September 2017
2000 Mathematics Subject Classification: 35Q35, 76D03

a key observation on the structure of the nonlinearity in the MHD equations and technical tools on Fourier multiplier operators such as the Hörmander–Mikhlin multiplier theorem. The result presented here appears to be the sharpest for the 2D MHD equations with partial magnetic diffusion.

1 Introduction

The magnetohydrodynamic (MHD) equations are the center piece of the magnetohydrodynamics. Since their initial derivation by the Nobel Laureate H. Alfvén in 1924, the MHD equations have played pivotal roles in the study of many phenomena in geophysics, astrophysics, cosmology, and engineering (see, e.g., [4, 13]). The standard incompressible MHD equations can be written as

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \quad (1.1)$$

where u denotes the velocity field, b the magnetic field, p the pressure, $\nu \geq 0$ the kinematic viscosity, and $\eta \geq 0$ the magnetic diffusivity. Equation (1.1) reflects the interaction between the velocity field and the magnetic field. They consist of a coupled system of the Navier–Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism.

The MHD equations are also of great interest in mathematics. Fundamental issues such as the global existence and regularity of solutions to the MHD equations have recently attracted considerable interest. Mathematically the MHD equations are not merely a combination of two parallel Navier–Stokes type equations but an interactive and integrated system. They contain richer structures than the Navier–Stokes equations and exploring these special structures can lead to interesting results that are not parallel to those for the Navier–Stokes equations.

Attention here is focused on the 2D MHD equations. When there is no kinematic dissipation or magnetic diffusion, namely (1.1) with $\nu = \eta = 0$, the MHD equations become inviscid and the global regularity problem appears to be out of reach at this moment. In contrast, when both the dissipation and the magnetic diffusion are present, namely (1.1) with $\nu > 0$ and $\eta > 0$, the MHD equations are fully dissipative and the global regularity problem in the 2D case can be solved following the approach for the 2D Navier–Stokes equations. It is natural to explore the intermediate equations that bridge the two extreme cases. The MHD equations with partial or fractional dissipation exactly fill this

gap. There have been significant recent developments on the MHD equations with partial or fractional dissipation. Important progress has been made (see, e.g, [1, 5–12, 14–18, 20–26, 28, 30, 31, 33, 36–51]).

One special partial dissipation case is the 2D resistive MHD equations, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \quad (1.2)$$

where $\eta > 0$ denotes the magnetic diffusivity (resistivity). Equation (1.2) is applicable when the fluid viscosity can be ignored while the role of resistivity is important such as in magnetic reconnection and magnetic turbulence. Magnetic reconnection refers to the breaking and reconnecting of oppositely directed magnetic field lines in a plasma and is at the heart of many spectacular events in our solar system such as solar flares and northern lights. The mathematical study of (1.2) may help understand the Sweet–Parker model arising in magnetic reconnection theory [32]. Although the global regularity problem on (1.2) is not completely solved at this moment, recent efforts on this problem have significantly advanced our understanding.

In certain physical regimes and under suitable scaling, the full Laplacian dissipation is reduced to a partial dissipation. One notable example is the Prandtl boundary layer equation in which only the vertical dissipation is included in the horizontal component (see, e.g., [35]). This article focuses on a system of the 2D MHD equations that is closely related to (1.2),

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = b \cdot \nabla b, \\ \partial_t b_1 + (u \cdot \nabla)b_1 + \eta \Lambda_2^{2\beta} b_1 = b \cdot \nabla u_1, \\ \partial_t b_2 + (u \cdot \nabla)b_2 + \eta \Lambda_1^{2\beta} b_2 = b \cdot \nabla u_2, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \end{cases} \quad (1.3)$$

where b_1 and b_2 denote the components of b ; $\eta > 0$ and $\beta > 0$ are real parameters. The fractional partial derivative operators Λ_1^γ and Λ_2^γ with $\gamma > 0$ are defined through the Fourier transform, namely

$$\widehat{\Lambda_1^\gamma f}(\xi_1, \xi_2) = |\xi_1|^\gamma \widehat{f}(\xi_1, \xi_2), \quad \widehat{\Lambda_2^\gamma f}(\xi_1, \xi_2) = |\xi_2|^\gamma \widehat{f}(\xi_1, \xi_2).$$

In addition, we also use Λ^σ with $\sigma > 0$ to denote the 2D fractional Laplace operator,

$$\widehat{\Lambda^\sigma f}(\xi) = |\xi|^\sigma \widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2).$$

In comparison with (1.2), (1.3) only has vertical fractional Laplacian diffusion, no horizontal diffusion in the b_1 equation and no vertical diffusion in the b_2 equation.

Our goal here is to show that, when $\beta > 1$, any sufficiently smooth initial data (u_0, b_0) leads to a unique global solution of (1.3). More precisely, we establish the following theorem.

Theorem 1.1. Consider the 2D MHD equations in (1.3) with $\eta > 0$ and $\beta > 1$. Assume $(u_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, and $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then (1.3) possesses a unique global solution (u, b) satisfying, for any $T > 0$,

$$(u, b) \in L^\infty(0, T; H^s(\mathbb{R}^2)), \quad b \in L^2(0, T; \dot{H}^{s+\beta}(\mathbb{R}^2)). \quad \square$$

The proof of this result takes advantage of the special structure in the nonlinear terms of the magnetic field equation. Even though the system contains only directional fractional magnetic diffusion, we are still able to establish the regularization in all directions. Theorem 1.1 improves previous work of Cao–Wu–Yuan [8] and of Jiu–Zhao [26]. [8] and [26] obtained via different approaches the global regularity of a more regularized system

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = b \cdot \nabla b, \\ \partial_t b + (u \cdot \nabla)b + \eta \Lambda^{2\beta} b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0. \end{cases} \quad (1.4)$$

for the case when $\beta > 1$. Equation (1.4) involves full fractional magnetic diffusion, while (1.3) involves only directional fractional diffusion. The improvement is not a trivial one.

