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Unique weak solutions of the magnetohydrodynamic equations with fractional dissipation

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This paper examines the existence and uniqueness of weak solutions to the d-dimensional magnetohydrodynamic (MHD) equations with fractional dissipation $(-\Delta)^\alpha u$ and fractional magnetic diffusion $(-\Delta)^\beta b$. The aim is at the uniqueness of weak solutions in the weakest possible inhomogeneous Besov spaces. We establish the local existence and uniqueness in the functional setting $u \in L^\infty(0,T;B_{2,1}^{d/2-2\alpha+1}(\mathbb{R}^d))$ and $b \in L^\infty(0,T;B_{2,1}^{d/2}\mathbb{R}^d))$ when $\alpha > 1/2$, $\beta \geq 0$ and $\alpha+\beta \geq 1$. The case when $\alpha = 1$ with $\nu > 0$ and $\eta = 0$ has previously been studied in [7, 19]. However, their approaches can not be directly extended to the fractional case when $\alpha < 1$ due to the breakdown of a bilinear estimate. By decomposing the bilinear term into different frequencies, we are able to obtain a suitable upper bound on the bilinear term for $\alpha < 1$, which allows us to close the estimates in the aforementioned Besov spaces.

KEYWORDS
Littlewood-Paley, local solution, magnetohydrodynamic equations, uniqueness

2010 MATHEMATICS SUBJECT CLASSIFICATION
35A05, 35Q35, 76D03

1 | INTRODUCTION

This paper examines the existence and uniqueness of weak solutions to the d-dimensional incompressible magnetohydrodynamic (MHD) equations with fractional dissipation,

$$
\begin{cases}
\partial_t u + u\cdot\nabla u + \nu(-\Delta)^\alpha u = -\nabla P + b\cdot\nabla b, \quad x\in\mathbb{R}^d,\ t>0,\\
\partial_t b + u\cdot\nabla b + \eta(-\Delta)^\beta b = b\cdot\nabla u, \quad x\in\mathbb{R}^d,\ t>0,\\
\nabla\cdot u = \nabla\cdot b = 0, \quad x\in\mathbb{R}^d,\ t>0,\\
u(x,0)=u_0(x), \quad b(x,0)=b_0(x),
\end{cases}
\tag{1.1}
$$

where u, P and b represent the velocity, the pressure and the magnetic field, respectively, and $\nu > 0$, $\eta > 0$, $\alpha \geq 0$ and $\beta \geq 0$ are real parameters. The fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\,\hat{f}(\xi),$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx.$$

The standard MHD equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. The justification for the study of this fractionally dissipated system can be made from several different perspectives. First, (1.1) represents a two-parameter family of systems and contains the MHD systems with standard Laplacian dissipation as special cases. (1.1) allows us to simultaneously examine a whole family of equations and potentially reveals how the properties of its solutions are related to the sizes of α and β. Second, the fractional diffusion operators can model the so-called anomalous diffusion, a much studied topic in physics, probability and finance (see, e.g., [1, 15]). In particular, (1.1) can model long-range diffusive interactions. Third, fractional dissipation has been used in turbulence modeling to control the effective range of the non-local dissipation and to make numerical resolutions more efficient (see, e.g., [14]).

The MHD equations have always been of great interest in mathematics. Mathematically rigorous foundational work has been laid in [10] and [22]. Recently, the MHD equations have gained renewed interests and there have been substantial developments on the well-posedness problem, especially when the MHD equations involve only partial or fractional dissipation. A summary on some of the recent results can be found in a review paper [27]. Roughly speaking, there are two different focuses on the well-posedness problem. One is the global existence and regularity of classical solutions while the other is the uniqueness of solutions in a weak functional setting. The first focus intends to establish the global existence of classical solutions with the smallest fractional indices. A general result on the global existence and uniqueness of classical solutions to the d-dimensional MHD equations with fractional dissipation can be found in [25] and [26]. A special consequence of this result asserts the global well-posedness of (1.1) when

$$\alpha \geq \frac{1}{2}+\frac{d}{4}, \quad \beta > 0, \quad \alpha+\beta \geq 1+\frac{d}{2}.$$

The case when $\alpha \geq 1+\frac{d}{2}$ with $\nu > 0$ and $\eta = 0$ is obtained by K. Yamazaki [31]. Logarithmic improvements of these fractional powers are also contained in [26] and [31]. A very recent work establishes the global well-posedness with only directional hyperviscosity [32]. Many more exciting results on the global regularity problem are available for the 2D case (see, e.g., [4–6, 8, 9, 11, 17, 18, 28–31, 33]).

The other focus on (1.1) is to establish the existence and uniqueness of local solutions in a weakest possible functional setting. There is a stream of progress in this direction on (1.1) with $\alpha = 1$, $\nu > 0$ and $\eta = 0$. Q. Jiu and D. Niu [16] proved the local well-poseness of (1.1) in the Sobolev space H^s with $s \geq 3$. Fefferman, McCormick, Robinson and Rodrigo were able to weaken the regularity assumption to $(u_0, b_0) \in H^s$ with $s > \frac{d}{2}$ in [12] and then to $u_0 \in H^{s-1-\epsilon}$ and $b_0 \in H^s$ with $s > \frac{d}{2}$ in [13]. Chemin, McCormick, Robinson and Rodrigo [7] made further improvement by assuming only $u_0 \in B_{2,1}^{\frac{d}{2}-1}$ and $b_0 \in B_{2,1}^{\frac{d}{2}}$. They obtained the local existence for $d = 2$ and 3, and the uniqueness for $d = 3$. R. Wan [24] obtained the uniqueness for $d = 2$. J. Li, W. Tan and Z. Yin [19] recently made an important progress by reducing the functional setting to homogeneous Besov space $u_0 \in \dot{B}_{p,1}^{\frac{d}{p}-1}$ and $b_0 \in \dot{B}_{p,1}^{\frac{d}{p}}$ with $p \in [1, 2d]$.

The aim of this paper is to establish the local existence and uniqueness when the initial data $u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)$ and $b_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)$ for the largest possible ranges of α and β. We remark that our result will not be just a parallel extension of the results for the case when $\alpha = 1$. As we shall explain in detail later, the proofs in [7] and [19] can not be directly extended to the case when $\alpha < 1$. Our main result can be stated as follows.

Theorem 1.1. *Let $d \geq 2$. Consider (1.1) with α and β satisfying*

$$\alpha > \frac{1}{2}, \quad \beta \geq 0, \quad \alpha+\beta \geq 1.$$

Assume the initial data (u_0, b_0) obeys $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ and

$$u_0 \in B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d), \qquad b_0 \in B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d). \tag{1.2}$$

Then, there exist $T > 0$ and a unique weak solution (u, b) of (1.1) on $[0, T]$ satisfying

$$u \in C([0,T]; B_{2,1}^{\frac{d}{2}+1-2\alpha}(\mathbb{R}^d)) \cap L^1(0,T; B_{2,1}^{\frac{d}{2}+1}(\mathbb{R}^d)), \tag{1.3}$$

$$b \in C([0,T]; B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)) \cap L^1(0,T; B_{2,1}^{\frac{d}{2}+2\beta}(\mathbb{R}^d)). \tag{1.4}$$

The Besov spaces in Theorem 1.1 are defined in Section 2. Theorem 1.1 with $\alpha = 1$, $\nu > 0$ and $\eta = 0$ recovers the results in some of the previous work. We describe the framework in the proof of Theorem 1.1 and explain why the approaches for the case $\alpha = 1$, $\nu > 0$ and $\eta = 0$ in [7] and [19] can not be directly extended to the case $\alpha < 1$. The existence of weak solutions stated in Theorem 1.1 is proven through a successive approximation process, which starts with the construction of a successive approximation sequence $(u^{(n)}, b^{(n)})$ satisfying

$$\begin{cases} u^{(1)} = S_2 u_0, \quad b^{(1)} = S_2 b_0, \\ \partial_t u^{(n+1)} + \nu(-\Delta)^\alpha u^{(n+1)} = \mathbb{P}(-u^{(n)} \cdot \nabla u^{(n+1)} + b^{(n)} \cdot \nabla b^{(n)}), \\ \partial_t b^{(n+1)} + \eta(-\Delta)^\beta b^{(n+1)} = -u^{(n)} \cdot \nabla b^{(n+1)} + b^{(n)} \cdot \nabla u^{(n)}, \\ \nabla \cdot u^{(n+1)} = 0, \quad \nabla \cdot b^{(n+1)} = 0, \\ u^{(n+1)}(x,0) = S_{n+1} u_0, \quad b^{(n+1)}(x,0) = S_{n+1} b_0, \end{cases}$$

where $\mathbb{P}$ is the standard Leray projection and S_j is the standard inhomogeneous low frequency cutoff function (see Section 2 for its definition). The next step is to define a suitable functional setting Y and show that, if (u_0, b_0) satisfies (1.2), then the sequence $(u^{(n)}, b^{(n)})$ is bounded uniformly in Y. The precise definition of Y is given in (3.2) in Section 3. The uniform boundedness is shown via an iterative process. We assume $(u^{(n)}, b^{(n)}) \in Y$ and show $(u^{(n+1)}, b^{(n+1)}) \in Y$. To do so, we need to evaluate

$$\|b^{(n)} \cdot \nabla b^{(n)}\|_{B^{\frac{d}{2}-2\alpha+1}_{2,1}(\mathbb{R}^d)}. \tag{1.5}$$

In the case when $\alpha = 1$, this term can be bounded suitably through the product estimate

$$\|b^{(n)} \cdot \nabla b^{(n)}\|_{B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)} \le \|b^{(n)} \otimes b^{(n)}\|_{B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)} \le C\, \|b^{(n)}\|^2_{B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)}$$

based on the following lemma (see, e.g., [2, p.90] or Lemma 2.6 in [19]).

