Z. Angew. Math. Phys. (2019) 70:153
© 2019 Springer Nature Switzerland AG
0044-2275/19/050001-13
*published online* September 20, 2019
https://doi.org/10.1007/s00033-019-1193-0

**Zeitschrift für angewandte Mathematik und Physik ZAMP**

# A class of global large solutions to the magnetohydrodynamic equations with fractional dissipation

Yichen Dai, Zhong Tan and Jiahong Wu

**Abstract.** The global existence and regularity problem on the magnetohydrodynamic (MHD) equations with fractional dissipation is not well understood for many ranges of fractional powers. This paper examines this open problem from a different perspective. We construct a class of large solutions to the $d$-dimensional ($d = 2, 3$) MHD equations with any fractional power. The process presented here actually assesses that an initial data near any function whose Fourier transform lives in a compact set away from the origin always leads to a unique and global solution.

**Mathematics Subject Classification.** 35A05, 35Q35, 76D03.

**Keywords.** Fractional dissipation, Large solutions, Magnetohydrodynamic equation.

## 1. Introduction

The magnetohydrodynamic (MHD) equations govern the motion of electrically conducting fluids such as plasmas, liquid metals and electrolytes (see, e.g., [9,23]). They are the centerpiece of the magnetohydrodynamics initiated by Alfvén [2]. The MHD equations consist of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Due to the nonlinear interaction between the fluid velocity and the magnetic field, the MHD equations can accommodate much richer phenomena than the Navier–Stokes equations alone. One significant example is that the magnetic field can actually stabilize the fluid motion [24].

The MHD equations have always been of great interest in mathematics. Mathematically rigorous foundational work has been laid by Duvaut and Lions [12] and Sermange and Temam [30]. Recently the MHD equations have gained renewed interests and there have been substantial developments on the well-posedness problem, especially when the MHD equations involve only partial or fractional dissipation. A summary on some of the recent results can be found in a review paper [36]. This paper focuses on the 2D and the 3D incompressible MHD equations with fractional dissipation,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu(-\Delta)^{\alpha} u = -\nabla P + b \cdot \nabla b, \quad x \in \mathbb{R}^d,\ t > 0, \\ \partial_t b + u \cdot \nabla b + \eta(-\Delta)^{\beta} b = b \cdot \nabla u, \quad x \in \mathbb{R}^d,\ t > 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $d = 2$ or 3, $u$, $P$ and $b$ represent the velocity, the pressure and the magnetic field, respectively, and $\nu > 0$, $\eta > 0$, $\alpha \geq 0$ and $\beta \geq 0$ are real parameters. The fractional Laplacian operator $(-\Delta)^{\alpha}$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^{\alpha} f}(\xi) = |\xi|^{2\alpha}\, \widehat{f}(\xi),$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} e^{-ix\cdot\xi}\, f(x)\, \mathrm{d}x.$$

The MHD equations with fractional dissipation given by (1.1) have recently attracted considerable interests, due to their mathematical importance and physical applications. Mathematically (1.1) represents a two-parameter family of systems and contains the MHD systems with standard Laplacian dissipation as special cases. (1.1) allows us to simultaneously examine a whole family of equations and potentially reveals how the solution properties are related to the sizes of $\alpha$ and $\beta$. Physically the fractional diffusion operators can model the anomalous diffusion and have now been widely used in turbulence modeling to control the effective range of the non-local dissipation (see, e.g., [1,16,17]).

A range of global well-posedness results on (1.1) have been obtained. [34] has shown that (1.1) is globally well posed if $\alpha$ and $\beta$ satisfy

$$\alpha \geq \frac{1}{2} + \frac{d}{4}, \quad \beta > 0, \quad \alpha + \beta \geq 1 + \frac{d}{2}. \tag{1.2}$$

Wu [35] was able to sharpen this result by replacing the fractional Laplacian operators by general Fourier multiplier operators. In particular, (1.1) with a $\frac{(-\Delta)^\alpha}{\log(I-\Delta)}u$ and $\frac{(-\Delta)^\beta}{\log(I-\Delta)}b$ for $\alpha$ and $\beta$ satisfying (1.2) is also globally well posed. Yamazaki obtained the global regularity for the case when $\alpha = 2$ and $\beta = 0$ and for a logarithmically reduced fractional dissipation [40]. A further logarithmic refinement was recently worked out by Yamazaki [41].

More global regularity results beyond those stated above are available for the 2D case. When $d = 2$, (1.1) with $\nu = 0$ and $\beta > 1$ was shown to always possess global classical solutions by [5] via the Besov space approach and later by [20] via the parabolic regularity estimates. The global regularity for the case when $\alpha > 0$ and $\beta = 1$ has also been resolved [13]. A significant improvement of [13] is the global regularity of (1.1) with $\beta = 1$ and $(-\Delta)^\alpha u$ replaced by $\log(I - \Delta)u$ [44]. Discovering and exploring a special structure in the nonlinear terms in the equation of the magnetic field, Dong et al. [10] and [11] were able to sharpen the results of [5,20] and [13] by removing half of the magnetic diffusion. The critical case $\nu = 0$ and $\beta = 1$ has so far resisted a complete resolution. A very recent work establishes the global well posedness with only directional hyperviscosity [42]. Many more exciting results on the global regularity problem are available for the 2D case (see, e.g., [3–5,10,11,13,19,20,32,33,37–40,44]). There is also important progress on the uniqueness of weak local solutions to the MHD equations with partial or fractional dissipation (see, e.g., [6,8,14,15,18,26]).

The issue of whether smooth solutions of the MHD equations (1.1) with large initial data can develop singularity in finite time is still a challenging open problem when $\alpha$ and $\beta$ are not in the ranges mentioned above. The perspective of this paper is different. Our goal here is to offer an effective approach of constructing large solutions of (1.1). A special consequence of our construction assesses that any initial data close to a function whose Fourier transform supported in a suitable domain away from the origin always leads to a unique global solution of (1.1). We now describe the construction in some detail. There are some differences between the 2D and the 3D cases, so we split our consideration into two cases, one for the 3D case and one for the 2D case, for the sake of clarity. We begin with the 3D case.

