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# The Littlewood–Paley decomposition for periodic functions and applications to the Boussinesq equations

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The Littlewood–Paley decomposition for functions defined on the whole space $\mathbb{R}^d$ and related Besov space techniques have become indispensable tools in the study of many partial differential equations (PDEs) with $\mathbb{R}^d$ as the spatial domain. This paper intends to develop parallel tools for the periodic domain $\mathbb{T}^d$. Taking advantage of the boundedness and convergence theory on the square-cutoff Fourier partial sum, we define the Littlewood–Paley decomposition for periodic functions via the square cutoff. We remark that the Littlewood–Paley projections defined via the circular cutoff in $\mathbb{T}^d$ with $d > 1$ in the literature do not behave well on the Lebesgue space $L^q$ except for $q = 2$. We develop a complete set of tools associated with this decomposition, which would be very useful in the study of PDEs defined on $\mathbb{T}^d$. As an application of the tools developed here, we study the periodic weak solutions of the $d$-dimensional Boussinesq equations with the fractional dissipation $(-\Delta)^\alpha \mathbf{u}$ and without thermal diffusion. We obtain two main results. The first assesses the global existence of $L^2$-weak solutions for any $\alpha > 0$ and the existence and uniqueness of the $L^2$-weak solutions when $\alpha \geq \frac{1}{2} + \frac{d}{4}$ for $d \geq 2$. The second establishes the zero thermal diffusion limit with an explicit convergence rate.

*Keywords*: Boussinesq equations; Littlewood–Paley; periodic domain; uniqueness; weak solutions

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## 1. Introduction

The Littlewood–Paley decomposition and Besov space techniques are important tools in the study of solutions of partial differential equations (PDEs) defined in the whole space $\mathbb{R}^d$. This paper intends to develop parallel tools for periodic functions. We define the Littlewood–Paley decomposition for periodic functions via the square cutoff of the Fourier series. As we explain in Sec. 2, the definition of the Littlewood–Paley decomposition in $\mathbb{T}^d$ with $d > 1$ generated in terms of the circular cutoff do not work well with the Lebesgue space $L^q$ except for $q = 2$. We develop a complete set of tools associated with this decomposition including Bernstein's inequalities, Bony's paraproducts and the Besov spaces on periodic domains. This decomposition and the corresponding tools are useful in the study of PDEs defined on the periodic box $\mathbb{T}^d$, and have an advantages over the classical Fourier series.

As an application of the tools developed here, we study the existence and uniqueness of weak solutions to the $d$-dimensional (d-D) Boussinesq system in a periodic domain $\mathbb{T}^d$,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nu(-\Delta)^\alpha \mathbf{u} - \nabla P + \theta \mathbf{e}_d, & \mathbf{x}\in\mathbb{T}^d, \quad t>0,\\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \gamma u_d, & \mathbf{x}\in\mathbb{T}^d, \quad t>0,\\ \nabla\cdot\mathbf{u} = 0, & \mathbf{x}\in\mathbb{T}^d, \quad t>0,\\ (\mathbf{u},\theta)|_{t=0} = (\mathbf{u}_0,\theta_0), & \mathbf{x}\in\mathbb{T}^d, \end{cases} \tag{1.1}$$

where $\mathbb{T}^d = [-\pi,\pi]^d$ denotes the periodic box, and $\mathbf{u}$, $P$ and $\theta$ represent the velocity field, the pressure and the temperature, respectively. Here $\nu > 0$ denotes the kinematic viscosity, $\mathbf{e}_d = (0,0,\dots,1)$ is the unit vector in the vertical direction, $\alpha > 0$ and $\gamma$ are real parameters, and $u_d$ is the $d$th component of $\mathbf{u}$.

Here the fractional Laplacian $(-\Delta)^\alpha f$ is defined via the Fourier modes of $f$,

$$\widehat{(-\Delta)^\alpha f}(\mathbf{k}) = |\mathbf{k}|^{2\alpha}\widehat{f}(\mathbf{k}).$$

For $f \in L^1(\mathbb{T}^d)$, the Fourier modes $\widehat{f}(\mathbf{k})$ with $\mathbf{k}\in\mathbb{Z}^d$ is given by

$$\widehat{f}(\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

More generally, for a distribution $f$, the $\mathbf{k}$th Fourier coefficient is given by

$$\widehat{f}(\mathbf{k}) = \langle f, e^{-i\mathbf{k}\cdot\mathbf{x}}\rangle.$$

More details can be found in [23]. Sometimes we also write $\Lambda = (-\Delta)^{\frac{1}{2}}$. When $\alpha = 1$ and $\gamma = 0$, (1.1) becomes the standard Boussinesq system. When $\theta = 0$, (1.1) reduces to the generalized Navier–Stokes equations.