The proof of Theorem 1.1 is not a simple generalization of those in the previous papers [8, 26]. Since (1.3) contains only partial fractional diffusion, some of the classical tools such as the maximal regularity type estimates for the 2D heat equation [2, 19, 27] can no longer be used here. The equation of b_1 in (1.3) involves only the vertical fractional diffusion, and in general we would not be able to obtain the smoothing of b_1 in the horizontal direction. However, the special nonlinear structure actually allows us to prove that the derivatives of b_1 with respect to x_1 are globally bounded. The key

observation is the following identity, thanks to $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$,

$$b \cdot \nabla u_1 - u \cdot \nabla b_1 = \partial_1(b_1 u_1) + \partial_2(b_2 u_1) - \partial_1(b_1 u_1) - \partial_2(u_2 b_1) = \partial_2(b_2 u_1 - u_2 b_1). \quad (1.5)$$

To make use of this special structure, we write b_1 in the integral form,

$$b_1(t) = g(t) *_2 b_{01} + \int_0^t g(t - \tau) *_2 (b \cdot \nabla u_1 - u \cdot \nabla b_1) d\tau, \quad (1.6)$$

where g denotes the 1D kernel functions associated with the Fourier multiplier $e^{-t|\xi_2|^{2\beta}}$, namely

$$g(x_2, t) = \int_{\mathbb{R}} e^{-t|\xi_2|^{2\beta}} e^{ix_2 \xi_2} d\xi_2.$$

The notation for the convolution here is given by

$$g(t) *_2 b_{01} = \int_{\mathbb{R}} g(x_2 - y_2, t) b_{01}(x_1, y_2) dy_2.$$

Equations (1.5) and (1.6) together allow us to obtain the control on $\partial_1 b_1$ and $\partial_1 \partial_2 b_1$. Similarly, we can control $\partial_2 b_2$ and $\partial_1 \partial_2 b_2$ even when the equation of b_2 involves no vertical dissipation. This explains how we take advantage of the special structure in the nonlinearity to control all the second-order derivatives of b . To control even higher-order derivatives of b , say $\Lambda^\sigma \Delta b$ with a fractional power $\sigma > 0$, we first make use of the directional diffusion to obtain the directional regularization and then use the Hörmander–Mikhlin multiplier theorem to obtain the regularization in directions in which the directional diffusion is missing. More technical details can be found in the proof of Theorem 1.1 in Section 2.

The rest of this article contains the proof of Theorem 1.1. Section 2 is divided into three subsections which successively provide more and more regular global bounds.

2 Proof of Theorem 1.1

This section proves Theorem 1.1. As we know, the core part of the proof is the global a priori bounds. For the sake of clarity, we divide this section into three subsections. The first subsection supplies the global H^1 -bound, which relies on the equations of the vorticity ω and the current density $j = \nabla \times b$. The second subsection proves the global bounds for $\|\nabla b\|_{L_t^\infty L^q}$ with any $1 < q < \infty$ and for $\|\Delta b\|_{L_t^1 L^q}$ and $\|\omega\|_{L_t^\infty L^q}$. The proof makes use of the special structure of the nonlinear terms and a lemma assessing the behavior of the 1D kernel function on Lebesgue spaces. The third subsection establishes the global

bounds for $\|\nabla j\|_{L_t^1 L^\infty}$ and $\|\omega\|_{L_t^\infty L^\infty}$. To prove these global bounds, the key is to show that $\|\Lambda^\sigma \Delta b\|_{L_t^1 L^q}$ is globally bounded for any $0 < \sigma < 2\beta - 2$. We need to overcome the difficulty due to the lack of full fractional diffusion. The strategy here is to first obtain the regularization along the direction of the diffusion and then make use of the Hörmander–Mikhlin multiplier theorem to obtain the regularization in other directions.

2.1 Global H^1 bound for (u, b)

This subsection proves that (u, b) admits the following global H^1 -bound.

Proposition 2.1. Assume (u_0, b_0) obeys the conditions stated in Theorem 1.1. Let (u, b) be the corresponding solution of (1.3). Then (u, b) satisfies

$$\|(u, b)\|_{L^2}^2 + 2\eta \int_0^t H(b)(\tau) \, d\tau = \|(u_0, b_0)\|_{L^2}^2, \tag{2.1}$$

$$\|(\omega, j)\|_{L^2}^2 + \eta \int_0^t H(\nabla b)(\tau) \, d\tau \leq C(1 + \|(\omega_0, j_0)\|_{L^2}) \exp(C(1+t)\|(u_0, b_0)\|_{L^2}^2), \tag{2.2}$$

where $C = C(\beta)$ is a constant and

$$H(b) = \|(\Lambda_2^\beta b_1, \Lambda_1^\beta b_2)\|_{L^2}^2. \tag{□}$$

Proof. The global L^2 bound is obvious. Dotting (1.3) with (u, b) , integrating by parts and using $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we obtain (2.1). To obtain the global H^1 bound, we use the equations of the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$,

$$\begin{cases} \partial_t \omega + (u \cdot \nabla) \omega = b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j + \eta(\Lambda_1^{2\beta} \partial_1 b_2 - \Lambda_2^{2\beta} \partial_2 b_1) = b \cdot \nabla \omega + Q(u, b), \end{cases} \tag{2.3}$$

where

$$Q(u, b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

Integrating by parts and using $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|(\omega, j)\|_{L^2}^2 + \eta H(\nabla b) = I, \tag{2.4}$$

where

$$H(\nabla b) = \|\Lambda_2^\beta \nabla b_1\|_{L^2}^2 + \|\Lambda_1^\beta \nabla b_2\|_{L^2}^2, \quad I = \int Q(u, b) j \, dx.$$

It suffices to estimate a typical term in I ,

$$I_1 = 2 \int \partial_1 b_1 \partial_2 u_1 j \, dx$$

By the boundedness of Zygmund–Calderon operators and a standard Sobolev inequality,

$$|I_1| \leq C \|j\|_{L^4}^2 \|\omega\|_{L^2} \leq \frac{\beta\eta}{64} \|\nabla j\|_{L^2}^2 + C \|j\|_{L^2}^2 \|\omega\|_{L^2}^2.$$

Due to the elementary inequality, for $\beta > 1$,

$$\xi_k^2 \leq \frac{\beta - 1}{\beta} + \frac{1}{\beta} \xi_k^{2\beta}, \quad k = 1, 2,$$

we have

$$\|j\|_{L^2}^2 \leq 2 (\|\partial_1 b_2\|_{L^2}^2 + \|\partial_2 b_1\|_{L^2}^2) \leq \frac{2(\beta - 1)}{\beta} \|b\|_{L^2}^2 + \frac{2}{\beta} H(b)$$

and

$$\|\nabla j\|_{L^2}^2 \leq \frac{2(\beta - 1)}{\beta} \|\nabla b\|_{L^2}^2 + \frac{2}{\beta} H(\nabla b).$$

Inserting the bounds above in (2.4) yields

$$\frac{d}{dt} \|(\omega, j)\|_{L^2}^2 + \eta H(\nabla b) \leq C \|j\|_{L^2}^2 (1 + \|\omega\|_{L^2}^2),$$

which yields the global H^1 bound

$$\|(\omega, j)\|_{L^2}^2 + \nu \int_0^t H(\nabla b) \, d\tau \leq C (1 + \|(\omega_0, j_0)\|_{L^2}^2) \exp \int_0^t \|j\|_{L^2}^2 \, d\tau.$$

This completes the proof of Proposition 2.1. ■

2.2 Global bounds on $\|\nabla b\|_{L_t^\infty L^q}$, $\|\Delta b\|_{L_t^1 L^q}$ and $\|\omega\|_{L_t^\infty L^q}$ with $1 < q < \infty$

This subsection uses the integral form of the equation of b to prove the following proposition.