To construct large solutions for the 3D MHD equations, we define two suitable vector fields $\phi_0 \in C_0^\infty(\mathbb{R}^3)$ and $\psi_0 \in C_0^\infty(\mathbb{R}^3)$ with their Fourier transforms satisfying

$$\widehat{\phi}_0(\xi) = \widehat{\psi}_0(\xi) = \left(\varepsilon^{-1} \log \frac{1}{\varepsilon}\right) \chi(\xi), \quad \xi \in \mathbb{R}^3, \tag{1.3}$$

where $0 < \varepsilon \leq 1$ is a small parameter depending on $\nu$ and $\eta$ and will be specified later. Here, $\chi$ is a smooth cutoff function,

$$\mathrm{supp}\chi \subset \mathcal{C} \quad \text{and } \chi = 1 \quad \text{on} \quad \mathcal{C}_1,$$

where $\mathcal{C}$ and $\mathcal{C}_1$ denote the annuli,

$$\mathcal{C} := \Big\{\xi \in \mathbb{R}^3 \,|\, |\xi_i - \xi_j| \le \varepsilon,\ i,j = 1,2,3,\ 1 \le |\xi|^2 \le 2\Big\},$$

$$\mathcal{C}_1 := \Big\{\xi \in \mathbb{R}^3 \,|\, |\xi_i - \xi_j| \le \varepsilon,\ i,j = 1,2,3,\ \frac{5}{4} \le |\xi|^2 \le \frac{7}{4}\Big\}.$$

We remark that the set $|\xi_i - \xi_j| \le \varepsilon$ can be realized by restricting $\xi \in \mathbb{R}^3$ to the cylinder of radius $\frac{\varepsilon}{\sqrt{3}}$ centered on the line $x_1 = x_2 = x_3$. Suppose $\xi \in \mathbb{R}^3$ satisfies, for any $r \in \mathbb{R}$,

$$x_1 = x_2 = x_3 = \frac{r}{3}, \quad \xi_1 + \xi_2 + \xi_3 = r, \tag{1.4}$$

$$(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2 \le \frac{1}{3}\varepsilon^2. \tag{1.5}$$

Then, it is not difficult to see that $|\xi_i - \xi_j| \le \varepsilon$, $i,j = 1,2,3$. In fact, we have from (1.4) and (1.5)

$$\begin{aligned}
&(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2 \\
&\quad = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2x_1(\xi_1 + \xi_2 + \xi_3) + 3x_1^2 \\
&\quad = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\frac{r}{3}r + \frac{r^2}{3} \\
&\quad = \xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{r^2}{3} \le \frac{1}{3}\varepsilon^2.
\end{aligned}$$

Then,

$$\begin{aligned}
&(\xi_1 - \xi_2)^2 + (\xi_1 - \xi_3)^2 + (\xi_2 - \xi_3)^2 \\
&\quad = 2(\xi_1^2 + \xi_2^2 + \xi_3^2) - 2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) \\
&\quad = 2(\xi_1^2 + \xi_2^2 + \xi_3^2) + (\xi_1^2 + \xi_2^2 + \xi_3^2 - r^2) \\
&\quad \le \varepsilon^2,
\end{aligned}$$

which implies $|\xi_i - \xi_j| \le \varepsilon$, $i,j = 1,2,3$. Our global existence and regularity result for the 3D MHD equations can be stated as follows.

**Theorem 1.1.** *Assume $\phi_0$ and $\psi_0$ are given by (1.3). Define $U_0$ and $B_0$ by*

$$U_0 = \nabla \times \phi_0, \qquad B_0 = \nabla \times \psi_0. \tag{1.6}$$

*Let $\nu > 0$, $\eta > 0$ and $\alpha \ge 0$ and $\beta \ge 0$. Let $s > \max\left\{\frac{5}{2} - \alpha, \frac{5}{2} - \beta, 1\right\}$. Consider the 3D MHD equations in (1.1) with the initial data*

$$u_0 = U_0 + v_0 \quad \textit{and} \quad b_0 = B_0 + h_0.$$

*If $\varepsilon$, $v_0 \in H^s(\mathbb{R}^3)$ and $h_0 \in H^s(\mathbb{R}^3)$ satisfy*

$$\varepsilon = C_1\ (\min\{\nu,\eta\})^{\frac{10}{9}} \qquad \textit{and} \quad \|v_0\|_{H^s} + \|h_0\|_{H^s} \le C_2 \min\{\nu,\eta\}$$

*for suitable constants $C_1 > 0$ and $C_2 > 0$, then (1.1) has a unique global solution $(u,b)$ satisfying*

$$u,\, b \in C([0,\infty); H^s(\mathbb{R}^3)), \quad \Lambda^\alpha u,\ \Lambda^\beta b \in L^2(0,\infty; L^2(\mathbb{R}^3)).$$

The initial data $(u_0, b_0)$ in Theorem 1.1 is not small. In fact,

$$\begin{aligned}
\|u_0\|_{L^2} &\ge \|U_0\|_{L^2} - \|v_0\|_{L^2} \\
&\ge \left[\int_{\mathbb{R}^3} |\xi|^2 |\widehat{\phi}_0(\xi)|^2\, d\xi\right]^{\frac{1}{2}} - \|v_0\|_{H^s}
\end{aligned}$$

$$\geq \left[\int_{\mathcal{C}_1} |\xi|^2 |\widehat{\phi}_0(\xi)|^2 \, d\xi\right]^{\frac{1}{2}} - \|v_0\|_{H^s}$$
$$\geq C \left(\varepsilon^{-1} \log \frac{1}{\varepsilon}\right) \varepsilon - \|v_0\|_{H^s}$$
$$\geq C \log \frac{1}{\varepsilon} - C_2 \max\{\nu, \eta\}, \tag{1.7}$$

which can be really large when $\varepsilon$ is take to be small. Similarly, any homogeneous $\dot{H}^s$ norm is not small and $\|b_0\|_{H^s}$ is also large. In addition, as we shall see from the proof, Theorem 1.1 remains valid when the annulus in the definition of $\mathcal{C}$ is replaced by any compact set supported away from the origin.

A similar result also holds for the 2D MHD equations in (1.1). There are some minor differences in the construction process. We define two scalar functions $\widetilde{\phi} \in C_0^\infty(\mathbb{R}^2)$ and $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^2)$ satisfying

$$\widehat{\widetilde{\phi}}(\xi) = \widehat{\widetilde{\psi}}(\xi) = \left(\varepsilon^{-\frac{1}{2}} \log \frac{1}{\varepsilon}\right) \chi_1(\xi), \quad \xi \in \mathbb{R}^2, \tag{1.8}$$

where $\chi_1$ is a smooth cutoff function,

$$\mathrm{supp}\chi_1 \subset \mathcal{D} \quad \text{and } \chi_1 = 1 \quad \text{on} \quad \mathcal{D}_1.$$

Here, $\mathcal{D}$ and $\mathcal{D}_1$ denote the annuli,

$$\mathcal{D} := \Big\{\xi \in \mathbb{R}^2 \,|\, |\xi_i - \xi_j| \leq \varepsilon,\, i, j = 1, 2,\, 1 \leq |\xi|^2 \leq 2\Big\},$$
$$\mathcal{D}_1 := \Big\{\xi \in \mathbb{R}^2 \,|\, |\xi_i - \xi_j| \leq \varepsilon,\, i, j = 1, 2,\, \frac{5}{4} \leq |\xi|^2 \leq \frac{7}{4}\Big\}.$$

We remark that it is not necessary for $\widetilde{\phi}$ and $\widetilde{\psi}$ to be the same. They are taken to be the same here for the sake of brevity. The global existence and regularity result for the 2D MHD equations can be stated as follows.