Lemma 1.2. *Let* $1 \le p \le \infty$, $s_1, s_2 \le \frac{d}{p}$ *and* $s_1 + s_2 > d\, \max\{0, \frac{2}{p} - 1\}$. *Then*

$$\|f g\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}(\mathbb{R}^d)} \le C\, \|f\|_{\dot{B}^{s_1}_{p,1}(\mathbb{R}^d)}\, \|g\|_{\dot{B}^{s_2}_{p,1}(\mathbb{R}^d)}.$$

However, when $\alpha < 1$, Lemma 1.2 does not appear to be applicable. We can still write (1.5) as

$$\|b^{(n)} \cdot \nabla b^{(n)}\|_{B^{\frac{d}{2}-2\alpha+1}_{2,1}(\mathbb{R}^d)} \le \|b^{(n)} \otimes b^{(n)}\|_{B^{\frac{d}{2}-2\alpha+2}_{2,1}(\mathbb{R}^d)} = \|b^{(n)} \otimes b^{(n)}\|_{B^{(\frac{d}{2}-\alpha+1)+(\frac{d}{2}-\alpha+1)-\frac{d}{2}}_{2,1}}.$$

For $\alpha < 1$, $s_1 = \frac{d}{2} - \alpha + 1$ and $s_2 = \frac{d}{2} - \alpha + 1$ no longer satisfy the condition

$$s_1, s_2 \le \frac{d}{2}$$

and Lemma 1.2 is not applicable. We are able to overcome this difficulty by performing a detailed analysis on different frequencies of this product and making full use of the available dissipation in the case when $\alpha < 1$. The suitable estimate obtained for this product allows us to conclude that $(u^{(n+1)}, b^{(n+1)})$ is indeed in Y. The next step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma and show the limit (u, b) is indeed a weak solution of (1.1). The uniqueness of weak solutions in the regularity class (1.3) and (1.4) can be established by directly working with the L^2-norm of the difference between any two weak solutions.

The rest of this paper is divided into three sections. Section 2 provides the definitions of the Besov spaces and related tools. In addition, we prove two bounds on triple products involving Fourier localized functions. These bounds are repeatedly used in the subsequent sections. Section 3 proves the existence part of Theorem 1.1 while Section 4 establishes the uniqueness part of Theorem 1.1.

2 | PREPARATION

This section serves as a preparation. We provide the definition of the Besov spaces and related facts to be used in the subsequent sections. More details can be found in several books and many papers (see, e.g., [2, 3, 20, 21, 23]). In addition, we prove bounds on triple products involving Fourier localized functions to be used extensively in the sections that follow.

We start with the partition of unit. Let $B(0,r)$ and $\mathcal{C}(0,r_1,r_2)$ denote the standard ball and the annulus, respectively,

$$B(0,r)=\left\{\xi\in\mathbb{R}^d\ :\ |\xi|\le r\right\},\qquad \mathcal{C}(0,r_1,r_2)=\left\{\xi\in\mathbb{R}^d\ :\ r_1\le|\xi|\le r_2\right\}.$$

There are two compactly supported smooth radial functions ϕ and ψ satisfying

$$\begin{aligned}&\operatorname{supp}\phi\subset B(0,4/3),\quad \operatorname{supp}\psi\subset\mathcal{C}(0,3/4,8/3),\\ &\phi(\xi)+\sum_{j\ge0}\psi(2^{-j}\xi)=1\qquad\text{for all }\xi\in\mathbb{R}^d.\end{aligned}\tag{2.1}$$

We use $\widetilde{h}$ and h to denote the inverse Fourier transforms of ϕ and ψ respectively,

$$\widetilde{h}=\mathcal{F}^{-1}\phi,\quad h=\mathcal{F}^{-1}\psi.$$

In addition, for notational convenience, we write $\psi_j(\xi)=\psi(2^{-j}\xi)$. By a simple property of the Fourier transform,

$$h_j(x):=\mathcal{F}^{-1}(\psi_j)(x)=2^{dj}\,h(2^jx).$$

The inhomogeneous dyadic block operator Δ_j are defined as follows

$$\begin{aligned}&\Delta_jf=0\qquad\text{for } j\le-2,\\ &\Delta_{-1}f=\widetilde{h}*f=\int_{\mathbb{R}^d}f(x-y)\widetilde{h}(y)\,dy,\\ &\Delta_jf=h_j*f=2^{dj}\int_{\mathbb{R}^d}f(x-y)\,h(2^jy)\,dy\qquad\text{for } j\ge0.\end{aligned}$$

The corresponding inhomogeneous low frequency cut-off operator S_j is defined by

$$S_jf=\sum_{k\le j-1}\Delta_kf.$$

For any function f in the usual Schwarz class $\mathcal{S}$, (2.1) implies

$$\widehat{f}(\xi)=\phi(\xi)\,\widehat{f}(\xi)+\sum_{j\ge0}\psi(2^{-j}\xi)\,\widehat{f}(\xi)\tag{2.2}$$

or, in terms of the inhomogeneous dyadic block operators,

$$f=\sum_{j\ge-1}\Delta_jf\quad\text{or}\quad \mathrm{Id}=\sum_{j\ge-1}\Delta_j,$$

where Id denotes the identity operator. More generally, for any F in the space of tempered distributions, denoted $\mathcal{S}'$, (2.2) still holds but in the distributional sense. That is, for $F\in\mathcal{S}'$,

$$F=\sum_{j\ge-1}\Delta_jF\quad\text{or}\quad \mathrm{Id}=\sum_{j\ge-1}\Delta_j\qquad\text{in}\quad\mathcal{S}'.\tag{2.3}$$

In fact, one can verify that

$$S_jF:=\sum_{k\le j-1}\Delta_kF\to F\quad as\quad j\to\infty\quad\text{in}\quad\mathcal{S}'.$$

(2.3) is referred to as the Littlewood-Paley decomposition for tempered distributions.

In terms of the inhomogeneous dyadic block operators, we can write the standard product in terms of the paraproducts, namely

$$F\,G=\sum_{|j-k|\le 2} S_{k-1}F\,\Delta_k G+\sum_{|j-k|\le 2}\Delta_k F\,S_{k-1}G+\sum_{k\ge j-1}\Delta_k F\,\widetilde{\Delta}_k G,$$

where $\widetilde{\Delta}_k=\Delta_{k-1}+\Delta_k+\Delta_{k+1}$. Due to $\nabla\cdot F=0$. This is the so-called Bony decomposition.

The inhomogeneous Besov space can be defined in terms of Δ_j specified as above.

Definition 2.1. The inhomogeneous Besov space $B^s_{p,q}$ with $1\le p,q\le\infty$ and $s\in\mathbb{R}$ consists of $f\in S'$ satisfying

$$\|f\|_{B^s_{p,q}}\equiv\|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q}<\infty.$$

Bernstein's inequality is a useful tool on Fourier localized functions and these inequalities trade derivatives for integrability. The following lemma provides Bernstein type inequalities for fractional derivatives.

Lemma 2.2. *Let $\alpha\ge 0$. Let $1\le p\le q\le\infty$.*

1) If f satisfies

$$supp\,\widehat{f}\subset\{\xi\in\mathbb{R}^d:\ |\xi|\le K2^j\},$$

for some integer j and a constant $K>0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)}\le C_1\,2^{2\alpha j+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$supp\,\widehat{f}\subset\{\xi\in\mathbb{R}^d:\ K_1 2^j\le|\xi|\le K_2 2^j\}$$

for some integer j and constants $0<K_1\le K_2$, then

$$C_1\,2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)}\le\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)}\le C_2\,2^{2\alpha j+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

Next we state and prove bounds for the triple products involving Fourier localized functions. These bounds will be used quite frequently in the proof of Theorem 1.1 in the subsequent section.