Proposition 2.2. Assume (u_0, b_0) obeys the conditions stated in Theorem 1.1. Let (u, b) be the corresponding solution of (1.3) with $\beta > 1$. Then (u, b) obeys the following global *a priori* bounds:

- (1) For
- $1 < q < \infty$
- and
- $t > 0$
- ,

$$\|\nabla b(t)\|_{L^q(\mathbb{R}^2)} \leq C(t, u_0, b_0), \quad \|j(t)\|_{L^q(\mathbb{R}^2)} \leq C(t, u_0, b_0). \quad (2.5)$$

A special consequence is the $L_t^\infty L^\infty(\mathbb{R}^2)$ bound for b ,

$$\|b\|_{L_t^\infty(L^\infty(\mathbb{R}^2))} \leq C(\|b\|_{L_t^\infty L^2} + \|\nabla b\|_{L_t^\infty L^q}) = C(t, u_0, b_0).$$

- (2) For any
- $1 < q < \infty$
- and
- $t > 0$
- ,

$$\|\omega\|_{L_t^\infty L^q} \leq C(t, u_0, b_0), \quad \|\nabla j\|_{L_t^1 L^q} \leq C(t, u_0, b_0). \quad (2.6)$$

□

To prove Proposition 2.2, we state a few properties for the kernel function associated with the 1D fractional heat operator.

Lemma 2.3. Let $\beta \geq 1$. Let $t > 0$ and denote by $g = g(x_2, t)$ the 1D inverse Fourier transform of $e^{-|\xi_2|^{2\beta} t}$, namely

$$g(x_2, t) = \int_{\mathbb{R}} e^{-t|\xi_2|^{2\beta}} e^{ix_2 \xi_2} d\xi_2. \quad (2.7)$$

Then g satisfies the following properties:

- (a) For any
- $t > 0$
- ,

$$g(x_2, t) = t^{-\frac{1}{2\beta}} g\left(\frac{x_2}{t^{\frac{1}{2\beta}}}, 1\right).$$

- (b) For any integer
- $m \geq 0$
- , any
- $1 \leq r \leq \infty$
- and any
- $t > 0$
- ,

$$\|\partial_{x_2}^m g(x_2, t)\|_{L^r(\mathbb{R})} \leq C t^{-\frac{m}{2\beta} - \frac{1}{2\beta}(1-\frac{1}{r})}, \quad (2.8)$$

which especially implies, for any $1 \leq p \leq q \leq \infty$ and $f \in L^p(\mathbb{R})$,

$$\|(\partial_{x_2}^m g(x_2, t)) *_2 f\|_{L^q(\mathbb{R})} \leq C t^{-\frac{m}{2\beta} - \frac{1}{2\beta}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R})}.$$

- (c) For any fractional
- $\sigma > 0$
- , any
- $1 \leq r \leq \infty$
- , and any
- $t > 0$
- ,

$$\|\Lambda_2^\sigma g(x_2, t)\|_{L^r(\mathbb{R})} \leq C t^{-\frac{\sigma}{2\beta} - \frac{1}{2\beta}(1-\frac{1}{r})},$$

where $\Lambda_2^\sigma g(x_2, t)$ is defined via the Fourier transform, for fixed $t > 0$,

$$\widehat{\Lambda_2^\sigma g}(x_2, t) = |\xi_2|^\sigma e^{-|\xi_2|^{2\beta} t}.$$

Especially, for any $1 \leq p \leq q \leq \infty$ and $f \in L^p(\mathbb{R})$,

$$\|(\Lambda_{x_2}^\sigma g(x_2, t)) *_2 f\|_{L^q(\mathbb{R})} \leq C t^{-\frac{\sigma}{2\beta} - \frac{1}{2\beta}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R})}. \quad \square$$

Proof of Lemma 2.3. (a) Follows directly from the definition of g in (2.7). To prove (b), we first show that the L^1 -norm of $g(x_2, 1)$ is finite. In fact,

$$(1 + x_2^2) g(x_2, 1) = \int e^{ix_2 \xi_2} (1 - \partial_{\xi_2}^2) e^{-|\xi_2|^{2\beta}} d\xi_2. \quad (2.9)$$

It is clear that, for $\beta \geq 1$, the right-hand side of (2.9) is finite. Thus,

$$|g(x_2, 1)| \leq C (1 + x_2^2)^{-1} \quad \text{and} \quad \|g(\cdot, 1)\|_{L^1(\mathbb{R})} \leq C.$$

According to (a), for any $t > 0$,

$$\|g(\cdot, t)\|_{L^1(\mathbb{R})} = \|g(\cdot, 1)\|_{L^1(\mathbb{R})} \leq C.$$

For any $t > 0$,

$$\|g(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|e^{-t|\xi_2|^{2\beta}}\|_{L^1(\mathbb{R})} = C t^{-\frac{1}{2\beta}},$$

where C is a constant independent of t . Therefore, for any $1 \leq r \leq \infty$, by a simple interpolation inequality,

$$\|g(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|g(\cdot, t)\|_{L^\infty(\mathbb{R})}^{1-\frac{1}{r}} \|g(\cdot, t)\|_{L^1(\mathbb{R})}^{\frac{1}{r}} \leq C t^{-\frac{1}{2\beta}(1-\frac{1}{r})}.$$

This proves (2.8) with $m = 0$. The general case $m > 0$ can be shown by repeating the process above with $\partial_{x_2}^m g(x_2, t)$ whose corresponding Fourier transform is $\xi_2^m e^{-|\xi_2|^{2\beta} t}$. The proof of the results in (c) for the fractional derivative σ is similar. We omit further details. ■