**Theorem 1.2.** *Assume $\widetilde{\phi}$ and $\widetilde{\psi}$ are given by (1.8). Define $\widetilde{U}$ and $\widetilde{B}$ by*

$$\widetilde{U} = \nabla^\perp \widetilde{\phi} := (\partial_2 \widetilde{\phi}, -\partial_1 \widetilde{\phi}), \qquad \widetilde{B} = \nabla^\perp \widetilde{\psi}. \tag{1.9}$$

*Let $\nu > 0$, $\eta > 0$ and $\alpha \geq 0$ and $\beta \geq 0$. Let $s > 2$. Consider the 2D MHD equations in (1.1) with the initial data*

$$u_0 = \widetilde{U} + v_0 \quad \text{and} \quad b_0 = \widetilde{B} + h_0.$$

*If $\varepsilon$, $v_0 \in H^s(\mathbb{R}^2)$ and $h_0 \in H^s(\mathbb{R}^2)$ satisfy*

$$\varepsilon = C_3 \,(\min\{\nu, \eta\})^{\frac{5}{2}} \quad \text{and} \quad \|v_0\|_{H^s} + \|h_0\|_{H^s} \leq C_4 \min\{\nu, \eta\}$$

*for suitable constant $C_3 > 0$ and $C_4 > 0$, then (1.1) has a unique global solution $(u, b)$ satisfying*

$$u,\, b \in C([0, \infty); H^s(\mathbb{R}^2)), \quad \Lambda^\alpha u,\, \Lambda^\beta b \in L^2(0, \infty; L^2(\mathbb{R}^2)).$$

Again the initial data $(u_0, b_0)$ is not small in $H^s(\mathbb{R}^2)$. As in (1.7),

$$\|u_0\|_{L^2},\, \|b_0\|_{L^2} \geq C \log \frac{1}{\varepsilon} - C_4 \max\{\nu, \eta\}.$$

Theorem 1.2 remains true if the annulus in the definition of $\mathcal{D}$ is changed to any compact set supported away from the origin.

We mention some related results. Lei et al. [25] constructed smooth large solutions to the 3D Navier–Stokes equations with the initial data close to a Beltrami flow. More information on the Beltrami flow can be found in [7,29]. Zhou–Zhu [45] obtained a class of large solutions to the 3D damped Euler near the Beltrami flow. Family of large solutions have also been obtained for the damped MHD equations and the

Hall-MHD equations (see [27,28,43]). Our construction presented here is somewhat different and does not involve Beltrami flow.

To prove Theorem 1.1, we seek a solution of the form

$$u = U + v, \quad b = B + h,$$

where $U$ and $B$ are the solutions of the corresponding linearized equations

$$\begin{cases} \partial_t U + \nu(-\Delta)^\alpha U = 0, \\ \partial_t B + \eta(-\Delta)^\beta B = 0, \\ U(x,0) = U_0(x), \quad B(x,0) = B_0(x). \end{cases} \tag{1.10}$$

By the definition of $U_0$ and $B_0$ in Theorem 1.1, $U$ and $B$ can be written as

$$U(t) = e^{-\nu(-\Delta)^\alpha t} U_0 = e^{-\nu(-\Delta)^\alpha t} \nabla \times \phi_0, \quad B(t) = e^{-\eta(-\Delta)^\beta t} \nabla \times \psi_0. \tag{1.11}$$

The equations of $(v, h)$ are given by

$$\begin{cases} \partial_t v + u \cdot \nabla v + v \cdot \nabla U + \nu(-\Delta)^\alpha v = -\nabla P + b \cdot \nabla h + h \cdot \nabla B + f, \\ \partial_t h + u \cdot \nabla h + v \cdot \nabla B + \eta(-\Delta)^\beta h = b \cdot \nabla v + h \cdot \nabla U + g, \\ \nabla \cdot v = \nabla \cdot h = 0, \\ v(x,0) = v_0(x), \quad h(x,0) = h_0(x), \end{cases} \tag{1.12}$$

where

$$f = -U \cdot \nabla U + B \cdot \nabla B \quad \text{and} \quad g = -U \cdot \nabla B + B \cdot \nabla U. \tag{1.13}$$

Then, it suffices to show that (1.12) has a unique global solution. Since the local well posedness follows from as standard procedure, we focus on the global bound via the bootstrap argument. Details on how to obtain suitable energy inequalities and how the bootstrap argument is applied are given in Sect. 2. The proof of Theorem 1.2 is similar and is sketched in Sect. 3.

The rest of this paper is divided into two sections. Section 2 proves Theorem 1.1, while Sect. 3 provides the proof for Theorem 1.2.

## 2. Proof for Theorem 1.1

This section proves Theorem 1.1. As we described in "Introduction", it suffices to establish that solutions of (1.12) remain bounded in $H^s$ for all time. This is verified by deriving suitable energy inequalities and applying the bootstrap argument. As a preparation, we first present some bounds on $U(t)$ and $B(t)$ given by (1.11) and $f$ and $g$ defined in (1.13).

**Lemma 2.1.** *Let $\phi_0$ and $\psi_0$ be given by (1.3), $U_0$ and $B_0$ by (1.6), $U(t)$ and $B(t)$ by (1.11), and $f$ and $g$ by (1.13). Then, the following estimates hold.*

(1) *For any $\sigma \geq 0$ and $2 \leq q \leq \infty$,*

$$\|\Lambda^\sigma \phi_0\|_{L^q(\mathbb{R}^3)}, \ \|\Lambda^\sigma \psi_0\|_{L^q(\mathbb{R}^3)} \leq C\, \varepsilon^{1-\frac{2}{q}} \log \frac{1}{\varepsilon},$$

$$\|\Lambda^\sigma U(t)\|_{L^q(\mathbb{R}^3)}, \ \|\Lambda^\sigma B(t)\|_{L^q(\mathbb{R}^3)} \leq C\, \varepsilon^{1-\frac{2}{q}} \log \frac{1}{\varepsilon}\, e^{-C_0 t},$$

*where $C_0 > 0$ is a constant.*

(2) *For any $s > \frac{3}{2}$,*

$$\|f\|_{H^s} + \|g\|_{H^s} \leq C\varepsilon\, e^{-2C_0 t} \left(\|\phi_0\|_{H^{s+2}}^2 + \|\psi_0\|_{H^{s+2}}^2\right). \tag{2.1}$$

*Proof of Lemma 2.1.* By Hausdorff–Young inequality,

$$\|\Lambda^\sigma \phi_0\|_{L^q(\mathbb{R}^3)} \le C\|\widehat{\Lambda^\sigma \phi}_0\|_{L^{\tilde q}(\mathbb{R}^3)} \le C\left(\varepsilon^{-1}\log\frac{1}{\varepsilon}\right)\varepsilon^{\frac{2}{\tilde q}} = C\,\varepsilon^{1-\frac{2}{q}}\log\frac{1}{\varepsilon}.$$