Lemma 2.3. *Let $j\ge 0$ be an integer. Let Δ_j be the inhomogeneous Littlewood-Paley localization operator. Let F be a divergence-free vector field. Then*

$$\left|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j H\,dx\right|\le C\,\|\Delta_j H\|_{L^2}\Bigg(2^j\sum_{m\le j-1}2^{\frac{d}{2}m}\|\Delta_m F\|_{L^2}\sum_{|j-k|\le 2}\|\Delta_k G\|_{L^2}$$
$$+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\sum_{m\le j-1}2^{(1+\frac{d}{2})m}\|\Delta_m G\|_{L^2}+\sum_{k\ge j-1}2^j\,2^{\frac{d}{2}k}\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^2}\Bigg)\tag{2.4}$$

and

$$\left|\int_{\mathbb{R}^d} \Delta_j(F\cdot\nabla G)\cdot\Delta_j G\,dx\right| \le C\,\|\Delta_j G\|_{L^2}\Bigg(\sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\,\|\Delta_m F\|_{L^2}\sum_{|j-k|\le 2}\|\Delta_k G\|_{L^2}$$
$$+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\sum_{m\le j}2^{(1+\frac{d}{2})m}\,\|\Delta_m G\|_{L^2}+\sum_{k\ge j-1}2^j\,2^{\frac{d}{2}k}\,\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^2}\Bigg). \tag{2.5}$$

Proof. By the paraproduct decomposition,

$$\Delta_j(F\cdot\nabla G)=\sum_{|j-k|\le 2}\Delta_j(S_{k-1}F\cdot\Delta_k\nabla G)+\sum_{|j-k|\le 2}\Delta_j(\Delta_k F\cdot S_{k-1}\nabla G)+\sum_{k\ge j-1}\Delta_j(\Delta_k F\cdot\nabla\widetilde{\Delta}_k G).$$

By Hölder's inequality and Bernstein's inequality in Lemma 2.2,

$$\left|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j H\,dx\right|\le\|\Delta_j H\|_{L^2}\Bigg(\sum_{|j-k|\le 2}2^k\,\|S_{k-1}F\|_{L^\infty}\,\|\Delta_k G\|_{L^2}+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\,\|S_{k-1}\nabla G\|_{L^\infty}$$
$$+\sum_{k\ge j-1}2^j\,\|\Delta_k F\|_{L^2}\|\widetilde{\Delta}_k G\|_{L^\infty}\Bigg),$$

where we have used $\nabla\cdot F=0$ in the last part. (2.4) then follows if we invoke the inequalities of the form

$$\|S_{k-1}F\|_{L^\infty}\le\sum_{m\le k-2}2^{\frac{d}{2}m}\,\|\Delta_m F\|_{L^2}. \tag{2.6}$$

To prove (2.5), we further write the first term as the sum of a commutator and two correction terms,

$$\Delta_j(F\cdot\nabla G)=\sum_{|j-k|\le 2}[\Delta_j,S_{k-1}F\cdot\nabla]\Delta_k G+\sum_{|j-k|\le 2}(S_{k-1}F-S_jF)\cdot\Delta_j\Delta_k\nabla G$$
$$+S_jF\cdot\nabla\Delta_j G+\sum_{|j-k|\le 2}\Delta_j(\Delta_k F\cdot S_{k-1}\nabla G)+\sum_{k\ge j-1}\Delta_j(\Delta_k F\cdot\nabla\widetilde{\Delta}_k G),$$

where $\widetilde{\Delta}_k=\Delta_{k-1}+\Delta_k+\Delta_{k+1}$. Due to $\nabla\cdot F=0$,

$$\int_{\mathbb{R}^d}S_jF\cdot\nabla\Delta_j G\cdot\Delta_j G\,dx=0.$$

By Hölder's inequality, Bernstein's inequality and a commutator estimate,

$$\left|\int_{\mathbb{R}^d}\Delta_j(F\cdot\nabla G)\cdot\Delta_j G\,dx\right|\le\|\Delta_j G\|_{L^2}\Bigg(\sum_{|j-k|\le 2}\|\nabla S_{k-1}F\|_{L^\infty}\|\Delta_k G\|_{L^2}+C\,2^{(1+\frac{d}{2})j}\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\,\|\Delta_j G\|_{L^2}$$
$$+\sum_{|j-k|\le 2}\|\Delta_k F\|_{L^2}\,\|S_{k-1}\nabla G\|_{L^\infty}+\sum_{k\ge j-1}2^j\,2^{\frac{d}{2}k}\,\|\Delta_k F\|_{L^2}\,\|\widetilde{\Delta}_k G\|_{L^2}\Bigg).$$

(2.5) then follows when we invoke similar inequalities as (2.6). This completes the proof of Lemma 2.3. □

3 | PROOF FOR THE EXISTENCE PART OF THEOREM 1.1

This section proves the existence part of Theorem 1.1. The approach is to construct a successive approximation sequence and show that the limit of a subsequence actually solves (1.1) in the weak sense.

Proof for the existence part of Theorem 1.1. We consider a successive approximation sequence $\{(u^{(n)}, b^{(n)})\}$ satisfying

$$\begin{cases} u^{(1)} = S_2 u_0, \quad b^{(1)} = S_2 b_0, \\ \partial_t u^{(n+1)} + \nu(-\Delta)^\alpha u^{(n+1)} = \mathbb{P}(-u^{(n)} \cdot \nabla u^{(n+1)} + b^{(n)} \cdot \nabla b^{(n)}), \\ \partial_t b^{(n+1)} + \eta(-\Delta)^\beta b^{(n+1)} = -u^{(n)} \cdot \nabla b^{(n+1)} + b^{(n)} \cdot \nabla u^{(n)}, \\ \nabla \cdot u^{(n+1)} = 0, \quad \nabla \cdot b^{(n+1)} = 0, \\ u^{(n+1)}(x,0) = S_{n+1} u_0, \quad b^{(n+1)}(x,0) = S_{n+1} b_0, \end{cases} \tag{3.1}$$

where $\mathbb{P}$ is the standard Leray projection. For

$$M = 2\left(\|u_0\|_{B^{\frac{d}{2}+1-2\alpha}_{2,1}} + \|b_0\|_{B^{\frac{d}{2}}_{2,1}} \right),$$

$T > 0$ being sufficiently small and $0 < \delta < 1$ (to be specified later), we set

$$Y \equiv \left\{ (u,b) \Big| \ \|u\|_{L^\infty(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1})} \le M, \quad \|b\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \le M, \|u\|_{L^1(0,T;\,B^{\frac{d}{2}+1}_{2,1})} \le \delta, \quad \|b\|_{L^1(0,T;\,B^{\frac{d}{2}+2\beta}_{2,1})} \le \delta \right\}. \tag{3.2}$$

Our goal is to show that $\{(u^{(n)}, b^{(n)})\}$ has a subsequence that converges to the weak solution of (1.1). This process consists of three main steps. The first step is to show that $(u^{(n)}, b^{(n)})$ is uniformly bounded in Y. The second step is to extract a strongly convergent subsequence via the Aubin-Lions Lemma while the last step is to show that the limit is indeed a weak solution of (1.1).

Our main effort is devoted to showing the uniform bound for $(u^{(n)}, b^{(n)})$ in Y. This is proven by induction. Clearly,

$$\|u^{(1)}\|_{L^\infty(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1})} \le M, \quad \|b^{(1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \le M.$$

If $T > 0$ is sufficiently small, then

$$\|u^{(1)}\|_{L^1(0,T;B^{\frac{d}{2}+1}_{2,1})} \le T\|S_2 u_0\|_{B^{\frac{d}{2}+1}_{2,1}} \le T\,C\,\|u_0\|_{B^{\frac{d}{2}+1-2\alpha}_{2,1}} \le \delta,$$

$$\|b^{(1)}\|_{L^1(0,T;B^{\frac{d}{2}+2\beta}_{2,1})} \le T\,\|S_2 b_0\|_{B^{\frac{d}{2}+2\beta}_{2,1}} \le T\,C\,\|b_0\|_{B^{\frac{d}{2}}_{2,1}} \le \delta.$$

Assuming that $(u^{(n)}, b^{(n)})$ obeys the bounds defined in Y, namely

$$\|u^{(n)}\|_{L^\infty(0,T;B^{\frac{d}{2}+1-2\alpha}_{2,1})} \le M, \quad \|b^{(n)}\|_{L^\infty(0,T;\,B^{\frac{d}{2}}_{2,1})} \le M,$$

$$\|u^{(n)}\|_{L^1(0,T;\,B^{\frac{d}{2}+1}_{2,1})} \le \delta, \quad \|b^{(n)}\|_{L^1(0,T;\,B^{\frac{d}{2}+2\beta}_{2,1})} \le \delta,$$

we prove that $(u^{(n+1)}, b^{(n+1)})$ obeys the same bound for suitably selected $T > 0$ and $\delta > 0$. For the sake of clarity, the proof of the four bounds is achieved in the following four subsections.

