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Global well-posedness and large-time decay for the 2D micropolar equations

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Abstract

This paper studies the global (in time) regularity and large time behavior of solutions to the 2D micropolar equations with only angular viscosity dissipation. Micropolar equations model a class of fluids with nonsymmetric stress tensor such as fluids consisting of particles suspended in a viscous medium. When there is no kinematic viscosity in the momentum equation, the global regularity problem is not easy due to the lack of suitable bounds on the derivatives. The idea here is to fully exploit the structure of the system and control the vorticity via the evolution equation of a combined quantity of the vorticity and the micro-rotation angular velocity. To understand the large time behavior, we overcome two main difficulties, the lack of kinematic viscosity and the presence of linear terms. Classical tools such as the Fourier splitting method of Schonbek and Kato's approach for the decay of small solutions do not apply here. We introduce a diagonalization process to eliminate the linear terms and rely on the uniform bounds for the first derivatives of the solutions to generate suitable decay rates.

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Keywords: Micropolar equation; Angular viscosity dissipation; Global well-posedness; Large-time behavior

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1. Introduction

The micropolar equations were first introduced in 1965 by C.A. Eringen to model micropolar fluids (see Eringen [9, Sections 1 and 6]). Micropolar fluids are fluids with microstructure. They belong to a class of fluids with nonsymmetric stress tensor (called polar fluids) and include, as a special case, the classical fluids modeled by the Navier–Stokes equations (see, e.g., [4,8–10, 16]). The system of the micropolar equations is a significant generalization of the Navier–Stokes equations covering many more phenomena such as fluids consisting of particles suspended in a viscous medium (see, e.g., [16,17,19]). The micropolar equations have been extensively studied and applied by many engineers and physicists.

The 3D micropolar equations can be written as

$$(3DMP) \quad \begin{cases} \partial_t u - (\nu+\kappa)\Delta u - 2\kappa\nabla\times w + u\cdot\nabla u + \nabla\pi = 0, \\ \nabla\cdot u = 0, \\ \partial_t w - \gamma\Delta w + 4\kappa w - \mu\nabla\nabla\cdot w - 2\kappa\nabla\times u + u\cdot\nabla w = 0, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the fluid velocity, $w(x,t)$ the field of microrotation representing the angular velocity of the rotation of the particles of the fluid, $\pi(x,t)$ the scalar pressure, ν denotes the Newtonian kinematic viscosity, κ the micro-rotation viscosity, and γ and μ the angular viscosities. The 3D micropolar equations reduce to the 2D micropolar equation when

$$u = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0), \quad w = (0,0,w_3(x_1,x_2,t)), \quad \pi = \pi(x_1,x_2,t).$$

More explicitly, the 2D micropolar equations can be written as

$$(2DMP) \quad \begin{cases} \partial_t u - (\nu+\kappa)\Delta u - 2\kappa\nabla\times w + u\cdot\nabla u + \nabla\pi = 0, \\ \nabla\cdot u = 0, \\ \partial_t w - \gamma\Delta w + 4\kappa w - 2\kappa\nabla\times u + u\cdot\nabla w = 0, \end{cases} \tag{1.2}$$

where we have written $u = (u_1, u_2)$ and w for w_3 for notational brevity. It is worth noting that, in the 2D case,

$$\Omega \equiv \nabla\times u = \partial_{x_1}u_2 - \partial_{x_2}u_1$$

is a scalar function representing the vorticity, and $\nabla\times w = \big(\partial_{x_2}w, -\partial_{x_1}w\big)$.

The micropolar equations are also mathematically significant. The well-posedness problem on the micropolar and closely related equations such as the magneto-micropolar equations has attracted considerable attention recently from the community of mathematical fluids (see, e.g., [2,5,12,16,21,22]). Generally speaking, the global regularity problem for the micropolar equations is easier than that for the corresponding incompressible magnetohydrodynamic equations and harder than that for the corresponding incompressible Boussinesq equations.

More recent efforts are focused on the 2D micropolar equations with partial dissipation. In [7] Dong and Zhang examined (1.2) without the micro-rotation viscosity, namely $\gamma = 0$. The global regularity problem for this partial dissipation case is not trivial due to the presence of the term $\nabla\times w$ in the velocity equation. Dong and Zhang in [7] observed that the combined quantity

$$\Omega - \frac{2\kappa}{\nu+\kappa} w$$

obeys a transport-diffusion equation, which allows the extraction of a global bound. Another partial dissipation case, (1.2) with $\nu = 0$, $\gamma > 0$, $\kappa > 0$ and $\kappa \neq \gamma$, was examined by Xue, who was able to obtain the global well-posedness in the frame work of Besov spaces [20]. We remark that the requirement $\kappa \neq \gamma$ in [20] is not crucial and it is not difficult to see that the global well-posedness remains valid even when $\kappa = \gamma$.

This paper aims at the partial dissipation case when (1.2) involves no velocity dissipation. More precisely, we study the existence and uniqueness of classical solutions to the 2D micropolar equation with only angular velocity dissipation

$$\begin{cases} \partial_t u + \kappa u - 2\kappa \nabla \times w + \nabla \pi + u \cdot \nabla u = 0, \\ \nabla \cdot u = 0, \\ \partial_t w - \gamma \Delta w + 4\kappa w - 2\kappa \nabla \times u + u \cdot \nabla w = 0. \end{cases} \tag{1.3}$$

We remark that the term κu does not play any significant role in the global regularity problem. It is kept in (1.3) simply to reflect the fact that the micropolar fluid motion requires the presence of the micro-rotational effect and micro-rotational inertia, namely $\kappa > 0$. We establish the following global existence and uniqueness result for (1.3).

Theorem 1.1. *Assume $(u_0, w_0) \in H^s(\mathbb{R}^2)$ $(s > 2)$ and $\nabla \cdot u_0 = 0$. Then (1.3) has a unique global solution (v, w) satisfying*

$$(u, w) \in C([0, \infty); H^s(\mathbb{R}^2)), \ \ w \in L^2(0, T; H^{s+1}(\mathbb{R}^2)), \ \ \forall\, T > 0.$$

We remark that the global regularity problem on (1.3) is not trivial. The difficulty is due to the dynamic micro-rotational term $\nabla \times w$ in the velocity equation. This term prevents us from obtaining the global L^∞-bound for the vorticity $\Omega = \nabla \times u$ directly from the vorticity equation,

$$\partial_t \Omega + \kappa \Omega + u \cdot \nabla \Omega + 2\kappa \Delta w = 0, \tag{1.4}$$

where we have used $\nabla \times (\nabla \times w) = -\Delta w$. The bound $\|\Omega(t)\|_{L^\infty}$ relies on Δw, namely,

$$\|\Omega(t)\|_{L^\infty} \le \|\Omega(0)\|_{L^\infty} + 2\kappa \int_0^t \|\Delta w(\tau)\|_{L^\infty}\, d\tau.$$

To overcome this difficulty, we make use of the angular viscosity dissipation $\gamma \Delta w$ to balance out the bad term $2\kappa \Delta w$ in (1.4). More precisely, we consider the sum of the vorticity and micro-rotation angular velocity

$$\mathcal{Z} = \Omega + \frac{2\kappa}{\gamma} w,$$

which satisfies

$$\partial_t \mathcal{Z} + u \cdot \nabla \mathcal{Z} + \left(\kappa - \frac{4\kappa^2}{\gamma}\right) \mathcal{Z} + \left(\frac{8\kappa^3}{\gamma^2} + \frac{6\kappa^2}{\gamma}\right) w = 0. \tag{1.5}$$

This equation of $\mathcal{Z}$ serves our purpose to obtain a global bound for Ω via the global bound for $\mathcal{Z}$. To bound $\|\mathcal{Z}\|_{L^\infty}$, we need a global bound for $\|w\|_{L^1_t L^\infty_x}$ according to (1.5). To obtain the global bound for $\|w\|_{L^1_t L^\infty_x}$, we first establish the global bound for $\|w\|_{L^1_t H^1_x}$ via energy estimate and then the global bound for $\|\Delta w\|_{L^2_t L^2_x}$ via the maximal regularity of the heat operator. More details can be found in Section 2.

The second purpose of this paper is to obtain explicit time decay rates for the following 2D micropolar equation

$$\begin{cases} \partial_t u + \kappa u - 2\kappa \nabla \times w + \nabla \pi + u \cdot \nabla u = 0, \\ \nabla \cdot u = 0, \\ \partial_t w - \gamma \Delta w - 2\kappa \nabla \times u + u \cdot \nabla w = 0. \end{cases} \tag{1.6}$$

(1.6) is obtained by removing the term $4\kappa\, w$ from (1.3). It is not difficult to understand that solutions to (1.3) decay exponentially. We ignore this term and consider (1.6) instead. We remark that Theorem 1.1 remains valid for (1.6).

We aim at developing an effective approach on large-time behavior for systems involving linear terms and with mixed damping and dissipation. As we know, linear terms are usually obstacles in the study of large-time behavior and in obtaining explicit decay rates. The diagonalization process presented here eliminates the linear terms in (1.6) and leads to an integral representation in terms of the nonlinear terms only. This representation allows us to derive the desired decay rates. This practice may be useful for more general decay problems.

We remark that, when the micropolar equation has full dissipation, namely (1.2) with $\nu > 0$, $\kappa > 0$ and $\gamma > 0$, effective approaches such as the Fourier splitting method of Schonbek [18] and Kato's method for small solutions [14] have been developed to obtain explicit decay rates. In fact, the L^2 time decay rate was obtained by Dong–Chen [6] for global solutions of the 2D micropolar equation (1.2) via the Fourier splitting method and by Chen–Price [3] for small solutions of the 3D micropolar equation (1.1) via Kato's method. However, when the velocity equation has no dissipation, the Fourier splitting method which relies on the dissipation in order to decompose the whole space into two time-dependent sub-domains does not apply. Furthermore, without smallness assumption, Kato's method [14] does not work due to the difficulty of constructing an iterative procedure. Some more recent new time decay methods such as the one by Guo–Wang [11] involving Sobolev space of negative indices and the one by [13] for dual equation technique do not apply to our circumstance.

In order to derive the decay estimates for (1.6), we make the assumption

$$\gamma > 4\kappa.$$

As we explain below, this condition is sharp and necessary. It allows us to derive uniform (in time) bounds for both the solution and its first derivatives. Especially we are able to show that, for any $0 \le s < t < \infty$,

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \le C \left(\|\nabla u(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2\right),$$

where C is a constant depending on the L^2-norm of (u_0, w_0) only (see (3.4) in Section 3 for details). This inequality leads to the large-time behavior for ∇u and ∇w,

$$(1+t)^{\frac{1}{2}} \|(\nabla u(t), \nabla w(t))\|_{L^2} \to 0, \quad t \to \infty,$$

which also serves as the first step in seeking the decay rate for $\|u(t)\|_{L^2}$. To obtain the decay rate, we take the inner product of the velocity equation with u and then represent $\|u(t)\|_{L^2}^2$ in an integral form, which leads to the decay estimate

$$(1+t)^{\frac{1}{2}} \ \|u(t)\|_{L^2} \to 0, \quad \text{ as } t \to \infty.$$

To summarize, we have obtained the following decay rates for $\|\nabla u\|_{L^2}$, $\|\nabla w(t)\|_{L^2}$ and $\|u(t)\|_{L^2}$.

Theorem 1.2. *Assume (u_0, w_0) are sufficiently regular, say, $(u_0, w_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, and $\nabla \cdot u_0 = 0$. Let (u, w) be the corresponding global solution of the system defined by (1.6). When κ and γ satisfy*

$$\gamma > 4\kappa, \tag{1.7}$$

then we have the following decay rates, as $t \to \infty$,

$$(1+t)^{\frac{1}{2}} \|(\nabla u(t), \nabla w(t))\|_{L^2} \to 0, \qquad (1+t)^{\frac{1}{2}} \|u(t)\|_{L^2} \to 0.$$

Due to the presence of the linear terms in (1.6), any direct approach such as energy estimates and the integral representation of w does not lead to the large-time behavior for $\|w(t)\|_{L^2}$. This forces us to eliminate the linear terms of (1.6) via a diagonalization process performed on the system of equations for the vorticity Ω in (1.4) and of w in (1.6), namely

$$\begin{cases} \partial_t \Omega + \kappa \Omega + u \cdot \nabla \Omega + 2\kappa \Delta w = 0, \\ \partial_t w - \gamma \Delta w - 2\kappa \nabla \times u + u \cdot \nabla w = 0. \end{cases} \tag{1.8}$$

To do so, we rewrite (1.8) in the Fourier space as

$$\begin{bmatrix} \partial_t \widehat{\Omega}(\xi) \\ \partial_t \widehat{w}(\xi) \end{bmatrix} = \begin{bmatrix} -\kappa & 2\kappa|\xi|^2 \\ 2\kappa & -\gamma|\xi|^2 \end{bmatrix} \begin{bmatrix} \widehat{\Omega}(\xi) \\ \widehat{w}(\xi) \end{bmatrix} + \begin{bmatrix} -\widehat{u \cdot \nabla \Omega}(\xi) \\ -\widehat{u \cdot \nabla w}(\xi) \end{bmatrix}, \tag{1.9}$$

where $\widehat{F}$ denotes the Fourier transform of F. To diagonalize the coefficient matrix, we seek the eigenvalues and eigenvectors. The eigenvalues satisfy the characteristic equation

$$\lambda^2 + (\kappa + \gamma|\xi|^2)\lambda + (\kappa\gamma - 4\kappa^2)|\xi|^2 = 0,$$

which is solved by

$$\lambda_{1,2} = \frac{-(\kappa + \gamma|\xi|^2) \pm \sqrt{(\kappa - \gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}{2}. \tag{1.10}$$

When the condition (1.7) holds, both eigenvalues $\lambda_{1,2} < 0$. If (1.7) is violated, the larger eigenvalue is zero or positive and even the solution of the linear part of (1.9) does not decay in time. This explains why (1.7) is necessary and sharp. This diagonalization process with the condition (1.7) allows us to eliminate the linear terms and obtain an integral representation of (1.9) in terms of the nonlinear terms only. More precisely, we obtain the following proposition.