Similarly,

$$\begin{aligned}\|\Lambda^\sigma U(t)\|_{L^q(\mathbb{R}^3)} &\le C\,\|\widehat{\Lambda^\sigma U}(t)\|_{L^{\tilde q}(\mathbb{R}^3)}\\ &\le C\left(\varepsilon^{-1}\log\frac{1}{\varepsilon}\right)\varepsilon^{\frac{2}{\tilde q}}\,e^{-C_0 t} = C\,\varepsilon^{1-\frac{2}{q}}\log\frac{1}{\varepsilon}\,e^{-C_0 t},\end{aligned}$$

where $\frac{1}{q}+\frac{1}{\tilde q}=1$. $\|\Lambda^\sigma B(t)\|_{L^q(\mathbb{R}^3)}$ obeys the same bound. To prove (2.1), we rewrite the terms in $f$ and $g$ so that each term contains a difference $\partial_i-\partial_j$ with $i,j=1,2,3$. Clearly, for $\phi=e^{-\nu(-\Delta)^\alpha t}\phi_0$ and $\psi=e^{-\eta(-\Delta)^\beta t}\psi_0$,

$$U=\nabla\times\phi=(\partial_2\phi_3-\partial_3\phi_2,\,\partial_3\phi_1-\partial_1\phi_3,\,\partial_1\phi_2-\partial_2\phi_1)$$

and

$$B=\nabla\times\psi=(\partial_2\psi_3-\partial_3\psi_2,\,\partial_3\psi_1-\partial_1\psi_3,\,\partial_1\psi_2-\partial_2\psi_1).$$

Then, direct calculations show that the first component of $-U\cdot\nabla U$ becomes

$$\begin{aligned}&-U\cdot\nabla U^1\\ &\quad=-\partial_2\phi_3\partial_1\partial_2\phi_3+\partial_1\phi_3\partial_2\partial_2\phi_3+\partial_2\phi_3\partial_1\partial_3\phi_2-\partial_1\phi_3\partial_2\partial_3\phi_2\\ &\qquad+\partial_3\phi_2\partial_1\partial_2\phi_3-\partial_1\phi_2\partial_3\partial_2\phi_3-\partial_3\phi_2\partial_1\partial_3\phi_2+\partial_1\phi_2\partial_3\partial_3\phi_2\\ &\qquad-\partial_3\phi_1\partial_2\partial_2\phi_3+\partial_2\phi_1\partial_3\partial_2\phi_3+\partial_3\phi_1\partial_2\partial_3\phi_2-\partial_2\phi_1\partial_3\partial_3\phi_2\\ &\quad=[(\partial_1-\partial_2)\phi_3\partial_1\partial_2\phi_3+\partial_1\phi_3\partial_2(\partial_2-\partial_1)\phi_3]\\ &\qquad+[(\partial_2-\partial_1)\phi_3\partial_1\partial_3\phi_2+\partial_1\phi_3\partial_3(\partial_1-\partial_2)\phi_2]\\ &\qquad+[(\partial_3-\partial_1)\phi_2\partial_1\partial_2\phi_3+\partial_1\phi_2\partial_2(\partial_1-\partial_3)\phi_3]\\ &\qquad+[(\partial_1-\partial_3)\phi_2\partial_1\partial_3\phi_2+\partial_1\phi_2\partial_3(\partial_3-\partial_1)\phi_2]\\ &\qquad+[(\partial_2-\partial_3)\phi_1\partial_2\partial_2\phi_3+\partial_2\phi_1\partial_2(\partial_3-\partial_2)\phi_3]\\ &\qquad+[(\partial_3-\partial_2)\phi_1\partial_2\partial_3\phi_2+\partial_2\phi_1\partial_3(\partial_2-\partial_3)\phi_2].\end{aligned}$$

Taking the $H^s$-norm yields,

$$\begin{aligned}\|-U\cdot\nabla U^1\|_{H^s}\le C\big(&\|(\partial_1-\partial_2)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_2-\partial_1)\phi\|_{H^{s+1}}\\ &+\|(\partial_2-\partial_1)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_1-\partial_2)\phi\|_{H^{s+1}}\\ &+\|(\partial_3-\partial_1)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_1-\partial_3)\phi\|_{H^{s+1}}\\ &+\|(\partial_1-\partial_3)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_3-\partial_1)\phi\|_{H^{s+1}}\\ &+\|(\partial_2-\partial_3)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_3-\partial_2)\phi\|_{H^{s+1}}\\ &+\|(\partial_3-\partial_2)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_2-\partial_3)\phi\|_{H^{s+1}}\big)\\ \le C\big(&\|(\partial_i-\partial_j)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_i-\partial_j)\phi\|_{H^{s+1}}\big),\end{aligned}$$

where $i,j$ in the last line are summed over $i,j=1,2,3$. Similarly,

$$\begin{aligned}\|f\|_{H^s}&\le C\big(\|(\partial_i-\partial_j)\phi\|_{H^s}\|\phi\|_{H^{s+2}}+\|\phi\|_{H^{s+1}}\|(\partial_i-\partial_j)\phi\|_{H^{s+1}}\\ &\qquad+\|(\partial_i-\partial_j)\psi\|_{H^s}\|\psi\|_{H^{s+2}}+\|\psi\|_{H^{s+1}}\|(\partial_i-\partial_j)\psi\|_{H^{s+1}}\big)\\ &\le Ce^{-2C_0t}\big(\|(\partial_i-\partial_j)\phi_0\|_{H^s}\|\phi_0\|_{H^{s+2}}+\|\phi_0\|_{H^{s+1}}\|(\partial_i-\partial_j)\phi_0\|_{H^{s+1}}\big)\\ &\qquad+Ce^{-2C_0t}\big(\|(\partial_i-\partial_j)\psi_0\|_{H^s}\|\psi_0\|_{H^{s+2}}+\|\psi_0\|_{H^{s+1}}\|(\partial_i-\partial_j)\psi_0\|_{H^{s+1}}\big)\end{aligned}$$

$$\begin{aligned}&\le C\varepsilon e^{-2C_0t}\big(\|\phi_0\|_{H^s}\|\phi_0\|_{H^{s+2}}+\|\phi_0\|_{H^{s+1}}\|\phi_0\|_{H^{s+1}}\big)\\&\quad+C\varepsilon e^{-2C_0t}\big(\|\psi_0\|_{H^s}\|\psi_0\|_{H^{s+2}}+\|\psi_0\|_{H^{s+1}}\|\psi_0\|_{H^{s+1}}\big)\\&\le C\varepsilon e^{-2C_0t}\big(\|\phi_0\|^2_{H^{s+2}}+\|\psi_0\|^2_{H^{s+2}}\big).\end{aligned}$$

The bound for $\|g\|_{H^s}$ can be similarly obtained. This proves (2.1). □

In addition, the following commutator and bilinear estimates involving fractional derivatives will be used (see, e.g., [21,22]).