3.1 | The estimate of $u^{(n+1)}$ in $B^{1+\frac{d}{2}-2\alpha}_{2,1}(\mathbb{R}^d)$

Let $j \ge 0$ be an integer. Applying Δ_j to the second equation in (3.1) and then dotting with $\Delta_j u^{(n+1)}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^2}^2 + \nu\|\Lambda^\alpha \Delta_j u^{(n+1)}\|_{L^2}^2 = A_1 + A_2, \tag{3.3}$$

where

$$A_1 = -\int \Delta_j(u^{(n)} \cdot \nabla u^{(n+1)}) \cdot \Delta_j u^{(n+1)}\, dx,$$

$$A_2 = \int \Delta_j(b^{(n)} \cdot \nabla b^{(n)}) \cdot \Delta_j u^{(n+1)}\, dx.$$

We remark that the projection operator $\mathbb{P}$ has been eliminated due to the divergence-free condition $\nabla \cdot u^{(n+1)} = 0$. The dissipative part admits a lower bound

$$\nu\|\Lambda^\alpha \Delta_j u^{(n+1)}\|_{L^2}^2 \geq C_0\, 2^{2\alpha j}\, \|\Delta_j u^{(n+1)}\|_{L^2}^2,$$

where $C_0 > 0$ is a constant. According to Lemma 2.3, A_1 can be bounded by

$$|A_1| \leq C\, \|\Delta_j u^{(n+1)}\|_{L^2}^2 \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2} + C\, \|\Delta_j u^{(n+1)}\|_{L^2}\, \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2}$$
$$+ C\, \|\Delta_j u^{(n+1)}\|_{L^2}\, 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k}\, \|\Delta_k u^{(n)}\|_{L^2}\, \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}.$$

Also by Lemma 2.3, A_2 is bounded by

$$|A_2| \leq C\, \|\Delta_j u^{(n+1)}\|_{L^2}\, 2^j\, \|\Delta_j b^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2} + C\, \|\Delta_j u^{(n+1)}\|_{L^2}\, \|\Delta_j b^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n)}\|_{L^2}$$
$$+ C\, \|\Delta_j u^{(n+1)}\|_{L^2}\, 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k}\, \|\Delta_k b^{(n)}\|_{L^2}\, \|\widetilde{\Delta}_k b^{(n)}\|_{L^2}.$$

Inserting the estimates above in (3.3) and eliminating $\|\Delta_j u^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^2} + C_0\, 2^{2\alpha j}\|\Delta_j u^{(n+1)}\|_{L^2} \leq J_1 + \cdots + J_6, \tag{3.4}$$

where

$$J_1 = C\, \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},$$

$$J_2 = C\, \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n+1)}\|_{L^2}$$

$$J_3 = C\, 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k}\, \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\, \|\Delta_k u^{(n)}\|_{L^2},$$

$$J_4 = C\, 2^j\, \|\Delta_j b^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2},$$

$$J_5 = C\, \|\Delta_j b^{(n)}\|_{L^2} \sum_{m\leq j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n)}\|_{L^2},$$

$$J_6 = C\, 2^j \sum_{k\geq j-1} 2^{\frac{d}{2}k}\, \|\Delta_k b^{(n)}\|_{L^2}\, \|\widetilde{\Delta}_k b^{(n)}\|_{L^2}.$$

Integrating (3.4) in time yields

$$\|\Delta_j u^{(n+1)}(t)\|_{L^2} \leq e^{-C_0\, 2^{2\alpha j} t}\, \|\Delta_j u_0^{(n+1)}\|_{L^2} + \int_0^t e^{-C_0\, 2^{2\alpha j}(t-\tau)}(J_1 + \cdots + J_6)\, d\tau. \tag{3.5}$$

Multiplying (3.5) by $2^{(1+\frac{d}{2}-2\alpha)j}$ and summing over j, we have

$$\|u^{(n+1)}(t)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} \le \|u_0^{(n+1)}\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}} + \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-C_0\, 2^{2\alpha j}(t-\tau)} (J_1 + \cdots + J_6)\, d\tau. \tag{3.6}$$

The terms on the right-hand side can be estimated as follows. Recalling the definition of J_1 above and using the inductive assumption on $u^{(n)}$, we have, for any $t \le T$,

$$\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-C_0\, 2^{2\alpha j}(t-\tau)}\, J_1\, d\tau \\
&\le C \int_0^t \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\
&= C\, \|u^{(n+1)}\|_{L^\infty\left(0,t;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \|u^{(n)}\|_{L^1\left(0,t;B_{2,1}^{1+\frac{d}{2}}\right)} \\
&\le C\, \|u^{(n+1)}\|_{L^\infty\left(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \|u^{(n)}\|_{L^1\left(0,T;B_{2,1}^{1+\frac{d}{2}}\right)} \\
&\le C\,\delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}$$

The term involving J_2 admits the same bound. In fact, by Young's inequality for series convolution,

$$\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-C_0\, 2^{2\alpha j}(t-\tau)}\, J_2\, d\tau \\
&\le C \int_0^t \sum_j 2^{(1+\frac{d}{2})j} \|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j} 2^{2\alpha(m-j)} 2^{(1+\frac{d}{2}-2\alpha)m} \|\Delta_m u^{(n+1)}(\tau)\|_{L^2}\, d\tau \\
&\le C \int_0^t \|u^{(n)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}}} \|u^{(n+1)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}}\, d\tau \\
&\le C\,\delta\, \|u^{(n+1)}\|_{L^\infty\left(0,t;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \\
&\le C\,\delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.
\end{aligned}$$

The term with J_3 is bounded by

$$\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \int_0^t e^{-C_0\, 2^{2\alpha j}(t-\tau)}\, J_3\, d\tau \\
&= \int_0^t \sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\, 2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2} \|\Delta_k u^{(n)}\|_{L^2}\, d\tau \\
&= C \int_0^t \sum_j \sum_{k\ge j-1} 2^{(2+\frac{d}{2}-2\alpha)(j-k)}\, 2^{(1+\frac{d}{2})k} \|\Delta_k u^{(n)}\|_{L^2} 2^{(1+\frac{d}{2}-2\alpha)k} \|\widetilde{\Delta}_k u^{(n+1)}\|_{L^2}\, d\tau \\
&\le C \int_0^t \|u^{(n)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}}} \|u^{(n+1)}(\tau)\|_{B_{2,1}^{1+\frac{d}{2}-2\alpha}}\, d\tau \\
&\le C\, \|u^{(n)}\|_{L^1\left(0,T;B_{2,1}^{1+\frac{d}{2}}\right)} \|u^{(n+1)}\|_{L^\infty\left(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \\
&\le C\,\delta\, \|u^{(n+1)}\|_{L^\infty\left(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)},
\end{aligned}$$

where we have used Young's inequality for convolution. Since $\alpha > \frac{1}{2}$, we choose $0 < \rho < \frac{1}{2}$ such that

$$\alpha - \frac{1}{2} - \rho > 0. \tag{3.7}$$

Using the elementary bound

$$2^{(2\alpha-4\rho)j}(t-\tau)^{\frac{2\alpha-4\rho}{2\alpha}}\, e^{-C_0\, 2^{2\alpha j}(t-\tau)} \le C, \tag{3.8}$$

we obtain

$$\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0\,2^{2\alpha j}(t-\tau)}\, J_4\, d\tau\\
&= \sum_j \int_0^t e^{-C_0\,2^{2\alpha j}(t-\tau)}\, 2^{(1+\frac{d}{2}-2\alpha)j}\, 2^j\, \|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2}\, d\tau\\
&\le C\,\|b^{(n)}\|_{L^\infty(0,t;B^{\frac{d}{2}}_{2,1})} \sum_j \int_0^t 2^{-(2\alpha-4\rho)j}(t-\tau)^{-\frac{2\alpha-4\rho}{2\alpha}}\, 2^{(1+\frac{d}{2}-2\alpha)j}\, 2^j\, \|\Delta_j b^{(n)}\|_{L^2}\, d\tau\\
&\le C\,\|b^{(n)}\|_{L^\infty(0,t;B^{\frac{d}{2}}_{2,1})} \int_0^t \sum_j 2^{-4\alpha j+2j+4\rho j}\, 2^{\frac{d}{2}j}\, \|\Delta_j b^{(n)}\|_{L^2}\, (t-\tau)^{-\frac{2\alpha-4\rho}{2\alpha}}\, d\tau\\
&\le C\,\|b^{(n)}\|^2_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \int_0^t (t-\tau)^{-\frac{2\alpha-4\rho}{2\alpha}}\, d\tau\\
&= \frac{C}{\rho}\, t^{\frac{2\rho}{\alpha}}\, \|b^{(n)}\|^2_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \le \frac{C}{\rho}\, T^{\frac{2\rho}{\alpha}}\, M^2.
\end{aligned}$$