Proof of Proposition 2.2. To prove (2.5), we start with the integral representations of b_1 and b_2 ,

$$b_1 = g(t) *_2 b_{01} + \int_0^t g(t - \tau) *_2 (b \cdot \nabla u_1 - u \cdot \nabla b_1) d\tau, \quad (2.10)$$

$$b_2 = h(t) *_1 b_{02} + \int_0^t h(t - \tau) *_1 (b \cdot \nabla u_2 - u \cdot \nabla b_2) d\tau, \quad (2.11)$$

where g and h denote the 1D kernel functions associated with the Fourier multiplier $e^{-t|\xi_2|^{2\beta}}$ and $e^{-t|\xi_1|^{2\beta}}$, namely

$$g(x_2, t) = \int_{\mathbb{R}} e^{-t|\xi_2|^{2\beta}} e^{ix_2\xi_2} d\xi_2, \quad h(x_1, t) = \int_{\mathbb{R}} e^{-t|\xi_1|^{2\beta}} e^{ix_1\xi_1} d\xi_1$$

and the convolution notations are defined as

$$g(t) *_2 b_{01} = \int_{\mathbb{R}} g(x_2 - y_2, t) b_{01}(x_1, y_2) dy_2,$$

$$h(t) *_1 b_{02} = \int_{\mathbb{R}} h(x_1 - y_1, t) b_{01}(y_1, x_2) dy_1.$$

To prove (2.5), it suffices to prove the bounds for $\partial_2 b_1$ and $\partial_1 b_2$. Since, if

$$\|\partial_2 b_1\|_{L^q} \leq C(t, u_0, b_0), \quad \|\partial_1 b_2\|_{L^q} \leq C(t, u_0, b_0),$$

then

$$\|j\|_{L^q} \leq C(t, u_0, b_0), \quad \|\nabla b\|_{L^q} \leq C \|j\|_{L^q} \leq C(t, u_0, b_0).$$

To show the bound for $\partial_2 b_1$, we write

$$b \cdot \nabla u_1 - u \cdot \nabla b_1 = \partial_1(b_1 u_1) + \partial_2(b_2 u_1) - \partial_1(b_1 u_1) - \partial_2(u_2 b_1) = \partial_2(b_2 u_1 - u_2 b_1) \tag{2.12}$$

and thus

$$\partial_2 b_1 = \partial_2(g(t) *_2 b_{01}) + \int_0^t \partial_2 \partial_2 g(t - \tau) *_2 (b_2 u_1 - u_2 b_1)(\tau) d\tau.$$

First, we take $L^q_{x_1}$ each side to obtain

$$\|\partial_2 b_1\|_{L^q_{x_1}} \leq |g(t)| *_2 \|\partial_2 b_{01}\|_{L^q_{x_1}} + \int_0^t |\partial_2 \partial_2 g(t - \tau)| *_2 \|(b_2 u_1 - u_2 b_1)(\tau)\|_{L^q_{x_1}} d\tau$$

We then take $L^q_{x_2}$ each side and apply Young’s inequality for convolution to obtain

$$\|\partial_2 b_1\|_{L^q} \leq \|g(t)\|_{L^1} \|\partial_2 b_{01}\|_{L^q} + \int_0^t \|\partial_2 \partial_2 g(t - \tau)\|_{L^1_{x_2}} \|b_2 u_1 - u_2 b_1\|_{L^q}(\tau) d\tau. \tag{2.13}$$

By Lemma 2.3,

$$\|\partial_2 \partial_2 g(t - \tau)\|_{L^1_{x_2}} \leq C (t - \tau)^{-\frac{1}{\beta}}, \tag{2.14}$$

where C is a constant depending on β only. By Hölder's inequality and Sobolev's inequality,

$$\|b_2 u_1 - u_2 b_1\|_{L^q} \leq \|u\|_{L^{2q}} \|b\|_{L^{2q}} \leq C (\|u\|_{L^2} + \|\omega\|_{L^2}) (\|b\|_{L^2} + \|j\|_{L^2}).$$

Inserting these estimates in (2.13) yields

$$\|\partial_2 b_1(t)\|_{L^q} \leq C \|\partial_2 b_{01}\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|u\|_{L_t^\infty L^2} + \|\omega\|_{L_t^\infty L^2}) (\|b\|_{L_t^\infty L^2} + \|j\|_{L_t^\infty L^2}).$$

Similarly, for any $t > 0$,

$$\|\partial_1 b_2(t)\|_{L^q} \leq C \|\partial_1 b_{02}\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|u\|_{L_t^\infty L^2} + \|\omega\|_{L_t^\infty L^2}) (\|b\|_{L_t^\infty L^2} + \|j\|_{L_t^\infty L^2}).$$

Consequently,

$$\begin{aligned} \|\nabla b\|_{L_t^\infty L^q} &\leq C \|j\|_{L_t^\infty L^q} \\ &\leq C \|\nabla b_0\|_{L^p} + C t^{1-\frac{1}{\beta}} (\|u\|_{L_t^\infty L^2} + \|\omega\|_{L_t^\infty L^2}) (\|b\|_{L_t^\infty L^2} + \|j\|_{L_t^\infty L^2}). \end{aligned}$$

An elementary Sobolev inequality then implies that, for any $q > 2$,

$$\|b\|_{L_t^\infty L^\infty} \leq C (\|b\|_{L_t^\infty L^2} + \|\nabla b\|_{L_t^\infty L^q}) = C(t, u_0, b_0).$$

Next we prove (2.6). To do so, we combine the estimates of $\|\omega\|_{L_t^\infty L^q}$ with $\|\nabla j\|_{L_t^1 L^q}$. It follows from the vorticity equation (see (2.3)) that

$$\|\omega(t)\|_{L^q} \leq \|\omega_0\|_{L^q} + \|b\|_{L_{x,t}^\infty} \int_0^t \|\nabla j(\tau)\|_{L^q} d\tau. \tag{2.15}$$

We then bound $\|\partial_2 \partial_2 b_1\|_{L^q}$ and $\|\partial_1 \partial_2 b_2\|_{L^q}$ in terms of $\|\omega\|_{L^q}$. Applying $\partial_2 \partial_2$ to (2.10) yields

$$\partial_2 \partial_2 b_1 = \partial_2 \partial_2 (g(t) *_2 b_{01}) + \int_0^t \partial_2 \partial_2 g(t - \tau) *_2 (b \cdot \nabla u_1 - u \cdot \nabla b_1) d\tau.$$

As in the proof of (2.13), we have

$$\|\partial_2 \partial_2 b_1\|_{L^q} \leq \|\partial_2 \partial_2 (g(t) *_2 b_{01})\|_{L^q} + \int_0^t \|\partial_2 \partial_2 g(t - \tau)\|_{L_{x,2}^1} \|b \cdot \nabla u_1 - u \cdot \nabla b_1\|_{L^q} d\tau.$$