The Littlewood–Paley decomposition and Besov space techniques have become an indispensable tool for PDEs on $\mathbb{R}^d$. Our intention is to introduce the concept of the Littlewood–Paley decomposition for periodic functions and derive related tools so that the periodic case and the whole space case can be treated in the same way.

The classical Fourier series expansion is not convenient for this purpose, especially when we estimate the $L^p$-norm of a series for $p \neq 2$.

In order to provide a suitable definition for the Littlewood–Paley decomposition for functions on $\mathbb{T}^d$, we exam partial sums of the Fourier series given by various kernel functions including the square Dirichlet kernel, the circular Dirichlet kernel, general $l_q$ Dirichlet kernel with $1 \leq q \leq \infty$, Fejér kernels and Riesz kernels. After taking into account of the convergence and boundedness properties as well as the easiness of being split into dyadic Fourier blocks, we choose the square cutoff to define the dyadic Fourier blocks. More precisely, we define the following localized Fourier projection operators as

$$\Delta_0 f(\mathbf{x}) = \sum_{\mathbf{k}\in A_0} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\Delta_j f(\mathbf{x}) = \sum_{\mathbf{k}\in A_j\setminus A_{j-1}} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}, \quad j \geq 1,\ \ j \in \mathbb{N},$$

where $A_j$'s are the $2^j$-sized blocks of $d$-dimensional integer lattice points,

$$A_j = \{\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d : |k_m| \leq 2^j,\ m = 1, 2, \ldots, d\}.$$

Since, for any $f \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$,

$$S_j f(\mathbf{x}) := \sum_{m=0}^{j-1} \Delta_m f(\mathbf{x}) = \sum_{\mathbf{k}\in A_{j-1}} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

converges to $f$ in $L^p(\mathbb{T}^d)$ for $1 < p < \infty$ and almost everywhere for $p = \infty$, we can write the Littlewood–Paley decomposition

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \Delta_k f(\mathbf{x}).$$

We show that $\Delta_j$ and $S_j$ defined above do share some of the crucial properties as their counterparts in the whole space case such as Bernstein type inequalities

$$c_1 2^{\alpha j} \|\Delta_j f\|_{L^p(\mathbb{T}^d)} \leq \|\Delta_j \Lambda^\alpha f\|_{L^p(\mathbb{T}^d)} \leq c_2 2^{\alpha j + jd(\frac{1}{q} - \frac{1}{p})} \|\Delta_j f\|_{L^q(\mathbb{T}^d)}.$$

More details can be found in Sec. 2. In addition, these operators allow us to develop similar tools for periodic functions. We can write a standard product into paraproducts and define the Besov type spaces $B^s_{p,q}(\mathbb{T}^d)$ for periodic functions. These tools are much more convenient and flexible than the classical Fourier series when we estimate solutions of PDEs defined in periodic domains.

The tools developed here are applied to study the solutions of (1.1) defined on $\mathbb{T}^d$. We provide some background information on (1.1). The Boussinesq equations model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream (see, e.g., the books by Gill [22], Pedlosky [49] and Majda [46], and the paper by Temam and Tribbia [54]). In addition, the Boussinesq system also plays an important role in the study of the Rayleigh–Benard convection, one of the most commonly studied convection phenomena (see, e.g., [52, 13, 17, 21]). The first equation in (1.1) reflects Newton's second law, with the left-hand side being the acceleration and the right-hand side being the forces due to viscosity, the pressure gradient and the buoyancy. The term $\theta\mathbf{e}_d$ models the buoyancy in the direction of gravitational force. The temperature difference generates density difference, which in turn generates the buoyancy force. The second equation in (1.1) simply states that the temperature is transported by the velocity field.

(1.1) involves fractional dissipation. Although the diffusion process is normally modeled by the standard Laplacian operator, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian arise. Flows in the middle atmosphere traveling upwards undergo changes due to the changes in atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled using the space fractional Laplacian (see [11, 22]).

Mathematically the Boussinesq system retains some key features of the 3D Euler and Navier–Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows [47]. (1.1) is a partially dissipated system with no thermal diffusion. Fundamental issues on the Boussinesq system with partial or fractional dissipation such as the global existence, uniqueness and regularity problem have attracted enormous interests during the last 15 years and significant progress has been made (see, e.g., [1–4, 7, 9, 10, 14, 15, 26, 27, 29, 32, 31, 33–35, 39–45, 57, 58, 60, 61, 63, 65, 66]).

By applying the tools developed here for periodic functions, we establish two main results for (1.1). The first is the global existence and uniqueness of weak solutions of (1.1) with initial data $\mathbf{u}_0 \in L^2(\mathbb{R}^d), \theta_0 \in L^2(\mathbb{R}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$. Our key point here is the uniqueness of solutions in a very weak setting for a partially dissipated system. The precise result is stated in the following theorem.