Since $J_5 \le J_4$, the term associated with J_5 in (3.6) admits the same bound,

$$\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0\,2^{2\alpha j}(t-\tau)}\, J_5\, d\tau \le \frac{C}{\rho}\, T^{\frac{2\rho}{\alpha}}\, M^2.$$

It remains to bound J_6. Invoking (3.7) and (3.8) again, we have

$$\begin{aligned}
&\sum_j 2^{(1+\frac{d}{2}-2\alpha)j}\int_0^t e^{-C_0\,2^{2\alpha j}(t-\tau)}\, J_6\, d\tau\\
&= C\int_0^t \sum_j 2^{(2+\frac{d}{2}-2\alpha)j} e^{-C_0\,2^{2\alpha j}(t-\tau)} \sum_{k\ge j-1} 2^{\frac{d}{2}k}\, \|\Delta_k b^{(n)}\|_{L^2}\, \|\widetilde{\Delta}_k b^{(n)}\|_{L^2}\, d\tau\\
&\le C\int_0^t \sum_j 2^{(2+\frac{d}{2}-2\alpha)j} 2^{-(2\alpha-4\rho)j}(t-\tau)^{-\frac{2\alpha-4\rho}{2\alpha}} \sum_{k\ge j-1} 2^{\frac{d}{2}k}\, \|\Delta_k b^{(n)}\|_{L^2}\, \|\widetilde{\Delta}_k b^{(n)}\|_{L^2}\, d\tau\\
&= C\int_0^t \sum_j 2^{(-4\alpha+2+4\rho)j}(t-\tau)^{-\frac{2\alpha-4\rho}{2\alpha}} \sum_{k\ge j-1} 2^{\frac{d}{2}(j-k)} 2^{\frac{d}{2}k}\, \|\Delta_k b^{(n)}\|_{L^2}\, 2^{\frac{d}{2}k}\, \|\widetilde{\Delta}_k b^{(n)}\|_{L^2}\, d\tau\\
&\le C\,\|b^{(n)}\|^2_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \int_0^t (t-\tau)^{-\frac{2\alpha-4\rho}{2\alpha}}\, d\tau\\
&= \frac{C}{\rho}\, t^{\frac{2\rho}{\alpha}}\, \|b^{(n)}\|^2_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \le \frac{C}{\rho}\, T^{\frac{2\rho}{\alpha}}\, M^2.
\end{aligned}$$

Collecting the bounds above and inserting them in (3.6), we find, for any $t \le T$,

$$\|u^{(n+1)}(t)\|_{B^{\frac{d}{2}+1-2\alpha}_{2,1}} \le \|u_0^{(n+1)}\|_{B^{1+\frac{d}{2}-2\alpha}_{2,1}} + C\,\delta\,\|u^{(n+1)}\|_{L^\infty\left(0,T;B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} + \frac{C}{\rho} T^{\frac{2\rho}{\alpha}} M^2, \tag{3.9}$$

where $0 < \rho < \frac{1}{2}$ satisfies (3.7).

3.2 | The estimate of $b^{(n+1)}$ in $B^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$

Applying Δ_j to the third equation in (3.1) and then dotting with $\Delta_j b^{(n+1)}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j b^{(n+1)}\|_{L^2}^2 + C_1\, 2^{2\beta j}\|\Delta_j b^{(n+1)}\|_{L^2}^2 \le B_1 + B_2, \tag{3.10}$$

where $C_1 > 0$ is a constant and

$$B_1 = -\int \Delta_j(u^{(n)} \cdot \nabla b^{(n+1)}) \cdot \Delta_j b^{(n+1)}\, dx,$$
$$B_2 = \int \Delta_j(b^{(n)} \cdot \nabla u^{(n)}) \cdot \Delta_j b^{(n+1)}\, dx.$$

By Lemma 2.3, we have

$$|B_1| \le C\,\|\Delta_j b^{(n+1)}\|_{L^2}^2 \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2} + C\,\|\Delta_j b^{(n+1)}\|_{L^2}\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n+1)}\|_{L^2}$$
$$+ C\,\|\Delta_j b^{(n+1)}\|_{L^2}\, 2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\widetilde{\Delta}_k b^{(n+1)}\|_{L^2}\,\|\Delta_k u^{(n)}\|_{L^2}$$

and

$$|B_2| \le C\,\|\Delta_j b^{(n+1)}\|_{L^2}\, 2^j\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2} + C\,\|\Delta_j b^{(n+1)}\|_{L^2}\,\|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2}$$
$$+ C\,\|\Delta_j b^{(n+1)}\|_{L^2}\, 2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\Delta_k b^{(n)}\|_{L^2}\,\|\widetilde{\Delta}_k u^{(n)}\|_{L^2}.$$

Inserting the estimates above in (3.10) and eliminating $\|\Delta_j b^{(n+1)}\|_{L^2}$ from both sides of the inequality, we obtain

$$\frac{d}{dt}\|\Delta_j b^{(n+1)}\|_{L^2} + C_1\, 2^{2\beta j}\|\Delta_j b^{(n+1)}\|_{L^2} \le K_1 + \cdots + K_6, \tag{3.11}$$

where

$$K_1 = C\,\|\Delta_j b^{(n+1)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},$$
$$K_2 = C\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m b^{(n+1)}\|_{L^2}$$
$$K_3 = C\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\widetilde{\Delta}_k b^{(n+1)}\|_{L^2}\,\|\Delta_k u^{(n)}\|_{L^2},$$
$$K_4 = C\,2^j\,\|\Delta_j u^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m}\|\Delta_m b^{(n)}\|_{L^2},$$
$$K_5 = C\,\|\Delta_j b^{(n)}\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m}\|\Delta_m u^{(n)}\|_{L^2},$$
$$K_6 = C\,2^j \sum_{k\ge j-1} 2^{\frac{d}{2}k}\,\|\Delta_k b^{(n)}\|_{L^2}\,\|\widetilde{\Delta}_k u^{(n)}\|_{L^2}.$$

Integrating (3.11) in time yields, for any $t \le T$,

$$\|\Delta_j b^{(n+1)}(t)\|_{L^2} \le e^{-C_1 2^{2\beta j} t} \|\Delta_j b_0^{(n+1)}\|_{L^2} + \int_0^t e^{-C_1 2^{2\beta j}(t-\tau)} (K_1 + \cdots + K_6)\, d\tau. \tag{3.12}$$

Multiplying (3.12) by $2^{\frac{d}{2}j}$ and summing over j, we have

$$\|b^{(n+1)}(t)\|_{B^{\frac{d}{2}}_{2,1}} \le \|b_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + \sum_j 2^{\frac{d}{2}j} \int_0^t e^{-C_1 2^{2\beta j}(t-\tau)} (K_1 + \cdots + K_6)\, d\tau. \tag{3.13}$$

The terms containing K_1 through K_6 on the right of (3.13) can be bounded suitably even without the dissipative factor. For this reason, we use the simple bound

$$e^{-C_0 2^{2\beta j}(t-\tau)} \le 1.$$

Since the estimates are similar to those in the previous subsection, we omit some details. The bounds are

$$\sum_j 2^{\frac{d}{2}j} \int_0^t K_1\, d\tau \le C \|b^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)} \le C\,\delta\, \|b^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})},$$

$$\sum_j 2^{\frac{d}{2}j} \int_0^t K_2\, d\tau \le C \|u^{(n)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)} \|b^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \le C\,\delta\, \|b^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}.$$