**Lemma 2.2.** *Let $s>0$. Let $p,p_1,p_3\in(1,\infty)$ and $p_2,p_4\in[1,\infty]$ satisfy*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

*Then, there exist constants C's such that*

$$\begin{aligned}&\|[J^s,F]G\|_{L^p}\le C\,\left(\|J^sF\|_{L^{p_1}}\,\|G\|_{L^{p_2}}+\|J^{s-1}G\|_{L^{p_3}}\,\|\nabla f\|_{L^{p_4}}\right),\\&\|J^s(F\,G)\|_{L^p}\le C\,\left(\|J^sF\|_{L^{p_1}}\,\|G\|_{L^{p_2}}+\|J^sG\|_{L^{p_3}}\,\|F\|_{L^{p_4}}\right),\end{aligned}$$

*where $J=(I-\Delta)^{\frac12}$.*

*Proof of Theorem 1.1.* Applying $J^s$ to (1.12) and dotting with $(J^sv,J^sh)$ yield

$$\frac12\frac{d}{dt}\big(\|v\|^2_{H^s}+\|h\|^2_{H^s}\big)+\nu\|v\|^2_{H^{s+\alpha}}+\eta\|h\|^2_{H^{s+\beta}}\le\sum_{m=1}^{7}I_m, \tag{2.2}$$

where, due to $\nabla\cdot v=0$, $\nabla\cdot U=0$, $\nabla\cdot h=0$ and $\nabla\cdot B=0$,

$$\begin{aligned}I_1&:=-\int[J^s,v\cdot\nabla]v\cdot J^sv\,dx-\int[J^s,v\cdot\nabla]h\cdot J^sh\,\mathrm{d}x,\\I_2&:=-\int[J^s,U\cdot\nabla]v\cdot J^sv\,dx-\int[J^s,U\cdot\nabla]h\cdot J^sh\,\mathrm{d}x,\\I_3&:=\int[J^s,h\cdot\nabla]h\cdot J^sv\,dx+\int[J^s,h\cdot\nabla]v\cdot J^sh\,\mathrm{d}x,\\I_4&:=\int[J^s,B\cdot\nabla]h\cdot J^sv\,dx+\int[J^s,B\cdot\nabla]v\cdot J^sh\,\mathrm{d}x,\\I_5&:=-\int J^s(v\cdot\nabla U)\cdot J^sv\,dx-\int J^s(v\cdot\nabla B)\cdot J^sh\,\mathrm{d}x,\\I_6&:=\int J^s(h\cdot\nabla B)\cdot J^sv\,dx+\int J^s(h\cdot\nabla U)\cdot J^sh\,\mathrm{d}x,\\I_7&:=\int J^sf\cdot J^sv\,dx+\int J^sg\cdot J^sh\,\mathrm{d}x.\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}|I_1|&\le C\,\|\nabla v\|_{L^\infty}\,\|v\|^2_{H^s}+C\,\|\nabla v\|_{L^\infty}\,\|h\|^2_{H^s}+C\,\|\nabla h\|_{L^\infty}\,\|v\|_{H^s}\,\|h\|_{H^s}\\&\le C\,\|v\|_{H^s}\,\|v\|^2_{H^{s+\alpha}}+C\,\|v\|_{H^{s+\alpha}}\,\|h\|_{H^{s+\beta}}\,\|h\|_{H^s}\\&\le C\,(\|v\|^2_{H^{s+\alpha}}+\|h\|^2_{H^{s+\beta}})\,(\|v\|_{H^s}+\|h\|_{H^s}),\end{aligned}$$

where we have used the Sobolev inequalities

$$\|\nabla v\|_{L^\infty}\le C\,\|v\|_{H^{s+\alpha}}\quad\text{and}\quad\|\nabla h\|_{L^\infty}\le C\,\|h\|_{H^{s+\beta}}$$

for $s > \max\{\frac{5}{2}-\alpha, \frac{5}{2}-\beta\}$. Applying Lemma 2.2 with $p_1 < \infty$ large and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{2}$ and Lemma 2.1,

$$\begin{aligned}|I_2| &\le C\,\|v\|_{H^s}\,(\|J^sU\|_{L^{p_1}}\,\|\nabla v\|_{L^{p_2}} + \|\nabla U\|_{L^\infty}\|v\|_{H^s})\\ &\quad + C\,\|h\|_{H^s}\,(\|J^sU\|_{L^{p_1}}\,\|\nabla h\|_{L^{p_2}} + \|\nabla U\|_{L^\infty}\|h\|_{H^s})\\ &\le C\,\varepsilon^{1-\frac{2}{p_1}}\left(\log\frac{1}{\varepsilon}\right)e^{-C_0t}\,(\|v\|_{H^s}^2+\|h\|_{H^s}^2),\end{aligned}$$

where we have used the Sobolev inequality $\|\nabla v\|_{L^{p_2}} \le C\,\|v\|_{H^s}$ for $s>1$ and $p_2>2$ but close to 2. For simplicity, we take $p_1 = 40$ (a concrete number is not crucial here). Using the simple fact that, for $0<\varepsilon\le 1$,

$$\varepsilon^{\frac{1}{20}}\left(\log\frac{1}{\varepsilon}\right)\le C,$$

we have

$$|I_2| \le C\,\varepsilon^{\frac{9}{10}}\,e^{-C_0t}\,(\|v\|_{H^s}^2+\|h\|_{H^s}^2).$$

$I_3$ can be similarly estimated as $I_1$,

$$|I_3| \le C\,(\|v\|_{H^{s+\alpha}}^2+\|h\|_{H^{s+\beta}}^2)\,(\|v\|_{H^s}+\|h\|_{H^s}).$$

$I_4$ can be similarly estimated as $I_2$,

$$|I_4| \le C\,\varepsilon^{\frac{9}{10}}\,e^{-C_0t}\,(\|v\|_{H^s}^2+\|h\|_{H^s}^2).$$

By Lemma 2.2,

$$\begin{aligned}|I_5| \le\ & C\,\|v\|_{H^s}\,(\|J^s\nabla U\|_{L^{p_1}}\,\|v\|_{L^{p_2}} + \|\nabla U\|_{L^\infty}\|v\|_{H^s})\\ &+ C\,\|v\|_{H^s}\,(\|J^s\nabla B\|_{L^{p_1}}\,\|h\|_{L^{p_2}} + \|\nabla B\|_{L^\infty}\|h\|_{H^s}).\end{aligned}$$

Applying Lemma 2.1 and following the estimates for $I_2$, we find

$$|I_5| \le C\,\varepsilon^{\frac{9}{10}}\,e^{-C_0t}\,(\|v\|_{H^s}^2+\|h\|_{H^s}^2).$$

$I_6$ can be estimated similarly as $I_5$,

$$|I_6| \le C\,\varepsilon^{\frac{9}{10}}\,e^{-C_0t}\,(\|v\|_{H^s}^2+\|h\|_{H^s}^2).$$

By Lemma 2.1,

$$\begin{aligned}|I_7| &\le \|f\|_{H^s}\,\|v\|_{H^s} + \|g\|_{H^s}\,\|h\|_{H^s}\\ &\le C\varepsilon e^{-2C_0t}\big(\|\phi_0\|_{H^{s+2}}^2+\|\psi_0\|_{H^{s+2}}^2\big)\sqrt{\|v\|_{H^s}^2+\|h\|_{H^s}^2}.\end{aligned}$$