Proposition 1.3. *The system in (1.8) can be represented in the following integral form (in the Fourier space),*

$$
\begin{aligned}
\widehat{\Omega}(\xi,t) = {} & e^{\lambda_1(\xi)t}\,\big(D_1(\xi)\,\widehat{\Omega}_0(\xi) - D_2(\xi)\,\widehat{w}_0(\xi)\big) \\
& + e^{\lambda_2(\xi)t}\,\big(D_3(\xi)\,\widehat{\Omega}_0(\xi) + D_2(\xi)\,\widehat{w}_0(\xi)\big) \\
& + \int_0^t e^{\lambda_1(\xi)(t-\tau)}\,\Big(-D_1(\xi)\,\widehat{u\cdot\nabla\Omega}(\xi,\tau) + D_2(\xi)\widehat{u\cdot\nabla w}(\xi,\tau)\Big)\,d\tau \\
& + \int_0^t e^{\lambda_2(\xi)(t-\tau)}\,\Big(-D_3(\xi)\,\widehat{u\cdot\nabla\Omega}(\xi,\tau) - D_2(\xi)\widehat{u\cdot\nabla w}(\xi,\tau)\Big)\,d\tau,
\end{aligned}
\tag{1.11}
$$

$$
\begin{aligned}
\widehat{w}(\xi,t) = {} & e^{\lambda_1(\xi)t}\,\big(-D_4(\xi)\,\widehat{\Omega}_0(\xi) + D_3(\xi)\,\widehat{w}_0(\xi)\big) \\
& + e^{\lambda_2(\xi)t}\,\big(D_4(\xi)\,\widehat{\Omega}_0(\xi) + D_1(\xi)\,\widehat{w}_0(\xi)\big) \\
& + \int_0^t e^{\lambda_1(\xi)(t-\tau)}\,\Big(D_4(\xi)\,\widehat{u\cdot\nabla\Omega}(\xi,\tau) - D_3(\xi)\widehat{u\cdot\nabla w}(\xi,\tau)\Big)\,d\tau \\
& + \int_0^t e^{\lambda_2(\xi)(t-\tau)}\,\Big(-D_4(\xi)\,\widehat{u\cdot\nabla\Omega}(\xi,\tau) - D_1(\xi)\widehat{u\cdot\nabla w}(\xi,\tau)\Big)\,d\tau,
\end{aligned}
\tag{1.12}
$$

where λ_1 denotes the eigenvalue in (1.10) with the negative sign in the front of the square-root sign and λ_2 with the positive sign, and D_1, D_2, D_3 and D_4 are given by

$$
\begin{aligned}
D_1(\xi) &= \frac{(\kappa-\gamma|\xi|^2) + \sqrt{(\kappa-\gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}{2\sqrt{(\kappa-\gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}, \\
D_2(\xi) &= \frac{2\kappa|\xi|^2}{\sqrt{(\kappa-\gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}, \\
D_3(\xi) &= \frac{-(\kappa-\gamma|\xi|^2) + \sqrt{(\kappa-\gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}{2\sqrt{(\kappa-\gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}, \\
D_4(\xi) &= \frac{2\kappa}{\sqrt{(\kappa-\gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}.
\end{aligned}
$$

This representation may appear to be complex, but it does not involve any linear terms and is suitable for extracting the desired decay rate for $\|w(t)\|_{L^2}$. Since λ_1 and λ_2 depend on ξ, we need to distinguish their behavior in different regions of ξ when estimating the terms in $\widehat{w}$. Nevertheless, we managed to obtain the following decay rate for $\|w(t)\|_{L^2}$.

Theorem 1.4. *Assume (u_0, w_0) are sufficiently regular, say, $(u_0, w_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, and $\nabla \cdot u_0 = 0$. Let (u, w) be the corresponding global solution of the system defined by (1.6). Assume (1.7) holds, namely*

$$\gamma > 4\kappa.$$

Assume the initial data w_0 satisfies

$$w_0 \in L^1(\mathbb{R}^2). \tag{1.13}$$

Then we have, as $t \to \infty$,

$$(1+t)^{\frac{1}{2}} \|w(t)\|_{L^2} \to 0.$$

The condition (1.13) can be replaced by more general ones such as

$$\|e^{\Delta t} w_0\|_{L^2} \le C\, t^{-\frac{1}{2}} \quad \text{for } t > 0. \tag{1.14}$$

To provide a more complete picture on the current status of the regularity results on the micropolar equations, we also present in the appendix the global existence and uniqueness results for the micropolar equations with full dissipation. The well-posedness result for the 3D micropolar equation extends the work of Fujita and Kato on the 3D Navier–Stokes equations to a nonlinearly coupled system. In the 2D case, we show that, any $u_0 \in L^2$ and $w_0 \in L^2$ generate a unique global solution. The result for the 2D micropolar equation involves the weakest initial data for which one can still deduce the uniqueness. We remark that Lukaszewicz in his monograph [16] studied the well-posedness problem on the 3D stationary as well as the time-dependent micropolar equations. The regularity assumptions on the initial data are different from ours.

Finally we remark that one can consider the global regularity and large-time decay problem for the 2D fractional dissipative micropolar equation

$$\begin{cases} \partial_t u + (\nu+\kappa)(-\Delta)^{\alpha} u - 2\kappa \nabla \times w + \nabla \pi + u \cdot \nabla u = 0, \\ \nabla \cdot u = 0, \\ \partial_t w + \gamma (-\Delta)^{\beta} w + 4\kappa w - 2\kappa \nabla \times u + u \cdot \nabla w = 0, \end{cases} \tag{1.15}$$

with

$$0 < \alpha, \beta < 1, \quad \alpha + \beta = 1.$$

This may not be an easy problem and is in our future study plan. We may need to fully explore the structure of this system.

The rest of this paper is divided into three sections and one appendix. The second section details the proof of Theorem 1.1 while the third section proves one of the decay theorems. Section 4 carries out the diagonalization process and establish the decay result for $\|w(t)\|_{L^2}$. The appendix provides the global existence and uniqueness results for the micropolar equations with full dissipation.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, the global existence and uniqueness of classic solutions for the 2D micropolar equation without velocity dissipation, namely (1.3).

The key component of the proof is the global *a priori* bound for (u, w) in H^s with $s > 2$. For the sake of clarity, we divide the estimates into several regularity levels.

2.1. Global H^1 estimate

We prove that any classical solution of (1.3) admits a global H^1-bound, as stated in the following proposition.

Proposition 2.1. *Assume u_0 and w_0 satisfy the conditions in Theorem 1.1. Then the corresponding solution (u, w) obeys, for any $0 < t < \infty$,*

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \gamma \int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau + 8\kappa \int_0^t \|w(\tau)\|_{L^2}^2 d\tau \le C\, e^{C\,t}, \tag{2.1}$$

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \gamma \int_0^t \|\Delta w(\tau)\|_{L^2}^2 \tau \le C\, e^{C\,t}\, e^{C\, e^{C\,t}}, \tag{2.2}$$

where C's are constants depending on κ, γ or $\|(u_0, w_0)\|_{H^1}$ only (their explicit dependence can be found in the proof).

Proof. Taking the L^2 inner product of (1.3) with (u, w), it is easy to verify

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\right) + \kappa\|u(t)\|_{L^2}^2 + \gamma\|\nabla w(t)\|_{L^2}^2 + 4\kappa\|w(t)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^2} \{2\kappa(\nabla \times w)\cdot u + 2\kappa(\nabla \times u)w\}\, dx \\
&\le 4\kappa\|u\|_{L^2}\|\nabla w\|_{L^2} \le \frac{8\kappa^2}{\gamma}\|u(t)\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla w(t)\|_{L^2}^2,
\end{aligned}$$

where we have used the following fact due to the divergence free of u

$$\int_{\mathbb{R}^2} (u\cdot\nabla u)\cdot u\, dx = 0, \qquad \int_{\mathbb{R}^2} (u\cdot\nabla w)\, w\, dx = 0.$$

Applying Gronwall inequality gives, for $0 < t < \infty$,

$$
\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \gamma \int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau + 8\kappa \int_0^t \|w(\tau)\|_{L^2}^2 d\tau \\
&\le e^{\frac{16\kappa^2}{\gamma} t} \left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 \right),
\end{aligned}
$$

which is (2.1). Taking the L^2 inner product of (1.3) with $(-\Delta u, -\Delta w)$ leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \right) + \kappa \|\nabla u(t)\|_{L^2}^2 + \gamma \|\Delta w(t)\|_{L^2}^2 + 4\kappa \|\nabla w(t)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^2} (2\kappa(\nabla \times w)\cdot(-\Delta u) + 2\kappa(\nabla \times u)(-\Delta w))\, dx + \int_{\mathbb{R}^2} u \cdot \nabla w \, (-\Delta w) dx \\
&\equiv I_1 + I_2,
\end{aligned}
\tag{2.3}
$$

where we have used the fact

$$
\int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \Delta u \, dx = 0.
$$

To estimate I_1, we integrate by parts and apply Hölder's inequality to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (2\kappa(\nabla \times w)\cdot(-\Delta u) + 2\kappa(\nabla \times u)(-\Delta w))\, dx \\
&\le 4\kappa \|\nabla u(t)\|_{L^2} \|\Delta w(t)\|_{L^2} \le \frac{16\kappa^2}{\gamma} \|\nabla u(t)\|_{L^2}^2 + \frac{\gamma}{4} \|\Delta w(t)\|_{L^2}^2.
\end{aligned}
$$

By Sobolev's inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} u \cdot \nabla w \, (-\Delta w) dx \\
&\le \left| \int_{\mathbb{R}^2} \nabla u \cdot \nabla w \, \nabla w dx \right| \le \|\nabla u\|_{L^2} \|\nabla w\|_{L^4}^2 \\
&\le \|\nabla u\|_{L^2} \|\nabla w\|_{L^2} \|\Delta w\|_{L^2} \le \frac{\gamma}{4} \|\Delta w(t)\|_{L^2}^2 + \frac{1}{\gamma} \|\nabla u\|_{L^2}^2 \|\nabla w\|_{L^2}^2.
\end{aligned}
$$

Inserting the bounds for I_1 and I_2 in (2.3) gives

$$\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\right)+\gamma\|\Delta\, w(t)\|_{L^2}^2$$
$$\le \frac{32\kappa^2+2}{\gamma}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\right)\left(\|\nabla w\|_{L^2}^2+1\right).$$

Gronwall inequality implies, for $0<t<\infty$,

$$\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2+\gamma\int_0^t\|\Delta w(\tau)\|_{L^2}^2\tau$$
$$\le \exp\left(\frac{32\kappa^2+2}{\gamma}\int_0^t\left(\|\nabla w(\tau)\|_{L^2}^2+1\right)d\tau\right)\left(\|\nabla u_0\|_{L^2}^2+\|\nabla w_0\|_{L^2}^2\right)$$
$$\le \exp\left\{\frac{32\kappa^2+2}{\gamma^2}e^{\frac{16\kappa^2}{\gamma}t}\left(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2\right)\right\}\exp\left\{\frac{32\kappa^2+2}{\gamma}t\right\}\left(\|\nabla u_0\|_{L^2}^2+\|\nabla w_0\|_{L^2}^2\right)$$
$$\triangleq C\,e^{C\,t}\,e^{C\,e^{C\,t}}.$$

This completes the proof of Proposition 2.1. □

2.2. $W^{2,q}$-bound for w and L^q-bound for $\Omega=\nabla\times u$ with $q\in(1,\infty)$

This subsection presents the global bound for $\|\Delta w\|_{L_t^2L_x^q}$ and $\|\Omega\|_{L_t^\infty L_x^q}$ with $q\in(1,\infty)$. To obtain these global bounds, we combine the maximal regularity property of the heat operator and energy estimates. We remark this step does not allow us to obtain the global bounds for $q=\infty$.