Furthermore, by Sobolev's inequality,

$$\begin{aligned} \|b \cdot \nabla u_1 - u \cdot \nabla b_1\|_{L^q} &\leq \|b\|_{L^\infty} \|\omega\|_{L^q} + \|u\|_{L^{2q}} \|\nabla b_1\|_{L^{2q}} \\ &\leq C \|\omega\|_{L^q} + C (\|u\|_{L^2} + \|\omega\|_{L^2}) (\|j\|_{L^2} + \|\nabla j\|_{L^2}). \end{aligned}$$

Taking L^1 in time on $[0, t]$, applying Young's inequality for convolution and invoking (2.14), we have

$$\|\partial_2 \partial_2 b_1\|_{L_t^1 L^q} \leq \|\partial_2 \partial_2 (g(t) * b_{01})\|_{L_t^1 L^q} + C \left(\|\omega\|_{L_t^1 L^q} + 1 \right) t^{1-\frac{1}{\beta}},$$

where $C = C(t, u_0, b_0)$ is bound for the norms of (u, b) obtained previously. Clearly, by Young's inequality for convolution,

$$\|\partial_2 \partial_2 (g(t) * b_{01})\|_{L_t^1 L^q} \leq \|\partial_2 \partial_2 g_2\|_{L_t^1 L_{x_2}^1} \|b_{01}\|_{L^q} \leq C t^{1-\frac{1}{\beta}} \|b_{01}\|_{L^q}.$$

Therefore,

$$\|\partial_2 \partial_2 b_1\|_{L_t^1 L^q} \leq C t^{1-\frac{1}{\beta}} \|b_{01}\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|\omega\|_{L_t^1 L^q} + 1).$$

Similarly,

$$\|\partial_1 \partial_1 b_2\|_{L_t^1 L^q} \leq C t^{1-\frac{1}{\beta}} \|b_{02}\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|\omega\|_{L_t^1 L^q} + 1).$$

To estimate $\|\partial_1 \partial_2 b_1\|_{L_t^1 L^q}$, we resort to the special structure of the nonlinear term, namely (2.12), which allows us to write $\partial_1 \partial_2 b_1$ as

$$\begin{aligned} \partial_1 \partial_2 b_1(t) &= \partial_1 \partial_2 (g(t) * b_{01}) + \int_0^t \partial_2 g(t-\tau) * \partial_1 (b \cdot \nabla u_1 - u \cdot \nabla b_1) \, d\tau \\ &= \partial_1 \partial_2 (g(t) * b_{01}) + \int_0^t \partial_2 g(t-\tau) * \partial_1 \partial_2 (b_2 u_1 - u_2 b_1) \, d\tau \\ &= \partial_1 \partial_2 (g(t) * b_{01}) + \int_0^t \partial_2 \partial_2 g(t-\tau) * \partial_1 (b_2 u_1 - u_2 b_1) \, d\tau. \end{aligned} \quad (2.16)$$

Then, as in the estimate of $\partial_2 \partial_2 b_1$, we have

$$\begin{aligned} \|\partial_1 \partial_2 b_1\|_{L_t^1 L^q} &\leq \|\partial_2 g(t)\|_{L_{x,t}^1} \|\partial_1 b_{01}\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|\omega\|_{L_t^1 L^q} + 1) \\ &\leq C t^{1-\frac{1}{2\beta}} \|\partial_1 b_{01}\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|\omega\|_{L_t^1 L^q} + 1). \end{aligned}$$

Similarly,

$$\|\partial_1 \partial_1 b_2\|_{L_t^1 L^q} \leq C t^{1-\frac{1}{2\beta}} \|\partial_2 b_{02}\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|\omega\|_{L_t^1 L^q} + 1).$$

Therefore, by $\nabla \cdot b = 0$,

$$\begin{aligned} \|\nabla j\|_{L_t^1 L^q} &= \|(\Delta b_2, -\Delta b_1)\|_{L_t^1 L^q} \\ &\leq \|\partial_2 \partial_2 b_1\|_{L_t^1 L^q} + \|\partial_1 \partial_2 b_2\|_{L_t^1 L^q} + \|\partial_1 \partial_1 b_2\|_{L_t^1 L^q} + \|\partial_1 \partial_2 b_1\|_{L_t^1 L^q} \\ &\leq C t^{1-\frac{1}{2\beta}} \|\nabla b_0\|_{L^q} + C t^{1-\frac{1}{\beta}} (\|\omega\|_{L_t^1 L^q} + 1). \end{aligned} \tag{2.17}$$

Then (2.15) and (2.17), together with Gronwall’s inequality implies (2.6). ■

2.3 Global bounds for $\|\nabla j\|_{L_t^1 L^\infty}$ and $\|\omega\|_{L_{x,t}^\infty}$ and proof of Theorem 1.1

This subsection proves that ω admits a global bound in $L_{x,t}^\infty$. This crucial global bound then ensures a global bound for $\|(u, b)\|_{H^s}$ for any $s > 0$.

Proposition 2.4. Assume (u_0, b_0) satisfies the conditions stated in Theorem 1.1. Let (u, b) be the corresponding solution of (1.3) with $\beta > 1$. Then, (u, b) admits the following global bounds, for any $0 < t < \infty$,

$$\|\nabla j\|_{L_t^1 L_x^\infty} \leq C(t, u_0, b_0), \quad \|\omega\|_{L_{x,t}^\infty} \leq C(t, u_0, b_0) \tag{2.18}$$

and

$$\|(u, b)\|_{H^s} \leq C(t, u_0, b_0). \tag{2.19}$$

□

To prove Proposition 2.4, we need the following Hörmander–Mikhlin multiplier theorem (see, e.g., [34, p. 96]).