By Lemma 2.1,

$$\|\phi_0\|_{H^{s+2}},\ \|\psi_0\|_{H^{s+2}} \le C\left(\log\frac{1}{\varepsilon}\right).$$

Therefore, if we use the simple fact that, for $0<\varepsilon\le 1$,

$$\varepsilon^{\frac{1}{20}}\left(\log\frac{1}{\varepsilon}\right)\le C,$$

we obtain

$$|I_7| \le C\,\varepsilon^{\frac{9}{10}}\,e^{-2C_0t}\sqrt{\|v\|_{H^s}^2+\|h\|_{H^s}^2}.$$

Inserting the estimates in (2.2) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|_{H^s}^2+\|h\|_{H^s}^2\right)+\left(\nu-C_5\left(\|v\|_{H^s}+\|h\|_{H^s}\right)\right)\|v\|_{H^{s+\alpha}}^2\\&\quad+\left(\eta-C_5\left(\|v\|_{H^s}+\|h\|_{H^s}\right)\right)\|h\|_{H^{s+\beta}}^2\\&\le C_6\,\varepsilon^{\frac{9}{10}}\,e^{-C_0t}\left(\|v\|_{H^s}^2+\|h\|_{H^s}^2\right)+C_7\,\varepsilon^{\frac{9}{10}}\,e^{-2C_0t}\sqrt{\|v\|_{H^s}^2+\|h\|_{H^s}^2}.\end{aligned}\tag{2.3}$$

We apply the bootstrap argument to (2.3) to establish that $\|v(t)\|_{H^s}+\|h(t)\|_{H^s}$ remains uniform bounded if $\|v_0\|_{H^s}+\|h_0\|_{H^s}$ is taken to be sufficiently small. The bootstrap argument starts with an ansatz that $\|v(t)\|_{H^s}+\|h(t)\|_{H^s}$ is bounded, say

$$\|v(t)\|_{H^s}+\|h(t)\|_{H^s}\le M$$

and shows that $\|v(t)\|_{H^s}+\|h(t)\|_{H^s}$ actually admits a smaller bound, say

$$\|v(t)\|_{H^s}+\|h(t)\|_{H^s}\le\frac{1}{2}M$$

when $\|v_0\|_{H^s}+\|h_0\|_{H^s}$ is sufficiently small. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book (see [31, p.21]). To apply the bootstrap argument to (2.3), we assume that

$$\|v(t)\|_{H^s}+\|h(t)\|_{H^s}\le M:=\frac{1}{2C_5}\min\{\nu,\eta\}.\tag{2.4}$$

Clearly, when (2.4) is fulfilled, we have

$$\nu-C_5\left(\|v\|_{H^s}+\|h\|_{H^s}\right)>0,\qquad\eta-C_5\left(\|v\|_{H^s}+\|h\|_{H^s}\right)>0.$$

It then follows from (2.3) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|v\|_{H^s}^2+\|h\|_{H^s}^2\right)\\&\quad\le C_6\,\varepsilon^{\frac{9}{10}}\,e^{-C_0t}\left(\|v\|_{H^s}^2+\|h\|_{H^s}^2\right)+C_7\,\varepsilon^{\frac{9}{10}}\,e^{-2C_0t}\sqrt{\|v\|_{H^s}^2+\|h\|_{H^s}^2}\end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\|v\|_{H^s}^2+\|h\|_{H^s}^2}\le C_6\,\varepsilon^{\frac{9}{10}}\,e^{-C_0t}\sqrt{\|v\|_{H^s}^2+\|h\|_{H^s}^2}+C_7\,\varepsilon^{\frac{9}{10}}\,e^{-2C_0t}.$$

By Gronwall's inequality,

$$\begin{aligned}\sqrt{\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2}&\le e^{C_6\,\varepsilon^{\frac{9}{10}}\int_0^t e^{-C_0\tau}\mathrm{d}\tau}\left(\sqrt{\|v_0\|_{H^s}^2+\|h_0\|_{H^s}^2}\right.\\&\qquad\left.+\int_0^t C_7\,\varepsilon^{\frac{9}{10}}\,e^{-2C_0\tau}\,\mathrm{d}\tau\right)\\&\le M_1\left(\|v_0\|_{H^s}+\|h_0\|_{H^s}\right)+M_1\varepsilon^{\frac{9}{10}},\end{aligned}\tag{2.5}$$

where

$$M_1=\max\left\{e^{C_6C_0^{-1}},\,\frac{1}{2}C_7C_0^{-1}e^{C_6C_0^{-1}}\right\}.$$

If $v_0, h_0$ and $\varepsilon$ satisfy

$$\|v_0\|_{H^s}+\|h_0\|_{H^s}\le\frac{1}{8\sqrt{2}M_1C_5}\min\{\nu,\eta\},\qquad\varepsilon\le\left(\frac{\min\{\nu,\eta\}}{8\sqrt{2}M_1C_5}\right)^{\frac{10}{9}},\tag{2.6}$$

then (2.5) implies

$$\sqrt{\|v(t)\|_{H^s}^2+\|h(t)\|_{H^s}^2}\le M_1\frac{\min\{\nu,\eta\}}{8\sqrt{2}M_1C_5}+M_1\frac{\min\{\nu,\eta\}}{8\sqrt{2}M_1C_5}=\frac{M}{2\sqrt{2}}.$$

That is,

$$\|v(t)\|_{H^s} + \|h(t)\|_{H^s} \le \sqrt{2}\sqrt{\|v(t)\|_{H^s}^2 + \|h(t)\|_{H^s}^2} \le \frac{M}{2}.$$

The bootstrap argument then implies that, for all $t > 0$,

$$\|v(t)\|_{H^s} + \|h(t)\|_{H^s} \le \frac{M}{2} = \frac{1}{4C_5}\min\{\nu, \eta\}$$

when $\|v_0\|_{H^s} + \|h_0\|_{H^s}$ and $\varepsilon$ satisfy (2.6). This proves Theorem 1.1. □

## 3. Proof of Theorem 1.2

This section proves Theorem 1.2. Since the proof shares many similarities with that for Theorem 1.1, we shall just provide the details for those parts that differ.

As a preparation of the proof, the following lemma provides upper bounds for $\widetilde{U}$, $\widetilde{B}$, $f$ and $g$ in the 2D case.