Proposition 2.2. *Assume that u_0 and w_0 satisfy the conditions in Theorem 1.1. Then the corresponding solution (u,w) admits the following global bounds, for any $q\in(1,\infty)$ and any $0<t<\infty$,*

$$\|\Delta w\|_{L^2(0,t;L^q)},\quad \|\Omega\|_{L^\infty(0,t;L^q)}\le C\,e^{C\,t}\,e^{C\,e^{C\,t}}, \tag{2.4}$$

where C's are constants depending only on q, κ, γ and $\|(u_0,w_0)\|_{H^2}$.

To prove this proposition, we recall the maximal regularity property for the heat kernel (see, e.g., [1], [15, p. 64]).

Lemma 2.3. *The operator A defined by*

$$Af(t)\equiv\int_0^t\Delta\,e^{\Delta(t-\tau)}\,f(\tau)\,d\tau$$

maps $L^p(0,T;L^q(\mathbb{R}^d))$ to $L^p(0,T;L^q(\mathbb{R}^d))$ for any $T\in(0,\infty]$ and $p,q\in(1,\infty)$.

We are now ready to prove Proposition 2.2.

Proof of Proposition 2.2. We write the second equation of (1.3) as

$$w(t) = e^{\gamma\Delta t} w_0 + \int_0^t e^{\gamma\Delta(t-\tau)} (2\kappa\nabla \times u - 4\kappa w - u\cdot\nabla w)\, d\tau.$$

Applying Lemma 2.3 with $p = 2$ and $2 \le q < \infty$ yields

$$\int_0^t \|\Delta w\|_{L^q}^2 d\tau \le C \int_0^t \left(\|\nabla \times u\|_{L^q}^2 + \|w\|_{L^q}^2 + \|u\cdot\nabla w\|_{L^q}^2\right) d\tau. \tag{2.5}$$

By (2.1), (2.2) and Sobolev embedding inequalities,

$$\int_0^t \|w(\tau)\|_{L^q}^2 d\tau \le C \int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau \le \frac{C}{\gamma} e^{\frac{16\kappa^2}{\gamma}t} \left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2\right) \le C\, e^{C\,t}$$

and

$$\begin{aligned}
&\int_0^t \|u\cdot\nabla w\|_{L^q}^2\, d\tau \le \int_0^t \|u\|_{L^{2q}}^2 \|\nabla w\|_{L^{2q}}^2\, d\tau \\
&\le C \int_0^t \|\nabla u\|_{L^2}^2 \|\Delta w\|_{L^2}^2 d\tau \le C \sup_{0\le s\le t} \|\nabla u(s)\|_{L^2}^2 \int_0^t \|\Delta w\|_{L^2}^2 ds \\
&\le C\, e^{C\,t}\, e^{C\,e^{C\,t}},
\end{aligned}$$

where we have invoked the global bound in Proposition 2.1. Inserting the two inequalities above in (2.5), we have

$$\int_0^t \|\Delta w\|_{L^q}^2 ds \le C \int_0^t \|\nabla \times u\|_{L^q}^2 ds \,+\, C\, e^{C\,t}\, e^{C\,e^{C\,t}}. \tag{2.6}$$

Since we do not have a global bound for $\int_0^t \|\nabla \times u\|_{L^q}^2 ds$, we need to estimate $\nabla \times u$ simultaneously. Writing $\Omega = \nabla \times u$ and applying the operator $\nabla\times$ to the velocity equation in (1.3), we obtain

$$\partial_t \Omega + \kappa\Omega + 2\kappa\Delta w + u\cdot\nabla\Omega = 0.$$

Multiplying by $|\Omega|^{q-2}\Omega$ and integrating on $\mathbb{R}^2$ lead to

$$\frac{1}{q}\frac{d}{dt}\|\Omega(t)\|_{L^q}^q + \kappa\|\Omega(t)\|_{L^q}^q$$
$$= 2\kappa \int_{\mathbb{R}^2} (-\Delta w)|\Omega|^{q-2}\Omega dx \ \le 2\kappa\|\Delta w\|_{L^q}\|\Omega\|_{L^q}^{q-1},$$

where we have used the fact

$$\int_{\mathbb{R}^2} (u\cdot\nabla\Omega)\ |\Omega|^{q-2}\Omega\ dx = 0.$$

Integrating the inequality above in time and invoking (2.6), we have

$$\|\Omega(t)\|_{L^q}^2 + \kappa\int_0^t \|\Omega(\tau)\|_{L^q}^2 d\tau \le \|\Omega_0\|_{L^q}^2 + C\int_0^t \left(\|\Delta w(\tau)\|_{L^q}^2 + \|\Omega(\tau)\|_{L^q}^2\right) d\tau$$
$$\le C\, e^{C\,t}\, e^{C\, e^{C\,t}} + C\int_0^t \|\Omega(\tau)\|_{L^q}^2\, d\tau \tag{2.7}$$

where we have used the fact that, for $s > 2$,

$$\|\Omega_0\|_{L^q} \le C\,\|u_0\|_{H^s}.$$

Applying Gronwall's inequality gives

$$\sup_{0<t<\infty}\|\Omega(t)\|_{L^q}^2 + \kappa\int_0^t \|\Omega(\tau)\|_{L^q}^2 d\tau \le C\,t\,e^{C\,t}\,e^{C\,e^{C\,t}} \ \text{ for } \ 2<q<\infty.$$

This global bound, together with (2.6), yields $W^{2,q}$-bound for w

$$\|\Delta w\|_{L^2(0,t;\,L^q(\mathbb{R}^2))} \le C\,t\,e^{C\,t}\,e^{C\,e^{C\,t}} \ \text{ for } \ 2\le q<\infty. \tag{2.8}$$

This completes the proof of Proposition 2.2. □

2.3. L^∞-bound for the vorticity $\nabla\times u$

This subsection makes use of the structure of the micropolar equation to obtain a global bound for the L^∞-norm of $\Omega = \nabla\times u$. As a consequence, we obtain the global H^s-bound for u and w.

Proposition 2.4. *Assume that u_0 and w_0 satisfy the conditions in Theorem 1.1. Then the corresponding solution (u, w) admits the following global bounds, for any $0<t<\infty$,*

$$\|\Omega(t)\|_{L^\infty} \le C\,t\,e^{C\,t}\,e^{C\,e^{C\,t}}, \qquad \|u(t)\|_{H^s},\ \|w(t)\|_{H^s} \le C\,e^{e^{te^{C\,t}}e^{C\,e^{C\,t}}}.$$

Proof. We set $\mathcal{Z} = \Omega + \frac{2\kappa}{\gamma} w$. By combing the equation of Ω

$$\partial_t \Omega + u \cdot \nabla \Omega + \kappa \Omega = -2\kappa \Delta w$$

and that of w, we find

$$\partial_t \mathcal{Z} + u \cdot \nabla \mathcal{Z} + \left(\kappa - \frac{4\kappa^2}{\gamma}\right) \mathcal{Z} + \left(\frac{8\kappa^3}{\gamma^2} + \frac{6\kappa^2}{\gamma}\right) w = 0. \tag{2.9}$$

Multiplying (2.9) by $|\mathcal{Z}|^{p-2}\mathcal{Z}$ with $2 \le p < \infty$ and integrating in $\mathbb{R}^2$, we have

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\mathcal{Z}(t)\|_{L^p}^p &\le \left| \int_{\mathbb{R}^2} \left[\left(\kappa - \frac{4\kappa^2}{\gamma}\right) \mathcal{Z} + \left(\frac{8\kappa^3}{\gamma^2} + \frac{6\kappa^2}{\gamma}\right) w \right] |\mathcal{Z}|^{p-2}\mathcal{Z} dx \right| \\ &\le c_1 \|\mathcal{Z}\|_{L^p}^p + c_2 \|w\|_{L^p} \|\mathcal{Z}\|_{L^p}^{p-1}, \end{aligned} \tag{2.10}$$

where we have used

$$\int_{\mathbb{R}^2} (u \cdot \nabla \mathcal{Z})\, |\mathcal{Z}|^{p-2}\mathcal{Z}\, dx = 0$$

and set

$$c_1 = \left|\kappa - \frac{4\kappa^2}{\gamma}\right|, \quad c_2 = \left|\frac{8\kappa^3}{\gamma^2} + \frac{6\kappa^2}{\gamma}\right|.$$

We simplify (2.10) and then integrate in time to obtain

$$\|\mathcal{Z}(t)\|_{L^p} \le \|\mathcal{Z}(0)\|_{L^p} + c_1 \int_0^t \|\mathcal{Z}(\tau)\|_{L^p} d\tau + c_2 \int_0^t \|w(\tau)\|_{L^p} d\tau, \quad 2 \le p < \infty. \tag{2.11}$$

Gronwall's inequality then implies

$$\|\mathcal{Z}(t)\|_{L^p} \le e^{c_1 t} \left(\|\mathcal{Z}(0)\|_{L^p} + c_2 \int_0^t \|w(\tau)\|_{L^p} d\tau \right).$$

Since c_1 and c_2 are independent of p, we obtain by letting $p \to \infty$,

$$\|\mathcal{Z}(t)\|_{L^\infty} \le e^{c_1 t} \left(\|\mathcal{Z}(0)\|_{L^\infty} + c_2 \int_0^t \|w(\tau)\|_{L^\infty} d\tau \right).$$

By the Gagliardo–Nirenberg inequality, (2.1) and (2.2), we obtain

$$\int_0^t \|w(\tau)\|_{L^\infty} d\tau \le \int_0^t \|w(\tau)\|_{L^2}^{1/2} \|\Delta w(\tau)\|_{L^2}^{1/2} d\tau \le C\, t\, e^{C\, t}\, e^{C\, e^{C\, t}}.$$

Consequently,

$$\|\mathcal{Z}(t)\|_{L^\infty} \le C\, t\, e^{C\, t}\, e^{C\, e^{C\, t}}$$

and

$$\|\Omega(t)\|_{L^\infty} \le \|\mathcal{Z}(t)\|_{L^\infty} + \|w(t)\|_{L^\infty} \le C\, t\, e^{C\, t}\, e^{C\, e^{C\, t}}.$$

To show the global bound for (u, w) in H^s, we start with the energy inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|w\|_{H^s}^2\right) + \kappa \|u\|_{H^s}^2 + \gamma \|\nabla w\|_{H^s}^2 \\
&= 2\kappa \int_{\mathbb{R}^2} \{(\nabla \times w)\cdot(-\Delta)^s u + (\nabla \times u)(-\Delta)^s w\}\, dx \\
&\quad + \int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla] u \cdot \Lambda^s u dx + \int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla] w \cdot \Lambda^s w dx,
\end{aligned}$$

where $\Lambda^s = (-\Delta)^{s/2}$ and $[a, b]$ is the standard commutator notation, namely $[a, b] = ab - ba$. The first two terms on the right can be bounded by

$$2\kappa \int_{\mathbb{R}^2} \{(\nabla \times w)\cdot(-\Delta)^s u + (\nabla \times u)(-\Delta)^s w\}\, dx \le \frac{8\kappa^2}{\gamma}\|u\|_{H^s}^2 + \frac{\gamma}{2}\|\nabla w\|_{H^s}^2.$$

Invoking the commutator estimate

$$\|[\Lambda^s, f]g\|_{L^p} \le C\|\nabla f\|_{L^q}\, \|\Lambda^{s-1} g\|_{L^r} + C\, \|\nabla^s f\|_{L^{q_1}}\, \|g\|_{L^{r_1}}$$

where $s > 0$, $p, r, q_1 \in (1, \infty)$, $q, r_1 \in [1, \infty]$ and $1/p = 1/q + 1/r = 1/q_1 + 1/r_1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla] u \cdot \Lambda^s u dx + \int_{\mathbb{R}^2} [\Lambda^s, u\cdot\nabla] w \cdot \Lambda^s w dx \\
&\le C\, (\|\nabla u\|_{L^\infty} + \|\nabla w\|_{L^\infty}) \left(\|u\|_{H^s}^2 + \|w\|_{H^s}^2\right).
\end{aligned}$$

Combining these estimates yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|w\|_{H^s}^2\right) + 2\kappa \|u\|_{H^s}^2 + \gamma \|\nabla w\|_{H^s}^2 \\
&\le C\, (1 + \|\nabla u\|_{L^\infty} + \|\nabla w\|_{L^\infty}) \left(\|u\|_{H^s}^2 + \|w\|_{H^s}^2\right).
\end{aligned}$$