**Theorem 1.1.** *Consider the d-D Boussinesq equations in* (1.1) *in the periodic domain* $\mathbb{T}^d$.

(1) *Let* $\alpha > 0$ *and* $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$ *with* $\nabla \cdot \mathbf{u}_0 = 0$. *Let* $T > 0$ *be arbitrarily fixed. Then* (1.1) *has a global weak solution* $(\mathbf{u}, \theta)$ *on* $[0, T]$ *satisfying*

$$\mathbf{u} \in C_w([0,T]; L^2) \cap L^2(0,T; \dot{H}^\alpha), \quad \theta \in C_w([0,T]; L^2) \cap L^\infty(0,T; L^2),$$

*where* $C_w([0,T]; L^2)$ *denotes the standard time continuous functions in the weak* $L^2$*-sense.*

(2) *Let* $\alpha \geq \frac{1}{2} + \frac{d}{4}$. *Assume* $\mathbf{u}_0 \in L^2(\mathbb{T}^d)$ *and* $\theta_0 \in L^2(\mathbb{T}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{T}^d)$ *with* $\nabla \cdot \mathbf{u}_0 = 0$. *Then* (1.1) *has a unique and global weak solution* $(\mathbf{u}, \theta)$ *satisfying*

$$\mathbf{u} \in C([0,T]; L^2) \cap L^2(0,T; \dot{H}^\alpha), \quad \mathbf{u} \in \widetilde{L}^1(0,T; B_{2,2}^{1+\frac{d}{2}}),$$

$$\theta \in C_w([0,T]; L^2) \cap L^\infty(0,T; L^2 \cap L^{\frac{4d}{d+2}}),$$

*where the definition of* $\widetilde{L}^1(0,T;\ B^{1+\frac{d}{2}}_{2,2})$ *can be found in Sec.* 2. *Especially,* $\mathbf{u}$ *satisfies*

$$\sup_{q\geq 2} \frac{1}{\sqrt{q}} \int_0^T \|\nabla \mathbf{u}(t)\|_{L^q}\, dt < \infty.$$

Theorem 1.1 assesses the global existence of weak solutions for any $\alpha > 0$ and any $L^2$ initial data, and the uniqueness when $\alpha \geq \frac{1}{2} + \frac{d}{4}$. A special consequence of Theorem 1.1 is the global existence of Leray–Hopf weak solutions to the $d$-D generalized Navier–Stokes equations with any $\alpha > 0$ and $\mathbf{u}_0 \in L^2(\mathbb{T}^d)$, and the uniqueness of weak solutions of the $d$-D Navier–Stokes equations with $\alpha \geq \frac{1}{2} + \frac{d}{4}$. Even though there is no thermal diffusion, the crucial step of passing to the limit in the thermal convection term still goes through.

Another significant consequence of Theorem 1.1 is the global $L^2$-stability of the hydrostatic balance for the 2D Boussinesq equations without thermal diffusion, namely (1.1) with $d = 2$, $\alpha = 1$ and $\gamma = 0$. In fluid mechanics, a fluid is said to be in hydrostatic balance when it is at rest, or when the flow velocity is constant over time. This occurs when external forces such as gravity are balanced by a pressure gradient force. Our atmosphere is mainly in hydrostatic balance. Mathematically the hydrostatic balance is represented by the steady state solution

$$\mathbf{u}_{\text{he}} = \mathbf{0}, \quad \theta_{\text{he}} = \lambda x_2, \quad \nabla P_{\text{he}} = \theta_{\text{he}}\, \mathbf{e}_2 \quad \text{or} \quad P_{\text{he}} = \frac{1}{2}\lambda x_2^2,$$

where $\lambda > 0$ is a parameter. The perturbation

$$\mathbf{w} = \mathbf{u} - \mathbf{u}_{\text{he}}, \quad Q = P - P_{\text{he}}, \quad \rho = \theta - \theta_{\text{he}} \tag{1.2}$$

satisfies

$$\begin{cases} \partial_t \mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{w} + \nabla Q = \nu\Delta\mathbf{w} + \rho\mathbf{e}_2, \\ \partial_t\rho + \mathbf{w}\cdot\nabla\rho + \lambda\, w_2 = 0, \\ \nabla\cdot\mathbf{w} = 0; \\ (\mathbf{w},\rho)(\mathbf{x},0) = (\mathbf{w}_0,\rho_0)(\mathbf{x}). \end{cases} \tag{1.3}$$

The global $L^2$-stability result on $(\mathbf{w},\rho)$ can be stated as follows.