The estimate for the term with K_3 is also similar,

$$\begin{aligned}\sum_j 2^{\frac{d}{2}j} \int_0^t K_3\, d\tau &\le C \int_0^t \sum_j \sum_{k\ge j-1} 2^{\frac{d}{2}k} \|\widetilde{\Delta}_k b^{(n+1)}\|_{L^2}\, 2^{(1+\frac{d}{2})(j-k)}\, 2^{(1+\frac{d}{2})k} \|\Delta_k u^{(n)}\|_{L^2}\, d\tau \\ &\le C \|u^{(n)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)} \|b^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \le C\,\delta\, \|b^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}.\end{aligned}$$

The terms related to K_4, K_5 and K_6 all obey the same bound

$$\begin{aligned}&\sum_j 2^{\frac{d}{2}j} \int_0^t K_4\, d\tau, \qquad \sum_j 2^{\frac{d}{2}j} \int_0^t K_5\, d\tau, \qquad \sum_j 2^{\frac{d}{2}j} \int_0^t K_6\, d\tau \\ &\le C \|b^{(n)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} \|u^{(n)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)} \le C\,\delta\, M.\end{aligned}$$

Collecting the estimates and inserting them in (3.13), we obtain, for any $t \le T$,

$$\|b^{(n+1)}(t)\|_{B^{\frac{d}{2}}_{2,1}} \le \|b_0^{(n+1)}\|_{B^{\frac{d}{2}}_{2,1}} + C\,\delta\, \|b^{(n+1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})} + C\,\delta\, M. \tag{3.14}$$

3.3 | The estimate of $\|u^{(n+1)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)}$

We multiply (3.5) by $2^{(1+\frac{d}{2})j}$, sum over j and integrate in time to obtain

$$\begin{aligned}\|u^{(n+1)}\|_{L^1\left(0,T;B^{1+\frac{d}{2}}_{2,1}\right)} \le &\int_0^T \sum_j 2^{(1+\frac{d}{2})j} e^{-C_0 2^{2\alpha j} t} \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt \\ &+ \int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0 2^{2\alpha j}(s-\tau)} (J_1 + \cdots + J_6)\, d\tau\, ds.\end{aligned}$$

We estimate the terms on the right and start with the first term.

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\, e^{-C_0\, 2^{2\alpha j} t}\, \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt = C \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \left(1 - e^{-C_0\, 2^{2\alpha j} T}\right) \|\Delta_j u_0^{(n+1)}\|_{L^2}. \tag{3.15}$$

Since $u_0 \in B_{2,1}^{1+\frac{d}{2}-2\alpha}$, it follows from the Dominated Convergence Theorem that

$$\lim_{T\to 0} \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \left(1 - e^{-C_0\, 2^{2\alpha j} T}\right) \|\Delta_j u_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j}\, e^{-C_0\, 2^{2\alpha j} t}\, \|\Delta_j u_0^{(n+1)}\|_{L^2}\, dt \le \frac{\delta}{4}.$$

Applying Young's inequality for the time convolution, we have

$$\begin{aligned}
&\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} J_1\, d\tau\, ds \\
&= C \int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau\, ds \\
&\le C \sum_j 2^{(1+\frac{d}{2})j} \int_0^T \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \int_0^T e^{-C_0\, 2^{2\alpha j} s}\, ds \\
&\le C\,(1 - e^{-C_2 T}) \int_0^T \sum_j 2^{(1+\frac{d}{2}-2\alpha)j} \|\Delta_j u^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\le C\,(1 - e^{-C_2 T})\, \|u^{(n+1)}\|_{L^\infty\left(0,T;\, B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} \|u^{(n)}\|_{L^1\left(0,T;\, B_{2,1}^{1+\frac{d}{2}}\right)} \\
&\le C\,\delta\,(1 - e^{-C_2 T})\, \|u^{(n+1)}\|_{L^\infty\left(0,T;\, B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)},
\end{aligned}$$

where we have used the fact that there exists $C_2 > 0$ satisfying, for $j \ge 0$,

$$\int_0^T e^{-C_0\, 2^{2\alpha j} s}\, ds \le C\, 2^{-2\alpha j}\, (1 - e^{-C_2 T}). \tag{3.16}$$

We remark that the functional setting here is the inhomogeneous Besov spaces and the index j is bounded below. This is the reason why there is $C_2 > 0$ satisfying (3.16). This can not be done for homogeneous Besov spaces. The terms with J_2 and J_3 can be similarly estimated and obey the same bound,

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} J_2\, d\tau\, ds \le C\,\delta\,(1 - e^{-C_2 T})\, \|u^{(n+1)}\|_{L^\infty\left(0,T;\, B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)},$$

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} J_3\, d\tau\, ds \le C\,\delta\,(1 - e^{-C_2 T})\, \|u^{(n+1)}\|_{L^\infty\left(0,T;\, B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)}.$$

By Young's inequality for the time convolution, we have

$$\begin{aligned}
&\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} J_4\, d\tau\, ds \\
&= C \int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} 2^j\, \|\Delta_j b^{(n)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m} \|\Delta_m b^{(n)}(\tau)\|_{L^2}\, d\tau\, ds \\
&\le C\,(1-e^{-C_2 T}) \int_0^T \sum_j 2^{(\frac{d}{2}+2-2\alpha)j}\, \|\Delta_j b^{(n)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m} \|\Delta_m b^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\le C\,(1-e^{-C_2 T}) \sup_{0\le t\le T} \|b^{(n)}(t)\|_{B^{\frac{d}{2}}_{2,1}} \int_0^T \|b^{(n)}(t)\|_{B^{\frac{d}{2}+2-2\alpha}_{2,1}}\, dt \\
&\le C\,(1-e^{-C_2 T})\, \|b^{(n)}\|_{L^\infty\left(0,T;\, B^{\frac{d}{2}}_{2,1}\right)} \|b^{(n)}\|_{L^1\left(0,T;\, B^{\frac{d}{2}+2\beta}_{2,1}\right)} \\
&= C\,(1-e^{-C_2 T})\, \delta\, M,
\end{aligned}$$

where we have used the condition that $\alpha+\beta\ge 1$. The other two terms involving J_5 and J_6 obey the same bound,

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} J_5\, d\tau\, ds \le C\,(1-e^{-C_2 T})\, \delta\, M,$$

$$\int_0^T \sum_j 2^{(1+\frac{d}{2})j} \int_0^s e^{-C_0\, 2^{2\alpha j}(s-\tau)} J_6\, d\tau\, ds \le C\,(1-e^{-C_2 T})\, \delta\, M.$$

Collecting the estimates above leads to

$$\|u^{(n+1)}\|_{L^1\left(0,T; B^{1+\frac{d}{2}}_{2,1}\right)} \le \frac{\delta}{4} + C\,\delta(1-e^{-C_2 T})\|u^{(n+1)}\|_{L^\infty\left(0,T;\, B^{1+\frac{d}{2}-2\alpha}_{2,1}\right)} + C\,(1-e^{-C_2 T})\,\delta\, M. \tag{3.17}$$

3.4 | The estimate of $\|b^{(n+1)}\|_{L^1\left(0,T;\, B^{\frac{d}{2}+2\beta}_{2,1}\right)}$

Multiplying (3.12) by $2^{(\frac{d}{2}+2\beta)j}$, summing over j and integrating in time, we have

$$\begin{aligned}
\|b^{(n+1)}\|_{L^1\left(0,T; B^{\frac{d}{2}+2\beta}_{2,1}\right)} &\le \int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j}\, e^{-C_1\, 2^{2\beta j} t}\, \|\Delta_j b_0^{(n+1)}\|_{L^2}\, dt \\
&\quad + \int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1\, 2^{2\beta j}(s-\tau)} (K_1+\cdots+K_6)\, d\tau\, ds.
\end{aligned}$$

The terms on the right can be bounded as follows. As in (3.15),

$$\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j}\, e^{-C_1\, 2^{2\beta j} t}\, \|\Delta_j b_0^{(n+1)}\|_{L^2}\, dt = C \sum_j 2^{\frac{d}{2}j}\left(1-e^{-C_1\, 2^{2\beta j} T}\right) \|\Delta_j b_0^{(n+1)}\|_{L^2}.$$

Since $b_0\in B^{\frac{d}{2}}_{2,1}$, it follows from the Dominated Convergence Theorem that

$$\lim_{T\to 0} \sum_j 2^{\frac{d}{2}j}\left(1-e^{-C_1\, 2^{2\beta j} T}\right) \|\Delta_j b_0^{(n+1)}\|_{L^2} = 0.$$

Therefore, we can choose T sufficiently small such that

$$\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j}\, e^{-C_1\, 2^{2\beta j} t}\, \|\Delta_j b_0^{(n+1)}\|_{L^2}\, dt \le \frac{\delta}{4}.$$