Invoking the logarithmic interpolation inequality

$$\|\nabla u\|_{L^\infty} \le C\,(1 + \|u\|_{L^2}) + \|\Omega\|_{L^\infty} \log(e + \|u\|_{H^s})$$

and applying the global bounds for the time integral of $\|\Delta w\|_{L^q}$ (as in Proposition 2.2) and for $\|\Omega\|_{L^\infty}$, we obtain the desired global H^s bound. This completes the proof of Proposition 2.4. □

With the global bounds in the previous propositions at our disposal, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is achieved via a standard procedure. One starts with the regularized micropolar equation, for small $\varepsilon > 0$,

$$\begin{cases} \partial_t u^\varepsilon - \varepsilon \Delta u^\varepsilon + \kappa u^\varepsilon - 2\kappa \nabla \times w^\varepsilon + \nabla \pi^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon = 0, \\ \nabla \cdot u^\varepsilon = 0, \\ \partial_t w^\varepsilon - \gamma \Delta w^\varepsilon + 4\kappa w^\varepsilon - 2\kappa \nabla \times u^\varepsilon + u^\varepsilon \cdot \nabla w^\varepsilon = 0, \\ (u^\varepsilon, w^\varepsilon)(x, 0) = (u_0 * \phi_\varepsilon, \;\; w_0 * \phi_\varepsilon) = (u_0^\varepsilon, w_0^\varepsilon) \end{cases} \tag{2.12}$$

where ϕ_ε is the standard mollifier, namely

$$0 \le \phi_\varepsilon(x) = \frac{1}{\varepsilon^2}\phi(\frac{|x|}{\varepsilon}), \;\; \phi \in C_0^\infty(\mathbb{R}^2), \; \mathrm{supp}\phi \subset \{x||x| < 1\}, \; \int \phi(x)dx = 1.$$

Following the lines as those in the proofs of Proposition 2.1, Proposition 2.2 and Proposition 2.4, we can establish the global bound, for any $t \in (0, \infty)$,

$$\|u^\varepsilon\|_{H^s}^2 + \|w^\varepsilon\|_{H^s}^2 + \gamma \int_0^t \|\nabla w^\varepsilon\|_{H^s}^2 \le C\, e^{e^{t e^{C\,t}} e^{C\, e^{C\,t}}}. \tag{2.13}$$

A standard compactness argument allows us to obtain the global existence of the classical solution (u, w) to (1.3). The uniqueness can be easily established. In fact, we show that any two solutions (u_1, w_1) and (u_2, w_2) to (1.3) must be the same. The difference (U, W) with $U = u_1 - u_2$ and $W = w_1 - w_2$ satisfies

$$\begin{cases} \partial_t U + \kappa U - 2\kappa \nabla \times W + \nabla \pi + U \cdot \nabla u_1 + u_2 \cdot \nabla U = 0, \\ \nabla \cdot U = 0, \\ \partial_t W - \gamma \Delta W + 4\kappa W - 2\kappa \nabla \times U + U \cdot \nabla w_1 + u_2 \cdot \nabla W = 0. \end{cases} \tag{2.14}$$

Taking the L^2 inner product of (U, W) with (2.14), we have

$$\begin{aligned} &\frac{d}{dt}\left(\|U\|_{L^2}^2 + \|W\|_{L^2}^2\right) + 2\kappa\|U\|_{L^2}^2 + 2\gamma\|\nabla W\|_{L^2}^2 + 8\kappa\|W\|_{L^2}^2 \\ &= 4\kappa \int_{\mathbb{R}^2} (\nabla \times W \cdot U + \nabla \times U W)dx \end{aligned}$$

$$
\begin{aligned}
& -2\int_{\mathbb{R}^2} (U\cdot\nabla u_1)\cdot U\,dx - 2\int_{\mathbb{R}^2}(U\cdot\nabla w_1)W\,dx\\
&\le 8\kappa\|U\|_{L^2}\|\nabla W\|_{L^2} + 2\|\nabla u_1\|_{L^\infty}\|U\|_{L^2}^2 + 2\|U\|_{L^2}\|\nabla w_1\|_{L^\infty}\|W\|_{L^2}\\
&\le \gamma\|\nabla W\|_{L^2}^2 + 2\,(\|\nabla u_1\|_{L^\infty} + \|\nabla w_1\|_{L^\infty})\,(\|U\|_{L^2}^2 + \|W\|_{L^2}^2)\\
&\le \gamma\|\nabla W\|_{L^2}^2 + 2\,(\|u_1\|_{H^s} + \|w_1\|_{H^s})\,(\|U\|_{L^2}^2 + \|W\|_{L^2}^2)\\
&\le \gamma\|\nabla W\|_{L^2}^2 + cte^{ct}e^{ce^{ct}}\,(\|U\|_{L^2}^2 + \|W\|_{L^2}^2).
\end{aligned}
$$

Gronwall's inequality then implies

$$
\|U(t)\|_{L^2}^2 + \|W(t)\|_{L^2}^2 \le \left(\|U_0\|_{L^2}^2 + \|W_0\|_{L^2}^2\right) e^{cte^{ct}e^{ce^{ct}}},
$$

which yields the uniqueness. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

This section proves Theorem 1.2, which provides large-time decay rates for $\|\nabla u(t)\|_{L^2}$, $\|\nabla w(t)\|_{L^2}^2$ and $\|u(t)\|_{L^2}$.

The following $L^p - L^q$ type estimate for the heat operator will be frequently used.

Lemma 3.1. *Let $1 \le p \le q \le \infty$. Let β be a multi-index. For any $t > 0$, the heat operator $e^{\Delta t}$ and $\partial_x^\beta e^{\Delta t}$ are bounded from L^p to L^q. Further, for any $f \in L^p(\mathbb{R}^d)$,*

$$
\|e^{\Delta t} f\|_{L^q(\mathbb{R}^d)} \le C_1\, t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\,\|f\|_{L^p(\mathbb{R}^d)}
$$

and

$$
\|\partial_x^\beta e^{\Delta t} f\|_{L^q(\mathbb{R}^d)} \le C_2\, t^{-\frac{|\beta|}{2}-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\,\|f\|_{L^p(\mathbb{R}^d)},
$$

where $C_1 = C_1(p,q)$ and $C_2 = C_2(\beta, p, q)$ are constants.

Proof of Theorem 1.2. We take the L^2-inner product of (1.6) with (u, w) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\right) + \kappa\|u(t)\|_{L^2}^2 + \gamma\|\nabla w(t)\|_{L^2}^2\\
&= \int_{\mathbb{R}^2} \{2\kappa(\nabla\times w)\cdot u + 2\kappa(\nabla\times u)w\}\,dx\\
&\le 4\kappa\|u\|_{L^2}\|\nabla w\|_{L^2} \le \frac{8\kappa^2}{\gamma+4\kappa}\|u(t)\|_{L^2}^2 + \frac{\gamma+4\kappa}{2}\|\nabla w(t)\|_{L^2}^2,
\end{aligned}
$$

which yields

$$
\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\right) + \frac{2\kappa(\gamma-4\kappa)}{\gamma+4\kappa}\|u(t)\|_{L^2}^2 + (\gamma-4\kappa)\,\|\nabla w(t)\|_{L^2}^2 \le 0.
$$

Since $\gamma > 4\kappa$, we obtain by integrating in time,

$$\int_s^t \|u(\tau)\|_{L^2}^2 d\tau \le \frac{\gamma + 4\kappa}{2\kappa(\gamma - 4\kappa)} \left(\|u(s)\|_{L^2}^2 + \|w(s)\|_{L^2}^2\right), \quad 0 \le s \le t \le \infty \tag{3.1}$$

and

$$\int_s^t \|\nabla w(\tau)\|_{L^2}^2 d\tau \le \frac{1}{\gamma - 4\kappa} \left(\|u(s)\|_{L^2}^2 + \|w(s)\|_{L^2}^2\right), \quad 0 \le s \le t \le \infty. \tag{3.2}$$

We further take the L^2-inner product of (1.6) with $(-\Delta u, -\Delta w)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right) + \kappa \|\nabla u(t)\|_{L^2}^2 + \gamma \|\Delta w(t)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^2} (2\kappa(\nabla \times w) \cdot (-\Delta u) + 2\kappa(\nabla \times u)(-\Delta w))\, dx - \int_{\mathbb{R}^2} u \cdot \nabla w\, (-\Delta w) dx \\
&= 4\kappa \int_{\mathbb{R}^2} (\nabla \times u)(-\Delta w) dx - \int_{\mathbb{R}^2} \nabla u \cdot \nabla w\, (\nabla w) dx.
\end{aligned}$$

By Young's inequality,

$$\left| 4\kappa \int_{\mathbb{R}^2} (\nabla \times u)(-\Delta w) dx \right| \le \frac{16\kappa^2}{\gamma + 12\kappa} \|\nabla u(t)\|_{L^2}^2 + \frac{\gamma + 12\kappa}{4} \|\Delta w(t)\|_{L^2}^2.$$

By Sobolev's inequality,

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \nabla u \cdot \nabla w\, (\nabla w) dx \right| &\le \|\nabla u\|_{L^2} \|\nabla w\|_{L^4}^2 \le \|\nabla u\|_{L^2} \|\nabla w\|_{L^2} \|\Delta w\|_{L^2} \\
&\le \frac{\gamma - 4\kappa}{4} \|\Delta w(t)\|_{L^2}^2 + \frac{1}{\gamma - 4\kappa} \|\nabla u\|_{L^2}^2 \|\nabla w\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right) + \frac{2\kappa(\gamma - 4\kappa)}{\gamma + 12\kappa} \|\nabla u(t)\|_{L^2}^2 + (\gamma - 4\kappa) \|\Delta w(t)\|_{L^2}^2 \\
&\le \frac{2}{\gamma - 4\kappa} \|\nabla w\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2).
\end{aligned} \tag{3.3}$$

Applying Gronwall's inequality and using (3.2), we have

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \\
&\le \exp\left\{ \frac{2}{\gamma - 4\kappa} \int_s^t \|\nabla w(\tau)\|_{L^2}^2 d\tau \right\} \left(\|\nabla u(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2 \right) \\
&\le \exp\left\{ \frac{2}{\gamma - 4\kappa} \int_0^t \|\nabla w(\tau)\|_{L^2}^2 d\tau \right\} \left(\|\nabla u(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2 \right) \\
&\le \exp\left\{ \frac{2}{(\gamma - 4\kappa)^2} (\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2) \right\} \left(\|\nabla u(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2 \right)
\end{aligned}
\tag{3.4}
$$

for $0 \le s \le t \le \infty$. Integrating (3.3) in time and applying (3.2) and (3.4), we have

$$
\begin{aligned}
&\int_0^\infty \|\nabla u(\tau)\|_{L^2}^2 d\tau \\
&\le \frac{\gamma + 12\kappa}{\kappa(\gamma - 4\kappa)^2} \int_0^\infty \|\nabla w(\tau)\|_{L^2}^2 \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2 \right) d\tau \\
&\le \frac{\gamma + 12\kappa}{\kappa(\gamma - 4\kappa)^2} \int_0^\infty \|\nabla w(\tau)\|_{L^2}^2 d\tau \\
&\quad \times \exp\left\{ \frac{2}{(\gamma - 4\kappa)^2} (\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2) \right\} \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 \right) \\
&\le \frac{\gamma + 12\kappa}{\kappa(\gamma - 4\kappa)^3} \left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 \right) \\
&\quad \times \exp\left\{ \frac{2}{(\gamma - 4\kappa)^2} (\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2) \right\} \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 \right).
\end{aligned}
\tag{3.5}
$$

Thus combining (3.2) and (3.5) gives

$$
\begin{aligned}
&\int_0^\infty \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2 \right) d\tau \le \left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 \right) \\
&\times \left\{ \frac{\gamma + 12\kappa}{\kappa(\gamma - 4\kappa)^3} \exp\left\{ \frac{2(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2)}{(\gamma - 4\kappa)^2} \right\} \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 \right) + \frac{1}{\gamma - 4\kappa} \right\}.
\end{aligned}
$$

A special consequence is that

$$\int_{\frac{t}{2}}^{t} \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2\right) d\tau \to 0 \quad \text{as} \quad t \to \infty. \tag{3.6}$$

Furthermore, by (3.4),

$$\frac{t}{2} \exp\left\{\frac{-2}{(\gamma - 4\kappa)^2}(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2)\right\} \left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right)$$
$$\le \int_{\frac{t}{2}}^{t} \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2\right) d\tau,$$

which yields the desired decay rate

$$(1+t)\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.7}$$

We now make use of this decay rate in (3.7) to derive the decay rate for $\|u(t)\|^2$. Taking the L^2-inner product of the velocity equation in (1.6) with u gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \kappa\|u(t)\|_{L^2}^2 = \int_{\mathbb{R}^2} 2\kappa(\nabla \times w)\cdot u dx \le 2\kappa\|u\|_{L^2}\|\nabla w\|_{L^2}.$$