**Lemma 3.1.** *Let $\widetilde{\phi}$ and $\widetilde{\psi}$ be given by (1.8), and $\widetilde{U}$ and $\widetilde{B}$ by (1.9). Let $\widetilde{U}(t)$ and $\widetilde{B}(t)$ be given by*

$$\widetilde{U}(t) = e^{-\nu(-\Delta)^\alpha t}\widetilde{U}_0, \qquad \widetilde{B}(t) = e^{-\eta(-\Delta)^\beta t}\widetilde{B}_0.$$

*Let $f$ and $g$ be given by (1.13), namely*

$$f = -\widetilde{U}\cdot\nabla\widetilde{U} + \widetilde{B}\cdot\nabla\widetilde{B} \quad \text{and} \quad g = -\widetilde{U}\cdot\nabla\widetilde{B} + \widetilde{B}\cdot\nabla\widetilde{U}.$$

*Then, the following estimates hold.*

(1) *For any $\sigma \ge 0$ and $2 \le q \le \infty$,*

$$\|\Lambda^\sigma\widetilde{\phi}\|_{L^q(\mathbb{R}^2)},\ \|\Lambda^\sigma\widetilde{\psi}\|_{L^q(\mathbb{R}^2)} \le C\,\varepsilon^{\frac12-\frac1q}\,\log\frac{1}{\varepsilon},$$

$$\|\Lambda^\sigma\widetilde{U}(t)\|_{L^q(\mathbb{R}^2)},\ \|\Lambda^\sigma\widetilde{B}(t)\|_{L^q(\mathbb{R}^2)} \le C\,\varepsilon^{\frac12-\frac1q}\,\log\frac{1}{\varepsilon}\,e^{-C_0 t}.$$

(2) *For any $s > 1$,*

$$\|f\|_{H^s} + \|g\|_{H^s} \le C\varepsilon\, e^{-2C_0 t}\left(\|\widetilde{\phi}\|_{H^{s+2}}^2 + \|\widetilde{\psi}\|_{H^{s+2}}^2\right). \tag{3.1}$$

*Proof.* The first part of the estimates can be similarly proven as in the proof of Lemma 2.1. To prove (3.1), we write

$$\widetilde{\phi} = e^{-\nu(-\Delta)^\alpha t}\widetilde{\phi}_0, \quad \widetilde{\psi} = e^{-\eta(-\Delta)^\beta t}\widetilde{\psi}_0$$

and rewrite the first component of $f$ as

$$\begin{aligned} f^1 &= -\widetilde{U}\cdot\nabla\widetilde{U}^1 + \widetilde{B}\cdot\nabla\widetilde{B}^1 \\ &= \partial_1\widetilde{\phi}\partial_2\partial_2\widetilde{\phi} - \partial_2\widetilde{\phi}\partial_1\partial_2\widetilde{\phi} - \partial_1\widetilde{\psi}\partial_2\partial_2\widetilde{\psi} + \partial_2\widetilde{\psi}\partial_1\partial_2\widetilde{\psi} \\ &= (\partial_1-\partial_2)\widetilde{\phi}\partial_2\partial_2\widetilde{\phi} + \partial_2\widetilde{\phi}\partial_2(\partial_2-\partial_1)\widetilde{\phi} + (\partial_2-\partial_1)\widetilde{\psi}\partial_2\partial_2\widetilde{\psi} + \partial_2\widetilde{\psi}\partial_2(\partial_1-\partial_2)\widetilde{\psi}. \end{aligned}$$

By Hölder's inequality and Sobolev embedding, for $s > 1$,

$$\begin{aligned} \|f^1\|_{H^s} &\le C\big(\|(\partial_1-\partial_2)\widetilde{\phi}\|_{H^s}\|\widetilde{\phi}\|_{H^{s+2}} + \|\widetilde{\phi}\|_{H^{s+1}}\|(\partial_2-\partial_1)\widetilde{\phi}\|_{H^{s+1}} \\ &\qquad + \|(\partial_1-\partial_2)\widetilde{\psi}\|_{H^s}\|\widetilde{\psi}\|_{H^{s+2}} + \|\widetilde{\psi}\|_{H^{s+1}}\|(\partial_2-\partial_1)\widetilde{\psi}\|_{H^{s+1}}\big) \\ &\le Ce^{-2C_0 t}\big(\|(\partial_1-\partial_2)\widetilde{\phi}\|_{H^s}\|\widetilde{\phi}\|_{H^{s+2}} + \|\widetilde{\phi}\|_{H^{s+1}}\|(\partial_2-\partial_1)\widetilde{\phi}\|_{H^{s+1}}\big) \\ &\qquad + Ce^{-2C_0 t}\big(\|(\partial_1-\partial_2)\widetilde{\psi}\|_{H^s}\|\widetilde{\psi}\|_{H^{s+2}} + \|\widetilde{\psi}\|_{H^{s+1}}\|(\partial_2-\partial_1)\widetilde{\psi}\|_{H^{s+1}}\big) \end{aligned}$$

$$\le C\varepsilon e^{-2C_0 t}\big(\|\widetilde{\phi}\|_{H^s}\|\widetilde{\phi}\|_{H^{s+2}} + \|\widetilde{\phi}\|_{H^{s+1}}\|\widetilde{\phi}\|_{H^{s+1}}\big)$$
$$+ C\varepsilon e^{-2C_0 t}\big(\|\widetilde{\psi}\|_{H^s}\|\widetilde{\psi}\|_{H^{s+2}} + \|\widetilde{\psi}\|_{H^{s+1}}\|\widetilde{\psi}\|_{H^{s+1}}\big)$$
$$\le C\varepsilon e^{-2C_0 t}\big(\|\widetilde{\phi}\|_{H^{s+2}}^2 + \|\widetilde{\psi}\|_{H^{s+2}}^2\big),$$

The second component of $f$ admits the same bound. Therefore,

$$\|f\|_{H^s} \le \|f^1\|_{H^s} + \|f^2\|_{H^s} \le C\varepsilon e^{-2C_0 t}\big(\|\widetilde{\phi}\|_{H^{s+2}}^2 + \|\widetilde{\psi}\|_{H^{s+2}}^2\big).$$

$\|g\|_{H^s}$ can be similarly estimated. This completes the proof of Lemma 3.1. □

The proof of Theorem 1.2 is close to that for Theorem 1.1 and we omit the details.

## Acknowledgements

Y. Dai was partially supported by China Scholarship Council. Z. Tan was partially supported by the National Natural Science Foundation of China Grant 11826025 and 11531010. J. Wu was partially supported by the NSF Grant DMS-1614246 and by the AT&T Foundation at the Oklahoma State University.