**Theorem 1.2.** *Consider* (1.3) *with* $(\mathbf{w}_0,\rho_0) \in L^2(\mathbb{T}^2)$. *Then* (1.3) *has a unique global weak solution* $(\mathbf{w},\rho)$ *obeying the following uniform global bound, for any* $t > 0$,

$$\|\mathbf{w}(t)\|_{L^2}^2 + \|\rho(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla\mathbf{w}(\tau)\|_{L^2}^2\, d\tau \leq \|(\mathbf{w}_0,\rho_0)\|_{L^2}^2.$$

*A special consequence is the global* $L^2$*-stability for any perturbations in* $L^2$.

The stability of the hydrostatic balance is a really important topic in the study of geophysical fluids (see, e.g., [46]). A few mathematically rigorous results are currently available (see, e.g., [18, 19, 53, 46]). The spatial domain in [18] is either a rectangle or more general Lipschitz domain with minor constraints. Doering *et al.* [18]

established the global existence and uniqueness in the functional setting $\mathbf{w}_0 \in H^2$ and $\rho_0 \in H^1$. More importantly, [18] obtained the global stability and the large-time behavior of the perturbation. A very recent work [53] further developed the stability theory on the hydrostatic balance by deriving the exact decay rates of the velocity fields and the precise final buoyancy distribution.

The second application of the Littlewood–Paley approach for periodic domains developed here is on the zero thermal diffusion limit of the Boussinesq equation with thermal diffusion

$$\begin{cases} \partial_t \mathbf{u}^{(\eta)} + \mathbf{u}^{(\eta)} \cdot \nabla \mathbf{u}^{(\eta)} = -\nu(-\Delta)^\alpha \mathbf{u}^{(\eta)} - \nabla P^{(\eta)} + \theta^{(\eta)} \mathbf{e}_d, & \mathbf{x} \in \mathbb{T}^d, \quad t > 0, \\ \partial_t \theta^{(\eta)} + \mathbf{u}^{(\eta)} \cdot \nabla \theta^{(\eta)} = \eta \, \Delta \, \theta^{(\eta)} + \gamma \, u_d, & \mathbf{x} \in \mathbb{T}^d, \quad t > 0, \\ \nabla \cdot \mathbf{u}^{(\eta)} = 0, \\ (\mathbf{u}^{(\eta)}, \theta^{(\eta)})|_{t=0} = (\mathbf{u}_0^{(\eta)}, \theta_0^{(\eta)}). \end{cases} \tag{1.4}$$

We show that the solution of (1.4) converges strongly to the corresponding solution of (1.1) with an explicit convergence rate as $\eta \to 0$. Due to the weak initial setup $\mathbf{u}_0^{(\eta)} \in L^2(\mathbb{T}^d)$, $\theta_0^{(\eta)} \in L^2(\mathbb{T}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{T}^d)$, we resort to lower regularity quantities and the Yudovich approach.

**Theorem 1.3.** *Let $d \geq 2$ and $\alpha \geq \frac{1}{2} + \frac{d}{4}$. Assume $\mathbf{u}_0, \theta_0, \mathbf{u}_0^{(\eta)}, \theta_0^{(\eta)}$ satisfy*

$$\mathbf{u}_0, \mathbf{u}_0^{(\eta)} \in L^2(\mathbb{T}^d), \quad \nabla \cdot \mathbf{u}_0 = 0, \quad \nabla \cdot \mathbf{u}_0^{(\eta)} = 0, \quad \theta_0, \theta_0^{(\eta)} \in L^2(\mathbb{T}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{T}^d).$$

*Let $(\mathbf{u}, \theta)$ and $(\mathbf{u}^{(\eta)}, \theta^{(\eta)})$ be the corresponding weak solutions of* (1.1) *and* (1.4), *respectively. Then the difference $(\widetilde{\mathbf{u}}, \widetilde{h})$ with*

$$\widetilde{\mathbf{u}} = \mathbf{u}^{(\eta)} - \mathbf{u}, \quad \widetilde{h} = h^{(\eta)} - h, \quad -\Delta h^{(\eta)} = \theta^{(\eta)}, \quad -\Delta h = \theta$$

*satisfies, for any $t > 0$,*

$$\|(\widetilde{\mathbf{u}}, \nabla \widetilde{h})(t)\|_{L^2}^2 \leq C M^{(1-e^{-C_0 t})} (\|(\widetilde{\mathbf{u}_0}, \nabla \widetilde{h_0})\|_{L^2}^2 + \eta M t)^{e^{-C_0 t}}, \tag{1.5}$$

*where $C$ is a pure constant, $M = \|\theta_0\|_{L^2}^2 + \|\theta_0^{(\eta)}\|_{L^2}^2$ and*

$$C_0 = C \int_0^t \left(1 + \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \mathbf{u}\|_{L^2}^2 + \frac{\|\nabla \mathbf{u}^{(\eta)}\|_{L^p}}{p}\right) d\tau < \infty.$$