The terms involving K_1 through K_6 can be estimated as follows. Applying Young's inequality for the time convolution, we have

$$\begin{aligned}
&\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1\, 2^{2\beta j}(s-\tau)} K_1\, d\tau\, ds \\
&= C \int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1\, 2^{2\beta j}(s-\tau)} \|\Delta_j b^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau\, ds \\
&\le C \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^T \|\Delta_j b^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \int_0^T e^{-C_1\, 2^{2\beta j} s}\, ds \\
&\le C\,(1-e^{-C_3 T}) \int_0^T \sum_j 2^{\frac{d}{2}j}\, \|\Delta_j b^{(n+1)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{(1+\frac{d}{2})m} \|\Delta_m u^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\le C\,(1-e^{-C_3 T})\, \|b^{(n+1)}\|_{L^\infty\left(0,T;\, B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T;\, B^{1+\frac{d}{2}}_{2,1}\right)} \\
&\le C\,\delta\,(1-e^{-C_3 T})\, \|b^{(n+1)}\|_{L^\infty\left(0,T;\, B^{\frac{d}{2}}_{2,1}\right)},
\end{aligned}$$

where $C_3 > 0$ is a constant. The two terms involving K_2 and K_3 admit the same bound,

$$\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1\, 2^{2\beta j}(s-\tau)} K_2\, d\tau\, ds \le C\,\delta\,(1-e^{-C_3 T})\, \|b^{(n+1)}\|_{L^\infty\left(0,T;\, B^{\frac{d}{2}}_{2,1}\right)},$$

$$\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1\, 2^{2\beta j}(s-\tau)} K_3\, d\tau\, ds \le C\,\delta\,(1-e^{-C_3 T})\, \|b^{(n+1)}\|_{L^\infty\left(0,T;\, B^{\frac{d}{2}}_{2,1}\right)}.$$

The terms containing K_4 is bounded by

$$\begin{aligned}
&\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1\, 2^{2\beta j}(s-\tau)} K_4\, d\tau\, ds \\
&= C \int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1\, 2^{2\beta j}(s-\tau)} 2^j\, \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m} \|\Delta_m b^{(n)}(\tau)\|_{L^2}\, d\tau\, ds \\
&\le C\,(1-e^{-C_3 T}) \int_0^T \sum_j 2^{(1+\frac{d}{2})j}\, \|\Delta_j u^{(n)}(\tau)\|_{L^2} \sum_{m\le j-1} 2^{\frac{d}{2}m} \|\Delta_m b^{(n)}(\tau)\|_{L^2}\, d\tau \\
&\le C\,(1-e^{-C_3 T}) \sup_{0\le t\le T} \|b^{(n)}(t)\|_{B^{\frac{d}{2}}_{2,1}} \int_0^T \|u^{(n)}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\, dt \\
&\le C\,(1-e^{-C_3 T}) \|b^{(n)}\|_{L^\infty\left(0,T;\, B^{\frac{d}{2}}_{2,1}\right)} \|u^{(n)}\|_{L^1\left(0,T;\, B^{1+\frac{d}{2}}_{2,1}\right)} \\
&= C\,(1-e^{-C_3 T})\delta\, M.
\end{aligned}$$

The terms with K_5 and K_6 obey the same bound,

$$\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1 2^{2\beta j}(s-\tau)} K_5 \, d\tau \, ds \le C(1-e^{-C_3 T})\delta M,$$

$$\int_0^T \sum_j 2^{(\frac{d}{2}+2\beta)j} \int_0^s e^{-C_1 2^{2\beta j}(s-\tau)} K_6 \, d\tau \, ds \le C(1-e^{-C_3 T})\delta M.$$

Collecting the estimates above, we conclude that

$$\|b^{(n+1)}\|_{L^1\left(0,T;B_{2,1}^{\frac{d}{2}+2\beta}\right)} \le \frac{\delta}{4} + C\,\delta\,(1-e^{-C_3 T})\,\|b^{(n+1)}\|_{L^\infty\left(0,T;\,B_{2,1}^{\frac{d}{2}}\right)} + C(1-e^{-C_3 T})\delta M. \tag{3.18}$$

The bounds in (3.9), (3.14), (3.17) and (3.18) allow us to conclude that, if we choose $T > 0$ sufficiently small and $\delta > 0$ suitably, then

$$\|u^{(n+1)}\|_{L^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-2\alpha})} \le M, \quad \|b^{(n+1)}\|_{L^\infty(0,T;\,B_{2,1}^{\frac{d}{2}})} \le M,$$

$$\|u^{(n+1)}\|_{L^1(0,T;\,B_{2,1}^{\frac{d}{2}+1})} \le \delta, \quad \|b^{(n+1)}\|_{L^1(0,T;\,B_{2,1}^{\frac{d}{2}+2\beta})} \le \delta.$$

In fact, if we choose T and δ satisfying

$$C\delta \le \frac{1}{4}, \qquad \frac{C}{\rho} T^{\frac{2\rho}{\alpha}} M \le \frac{1}{4},$$

then (3.9) implies

$$\|u^{(n+1)}\|_{L^\infty(0,T;\,B_{2,1}^{\frac{d}{2}+1-2\alpha})} \le \frac{1}{2}M + \frac{1}{4}\|u^{(n+1)}\|_{L^\infty\left(0,T;B_{2,1}^{1+\frac{d}{2}-2\alpha}\right)} + \frac{1}{4}M$$

or

$$\|u^{(n+1)}\|_{L^\infty(0,T;\,B_{2,1}^{\frac{d}{2}+1-2\alpha})} \le M.$$

Similarly, if $C\delta \le \frac{1}{4}$, then (3.14) states

$$\|b^{(n+1)}\|_{L^\infty(0,T;\,B_{2,1}^{\frac{d}{2}})} \le M.$$

According to (3.17) and (3.18), if we choose T sufficiently small such that

$$C(1-e^{-C_2 T})\,\delta \le \frac{1}{2}, \qquad C(1-e^{-C_3 T})\,\delta \le \frac{1}{2},$$

$$C(1-e^{-C_2 T})\,M \le \frac{1}{4}, \qquad C(1-e^{-C_3 T})\,M \le \frac{1}{4},$$

then

$$\|u^{(n+1)}\|_{L^1(0,T;\,B_{2,1}^{\frac{d}{2}+1})} \le \delta, \quad \|b^{(n+1)}\|_{L^1(0,T;\,B_{2,1}^{\frac{d}{2}+2\beta})} \le \delta.$$

These uniform bounds allow us to extract a weakly convergent subsequence. That is, there is $(u, b) \in Y$ such that a subsequence of $(u^{(n)}, b^{(n)})$ (still denoted by $(u^{(n)}, b^{(n)})$) satisfies

$$u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;B_{2,1}^{\frac{d}{2}+1-2\alpha}) \cap L^1(0,T;\,B_{2,1}^{\frac{d}{2}+1}),$$

$$b^{(n)} \overset{*}{\rightharpoonup} b \quad \text{in} \quad L^\infty(0,T;B_{2,1}^{\frac{d}{2}}) \cap L^1(0,T;\,B_{2,1}^{\frac{d}{2}+2\beta}).$$

In order to show that (u, b) is a weak solution of (1.1), we need to further extract a subsequence which converges strongly to (u, b). This is done via the Aubin-Lions Lemma. We can show by making use of the equations in (3.1) that $(\partial_t u^{(n)}, \partial_t b^{(n)})$ is uniformly bounded in

$$\partial_t u^{(n)} \in L^1(0,T; B_{2,1}^{\frac{d}{2}-2\alpha+1}) \cap L^2(0,T; B_{2,1}^{\frac{d}{2}+1-3\alpha}),$$

$$\partial_t b^{(n)} \in L^1(0,T; B_{2,1}^{\frac{d}{2}}) \cap L^2(0,T; B_{2,1}^{\frac{d}{2}-\beta}).$$

Since we are in the case of the whole space $\mathbb{R}^d$, we need to combine Cantor's diagonal process with the Aubin-Lions Lemma to show that a subsequence of the weakly convergent subsequence, still denoted by $(u^{(n)}, b^{(n)})$, has the following strongly convergent property,

$$(u^{(n)}, b^{(n)}) \to (u, b) \qquad \text{in} \quad L^2(0,T; B_{2,1}^{\gamma}(Q)),$$

where $\frac{d}{2} - \alpha \le \gamma < \frac{d}{2}$ and $Q \subset \mathbb{R}^d$ is a compact subset. This strong convergence property would allow us to show that (u, b) is indeed a weak solution of (1.1). This process is routine and we omit the details. This completes the proof for the existence part of Theorem 1.1.

4 | PROOF FOR THE UNIQUENESS PART OF THEOREM 1.1

This section proves the uniqueness part of Theorem 1.1.