## References

[1] Abe, S., Thurner, S.: Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion. Physica A **356**, 403–407 (2005)
[2] Alfvén, H.: Existence of electromagnetic–hydrodynamic waves. Nature **150**, 405–406 (1942)
[3] Cao, C., Regmi, D., Wu, J.: The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. J. Differ. Equ. **254**, 2661–2681 (2013)
[4] Cao, C., Wu, J.: Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. **226**, 1803–1822 (2011)
[5] Cao, C., Wu, J., Yuan, B.: The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion. SIAM J. Math. Anal. **46**, 588–602 (2014)
[6] Chemin, J., McCormick, D., Robinson, J., Rodrigo, J.: Local existence for the non-resistive MHD equations in Besov spaces. Adv. Math. **286**, 1–31 (2016)
[7] Constantin, P., Majda, A.: The Beltrami spectrum for incompressible fluid flows. Commun. Math. Phys. **115**, 435–456 (1988)
[8] Dai, Y., Ji, R., Wu, J.: Unique weak solutions of the magnetohydrodynamic equation with fractional dissipation. submitted
[9] Davidson, P.A.: An Introduction to Magnetohydrodynamics. Cambridge University Press, Cambridge (2001)
[10] Dong, B., Jia, Y., Li, J., Wu, J.: Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion. J. Math. Fluid Mech. **20**, 1541–1565 (2018)
[11] Dong, B., Li, J., Wu, J.: Global regularity for the 2D MHD equations with partial hyperresistivity. International Math Research Notices. rnx240 (2018) https://doi.org/10.1093/imrn/rnx240
[12] Duvaut, G., Lions, J.L.: Inequations en thermoelasticite et magnetohydrodynamique. Arch. Ration. Mech. Anal. **46**, 241–279 (1972)
[13] Fan, J., Malaikah, H., Monaquel, S., Nakamura, G., Zhou, Y.: Global Cauchy problem of 2D generalized MHD equations. Monatsh. Math. **175**, 127–131 (2014)
[14] Fefferman, C., McCormick, D., Robinson, J., Rodrigo, J.: Higher order commutator estimates and local existence for the non-resistive MHD equations and related models. J. Funct. Anal. **267**, 1035–1056 (2014)
[15] Fefferman, C., McCormick, D., Robinson, J., Rodrigo, J.: Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces. Arch. Ration. Mech. Anal. **223**, 677–691 (2017)
[16] Frisch, U., Kurien, S., Pandit, R., Pauls, W., Ray, S., Wirth, A., Zhu, J.: Hyperviscosity, Galerkin truncation, and bottlenecks in turbulence. Phys. Rev. Lett. **101**, 264–502 (2008)
[17] Jara, M.: Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps. Commun. Pure Appl. Math. **62**, 198–214 (2009)

[18] Jiu, Q., Niu, D.: Mathematical results related to a two-dimensional magneto-hydrodynamic equations. Acta Math. Sci. Ser. B Engl. Ed. **26**, 744–756 (2006)
[19] Jiu, Q., Niu, D., Wu, J., Xu, X., Yu, H.: The 2D magnetohydrodynamic equations with magnetic diffusion. Nonlinearity **28**, 3935–3955 (2015)
[20] Jiu, Q., Zhao, J.: Global regularity of 2D generalized MHD equations with magnetic diffusion. Z. Angew. Math. Phys. **66**, 677–687 (2015)
[21] Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. Commun. Pure. Appl. Math. **41**, 891–907 (1988)
[22] Kenig, C., Ponce, G., Vega, L.: Well-posedness of the initial value problem for the Korteweg-de-Vries equation. J. Am. Math. Soc. **4**, 323–347 (1991)
[23] Laudau, L., Lifshitz, E.: Electrodynamics of Continuous Media, 2nd edn. Pergamon, New York (1984)
[24] Lin, H., Ji, R., Wu, J., Yan, L.: Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation. submitted for publication
[25] Lei, Z., Lin, F., Zhou, Y.: Structure of helicity and global solutions of incompressible Navier–Stokes equation. Arch. Ration. Mech. Anal. **218**, 1417–1430 (2015)
[26] Li, J., Tan, W., Yin, Z.: Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces. Adv. Math. **317**, 786–798 (2017)
[27] Li, J., Yang, M., Yu, Y.: A class large solution of the 2D MHD equations with velocity and magnetic damping. J. Math. Phys. **60**, 031503 (2019)
[28] Li, J., Yu, Y., Zhu, W.: A class large solution of the 3D Hall-magnetohydrodynamic equations. arXiv:1903.02299
[29] Majda, A., Bertozzi, A.: Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2002)
[30] Sermange, M., Temam, R.: Some mathematical questions related to the MHD equations. Commun. Pure Appl. Math. **36**, 635–664 (1983)
[31] Tao, T.: Nonlinear Dispersive Equations: Local and Global Analysis. CBMS Regional Conference Series in Mathematics, Providence. American Mathematical Society, RI (2006)
[32] Wan, R.: On the uniqueness for the 2D MHD equations without magnetic diffusion. Nonlinear Anal. Real World Appl. **30**, 32–40 (2016)
[33] Wu, J., Xu, X., Ye, Z.: Global smooth solutions to the n-dimensional damped models of incompressible fluid mechanics with small initial datum. J. Nonlinear Sci. **25**, 157–192 (2015)
[34] Wu, J.: Generalized MHD equations. J. Differ. Equ. **195**, 284–312 (2003)
[35] Wu, J.: Global regularity for a class of generalized magnetohydrodynamic equations. J. Math. Fluid Mech. **13**, 295–305 (2011)
[36] Wu, J.: The 2D magnetohydrodynamic equations with partial or fractional dissipation. Lectures on the analysis of nonlinear partial differential equations. Morningside Lectures on Mathematics, Part 5, MLM5, pp. 283–332. International Press, Somerville, MA (2018)
[37] Yamazaki, K.: On the global well-posedness of N-dimensional generalized MHD system in anisotropic spaces. Adv. Differ. Equ. **19**, 201–224 (2014)
[38] Yamazaki, K.: Remarks on the global regularity of the two-dimensional magnetohydrodynamics system with zero dissipation. Nonlinear Anal. **94**, 194–205 (2014)
[39] Yamazaki, K.: On the global regularity of two-dimensional generalized magnetohydrodynamics system. J. Math. Anal. Appl. **416**, 99–111 (2014)
[40] Yamazaki, K.: Global regularity of logarithmically supercritical MHD system with zero diffusivity. Appl. Math. Lett. **29**, 46–51 (2014)
[41] Yamazaki, K.: Global regularity of logarithmically supercritical MHD system with improved logarithmic powers. Dyn. Partial Differ. Equ. **15**, 147–173 (2018)
[42] Yang, W., Jiu, Q., Wu, J.: The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation. J. Differ. Equ. **266**, 630–652 (2019)
[43] Zhang, H.: Global large smooth solutions for 3-D Hall-magnetohydrodynamics. arXiv:1903.03212v1
[44] Yuan, B., Zhao, J.: Global regularity of 2D almost resistive MHD equations. Nonlinear Anal. Real World Appl. **41**, 53–65 (2018)
[45] Zhou, Yi, Zhu, Yi: A class of large solutions to the 3D incompressible MHD and Euler equations with damping. Acta Math. Sin. (Engl. Ser.) **34**, 63–78 (2018)

Yichen Dai
School of Mathematical Sciences
Xiamen University
Xiamen 361005 Fujian
People's Republic of China
e-mail: daiyichen@stu.xmu.edu.cn

Yichen Dai and Jiahong Wu
Department of Mathematics
Oklahoma State University
Stillwater OK74078
USA
e-mail: jiahong.wu@okstate.edu

Zhong Tan
School of Mathematical Sciences and Fujian Provincial Key Laboratory
on Mathematical Modeling and Scientific Computing
Xiamen University
Xiamen 361005
People's Republic of China
e-mail: tan85@xmu.edu.cn

(Received: April 29, 2019; revised: August 27, 2019)