We summarize closely related previous results on the Boussinesq equations without thermal diffusion to clarify how our theorems are different. The study of the global well-posedness of the 2D Boussinesq equations, namely (1.1) with $d = 2$ and $\alpha = 1$ in the whole space was initiated in the papers of Hou–Li [29] and of Chae [12], in which the global and unique solutions were obtained for the initial data $(\mathbf{u}_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$. The global existence and uniqueness of solutions to (1.1) with $d = 3$ and $\alpha \geq \frac{5}{4}$ was investigated by several researchers (see, e.g., [36, 50, 59, 62, 64]). The regularity assumptions on the initial data in these papers are $(\mathbf{u}_0, \theta_0) \in H^s(\mathbb{R}^3)$ with $s > \frac{5}{2}$ ($s > \frac{5}{4}$ in [36]). The 2D Boussinesq system in a bounded domain with the Dirichlet boundary condition was first studied by Lai, Pan and Zhao [39] and the global existence and uniqueness was obtained in the functional setting $(\mathbf{u}_0, \theta_0) \in H^3(\mathbb{R}^2)$. Danchin and Paicu [14] extended the Fujita-Kato result for the Naviver–Stokes equations to the Boussinesq system and, as a special consequence, obtained the well-posedness of the finite energy solutions for the 2D Boussinesq equations [14]. The paper [40] seriously sought the uniqueness of solutions of (1.1) in a weak setup. They were able to show, among many other results, that $\mathbf{u}_0 \in H^1(\mathbb{T}^2)$ and $\theta_0 \in L^2(\mathbb{T}^2)$ lead to a unique and global strong solution of (1.1). For the bounded domain $\Omega$ with Dirichlet boundary conditions, the work of He [25] further reduced the regularity assumption to $(\mathbf{u}_0, \theta_0) \in L^2(\Omega)$ and still managed to show the uniqueness. There are many more interesting results on the existence and uniqueness of the solutions to (1.1) with intermediate regularity settings (see, e.g., [32, 31, 33, 37]). The zero thermal diffusion limit of very weak solutions to the Boussinesq equations has not been investigated.

Due to the importance of bounded domains in practical applications, we devote two paragraphs exclusively to the Boussinesq equations in bounded domains. The first paragraph describes some existing well-posedness theories on the Boussinesq equations with standard Laplacian dissipation in bounded domains. The second paragraph outlines how one can build an existence and uniqueness theory as in Theorem 1.1 on the Boussinesq equations with fractional dissipation in a bounded domain with the Dirichlet boundary condition. The 2D Boussinesq equations with only viscous dissipation in a bounded domain with the Dirichlet boundary condition were first studied by Lai, Pan and Zhao [39]. They established the global existence and uniqueness of solutions in the Sobolev space $H^3$. The same Boussinesq equations with the stress-free boundary condition were considered by Doering, Wu, Zhao and Zheng [18] and were shown to possess a unique global solution when the velocity is in $H^2$ and the temperature in $H^1$. Zhao examined the 2D Boussinesq system without viscous dissipation but with thermal diffusion in a bounded domain and was able to obtain the global $H^3$ solutions [66]. He developed some very useful tools for the Navier–Stokes equations in general domains including bounded ones such as some new Brezis–Waigner type inequalities [25]. He then applied these tools to the 2D Boussinesq equations with viscous Laplacian dissipation in bounded domains to obtain the existence and uniqueness on solutions in a very weak setting. Many more interesting results on the Boussinesq equations in bounded domains can be found in [30, 32, 31, 33, 37].

To study the Boussinesq equations with fractional viscous dissipation in a bounded domain $\Omega$, we need to define the fractional Laplacian operator first. When the Dirichlet boundary condition is prescribed, the fractional Laplacian operator is defined through the eigenvalues and eigenfunctions of the classical Stokes operator $-\mathbb{P}\Delta$ (with $\mathbb{P}$ being the Leray projection), namely

$$(-\Delta)^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha (u, w_j) w_j,$$

where $\lambda_j$ and $w_j$ with $j = 1, 2, \ldots$ are the eigenvalues and eigenfunctions, and $(u, w_j)$ denotes the inner product in $L^2(\Omega)$. The existence of weak solutions can be approached via the Galerkin approximation. After applying suitable energy estimates, we will be able to establish the first part of Theorem 1.1. We also believe the second part of Theorem 1.1 can be validated for bounded domains with the Dirichlet boundary condition by suitably modifying the proof of Theorem 1.1 described below. We leave the details to a future work.