Integrating in time yields

$$\begin{aligned}
\|u(t)\|_{L^2}^2 &\le e^{-2\kappa t}\|u_0\|_{L^2}^2 + 4\kappa\int_0^t e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2} ds \\
&= e^{-2\kappa t}\|u_0\|_{L^2}^2 + 4\kappa\int_0^{\frac{t}{2}} e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2} ds \\
&\quad + 4\kappa\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2} ds.
\end{aligned} \tag{3.8}$$

The first time integral decays exponentially in time, more precisely,

$$4\kappa\int_0^{\frac{t}{2}} e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2} ds$$
$$\le 4\kappa e^{-\kappa t}\int_0^{\frac{t}{2}} \|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2} ds$$

$$\leq 4\kappa e^{-\kappa t}\left(\int_0^{\frac{t}{2}}\|\nabla w(s)\|_{L^2}^2 ds\right)^{\frac{1}{2}}\left(\int_0^{\frac{t}{2}}\|u(s)\|_{L^2}^2 ds\right)^{\frac{1}{2}}$$

$$\leq C\sqrt{\frac{\gamma+4\kappa}{2\kappa(\gamma-4\kappa)^2}}\left(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2\right)e^{-\kappa t}\leq C\,e^{-\kappa t},$$

where C depends on κ, γ and $\|(u_0, w_0)\|_{L^2}$. (3.8) is then reduced to

$$\|u(t)\|_{L^2}^2 \leq C\,e^{-\kappa t} + 2\kappa\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2} ds. \tag{3.9}$$

Multiplying (3.9) by $(1+t)$ yields

$$(1+t)\|u(t)\|_{L^2}^2 \leq C\,(1+t)e^{-\kappa t}$$
$$+\,2\kappa(1+t)\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}(1+s)^{-1}\left((1+s)^{\frac{1}{2}}\|u(s)\|_{L^2}(1+s)^{\frac{1}{2}}\|\nabla w(s)\|_{L^2}\right)ds,$$

which would allow us to show that

$$(1+t)^{\frac{1}{2}}\,\|u(t)\|_{L^2}\to 0 \quad \text{as} \quad t\to\infty. \tag{3.10}$$

We first show that $(1+t)^{\frac{1}{2}}\,\|u(t)\|_{L^2}\leq C$ for all $t\geq 0$ and then show (3.10). Writing

$$\mathcal{M}(t) = \sup_{0\leq s\leq t}\left\{(1+s)^{\frac{1}{2}}\|u(s)\|_{L^2}\right\}$$

and using the uniform bounds $(1+t)e^{-\kappa t}\leq C$ and $(1+t)^{\frac{1}{2}}\|\nabla w(t)\|_{L^2}\leq C$, we have

$$\mathcal{M}^2(t)\leq C\,+2\kappa(1+t)\mathcal{M}(t)\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}(1+s)^{-1}ds$$
$$\leq C\,+C\,\mathcal{M}(t)\leq \frac{1}{2}\mathcal{M}^2(t)+C,$$

which implies the uniform bound $\mathcal{M}(t)\leq C$ for all $t\geq 0$. We then show (3.10). If we use this uniform bound, we have

$$(1+t)\|u(t)\|_{L^2}^2\leq C\,(1+t)e^{-\kappa t}+2\kappa(1+t)\mathcal{M}(t)\,\mathcal{N}(t)\int_{\frac{t}{2}}^{t} e^{-2\kappa(t-s)}(1+s)^{-1}ds$$
$$\leq C\,(1+t)e^{-\kappa t}+C\,\mathcal{N}(t)\to 0 \quad \text{as } t\to\infty,$$

where we have used the fact (see (3.7)) that

$$\mathcal{N}(t) \equiv \sup_{\frac{t}{2}\le s\le t} \left\{(1+t)^{\frac{1}{2}}\|\nabla w(t)\|_{L^2}\right\} \to 0 \quad \text{as } t\to\infty.$$

This completes the proof of Theorem 1.2. □

4. Proofs of Proposition 1.3 and Theorem 1.4

This section is devoted to the proofs of Proposition 1.3 and Theorem 1.4. We first explain why it is hard to derive a decay rate for $\|w(t)\|_{L^2}$ via the standard approach. In fact, if we write the equation of w in (1.6) as

$$w(t) = e^{\gamma t\Delta} w_0 + 2\kappa \int_0^t e^{\gamma(t-s)\Delta}\nabla\times u(s)\,ds - \int_0^t e^{\gamma(t-s)\Delta}(u\cdot\nabla w)(s)\,ds$$

and estimate $\|w(t)\|_{L^2}$ directly, we would have trouble extracting an explicit decay rate for the linear term

$$\int_0^t e^{\gamma(t-s)\Delta}\nabla\times u(s)\,ds$$

if we only know that $(1+t)^{\frac{1}{2}}\,\|\nabla\times u(t)\|_{L^2}\to 0$ as $t\to\infty$. In order to obtain a decay rate for $\|w(t)\|_{L^2}$, we need to diagonalize the system in (1.8), namely

$$\begin{cases}\partial_t\Omega + \kappa\Omega + u\cdot\nabla\Omega + 2\kappa\Delta w = 0,\\ \partial_t w - \gamma\Delta w - 2\kappa\nabla\times u + u\cdot\nabla w = 0\end{cases} \tag{4.1}$$

to remove the linear part. The process appears to be complex, but it offers a general framework for handling similar and more general situations. The details are in the proof of Proposition 1.3.

Proof of Proposition 1.3. We rewrite (4.1) in the Fourier space as

$$\begin{bmatrix}\partial_t\widehat{\Omega}(\xi)\\ \partial_t\widehat{w}(\xi)\end{bmatrix} = \begin{bmatrix}-\kappa & 2\kappa|\xi|^2\\ 2\kappa & -\gamma|\xi|^2\end{bmatrix}\begin{bmatrix}\widehat{\Omega}(\xi)\\ \widehat{w}(\xi)\end{bmatrix} + \begin{bmatrix}-\widehat{u\cdot\nabla\Omega}(\xi)\\ -\widehat{u\cdot\nabla w}(\xi)\end{bmatrix}, \tag{4.2}$$

where we have suppressed the t-variable for notational brevity. The corresponding characteristic equation

$$\lambda^2 + (\kappa+\gamma|\xi|^2)\lambda + (\kappa\gamma - 4\kappa^2)|\xi|^2 = 0,$$

whose roots are given by

$$\lambda_{1,2} = \frac{-(\kappa + \gamma|\xi|^2) \pm \sqrt{(\kappa + \gamma|\xi|^2)^2 - 4(\kappa\gamma - 4\kappa^2)|\xi|^2}}{2}$$
$$= \frac{-(\kappa + \gamma|\xi|^2) \pm \sqrt{(\kappa - \gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}}{2}.$$

λ_1 denotes the one with the negative sign in the front of the square root sign. It is easy to check that, if $\gamma > 4\kappa$,

$$\lambda_{1,2} < 0.$$

Otherwise one of the eigenvalues may be zero or positive. The associated eigenvectors are given by

$$\mathbf{f} = \begin{bmatrix} 2\kappa|\xi|^2 \\ \lambda_1 + \kappa \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 2\kappa|\xi|^2 \\ \lambda_2 + \kappa \end{bmatrix}.$$

f and **g** are independent and

$$\begin{bmatrix} -\kappa & 2\kappa|\xi|^2 \\ 2\kappa & -\gamma|\xi|^2 \end{bmatrix} [\mathbf{f}\ \ \mathbf{g}] = [\mathbf{f}\ \ \mathbf{g}] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

The inverse of the matrix $[\mathbf{f}\ \ \mathbf{g}]$ is given by

$$[\mathbf{f}\ \ \mathbf{g}]^{-1} = \frac{1}{2\kappa|\xi|^2\sqrt{(\kappa - \gamma|\xi|^2)^2 + 16\kappa^2|\xi|^2}} \begin{bmatrix} \lambda_2 + \kappa & -2\kappa|\xi|^2 \\ -(\lambda_1 + \kappa) & 2\kappa|\xi|^2 \end{bmatrix}. \tag{4.3}$$

If we define

$$\begin{bmatrix} \widehat{A} \\ \widehat{B} \end{bmatrix} = [\mathbf{f}\ \ \mathbf{g}]^{-1} \begin{bmatrix} \widehat{\Omega} \\ \widehat{w} \end{bmatrix} \tag{4.4}$$

Then $\widehat{A}$ and $\widehat{B}$ satisfy

$$\begin{bmatrix} \partial_t \widehat{A}(\xi) \\ \partial_t \widehat{B}(\xi) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \widehat{A}(\xi) \\ \widehat{B}(\xi) \end{bmatrix} + [\mathbf{f}\ \ \mathbf{g}]^{-1} \begin{bmatrix} -\widehat{u \cdot \nabla\Omega}(\xi) \\ -\widehat{u \cdot \nabla w}(\xi) \end{bmatrix}.$$

Invoking (4.3), we have

$$\partial_t \widehat{A}(\xi) = \lambda_1\ \widehat{A}(\xi) + \widehat{F_1}(\xi),$$
$$\partial_t \widehat{B}(\xi) = \lambda_2\ \widehat{B}(\xi) + \widehat{F_2}(\xi), \tag{4.5}$$

where

$$\widehat{F}_1(\xi) = \frac{-(\lambda_2+\kappa)\widehat{u\cdot\nabla\Omega}(\xi) + 2\kappa|\xi|^2\,\widehat{u\cdot\nabla w}(\xi)}{2\kappa|\xi|^2\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}},$$

$$\widehat{F}_2(\xi) = \frac{(\lambda_1+\kappa)\widehat{u\cdot\nabla\Omega}(\xi) - 2\kappa|\xi|^2\,\widehat{u\cdot\nabla w}(\xi)}{2\kappa|\xi|^2\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}}.$$

We further write (4.5) in the integral form,

$$\widehat{A}(\xi,t) = e^{\lambda_1(\xi)t}\widehat{A}(\xi,0) + \int_0^t e^{\lambda_1(\xi)(t-\tau)}\,\widehat{F}_1(\xi,\tau)\,d\tau, \tag{4.6}$$

$$\widehat{B}(\xi,t) = e^{\lambda_2(\xi)t}\widehat{B}(\xi,0) + \int_0^t e^{\lambda_2(\xi)(t-\tau)}\,\widehat{F}_2(\xi,\tau)\,d\tau, \tag{4.7}$$

where, according to (4.4),

$$\widehat{A}(\xi,0) = \frac{(\lambda_2+\kappa)\widehat{\Omega}_0(\xi) - 2\kappa|\xi|^2\,\widehat{w}_0(\xi)}{2\kappa|\xi|^2\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}},$$

$$\widehat{B}(\xi,0) = \frac{-(\lambda_1+\kappa)\widehat{\Omega}_0(\xi) + 2\kappa|\xi|^2\,\widehat{w}_0(\xi)}{2\kappa|\xi|^2\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}}.$$

We can then find $\widehat{\Omega}$ and $\widehat{w}$ via (4.4),

$$\begin{bmatrix}\widehat{\Omega}\\ \widehat{w}\end{bmatrix} = [\mathbf{f}\ \ \mathbf{g}]\begin{bmatrix}\widehat{A}\\ \widehat{B}\end{bmatrix} = \begin{bmatrix}2\kappa|\xi|^2(\widehat{A}+\widehat{B})\\ (\lambda_1+\kappa)\widehat{A} + (\lambda_2+\kappa)\widehat{B}\end{bmatrix}. \tag{4.8}$$

To obtain a more explicit representation for $\widehat{\Omega}$ and $\widehat{w}$, we give more explicit representations of $\lambda_1+\kappa$, $\lambda_2+\kappa$, $\widehat{A}(\xi,0)$, $\widehat{B}(\xi,0)$, $\widehat{F}_1(\xi)$ and $\widehat{F}_2(\xi)$,

$$\lambda_1+\kappa = \frac{\kappa-\gamma|\xi|^2}{2} - \frac{1}{2}\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2},$$

$$\lambda_2+\kappa = \frac{\kappa-\gamma|\xi|^2}{2} + \frac{1}{2}\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2},$$

$$\widehat{A}(\xi,0) = \frac{1}{4\kappa|\xi|^2}\left(\frac{\kappa-\gamma|\xi|^2}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}} + 1\right)\widehat{\Omega}_0(\xi) - \frac{\widehat{w}_0(\xi)}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}},$$

$$\widehat{B}(\xi,0) = \frac{-1}{4\kappa|\xi|^2}\left(\frac{\kappa-\gamma|\xi|^2}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}} - 1\right)\widehat{\Omega}_0(\xi) + \frac{\widehat{w}_0(\xi)}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}},$$

$$\widehat{F}_1(\xi) = \frac{-1}{4\kappa|\xi|^2}\left(\frac{\kappa-\gamma|\xi|^2}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}}+1\right)\widehat{u\cdot\nabla\Omega}(\xi)$$
$$+\frac{\widehat{u\cdot\nabla w}(\xi)}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}},$$

$$\widehat{F}_2(\xi) = \frac{1}{4\kappa|\xi|^2}\left(\frac{\kappa-\gamma|\xi|^2}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}}-1\right)\widehat{u\cdot\nabla\Omega}(\xi)$$
$$-\frac{\widehat{u\cdot\nabla w}(\xi)}{\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}}.$$

Combining the formulas above and (4.6), (4.7) and (4.8), we are then led to the desired representation in Proposition 1.3. This completes the proof of Proposition 1.3. □

We now prove Theorem 1.4.