Lemma 2.5. Let m be a bounded function on \mathbb{R}^d which is smooth except possibly at the origin, and such that

$$|\nabla^k m(\xi)| \leq C |\xi|^{-k}, \quad 0 \leq k \leq \frac{d}{2} + 1.$$

Then m is an L^p multiplier for all $1 < p < \infty$, or the operator T_m defined by

$$\widehat{T_m f} = m \widehat{f}, \quad f \in L^2 \cap L^p,$$

is bounded from $L^2 \cap L^p$ to $L^2 \cap L^p$. □

Proof of Proposition 2.4. Due to the embedding inequality, for any $q > \frac{2}{\sigma}$,

$$\|\nabla j\|_{L^\infty} \leq C (\|\nabla j\|_{L^2} + \|\Lambda^\sigma \nabla j\|_{L^q}),$$

it suffices to show that, for some $\sigma > 0$ and for all $2 \leq q < \infty$,

$$\|\Lambda^\sigma \nabla j\|_{L_t^1 L^q} < \infty. \quad (2.20)$$

We first show, for $0 < \sigma < 2\beta - 2$,

$$\|\Lambda_2^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^q} \leq C(t, u_0, b_0) < \infty, \quad \|\Lambda_1^\sigma \partial_1 \partial_1 b_2\|_{L_t^1 L^q} \leq C(t, u_0, b_0) < \infty. \quad (2.21)$$

Applying $\Lambda_2^\sigma \partial_2 \partial_2$ to the integral representation of b_1 in (2.10), taking the norm in $L_t^1 L^q$ and using Young's inequality for convolution, we obtain

$$\|\Lambda_2^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^q} \leq \|\Lambda_2^\sigma \partial_2 \partial_2 g\|_{L_t^1 L_x^1} \|b_{01}\|_{L^q} + \|\Lambda_2^\sigma \partial_2 \partial_2 g\|_{L_t^1 L_x^1} \|b \cdot \nabla u_1 - u \cdot \nabla b_1\|_{L_t^1 L_x^q}.$$

According to Lemma 2.3,

$$\|\Lambda_2^\sigma \partial_2 \partial_2 g\|_{L_x^1} \leq C t^{-\frac{2+\sigma}{2\beta}}.$$

Therefore, for $0 < \sigma < 2\beta - 2$,

$$\|\Lambda_2^\sigma \partial_2 \partial_2 g\|_{L_t^1 L_x^1} = C t^{\frac{2\beta-(2+\sigma)}{2\beta}}.$$

By Hölder's inequality and Sobolev's inequality,

$$\|b \cdot \nabla u_1 - u \cdot \nabla b_1\|_{L_t^1 L_x^q} \leq C(t, u_0, b_0).$$

Therefore,

$$\|\Lambda_2^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^q} \leq C(t, u_0, b_0).$$

Similarly,

$$\|\Lambda_1^\sigma \partial_1 \partial_1 b_2\|_{L_t^1 L^q} \leq C(t, u_0, b_0).$$

Therefore, (2.21) holds. Making use of the structure of the nonlinearity, we can also show that

$$\|\Lambda_2^\sigma \partial_1 \partial_2 b_1\|_{L_t^1 L^q} \leq C(t, u_0, b_0), \quad \|\Lambda_1^\sigma \partial_1 \partial_2 b_2\|_{L_t^1 L^q} \leq C(t, u_0, b_0). \quad (2.22)$$

In fact, applying $\Lambda_2^\sigma \partial_1 \partial_2$ to (2.10), writing $\partial_1 \partial_2 b_1$ as in (2.16) and taking the norm in $L_t^1 L^q$, we obtain

$$\|\Lambda_2^\sigma \partial_1 \partial_2 b_1\|_{L_t^1 L^q} \leq \|\Lambda_2^\sigma \partial_2 g\|_{L_t^1 L_x^1} \|\partial_1 b_{01}\|_{L^q} + \|\Lambda_2^\sigma \partial_2 \partial_2 g\|_{L_t^1 L_x^1} \|\partial_1 (b_2 u_1 - u_2 b_1)\|_{L_t^1 L_x^q} \leq C < \infty.$$

Therefore, (2.22) holds. Next we show that

$$\|\Lambda_1^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^q} < \infty, \quad \|\Lambda_2^\sigma \partial_1 \partial_1 b_2\|_{L_t^1 L^q} < \infty. \tag{2.23}$$

It appears that we cannot prove (2.23) in the same way as (2.21) and (2.22). The main reason is that, when we apply the operator $\Lambda_1^\sigma \partial_2 \partial_2$ to the integral representation of b_1 in (2.10), the part Λ_1^σ has to be applied to $b \cdot \nabla u_1 - u \cdot \nabla b_1$, but unfortunately we have no control on $\Lambda_1^\sigma \nabla u_1$. Instead we prove (2.23) using the Hörmander–Mikhlin multiplier theorem stated in Lemma 2.5. More precisely, due to the simple inequality

$$|\xi_1|^\sigma |\xi_2|^2 \leq \frac{2}{2 + \sigma} (\xi_2^2)^{1 + \frac{\sigma}{2}} + \frac{\sigma}{2 + \sigma} (\xi_1^2)^{1 + \frac{\sigma}{2}},$$

Plancherel’s theorem and the global bounds in (2.21) and (2.22) imply that

$$\|\Lambda_1^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^2} \leq C \left(\|\Lambda_2^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^2} + \|\Lambda_1^\sigma \partial_1 \partial_2 b_2\|_{L_t^1 L^2} \right) \leq C(t, u_0, b_0).$$

Define the Fourier multiplier operator T_m by

$$\widehat{T_m f}(\xi) = m(\xi) \widehat{f}(\xi), \quad m(\xi) = \frac{|\xi_1|^\sigma |\xi_2|^2}{(\xi_2^2)^{1 + \frac{\sigma}{2}} + (\xi_1^2)^{1 + \frac{\sigma}{2}}}.$$

It is easy to check that m obeys the conditions of Lemma 2.5. It then follows from Lemma 2.5 that

$$\begin{aligned} \|\Lambda_1^\sigma \partial_2 \partial_2 b_1\|_{L^q} &= \|T_m \Lambda_2^\sigma \partial_2 \partial_2 b_1 + T_m \Lambda_1^\sigma \partial_1 \partial_2 b_2\|_{L^q} \\ &\leq C \left(\|\Lambda_2^\sigma \partial_2 \partial_2 b_1\|_{L^q} + \|\Lambda_1^\sigma \partial_1 \partial_2 b_2\|_{L^q} \right). \end{aligned}$$

Therefore, the global bounds in (2.21) and (2.22) implies

$$\|\Lambda_1^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^q} \leq C \left(\|\Lambda_2^\sigma \partial_2 \partial_2 b_1\|_{L_t^1 L^q} + \|\Lambda_1^\sigma \partial_1 \partial_2 b_2\|_{L_t^1 L^q} \right) \leq C(t, u_0, b_0).$$

Similarly,

$$\|\Lambda_2^\sigma \partial_1 \partial_1 b_2\|_{L_t^1 L^q} \leq C(t, u_0, b_0).$$