Proof. Assume that $(u^{(1)}, b^{(1)})$ and $(u^{(2)}, b^{(2)})$ are two solutions of (1.1) in the regularity class in (1.3) and (1.4). Their difference $(\widetilde{u}, \widetilde{b})$ with

$$\widetilde{u} = u^{(2)} - u^{(1)}, \qquad \widetilde{b} = b^{(2)} - b^{(1)}$$

satisfies

$$\begin{cases} \partial_t \widetilde{u} + \nu(-\Delta)^{\alpha}\widetilde{u} = -\mathbb{P}(u^{(2)} \cdot \nabla\widetilde{u} + \widetilde{u} \cdot \nabla u^{(1)}) + \mathbb{P}(b^{(2)} \cdot \nabla\widetilde{b} + \widetilde{b} \cdot \nabla b^{(1)}), \\ \partial_t \widetilde{b} + \eta(-\Delta)^{\beta}\widetilde{b} = -u^{(2)} \cdot \nabla\widetilde{b} - \widetilde{u} \cdot \nabla b^{(1)} + b^{(2)} \cdot \nabla\widetilde{u} + \widetilde{b} \cdot \nabla u^{(1)}, \\ \nabla \cdot \widetilde{u} = \nabla \cdot \widetilde{b} = 0, \\ \widetilde{u}(x,0) = 0, \quad \widetilde{b}(x,0) = 0. \end{cases} \tag{4.1}$$

In the case when $\alpha = 1$ and $\beta = 0$, the uniqueness has been obtained in [7, 19, 24]. We focus on the case when $\alpha > 1/2$, $\beta > 0$ and $\alpha + \beta \ge 1$. We estimate the difference $(\widetilde{u}, \widetilde{b})$ in L^2. Dotting (4.1) by $(\widetilde{u}, \widetilde{b})$ and applying the divergence-free condition, we find

$$\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2) + \nu\|\Lambda^{\alpha}\widetilde{u}\|_{L^2}^2 + \eta\|\Lambda^{\beta}\widetilde{b}\|_{L^2}^2 = L_1 + L_2 + L_3 + L_4 + L_5,$$

where

$$L_1 = -\int \widetilde{u} \cdot \nabla u^{(1)} \cdot \widetilde{u}\, dx,$$

$$L_2 = \int b^{(2)} \cdot \nabla\widetilde{b} \cdot \widetilde{u}\, dx + \int b^{(2)} \cdot \nabla\widetilde{u} \cdot \widetilde{b}\, dx,$$

$$L_3 = \int \widetilde{b} \cdot \nabla b^{(1)} \cdot \widetilde{u}\, dx,$$

$$L_4 = -\int \widetilde{u} \cdot \nabla b^{(1)} \cdot \widetilde{b}\, dx,$$

$$L_5 = \int \widetilde{b} \cdot \nabla u^{(1)} \cdot \widetilde{b}\, dx.$$

Due to $\nabla \cdot b^{(2)} = 0$, we find $L_2 = 0$ after integration by parts. We remark that $L_3 + L_4$ is not necessarily zero. By Hölder's inequality,

$$|L_1| \leq \|\nabla u^{(1)}\|_{L^\infty} \|\widetilde{u}\|_{L^2}^2 \leq C\,\|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}} \|\widetilde{u}\|_{L^2}^2.$$

To bound L_3, we set

$$\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{d}, \qquad \frac{1}{q} = \frac{\alpha}{d}$$

and apply Hölder's inequality to obtain

$$|L_3| \leq \|\widetilde{b}\|_{L^2} \|\nabla b^{(1)}\|_{L^q} \|\widetilde{u}\|_{L^p} \leq C\,\|\widetilde{b}\|_{L^2} \|b^{(1)}\|_{B^{\frac{d}{2}+\beta}_{2,1}} \|\Lambda^\alpha \widetilde{u}\|_{L^2} \leq \frac{\nu}{4}\|\Lambda^\alpha \widetilde{u}\|_{L^2} + C\,\|b^{(1)}\|^2_{B^{\frac{d}{2}+\beta}_{2,1}} \|\widetilde{b}\|_{L^2}^2,$$

where we have used the inequalities, due to $\alpha + \beta \geq 1$,

$$\begin{aligned}
\|\widetilde{u}\|_{L^p} &\leq C\,\|\Lambda^\alpha \widetilde{u}\|_{L^2},\\
\|\nabla b^{(1)}\|_{L^q} &\leq \sum_{j\geq -1} \|\Delta_j \nabla b^{(1)}\|_{L^q} \leq C \sum_{j\geq -1} 2^{j+dj(\frac{1}{2}-\frac{1}{q})}\,\|\Delta_j b^{(1)}\|_{L^2}\\
&\leq C \sum_{j\geq -1} 2^{\frac{d}{2}j+(1-\frac{d}{q})j}\,\|\Delta_j b^{(1)}\|_{L^2} \leq C \sum_{j\geq -1} 2^{\frac{d}{2}j+\beta j}\,\|\Delta_j b^{(1)}\|_{L^2}\\
&\leq C\,\|b^{(1)}\|_{B^{\frac{d}{2}+\beta}_{2,1}}.
\end{aligned}$$

L_4 obeys exactly the same bound,

$$|L_4| \leq \frac{\nu}{4}\|\Lambda^\alpha \widetilde{u}\|_{L^2} + C\,\|b^{(1)}\|^2_{B^{\frac{d}{2}+\beta}_{2,1}} \|\widetilde{b}\|_{L^2}^2.$$

The last term L_5 can be bounded as L_1,

$$|L_5| \leq C\,\|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}} \|\widetilde{b}\|_{L^2}^2.$$

Combining these estimates leads to

$$\begin{aligned}
&\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2) + \nu\|\Lambda^\alpha \widetilde{u}\|_{L^2}^2 + 2\eta\|\Lambda^\beta \widetilde{b}\|_{L^2}^2\\
&\leq C\,\|u^{(1)}\|_{B^{1+\frac{d}{2}}_{2,1}} (\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2) + C\,\|b^{(1)}\|^2_{B^{\frac{d}{2}+\beta}_{2,1}} \|\widetilde{b}\|_{L^2}^2. \qquad (4.2)
\end{aligned}$$

Since $(u^{(1)}, b^{(1)})$ is in the regularity class (1.3) and (1.4), we have

$$\int_0^T \|u^{(1)}(t)\|_{B^{1+\frac{d}{2}}_{2,1}}\,dt < \infty.$$

In addition, by Hölder's inequality,

$$\begin{aligned}
\|b^{(1)}\|_{B^{\frac{d}{2}+\beta}_{2,1}} &= \sum_j 2^{(\frac{d}{2}+\beta)j}\,\|\Delta_j b^{(1)}\|_{L^2} = \sum_j 2^{\frac{d}{4}j}\|\Delta_j b^{(1)}\|_{L^2}^{\frac{1}{2}}\,2^{\frac{d}{4}j+\beta j}\|\Delta_j b^{(1)}\|_{L^2}^{\frac{1}{2}}\\
&\leq \left(\sum_j 2^{\frac{d}{2}j}\,\|\Delta_j b^{(1)}\|_{L^2}\right)^{\frac{1}{2}} \left(\sum_j 2^{(\frac{d}{2}+2\beta)j}\,\|\Delta_j b^{(1)}\|_{L^2}\right)^{\frac{1}{2}}\\
&= \|b^{(1)}\|^{\frac{1}{2}}_{B^{\frac{d}{2}}_{2,1}}\,\|b^{(1)}\|^{\frac{1}{2}}_{B^{\frac{d}{2}+2\beta}_{2,1}}. \qquad (4.3)
\end{aligned}$$

Therefore,

$$\int_0^T \|b^{(1)}(t)\|^2_{B^{\frac{d}{2}+\beta}_{2,1}}\,dt \le \int_0^T \|b^{(1)}(t)\|_{B^{\frac{d}{2}}_{2,1}}\|b^{(1)}(t)\|_{B^{\frac{d}{2}+2\beta}_{2,1}}\,dt \le \|b^{(1)}\|_{L^\infty(0,T;B^{\frac{d}{2}}_{2,1})}\|b^{(1)}\|_{L^1(0,T;B^{\frac{d}{2}+2\beta}_{2,1})} < \infty. \tag{4.4}$$

Applying Gronwall's inequality to (4.2) and invoking (4.3) and (4.4), we obtain

$$\|\widetilde{u}\|_{L^2} = \|\widetilde{b}\|_{L^2} = 0,$$

which leads to the desired uniqueness. This completes the proof of the uniqueness part of Theorem 1.1. □

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