Proof of Theorem 1.4. We now use (1.12) in Proposition 1.3 to extract the decay rate for $\|w(t)\|_{L^2}$. To do so, we note the following uniform bounds,

$$|D_1(\xi)| \le 1, \quad |D_3(\xi)| \le 1, \quad |D_4(\xi)| \le C(\kappa,\gamma),$$

where $C(\kappa,\gamma)$ is a constant depending on κ and γ only. The bound for D_1 and D_3 is obvious and the bound for D_4 can be seen as follows,

$$D_4(\xi) \le \begin{cases} 4, & \text{if } \gamma|\xi|^2 \le \kappa/2, \\ \sqrt{\gamma/(2\kappa)}, & \text{if } \gamma|\xi|^2 > \kappa/2. \end{cases}$$

We now estimate the terms in the representation of $\widehat{w}$ in (1.12). We start with the first term. Using the fact that

$$\lambda_1 \le -\frac{\kappa+\gamma|\xi|^2}{2},$$

we have

$$\left\| e^{\lambda_1(\xi)t}\left(-D_4(\xi)\,\widehat{\Omega}_0(\xi)+D_3(\xi)\,\widehat{w}_0(\xi)\right)\right\|_{L^2}$$
$$\le C(\kappa,\gamma)\, e^{-\frac{\kappa}{2}t}\left\| e^{-\frac{|\xi|^2}{2}t}\left(|\widehat{\Omega}_0(\xi)|+|\widehat{w}_0(\xi)|\right)\right\|_{L^2}$$
$$\le C(\kappa,\gamma)\,(\|u_0\|_{L^2}+\|w_0\|_{L^2})\, e^{-\frac{\kappa}{2}t},$$

where we have used the fact that

$$|\widehat{\Omega}_0(\xi)| \le |\xi|\,|\widehat{u}_0(\xi)|.$$

To estimate the second term in (1.12), we note that

$$\begin{aligned}\lambda_2(\xi) &= \frac{-(\kappa+\gamma|\xi|^2)+\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}}{2} \\ &= \frac{2\kappa\,(4\kappa-\gamma)\,|\xi|^2}{\kappa+\gamma|\xi|^2+\sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}} \\ &\le \frac{\kappa\,(4\kappa-\gamma)\,|\xi|^2}{\kappa+\gamma|\xi|^2},\end{aligned}$$

where we have used the following facts in the last inequality,

$$4\kappa<\gamma, \qquad \kappa+\gamma|\xi|^2 \ge \sqrt{(\kappa-\gamma|\xi|^2)^2+16\kappa^2|\xi|^2}.$$

Therefore, for small $|\xi|$, λ_2 behaves like the heat operator, and for large $|\xi|$, it behaves like the exponential decay operator. More precisely,

$$\lambda_2(\xi) \le \begin{cases} \frac{1}{2}(4\kappa-\gamma)|\xi|^2, & \text{if } \gamma|\xi|^2\le\kappa, \\ \frac{\kappa(4\kappa-\gamma)}{2\gamma}, & \text{if } \gamma|\xi|^2>\kappa. \end{cases} \tag{4.9}$$

Therefore,

$$\begin{aligned}&\|e^{\lambda_2(\xi)t}\,D_4(\xi)\,\widehat{\Omega}_0(\xi)\|_{L^2} \\ &\le \|e^{\lambda_2(\xi)t}\,\widehat{\Omega}_0(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2\le\kappa\})} + \|e^{\lambda_2(\xi)t}\,\widehat{\Omega}_0(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})} \\ &\le \left\|e^{\frac{1}{2}(4\kappa-\gamma)|\xi|^2\,t}|\xi|\,|\widehat{u}_0(\xi)|\right\|_{L^2(\{\xi:\,\gamma|\xi|^2\le\kappa\})} + \left\|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,t}|\widehat{\Omega}_0(\xi)|\right\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})} \\ &\le C\,t^{-\frac{1}{2}}\,\|u_0\|_{L^2} + C\,e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,t}\,\|\Omega_0\|_{L^2} \le C\,t^{-\frac{1}{2}}\,\|u_0\|_{H^1}\end{aligned}$$

for $t\ge 1$. We can bound $\|e^{\lambda_2(\xi)t}\,D_1(\xi)\,\widehat{w}_0(\xi)\|_{L^2}$ similarly. In fact,

$$\begin{aligned}&\|e^{\lambda_2(\xi)t}\,D_1(\xi)\,\widehat{w}_0\|_{L^2} \\ &\le \|e^{\lambda_2(\xi)t}\,\widehat{w}_0(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2\le\kappa\})} + \|e^{\lambda_2(\xi)t}\,\widehat{w}_0(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})} \\ &\le \left\|e^{\frac{1}{2}(4\kappa-\gamma)|\xi|^2\,t}|\widehat{w}_0(\xi)|\right\|_{L^2(\{\xi:\,\gamma|\xi|^2\le\kappa\})} + \left\|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,t}|\widehat{w}_0(\xi)|\right\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})}\end{aligned}$$

for $t\ge 1$. When the initial data $w_0\in L^1(\mathbb{R}^2)$, or more generally, w_0 satisfies, for $t>0$,

$$\|e^{\Delta t}w_0\|_{L^2}\le C\,t^{-\frac{1}{2}}, \tag{4.10}$$

we can easily show that

$$\|e^{\lambda_2(\xi)t}\,D_1(\xi)\,\widehat{w}_0(\xi)\|_{L^2}\le C\,t^{-\frac{1}{2}}.$$

We now turn to the third term in (1.12). Splitting the time integral into two parts, we have

$$
\begin{aligned}
&\left\| \int_0^t e^{\lambda_1(\xi)(t-\tau)}\, D_4(\xi)\, \widehat{u\cdot\nabla\Omega}(\xi)\, d\tau \right\|_{L^2} \\
&\le \int_0^t \left\| e^{\lambda_1(\xi)(t-\tau)}\, \widehat{u\cdot\nabla\Omega}(\xi,\tau) \right\|_{L^2} d\tau \\
&\le \int_0^t \left\| e^{-\frac{1}{2}(\kappa+\gamma|\xi|^2)(t-\tau)}\, \widehat{u\cdot\nabla\Omega}(\xi,\tau) \right\|_{L^2} d\tau \\
&= \int_0^{\frac{t}{2}} \cdots + \int_{\frac{t}{2}}^t \cdots .
\end{aligned} \tag{4.11}
$$

The first time integral decays exponentially. In fact, by Lemma 3.1,

$$
\begin{aligned}
&\int_0^{\frac{t}{2}} \left\| e^{-\frac{1}{2}(\kappa+\gamma|\xi|^2)(t-\tau)}\, \widehat{u\cdot\nabla\Omega}(\xi,\tau) \right\|_{L^2} d\tau \\
&\le e^{-\frac{1}{4}\kappa t} \int_0^{\frac{t}{2}} \left\| e^{-\frac{1}{2}\gamma|\xi|^2(t-\tau)}\, \widehat{u\cdot\nabla\Omega}(\xi,\tau) \right\|_{L^2} d\tau \\
&\le C\, e^{-\frac{1}{4}\kappa t} \int_0^{\frac{t}{2}} (t-\tau)^{-1} \|u(\tau)\|_{L^2}\, \|\Omega(\tau)\|_{L^2}\, d\tau \\
&\le C\, e^{-\frac{1}{4}\kappa t}\, (1+t)^{-1} \ln(1+t),
\end{aligned} \tag{4.12}
$$

where we have used the uniform bounds

$$
(1+t)^{\frac{1}{2}} \|u(t)\|_{L^2} \le C \quad \text{and} \quad (1+t)^{\frac{1}{2}} \|\Omega(t)\|_{L^2} \le C.
$$

We now estimate the second time integral in (4.11). For any $\epsilon > 0$ small, by Lemma 3.1,

$$
\begin{aligned}
&\int_{\frac{t}{2}}^t e^{-\frac{\kappa}{2}(t-\tau)} \left\| e^{-\frac{1}{2}\gamma|\xi|^2(t-\tau)} \widehat{u\cdot\nabla\Omega}(\xi,\tau) \right\|_{L^2} d\tau \\
&\le C \int_{\frac{t}{2}}^t e^{-\frac{\kappa}{2}(t-\tau)}\, (t-\tau)^{-1+\epsilon}\, \|u(\tau)\,\Omega(\tau)\|_{L^{\frac{1}{1-\epsilon}}}\, d\tau
\end{aligned}
$$

$$\leq C \int_{\frac{t}{2}}^{t} e^{-\frac{\kappa}{2}(t-\tau)}\,(t-\tau)^{-1+\epsilon}\,\|\Omega(\tau)\|_{L^2}^{1+2\epsilon}\,\|u(\tau)\|_{L^2}^{1-2\epsilon}\,d\tau$$

$$\leq C\,t^{-1+\epsilon}, \tag{4.13}$$

where $C = C(\epsilon, \kappa, \gamma)$ is a constant. The estimate for the part

$$\int_0^t e^{\lambda_1(\xi)(t-\tau)}\,D_3(\xi)\widehat{u\cdot\nabla w}(\xi)\,d\tau$$

is very similar, although the decay rate is not as fast. As in (4.12), we have

$$\int_0^{\frac{t}{2}} \|e^{\lambda_1(\xi)(t-\tau)}\,D_3(\xi)\widehat{u\cdot\nabla w}(\xi)\|_{L^2}\,d\tau \leq C\,e^{-\frac{1}{4}\kappa t}\,(1+t)^{-\frac{1}{2}}.$$

As in (4.13),

$$\int_{\frac{t}{2}}^{t} \|e^{\lambda_1(\xi)(t-\tau)}\,D_3(\xi)\widehat{u\cdot\nabla w}(\xi)\|_{L^2}\,d\tau$$

$$\leq C \int_{\frac{t}{2}}^{t} e^{-\frac{\kappa}{2}(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,\|u(\tau)\|_{L^2}\,\|\nabla w(\tau)\|_{L^2}\,d\tau$$

$$\leq C \int_{\frac{t}{2}}^{t} e^{-\frac{\kappa}{2}(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-1}\,d\tau$$

$$\leq C\,t^{-\frac{1}{2}} \quad \text{for } t \geq 1.$$

We now estimate the last term in (1.12). We start with

$$\int_0^t \|e^{\lambda_2(\xi)(t-\tau)}\,D_4(\xi)\,\widehat{u\cdot\nabla\Omega}(\xi)\|_{L^2}\,d\tau.$$

Realizing the bound for λ_2 in (4.9), we divide the integral above into two parts,

$$\int_0^t \|e^{\lambda_2(\xi)(t-\tau)}\, D_4(\xi)\, \widehat{u\cdot\nabla\Omega}(\xi)\|_{L^2}\, d\tau$$
$$= \int_0^t \|e^{\lambda_2(\xi)(t-\tau)}\, D_4(\xi)\, \widehat{u\cdot\nabla\Omega}(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2\le\kappa\})}\, d\tau$$
$$+ \int_0^t \|e^{\lambda_2(\xi)(t-\tau)}\, D_4(\xi)\, \widehat{u\cdot\nabla\Omega}(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})}\, d\tau$$
$$\le \int_0^t \|e^{\frac{1}{2}(4\kappa-\gamma)|\xi|^2\,(t-\tau)}\widehat{u\cdot\nabla\Omega}(\xi)\|_{L^2}\, d\tau$$
$$+ \int_0^t \|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,(t-\tau)}\, D_4(\xi)|\xi|\,\widehat{u\,\Omega}(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})}\, d\tau. \tag{4.14}$$

As in (4.12) and (4.13), we have, for any $\epsilon > 0$ small,

$$\int_0^t \|e^{\frac{1}{2}(4\kappa-\gamma)|\xi|^2\,(t-\tau)}\widehat{u\cdot\nabla\Omega}(\xi)\|_{L^2}\, d\tau \le C\,(1+t)^{-1}\,\ln(1+t) + C(\epsilon,\kappa,\gamma)\,(1+t)^{-1+\epsilon}.$$