The proof of Theorem 1.1 consists of two main parts. The first part is the global existence of weak solutions for any $\alpha > 0$ and $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$. This is done by showing the global existence of smooth solutions $(\mathbf{u}^{(n)}, \theta^{(n)})$ to a sequence of approximate systems, establishing global uniform bounds for $(\mathbf{u}^{(n)}, \theta^{(n)})$, obtaining the $L^2$-convergence for a subsequence of $\mathbf{u}^{(n)}$ and passing to the limit. Due to the lack of thermal diffusion, there is no strong convergence in $\theta^{(n)}$. However, we can still pass the limit. When $\alpha \geq \frac{1}{2} + \frac{d}{4}$, the weak solution is unique. Due to the weak regularity setting, $\mathbf{u}$ is not Lipschitz and the corresponding vorticity is not necessarily bounded. The proof makes use of the following smoothing property of the velocity

$$\|\mathbf{u}\|_{\widetilde{L}^1(0,T;B_{2,2}^{1+\frac{d}{2}})} \leq C(T, \|\mathbf{u}_0\|_{L^2}, \|\theta_0\|_{L^2 \cap L^{\frac{4d}{d+2}}}), \tag{1.6}$$

and a special consequence of (1.6). The definition of $\widetilde{L}^1(0, T;\, B_{2,2}^{1+\frac{d}{2}})$ is provided in Sec. 2. (1.6) is proven via the Littlewood–Paley decomposition and Besov spaces techniques developed here for periodic domains. The proof for the coincidence of two weak solutions $(\mathbf{u}^{(1)}, \theta^{(1)})$ and $(\mathbf{u}^{(2)}, \theta^{(2)})$ is based on the bounds for the $L^2$-norms of the differences

$$\|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}\|_{L^2} + \|\nabla h^{(1)} - \nabla h^{(2)}\|_{L^2}$$

where $h^{(1)}$ and $h^{(2)}$ satisfy

$$-\Delta h^{(i)} = \theta^{(i)}, \quad i = 1, 2.$$

Due to the lack of thermal diffusion and the weak regularity of $\theta$, it is not possible to bound the difference $\|\theta^{(1)} - \theta^{(2)}\|_{L^2}$. The introduction of $h^{(1)}$ and $h^{(2)}$ reduce the regularity requirements and helps facilitate the proof. Such lower regularity variables have previously been used in [25, 40].

To prove Theorem 1.3 and compare the solutions $(\mathbf{u}^{(\eta)}, \theta^{(\eta)})$ of (1.4) and $(\mathbf{u}, \theta)$ of (1.1), we make use of the lower regularity quantities $h^{(\eta)}$ and $h$ satisfying

$$-\Delta h^{(\eta)} = \theta^{(\eta)}, \quad -\Delta h = \theta$$

and estimate the difference

$$\|(\mathbf{u}^{(\eta)} - \mathbf{u})(t)\|_{L^2}^2 + \|(\nabla h^{(\eta)} - \nabla h)(t)\|_{L^2}^2$$

via Yudovich techniques.

The rest of this paper is divided into three sections and two appendices. The second section introduces the concept of the Littlewood–Paley decomposition for periodic functions, proves some crucial properties of dyadic Fourier blocks, defines Besov type spaces and derives related properties. Section 3 proves Theorem 1.1. Due to the length of the proof for the global existence of weak solutions, the proof of this part is given in one of the appendices. Section 4 proves Theorem 1.3. The first appendix proves the global existence of weak solutions while the second appendix provides the definitions of the Littlewood–Paley decomposition for $\mathbb{R}^d$, the Besov spaces and an Osgood type inequality to be used in the subsequent sections.

## 2. Littlewood–Paley Decomposition for Periodic Functions

The purpose of this section is to introduce the concepts of Fourier dyadic blocks and the Littlewood–Paley decomposition for periodic functions, and develop associated tools that are useful for the study of solutions of PDEs in $\mathbb{T}^d$ with $d \geq 2$. This section starts with a review of partial sums for the Fourier series characterized by various kernel functions including the square Dirichlet kernel, the circular Dirichlet kernel, general $l_q$ Dirichlet kernel with $1 \leq q \leq \infty$, Fejér kernels and Riesz kernels. After examining the boundedness and convergence properties for these Fourier cutoffs, it appears that the square cutoff is the most suitable for the definition of the Littlewood–Paley blocks. We derive the properties of the associated localization operators such as Bernstein's inequalities. We define Besov type spaces and identify them with some of the standard Sobolev spaces.