This proves (2.23). We can also prove in a similar fashion that

$$\|\Lambda_1^\sigma \partial_1 \partial_2 b_1\|_{L_t^1 L^q} < \infty, \quad \|\Lambda_2^\sigma \partial_1 \partial_2 b_2\|_{L_t^1 L^q} < \infty. \quad (2.24)$$

It is clear that (2.21), (2.22), (2.23), and (2.24) imply (2.20). We thus have obtained

$$\|\nabla j\|_{L_t^1 L^\infty} \leq C(t, u_0, b_0).$$

Furthermore, the vorticity equation implies

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^t \|b(\tau)\|_{L^\infty} \|\nabla j(\tau)\|_{L^\infty} d\tau \leq C(t, u_0, b_0).$$

This completes the proof of (2.18). Once we have the global bounds

$$\|\omega\|_{L_t^1 L_x^\infty} \leq C(t, u_0, b_0), \quad \|j\|_{L_t^1 L_x^\infty} \leq C(t, u_0, b_0),$$

the global bound in (2.19) then follows from a standard procedure (see, e.g., [3]). This completes the proof of Proposition 2.4. \blacksquare

We finally provide the proof of Theorem 1.1.

Proof of Theorem 1.1. Once the global *a priori* bounds is at our disposal, the proof can be achieved via a standard procedure. First we seek the solution of a regularized system. We begin by introducing a few notation. For $\varepsilon > 0$, we denote by ϕ_ε the standard mollifier, namely

$$\phi_\varepsilon(x) = \varepsilon^{-2} \phi(\varepsilon^{-1}|x|)$$

with

$$\phi \in C_0^\infty(\mathbb{R}^2), \quad \phi(x) = \phi(|x|), \quad \text{supp } \phi \subset \{x \mid |x| < 1\}, \quad \int_{\mathbb{R}^2} \phi(x) dx = 1.$$

For any locally integrable function v , define the mollification $\mathcal{J}_\varepsilon v$ by

$$\mathcal{J}_\varepsilon v = \phi_\varepsilon * v.$$

Let \mathbb{P} denote the Leray projection operator (on to divergence-free vector fields). We seek a solution $(u^\varepsilon, b^\varepsilon)$ of the system

$$\begin{cases} \partial_t u^\varepsilon + \mathbb{P} \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla(\mathcal{J}_\varepsilon u^\varepsilon)) = \mathbb{P} \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon b^\varepsilon) \cdot \nabla(\mathcal{J}_\varepsilon b^\varepsilon)), \\ \partial_t b_1^\varepsilon + \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla(\mathcal{J}_\varepsilon b_1^\varepsilon)) + \eta \mathcal{J}_\varepsilon^2 \Lambda_2^{2\beta} b_1^\varepsilon = \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon b^\varepsilon) \cdot \nabla(\mathcal{J}_\varepsilon u_1^\varepsilon)), \\ \partial_t b_2^\varepsilon + \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla(\mathcal{J}_\varepsilon b_2^\varepsilon)) + \eta \mathcal{J}_\varepsilon^2 \Lambda_1^{2\beta} b_2^\varepsilon = \mathcal{J}_\varepsilon((\mathcal{J}_\varepsilon b^\varepsilon) \cdot \nabla(\mathcal{J}_\varepsilon u_2^\varepsilon)), \\ \nabla \cdot u^\varepsilon = \nabla \cdot b^\varepsilon = 0, \\ (u^\varepsilon, w^\varepsilon)(x, 0) = (u_0 * \phi_\varepsilon, b_0 * \phi_\varepsilon) = (u_0^\varepsilon, b_0^\varepsilon). \end{cases}$$

Following the lines as those in the proofs of Propositions 2.1, 2.2, and 2.4, we can establish the global bound, for any $t \in (0, \infty)$,

$$\|u^\varepsilon(t)\|_{H^s}^2 + \|b^\varepsilon(t)\|_{H^s}^2 \leq C(t, u_0, b_0).$$

A standard compactness argument allows us to obtain the global existence of the classical solution (u, b) to (1.3). The uniqueness can also be easily established. We omit further details. This completes the proof of Theorem 1.1. \blacksquare

Funding

This work was supported by the NNSFC [grant nos 11271019 and 11571240 to B.D.; 11201181 to J.L., in part]; NSF grant DMS [1614246 to J.W.], AT&T Foundation at Oklahoma State University, and by NNSFC [grant no. 11471103 to J.W.], in part (a grant awarded to B. Yuan).

References

- [1] Agapito, R. and M. Schonbek. "Non-uniform decay of MHD equations with and without magnetic diffusion." *Comm. Partial Differential Equations* 32 (2007): 1791–812.
- [2] Amann, H. "Maximal regularity for nonautonomous evolution equations." *Adv. Nonlinear Stud.* 4 (2004): 417–30.
- [3] Beale, J., T. Kato, and A. Majda. "Remarks on the breakdown of smooth solutions for the 3-D Euler equations." *Commun. Math. Phys.* 94 (1984): 61–6.
- [4] Biskamp, D. *Nonlinear Magnetohydrodynamics*. Cambridge: Cambridge University Press, 1993.
- [5] Cao, C., D. Regmi, and J. Wu. "The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion." *J. Differential Equations* 254 (2013): 2661–81.
- [6] Cao, C., D. Regmi, J. Wu, and X. Zheng. "Global regularity for the 2D magnetohydrodynamics equations with horizontal dissipation and horizontal magnetic diffusion." Preprint.
- [7] Cao, C. and J. Wu. "Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion." *Adv. Math.* 226 (2011): 1803–22.