To estimate the second part, we invoke the bound, for $\gamma|\xi|^2 > \kappa$,

$$D_4(\xi)\,|\xi|^2 \le C(\kappa,\gamma).$$

For any $\epsilon > 0$ small, by the Hardy–Littlewood–Sobolev inequality,

$$\int_0^t \|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,(t-\tau)}\, D_4(\xi)|\xi|^{2-2\epsilon}\,|\xi|^{-1+2\epsilon}\,\widehat{u\,\Omega}(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})}\, d\tau$$
$$\le C\int_0^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,(t-\tau)}\,\||\xi|^{-1+2\epsilon}\,\widehat{u\,\Omega}(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})}\, d\tau$$
$$\le C\int_0^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,(t-\tau)}\,\|u(\tau)\,\Omega(\tau)\|_{L^{\frac{1}{1-\epsilon}}}\, d\tau$$
$$\le C\int_0^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\,(t-\tau)}\,\|u(\tau)\|_{L^2}^{1-2\epsilon}\,\|\Omega(\tau)\|_{L^2}^{1+2\epsilon}\, d\tau$$

$$
\begin{aligned}
&\le C\, e^{\frac{\kappa(4\kappa-\gamma)}{4\gamma}\, t} \int_0^{\frac{t}{2}} \|u(\tau)\|_{L^2}^{1-2\epsilon}\, \|\Omega(\tau)\|_{L^2}^{1+2\epsilon}\, d\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\, (t-\tau)}\, \tau^{-1}\, d\tau \\
&\le C\, e^{\frac{\kappa(4\kappa-\gamma)}{4\gamma}\, t} + C\, t^{-1},
\end{aligned}
\tag{4.15}
$$

where we have used the facts that

$$
\begin{aligned}
&4\kappa < \gamma, \quad t^{\frac{1}{2}}\, \|u(t)\|_{L^2} \le C, \quad t^{\frac{1}{2}}\, \|\Omega(t)\|_{L^2} \le C, \\
&\int_0^{\infty} \|u(t)\|_{L^2}^2\, dt \le C, \quad \int_0^{\infty} \|\Omega(t)\|_{L^2}^2\, dt \le C.
\end{aligned}
$$

The L^2-norm of the term

$$
\int_0^t e^{\lambda_2(\xi)(t-\tau)}\, D_1(\xi) \widehat{u \cdot \nabla w}(\xi)\, d\tau
$$

can be similarly bounded. As in (4.14), we have

$$
\begin{aligned}
&\int_0^t \|e^{\lambda_2(\xi)(t-\tau)} D_1(\xi) \widehat{u \cdot \nabla w}(\xi)\|_{L^2}\, d\tau \\
&\le \int_0^t \|e^{\frac{1}{2}(4\kappa-\gamma)|\xi|^2\, (t-\tau)} \widehat{u \cdot \nabla w}(\xi)\|_{L^2}\, d\tau \\
&\quad + \int_0^t \|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\, (t-\tau)}\, D_1(\xi)\, \widehat{u \cdot \nabla w}(\xi)\|_{L^2(\{\xi:\, \gamma|\xi|^2>\kappa\})}\, d\tau.
\end{aligned}
$$

As in (4.12) and (4.13),

$$
\int_0^t \|e^{\frac{1}{2}(4\kappa-\gamma)|\xi|^2\, (t-\tau)} \widehat{u \cdot \nabla w}(\xi)\|_{L^2}\, d\tau \le C\, (1+t)^{-\frac{1}{2}}.
$$

Finally we show that, for any small $\epsilon > 0$,

$$
\int_0^t \|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}\, (t-\tau)}\, D_1(\xi)\, \widehat{u \cdot \nabla w}(\xi)\|_{L^2(\{\xi:\, \gamma|\xi|^2>\kappa\})}\, d\tau \le C\, t^{-1+\epsilon}.
\tag{4.16}
$$

Due to the fact that $|D_1| \le 1$ and by Plancherel's theorem,

$$\int_0^t \|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}(t-\tau)} D_1(\xi)\,\widehat{u\cdot\nabla w}(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})}\,d\tau$$
$$\le \int_0^t \|e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}(t-\tau)}\,\widehat{u\cdot\nabla w}(\xi)\|_{L^2(\{\xi:\,\gamma|\xi|^2>\kappa\})}\,d\tau$$
$$\le \int_0^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}(t-\tau)}\,\|u\cdot\nabla w\|_{L^2}\,d\tau.$$

We choose $r_1 = \frac{1}{\epsilon}$ and $r_2 = \frac{2}{1-2\epsilon}$. Clearly $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{2}$. By Hölder's inequality and Sobolev's inequality,

$$\int_0^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}(t-\tau)}\,\|u\cdot\nabla w\|_{L^2}\,d\tau$$
$$\le \int_0^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}(t-\tau)}\,\|u\|_{L^{r_1}}\,\|\nabla w\|_{L^{r_2}}\,d\tau$$
$$\le C\int_0^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}(t-\tau)}\,\|u\|_{L^2}^{\frac{2}{r_1}}\,\|\nabla u\|_{L^2}^{1-\frac{2}{r_1}}\,\|\nabla w\|_{L^2}^{\frac{2}{r_2}}\,\|\Delta w\|_{L^2}^{1-\frac{2}{r_2}}\,d\tau$$
$$\le C\,e^{\frac{\kappa(4\kappa-\gamma)}{4\gamma}t}\int_0^{\frac{t}{2}} \|u\|_{L^2}^{\frac{2}{r_1}}\,\|\nabla u\|_{L^2}^{1-\frac{2}{r_1}}\,\|\nabla w\|_{L^2}^{\frac{2}{r_1}}\,\|\Delta w\|_{L^2}^{1-\frac{2}{r_2}}\,d\tau$$
$$+\,C\int_{\frac{t}{2}}^t e^{\frac{\kappa(4\kappa-\gamma)}{2\gamma}(t-\tau)}\,\tau^{-\frac{1}{2}}\,\tau^{-\frac{1}{r_2}}\,\|\Delta w(\tau)\|_{L^2}^{1-\frac{2}{r_2}}\,d\tau$$
$$\le C\,e^{\frac{\kappa(4\kappa-\gamma)}{4\gamma}t} + C\,t^{-1+\epsilon},$$

which verifies (4.16). Here we have used the facts that

$$4\kappa<\gamma,\quad t^{\frac{1}{2}}\|u(t)\|_{L^2}\le C,\quad t^{\frac{1}{2}}\|\nabla u(t)\|_{L^2}\le C,$$
$$t^{\frac{1}{2}}\|\nabla w(t)\|_{L^2}\le C,\quad \int_0^\infty \|\Delta w(t)\|_{L^2}^2\,dt\le C.$$

Collecting the estimates for the terms in (1.12), we conclude that, for $t\ge 1$,

$$\|w(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}$$

if the initial data w_0 satisfies $w_0 \in L^1$ or more generally (1.14). This completes the proof of Theorem 1.4. □

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Appendix A. The micropolar equations with full dissipation

This appendix includes the global existence and uniqueness results for the micropolar equations with full dissipation. For the 3D equations, any small initial data $u_0 \in \mathring{H}^{\frac{1}{2}}$ and $w_0 \in \mathring{H}^{\frac{1}{2}}$ lead to a unique global solution. The solution in this functional setting is stable. In the 2D case, any $u_0 \in L^2$ and $w_0 \in L^2$ generate a unique global solution. The well-posedness result for the 3D micropolar equation extends the work of Fujita and Kato on the 3D Navier–Stokes equations to a nonlinearly coupled system. The result for the 2D micropolar equation involves the weakest initial data for which one can still deduce the uniqueness.

We start with the 3D result.

Theorem A.1. *Consider the 3D micropolar equations (1.1) with $\nu > 0$, $\kappa > 0$ and $\gamma > 0$. Assume that $u_0 \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $w_0 \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^3)$. The following results hold:*

(a) *There exists a constant $C_0 > 0$ such that, if*

$$\|(u_0, w_0)\|_{\mathring{H}^{\frac{1}{2}}} \le C_0 \min\{\nu, \gamma\}, \tag{A.1}$$

then (1.1) has a unique global solution (u, w) satisfying, for any $T > 0$,

$$u, w \in C([0,T]; \mathring{H}^{\frac{1}{2}}) \cap L^2([0,T]; \mathring{H}^{\frac{3}{2}}). \tag{A.2}$$

As a special consequence, $u, w \in L^q([0,T]; \mathring{H}^{\frac{1}{2}+\frac{2}{q}})$ with

$$\|(u,w)\|_{L^q(0,T;\mathring{H}^{\frac{1}{2}+\frac{2}{q}})} \le \|(u,w)\|^{(1-\frac{2}{q})}_{L^\infty(0,T;\mathring{H}^{\frac{1}{2}})} \|(u,w)\|^{\frac{2}{q}}_{L^2(0,T;\mathring{H}^{\frac{3}{2}})} \tag{A.3}$$

(b) *The solution given by (a) is stable in the sense that any two solutions $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$ obey the estimate*

$$
\begin{aligned}
&\|(u^{(1)}(t), w^{(1)}(t)) - (u^{(2)}(t), w^{(2)}(t))\|^2_{\mathring{H}^{\frac{1}{2}}} \\
&+ (\kappa/2+\nu)\int_0^t \|u^{(1)}(\tau) - u^{(2)}(\tau)\|^2_{\mathring{H}^{\frac{3}{2}}}\, d\tau + \gamma \int_0^t \|w^{(1)}(\tau) - w^{(2)}(\tau)\|^2_{\mathring{H}^{\frac{3}{2}}}\, d\tau \\
&\quad \le \|(u^{(1)}(0), w^{(1)}(0)) - (u^{(2)}(0), w^{(2)}(0))\|^2_{\mathring{H}^{\frac{1}{2}}}\, e^{C\int_0^t (1+\|\nabla u^{(1)}\|^4_{L^2} + \|\nabla w^{(1)}\|^4_{L^2})\, d\tau}.
\end{aligned}
\tag{A.4}
$$

Note that, due to (A.3) with $q = 4$, the right-hand side of (A.4) is bounded.

Proof. For the sake of conciseness, we shall just provide the key component of the proof for (a), namely the global *a priori* bound for $\|(u,w)\|_{\mathring{H}^{\frac{1}{2}}}$. To obtain the desired global bound, we perform the energy estimate to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{\mathring{H}^{\frac{1}{2}}} + \|w\|^2_{\mathring{H}^{\frac{1}{2}}}\right) + (\nu+\kappa)\|\nabla u\|^2_{\mathring{H}^{\frac{1}{2}}} \\
&\qquad + \gamma\,\|\nabla w\|^2_{\mathring{H}^{\frac{1}{2}}} + 4\kappa\|w\|^2_{\mathring{H}^{\frac{1}{2}}} + \mu\|\nabla\cdot w\|^2_{\mathring{H}^{\frac{1}{2}}} \\
&= 2\kappa\int \Lambda^{\frac{1}{2}}\nabla\times w\cdot\Lambda^{\frac{1}{2}}u + \int \Lambda^{\frac{1}{2}}(u\cdot\nabla u)\cdot\Lambda^{\frac{1}{2}}u \\
&\quad + 2\kappa\int \Lambda^{\frac{1}{2}}(\nabla\times u)\cdot\Lambda^{\frac{1}{2}}w - \int \Lambda^{\frac{1}{2}}(u\cdot\nabla w)\cdot\Lambda^{\frac{1}{2}}w
\end{aligned}
\tag{A.5}
$$

The terms on the right can be bounded as follows. By Hölder's inequality,

$$
\left|\int \Lambda^{\frac{1}{2}}\nabla\times w\cdot\Lambda^{\frac{1}{2}}u\right|, \quad \left|\int \Lambda^{\frac{1}{2}}\nabla\times u\cdot\Lambda^{\frac{1}{2}}w\right| \le \|w\|_{\mathring{H}^{\frac{1}{2}}}\,\|\nabla u\|_{\mathring{H}^{\frac{1}{2}}}.
$$

Therefore,

$$
2\kappa\left|\int \Lambda^{\frac{1}{2}}\nabla\times w\cdot\Lambda^{\frac{1}{2}}u\right| + 2\kappa\left|\int \Lambda^{\frac{1}{2}}\nabla\times u\cdot\Lambda^{\frac{1}{2}}w\right| \le 4\kappa\,\|w\|^2_{\mathring{H}^{\frac{1}{2}}} + \kappa\,\|\nabla u\|^2_{\mathring{H}^{\frac{1}{2}}}.
$$

By Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
\left|\int \Lambda^{\frac{1}{2}}(u\cdot\nabla u)\cdot\Lambda^{\frac{1}{2}}u\right| &\le \|\Lambda^{\frac{3}{2}}u\|_{L^2}\,\|\Lambda^{-\frac{1}{2}}(u\cdot\nabla u)\|_{L^2} \\
&\le C\,\|\Lambda^{\frac{3}{2}}u\|_{L^2}\,\|u\cdot\nabla u\|_{L^{\frac{3}{2}}} \\
&\le C\,\|\Lambda^{\frac{3}{2}}u\|_{L^2}\,\|u\|_{L^6}\,\|\nabla u\|_{L^2} \\
&\le C\,\|\Lambda^{\frac{3}{2}}u\|_{L^2}\,\|\nabla u\|^2_{L^2} \\
&\le C\,\|\Lambda^{\frac{1}{2}}u\|_{L^2}\,\|\Lambda^{\frac{3}{2}}u\|^2_{L^2}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}\left|\int \Lambda^{\frac{1}{2}}(u\cdot\nabla w)\cdot\Lambda^{\frac{1}{2}}w\right| &\le C\,\|\Lambda^{\frac{3}{2}}w\|_{L^2}\,\|\Lambda^{-\frac{1}{2}}(u\cdot\nabla w)\|_{L^2}\\ &\le C\,\|\Lambda^{\frac{3}{2}}w\|_{L^2}\,\|u\cdot\nabla w\|_{L^{\frac{3}{2}}}\\ &\le C\,\|\Lambda^{\frac{3}{2}}w\|_{L^2}\,\|u\|_{L^6}\,\|\nabla w\|_{L^2}\\ &\le C\,\|\Lambda^{\frac{3}{2}}w\|_{L^2}\,\|\nabla u\|_{L^2}\,\|\nabla w\|_{L^2}\\ &\le C\,\|\Lambda^{\frac{1}{2}}u\|_{L^2}^{\frac{1}{2}}\,\|\Lambda^{\frac{3}{2}}u\|_{L^2}^{\frac{1}{2}}\,\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{1}{2}}\,\|\Lambda^{\frac{3}{2}}w\|_{L^2}^{\frac{3}{2}}\\ &\le C\,(\|\Lambda^{\frac{1}{2}}u\|_{L^2}+\|\Lambda^{\frac{1}{2}}w\|_{L^2})(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}w\|_{L^2}^2)\\ &\le C\sqrt{\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2}\,(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}w\|_{L^2}^2).\end{aligned}$$

Combining these estimates yields

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|_{\mathring{H}^{\frac{1}{2}}}^2+\|w\|_{\mathring{H}^{\frac{1}{2}}}^2\right)+\mu\,\|\nabla\cdot w\|_{L^2}^2\\ &\quad+\left(2\min\{\nu,\gamma\}-C\sqrt{\|u\|_{\mathring{H}^{\frac{1}{2}}}^2+\|w\|_{\mathring{H}^{\frac{1}{2}}}^2}\right)(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}w\|_{L^2}^2)\le 0.\end{aligned}$$

This inequality indicates that, if the initial data (u_0, w_0) satisfies

$$2\min\{\nu,\gamma\}-C\sqrt{\|u_0\|_{\mathring{H}^{\frac{1}{2}}}^2+\|w_0\|_{\mathring{H}^{\frac{1}{2}}}^2}>0,$$

the corresponding solution will remain so for all time.

To show the stability estimate, we consider the difference $(\widetilde{u}, \widetilde{w})$ between two solutions $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$, which satisfies

$$\begin{aligned}&\partial_t\widetilde{u}+\widetilde{u}\cdot\nabla u^{(1)}+u^{(2)}\cdot\nabla\widetilde{u}+\nabla\widetilde{p}-2\kappa\nabla\times\widetilde{w}=(\kappa+\nu)\Delta\widetilde{u},\\ &\partial_t\widetilde{w}+\widetilde{u}\cdot\nabla w^{(1)}++u^{(2)}\cdot\nabla\widetilde{w}+4\kappa\widetilde{w}-2\kappa\nabla\times\widetilde{u}=\gamma\Delta\widetilde{w},\end{aligned}$$

where $\widetilde{p}$ denotes the corresponding difference between the pressures. Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|_{\mathring{H}^{\frac{1}{2}}}^2+(\kappa+\nu)\|\nabla\widetilde{u}\|_{\mathring{H}^{\frac{1}{2}}}^2=-\langle\widetilde{u}\cdot\nabla u^{(1)},\widetilde{u}\rangle_{\mathring{H}^{\frac{1}{2}}}+2\kappa\langle\nabla\times\widetilde{w},\widetilde{u}\rangle_{\mathring{H}^{\frac{1}{2}}},\\ &\frac{1}{2}\frac{d}{dt}\|\widetilde{w}\|_{\mathring{H}^{\frac{1}{2}}}^2+\gamma\,\|\nabla\widetilde{w}\|_{\mathring{H}^{\frac{1}{2}}}^2+4\kappa\|\widetilde{w}\|_{\mathring{H}^{\frac{1}{2}}}^2=-\langle\widetilde{u}\cdot\nabla w^{(1)},\widetilde{w}\rangle_{\mathring{H}^{\frac{1}{2}}}+2\kappa\langle\nabla\times\widetilde{u},\widetilde{w}\rangle_{\mathring{H}^{\frac{1}{2}}},\end{aligned}$$

where

$$\langle F, G\rangle_{\mathring{H}^s}=\frac{1}{2}\int|\xi|^{2s}(\widehat{F}(\xi)\,\overline{\widehat{G}(\xi)}+\overline{\widehat{F}(\xi)}\,\widehat{G}(\xi))\,d\xi.$$

We estimate the terms on the right-hand side as follows.

$$\begin{aligned}
\left|\langle \widetilde{u}\cdot\nabla u^{(1)}, \widetilde{u}\rangle_{\mathring{H}^{\frac12}}\right| &\le \|\Lambda^{-\frac12}(\widetilde{u}\cdot\nabla u^{(1)})\|_{L^2}\,\|\widetilde{u}\|_{\mathring{H}^{\frac32}}\\
&\le C\,\|\widetilde{u}\cdot\nabla u^{(1)})\|_{L^{\frac32}}\,\|\widetilde{u}\|_{\mathring{H}^{\frac32}}\\
&\le C\,\|\widetilde{u}\|_{L^6}\,\|\nabla u^{(1)}\|_{L^2}\,\|\widetilde{u}\|_{\mathring{H}^{\frac32}}\\
&\le C\,\|\nabla\widetilde{u}\|_{L^2}\,\|\nabla u^{(1)}\|_{L^2}\,\|\widetilde{u}\|_{\mathring{H}^{\frac32}}\\
&\le C\,\|\widetilde{u}\|_{\mathring{H}^{\frac12}}\,\|\nabla u^{(1)}\|_{L^2}\,\|\widetilde{u}\|^{\frac32}_{\mathring{H}^{\frac32}}\\
&\le \frac{\kappa+\nu}{4}\|\widetilde{u}\|^2_{\mathring{H}^{\frac32}} + C\,\frac{1}{\kappa+\nu}\|\widetilde{u}\|^2_{\mathring{H}^{\frac12}}\,\|\nabla u^{(1)}\|^4_{L^2}.
\end{aligned}$$

The term $\langle \widetilde{u}\cdot\nabla w^{(1)}, \widetilde{w}\rangle_{\mathring{H}^{\frac12}}$ can be bounded similarly,

$$\left|\langle \widetilde{u}\cdot\nabla w^{(1)}, \widetilde{w}\rangle_{\mathring{H}^{\frac12}}\right| \le \frac{\gamma}{2}\|\widetilde{w}\|_{\mathring{H}^{\frac32}} + \frac{\kappa+\nu}{4}\|\widetilde{u}\|^2_{\mathring{H}^{\frac32}} + C\,\|\widetilde{u}\|^2_{\mathring{H}^{\frac12}}\,\|\nabla w^{(1)}\|^4_{L^2}.$$

In addition,

$$2\kappa\,\langle\nabla\times\widetilde{w},\widetilde{u}\rangle_{\mathring{H}^{\frac12}},\quad 2\kappa\,\langle\nabla\times\widetilde{u},\widetilde{w}\rangle_{\mathring{H}^{\frac12}} \le \frac{\kappa}{16}\|\widetilde{u}\|_{\mathring{H}^{\frac32}} + 16\kappa\,\|\widetilde{w}\|^2_{\mathring{H}^{\frac12}}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\left(\|\widetilde{u}\|^2_{\mathring{H}^{\frac12}} + \|\widetilde{w}\|^2_{\mathring{H}^{\frac12}}\right) + (\kappa/2+\nu)\|\widetilde{u}\|^2_{\mathring{H}^{\frac32}} + \gamma\,\|\widetilde{w}\|^2_{\mathring{H}^{\frac32}}\\
&\quad\le 12\kappa\,\|\widetilde{w}\|^2_{\mathring{H}^{\frac12}} + C\,(\|\nabla u^{(1)}\|^4_{L^2} + \|\nabla w^{(1)}\|^4_{L^2})\,\|\widetilde{u}\|^2_{\mathring{H}^{\frac12}}.
\end{aligned}$$

Gronwall's inequality then implies (A.4). This completes the proof of Theorem A.1. □

We now turn to the 2D micropolar equation given by (1.2). The result presented here states that any L^2 data (u_0, w_0) yields a unique global solution.

Theorem A.2. *Assume $u_0 \in L^2(\mathbb{R}^2)$ with $\nabla\cdot u_0 = 0$ and $w_0 \in L^2(\mathbb{R}^2)$. Then (1.2) has a unique global solution (u, w) satisfying*

$$(u, w) \in L^\infty(0,\infty; L^2(\mathbb{R}^2)) \cap L^2(0,\infty; \mathring{H}^1(\mathbb{R}^2)). \tag{A.6}$$

In addition, for any $t_0 > 0$,

$$(u, w) \in C^\infty(\mathbb{R}^2 \times [t_0,\infty)).$$

Proof. The proof of this theorem follows from standard approach. (A.6) is a consequence of a simple energy estimate. We briefly indicate the uniqueness. Consider the difference $(\widetilde{u}, \widetilde{w})$ between two solutions $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$,

$$\partial_t \widetilde{u} + \widetilde{u} \cdot \nabla u^{(1)} + u^{(2)} \cdot \nabla \widetilde{u} + \nabla \widetilde{p} = (\kappa + \nu)\Delta \widetilde{u} + 2\kappa \nabla \times \widetilde{w},$$
$$\partial_t \widetilde{w} + \widetilde{u} \cdot \nabla w^{(1)} + u^{(2)} \cdot \nabla \widetilde{w} = \gamma \Delta \widetilde{w} + 2\kappa \nabla \times \widetilde{u}.$$

Due to the divergence-free condition and integrations by parts,

$$\begin{aligned}
&\frac{d}{dt}\left(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{w}\|_{L^2}^2\right) + 2(\kappa+\nu)\|\nabla \widetilde{u}\|_{L^2}^2 + 2\gamma \|\nabla \widetilde{w}\|_{L^2}^2 + 4\kappa \|\widetilde{w}\|_{L^2}^2 \\
&\quad = -\int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u} + 2\kappa \int (\nabla \times \widetilde{w}) \cdot \widetilde{u} - \int \widetilde{u} \cdot \nabla w^{(1)} \cdot \widetilde{w} + 2\kappa \int (\nabla \times \widetilde{u}) \cdot \widetilde{w}.
\end{aligned}$$

The terms on the right can be bounded as follows.

$$\begin{aligned}
\left| \int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u} \right| &\le \|\nabla u^{(1)}\|_{L^2} \, \|\widetilde{u}\|_{L^4}^2 \\
&\le \|\nabla u^{(1)}\|_{L^2} \, \|\widetilde{u}\|_{L^2} \, \|\nabla \widetilde{u}\|_{L^2} \\
&\le \frac{1}{4}(\kappa+\nu)\|\nabla \widetilde{u}\|_{L^2}^2 + C \, \|\nabla u^{(1)}\|_{L^2}^2 \, \|\widetilde{u}\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$\left| \int \widetilde{u} \cdot \nabla w^{(1)} \cdot \widetilde{w} \right| \le \frac{1}{4}(\kappa+\nu)\|\nabla \widetilde{u}\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla \widetilde{w}\|_{L^2}^2 + C \, \|\nabla w^{(1)}\|_{L^2}^2 \left(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{w}\|_{L^2}^2\right).$$

In addition,

$$2\kappa \int (\nabla \times \widetilde{w}) \cdot \widetilde{u} + 2\kappa \int (\nabla \times \widetilde{u}) \cdot \widetilde{w} \le \frac{\kappa}{2}\|\nabla \widetilde{u}\|_{L^2}^2 + C \, \|\widetilde{w}\|_{L^2}^2.$$

Combining the estimates above and applying Gronwall's inequality yield the desired uniqueness. This completes the proof of Theorem A.2. □

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