Let $d \geq 2$ and $d \in \mathbb{N}$. Let $\mathbb{T}^d = [-\pi, \pi]^d$. For $f \in L^1(\mathbb{T}^d)$, the Fourier modes $\widehat{f}(\mathbf{k})$ with $\mathbf{k} \in \mathbb{Z}^d$ is given by

$$\widehat{f}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}.$$

More generally, for any $f \in \mathcal{S}$, the set of all distributions on $\mathbb{T}^d$, the Fourier coefficient $\widehat{f}(\mathbf{k})$ is defined by $\widehat{f}(\mathbf{k}) = \langle f, e^{-i\mathbf{k}\cdot\mathbf{x}} \rangle$ (see, e.g., [23, 56]). The Fourier series of $f$ at $\mathbf{x} \in \mathbb{T}^d$ is the series

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.1}$$

One prominent issue is whether or not (2.1) converges. The convergence of the Fourier series in the one-dimensional case has been thoroughly studied and the classical convergence result is stated in the following lemma (see, e.g., [23, 56]).

**Lemma 2.1.** *Assume $f \in L^p(\mathbb{T})$ with $1 < p < \infty$. Then the partial sum*

$$S_N f(x) = \sum_{|k|\leq N} \widehat{f}(k) e^{ikx}$$

*satisfies*

$$\|S_N f\|_{L^p} \leq C_p \|f\|_{L^p} \tag{2.2}$$

*and*

$$\|S_N f - f\|_{L^p} \to 0 \quad \text{as } N \to \infty. \tag{2.3}$$

Let $D_N$ denote the Dirichlet kernel, namely

$$D_N(x) := \sum_{|k|\le N} e^{ikx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

The partial sum $S_Nf(x)$ can be written as the convolution of the Dirichlet kernel with $f$,

$$S_Nf(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-y)f(y)dy.$$

However, the $L^1$-norm of $D_N$ grows at the order of $\log N$, more precisely,

$$\frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \le \|D_N\|_{L^1(\mathbb{T})} \le 2 + \frac{\pi}{4} + \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}.$$

(2.2) does not trivially follow from Young's inequality applied to $S_Nf = D_N * f$.

(2.2) and (2.3) may not hold for $p=1$ or $p=\infty$. In fact, there exist a continuous function $f$ on $\mathbb{T}$ and an $x_0 \in \mathbb{T}$ such that the sequence

$$\limsup_{N\to\infty} |D_N * f|(x_0) = \infty.$$

There are also explicit integrable functions $f$ on $\mathbb{T}$ such that the corresponding Fourier series does not converge in $L^1(\mathbb{T})$.

We now consider multi-dimensional partial sums. Let $d \ge 2$. the partial sum can be defined in many ways. Two of the most natural ones are the partial sum with square-cutoff and the partial sum with circular cutoff, namely

$$S_Nf = \sum_{|k_1|\le N,\ldots,|k_d|\le N} \widehat{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = D_N * f \tag{2.4}$$

and

$$\widetilde{S}_Nf = \sum_{|\mathbf{k}|=\sqrt{k_1^2+\cdots+k_d^2}\le N} \widehat{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = \widetilde{D}_N * f, \tag{2.5}$$

where $D_N$ denotes the $d$-dimensional square Dirichlet kernel and $\widetilde{D}_N$ the circular Dirichlet kernel,

$$D_N(\mathbf{x}) = \sum_{|k_1|\le N,\ldots,|k_d|\le N} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \widetilde{D}_N(\mathbf{x}) = \sum_{|\mathbf{k}|\le N} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

These two partial sums have different convergence properties, as stated in the following lemmas (see, e.g., [23, 56]). The partial sum defined via the square-cutoff is bounded on any $L^p$ for $1<p<\infty$ and converges to the original function in $L^p$.

**Lemma 2.2.** *Let $d \ge 1$. The partial sum with the square cutoff $S_Nf$ satisfies, for any $f \in L^p(\mathbb{T}^d)$ with $1<p<\infty$,*

$$\|S_Nf\|_{L^p} \le C_p\|f\|_{L^p} \quad \textit{and} \quad \|S_Nf - f\|_{L^p} \to 0 \quad \textit{as } N\to\infty. \tag{2.6}$$

(2.6) *is false for* $p = 1$ *and for* $p = \infty$. *In addition, if* $f \in L^p(\mathbb{T}^d)$ *with* $1 < p \leq \infty$, *then*

$$S_N f \to f, \quad a.e.\ as\ N \to \infty.$$

The partial sum defined via the circular cutoff in $\mathbb{T}^d$ with $d \geq 2$ is not bounded on $L^p$ except for $p = 2$ and is not known to converge to the original function except for $p = 2$.

**Lemma 2.3.** *Let* $d \geq 2$ *and* $f \in L^2(\mathbb{T}^d)$. *Then*

$$\|\widetilde{S}_N f\|_{L^2} \leq \|f\|_{L^2} \quad and \quad \|\widetilde{S}_N f - f\|_{L^2} \to 0 \quad as\ N \to \infty. \tag{2.7}$$

(2.7) *is false if we change* $L^2$ *to* $L^p$ *with* $p \neq 2$.