- [8] Cao, C., J. Wu, and B. Yuan. "The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion." *SIAM J. Math. Anal.* 46 (2014): 588–602.
- [9] Cai, Y. and Z. Lei. "Global well-posedness of the incompressible magnetohydrodynamics." arXiv: 1605.00439 [math.AP] 2 May 2016.
- [10] Chemin, J.-Y., D. S. McCormick, J. C. Robinson, and J. L. Rodrigo. "Local existence for the non-resistive MHD equations in Besov spaces." *Adv. Math.* 286 (2016): 1–31.
- [11] Chen, G.-Q. and D. Wang. "Global solutions of nonlinear magnetohydrodynamics with large initial data." *J. Differential Equations* 182 (2002): 344–76.
- [12] Constantin, P. "Lagrangian-Eulerian methods for uniqueness in hydrodynamic systems." *Adv. Math.* 278 (2015): 67–102.
- [13] Davidson, P. A. *An Introduction to Magnetohydrodynamics*. Cambridge, England: Cambridge University Press, 2001.
- [14] Du, L. and D. Zhou. "Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion." *SIAM J. Math. Anal.* 47 (2015): 1562–89.
- [15] Fan, J., H. Malaikah, S. Monaque, G. Nakamura and Y. Zhou. "Global cauchy problem of 2D generalized MHD equations." *Monatsh. Math.* 175 (2014): 127–31.
- [16] Fefferman, C. L., D. S. McCormick, J. C. Robinson, and J. L. Rodrigo. "Higher order commutator estimates and local existence for the non-resistive MHD equations and related models." *J. Funct. Anal.* 267 (2014): 1035–56.
- [17] Fefferman, C. L., D. S. McCormick, J. C. Robinson, and J. L. Rodrigo. "Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces." *Arch. Ration. Mech. Anal.* 223 (2017): 677–91.
- [18] He, L., L. Xu, and P. Yu. "On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves." arXiv:1603.08205 [math.AP] 27 Mar 2016.
- [19] Hittmeir, S. and S. Merino-Aceituno. "Kinetic derivation of fractional Stokes and Stokes-Fourier systems." arXiv: 1408.6400v2 [math-ph] 30 Sep 2014.
- [20] Hu, X. "Global existence for two dimensional compressible magnetohydrodynamic flows with zero magnetic diffusivity." arXiv: 1405.0274v1 [math.AP] 1 May 2014.
- [21] Hu, X. and F. Lin. "Global existence for two dimensional incompressible magnetohydrodynamic flows with zero magnetic diffusivity." arXiv: 1405.0082v1 [math.AP] 1 May 2014.
- [22] Hu, X. and D. Wang. "Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows." *Arch. Ration. Mech. Anal.* 197 (2010): 203–38.
- [23] Jiu, Q. and D. Niu. "Mathematical results related to a two-dimensional magnetohydrodynamic equations." *Acta Math. Sci. Ser. B Engl. Ed.* 26 (2006): 744–56.
- [24] Jiu, Q., D. Niu, J. Wu, X. Xu, and H. Yu. "The 2D magnetohydrodynamic equations with magnetic diffusion." *Nonlinearity* 28 (2015): 3935–55.
- [25] Jiu, Q. and J. Zhao. "A remark on global regularity of 2D generalized magnetohydrodynamic equations." *J. Math. Anal. Appl.* 412 (2014): 478–84.

- [26] Jiu, Q. and J. Zhao. "Global regularity of 2D generalized MHD equations with magnetic diffusion." *Z. Angew. Math. Phys.* 66 (2015): 677–87.
- [27] Kato, T. and G. Ponce. "Commutator estimates and the Euler and the Navier–Stokes equations." *Comm. Pure Appl. Math.* 41 (1988): 891–907.
- [28] Lei, Z. and Y. Zhou. "BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity." *Discrete Contin. Dyn. Syst.* 25 (2009): 575–83.
- [29] Lemarie-Rieusset, P. G. *Recent Developments in the Navier–Stokes Problem*. Chapman Hall/CRC Research Notes in Mathematics Series. New York: CRC Press, 2002.
- [30] Lin, F., L. Xu, and P. Zhang. "Global small solutions to an MHD-type system: the three-dimensional case." *Comm. Pure Appl. Math.* 67 (2014): 531–80.
- [31] Lin, F., L. Xu, and P. Zhang. "Global small solutions to 2-D incompressible MHD system." *J. Differential Equations* 259 (2015): 5440–85.
- [32] Priest, E. and T. Forbes. *Magnetic Reconnection, MHD Theory and Applications*. Cambridge: Cambridge University Press, 2000.
- [33] Ren, X., J. Wu, Z. Xiang, and Z. Zhang. "Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion." *J. Functional Analysis* 267 (2014): 503–41.
- [34] Stein, E. *Singular Integrals and Differentiability Properties of Functions*. Princeton, NJ: Princeton University Press, 1970.
- [35] Stewartson, K. "On asymptotic expansions in the theory of boundary layers." *Stud. Appl. Math.* 36 (1957): 173–91.
- [36] Tran, C., X. Yu, and Z. Zhai. "On global regularity of 2D generalized magnetohydrodynamic equations." *J. Differential Equations* 254 (2013): 4194–216.
- [37] Wei, D. and Z. Zhang. "Global well-posedness of the MHD equations in a homogeneous magnetic field." arXiv:1607.04397 [math.AP] 15 Jul 2016.
- [38] Wu, J. "Generalized MHD equations." *J. Differential Equations* 195 (2003): 284–312.
- [39] Wu, J. "Regularity criteria for the generalized MHD equations." *Comm. Partial Differential Equations* 33 (2008): 285–306.
- [40] Wu, J. "Global regularity for a class of generalized magnetohydrodynamic equations." *J. Math. Fluid Mech.* 13 (2011): 295–305.
- [41] Wu, J. and Y. Wu. "Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion." *Adv. Math.* 310 (2017): 759–888.
- [42] Wu, J., Y. Wu, and X. Xu. "Global small solution to the 2D MHD system with a velocity damping term." *SIAM J. Math. Anal.* 47 (2015): 2630–56.
- [43] Wu, J. and P. Zhang. "The global regularity problem on the 2D magnetohydrodynamic equations with magnetic diffusion only." Work in progress.
- [44] Yamazaki, K. "On the global well-posedness of N-dimensional generalized MHD system in anisotropic spaces." *Adv. Differential Equations* 19 (2014): 201–24.
- [45] Yamazaki, K. "Remarks on the global regularity of the two-dimensional magnetohydrodynamics system with zero dissipation." *Nonlinear Anal.* 94 (2014): 194–205.
- [46] Yamazaki, K. "On the global regularity of two-dimensional generalized magnetohydrodynamics system." *J. Math. Anal. Appl.* 416 (2014): 99–111.

- [47] Yamazaki, K. "Global regularity of logarithmically supercritical MHD system with zero diffusivity." *Appl. Math. Lett.* 29 (2014): 46–51.
- [48] Yamazaki, K. "Global regularity of N-dimensional generalized MHD system with anisotropic dissipation and diffusion." *Nonlinear Anal.* 122 (2015): 176–91.
- [49] Yuan, B. and L. Bai. "Remarks on global regularity of 2D generalized MHD equations." *J. Math. Anal. Appl.* 413 (2014): 633–40.
- [50] Zhai, C. and T. Zhang. "Global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system with non-equilibrium background magnetic field." *J. Differential Equations* 261 (2016): 3519–50.
- [51] Zhang, T. "An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system." arXiv:1404.5681v1 [math.AP] 23 Apr 2014.