Other partial sums include the $l^q$-partial sum $S_N^q f$ defined by

$$S_N^q f(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^d, \|\mathbf{k}\|_{l^q} \leq N} \widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} D_N^q(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \tag{2.8}$$

where $l^q$-Dirichlet kernel $D_N^q$ is given by

$$D_N^q(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d, \|\mathbf{k}\|_{l^q} \leq N} e^{i\mathbf{k}\cdot\mathbf{x}}$$

with

$$\|\mathbf{k}\|_{l^q} := \begin{cases} \left( \sum_{m=1}^{d} |k_m|^q \right)^{\frac{1}{q}}, & 1 \leq q < \infty; \\ \max_{m=1,2,\ldots,d} |k_m|, & q = \infty. \end{cases}$$

The following lemma tells us what we know about the convergence of $l^q$-partial sum $S_N^q f$, besides the convergence results stated in Lemmas 2.2 and 2.3 (see [56]).

**Lemma 2.4.** *Let* $q = 1$ *or* $\infty$. *Assume* $f \in L^p(\mathbb{T}^d)$ *with* $1 < p < \infty$. *Then*

$$\|S_N^q f\|_{L^p} \leq C_p \|f\|_{L^p} \quad and \quad \|S_N^q f - f\|_{L^p} \to 0 \quad as\ N \to \infty.$$

We also mention the partial sums defined in terms of the Fejér and the Riesz kernels including the Bochner–Riesz kernels. Let $f \in L^1(\mathbb{T}^d)$ and let $1 \leq q \leq \infty$. The partial sum via the $l^q$-Fejér kernel is given by

$$\begin{aligned} \sigma_N^q f(\mathbf{x}) &= \sum_{\mathbf{k} \in \mathbb{Z}^d, \|\mathbf{k}\|_{l^q} \leq N} \left(1 - \frac{\|\mathbf{k}\|_{l^q}}{N}\right) \widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F_N^q(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \end{aligned} \tag{2.9}$$

where

$$F_N^q(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d, \|\mathbf{k}\|_{l^q} \leq N} \left(1 - \frac{\|\mathbf{k}\|_{l^q}}{N}\right) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

For $q = 1$ and $q = \infty$, the Fejér kernels are the arithmetic means of the $l^q$ kernels,

$$F_N^q(\mathbf{x}) = \frac{1}{N} \sum_{j=0}^{N-1} S_N^q(\mathbf{x}).$$

Let $1 \le \gamma < \infty$ and $0 \le \beta < \infty$. The partial sum via the Riesz kernels is given by

$$\sigma_N^{q,\gamma,\beta} f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d, \|\mathbf{k}\|_{l^q} \le N} \left(1 - \left(\frac{\|\mathbf{k}\|_{l^q}}{N}\right)^{\gamma}\right)^{\beta} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F_N^{q,\gamma,\beta}(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \tag{2.10}$$

where the Riesz kernels are given by

$$F_N^{q,\gamma,\beta}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d, \|\mathbf{k}\|_{l^q} \le N} \left(1 - \left(\frac{\|\mathbf{k}\|_{l^q}}{N}\right)^{\gamma}\right)^{\beta} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

When $q = \gamma = 2$ and $\beta > 0$, the Riesz kernel reduces to the Bochner–Riesz kernel.

The convergence of these partial sums have been investigated by many authors [20, 24, 28, 38, 56]) and the following results are relevant to our study.

**Lemma 2.5.** *If $q = 1, \infty$, $1 \le \gamma < \infty$ and $\beta > 0$, then the $L^1$-norm of $F_N^{q,\gamma,\beta}$ is bounded uniformly in terms of $N$,*

$$\|F_N^{q,\gamma,\beta}\|_{L^1(\mathbb{T}^d)} \le C,$$

*where $C$ is independent of $N$. If $q = 2$, then the same holds for $\beta > \frac{d-1}{2}$.*

**Lemma 2.6.** *Let $q = 1$ or $\infty$. Let $1 \le \gamma < \infty$ and $\beta > 0$. Assume $f \in L^p(\mathbb{T}^d)$ with $1 \le p < \infty$. Then*

$$\|\sigma_N^{q,\gamma,\beta} f\|_{L^p} \le C_p \|f\|_{L^p} \quad \text{and} \quad \|\sigma_N^{q,\gamma,\beta} f - f\|_{L^p} \to 0 \quad \text{as } N \to \infty.$$

*If $q = 2$, the same holds for $\beta > \frac{d-1}{2}$.*

The review of the convergence results and properties above for various partial sums of the Fourier series allows us to choose the suitable cutoff in the definition of the Littlewood–Paley blocks (or the localized Fourier projections). The circular cutoff defined in (2.5) is not suitable since it does not have the desired property

$$\|\widetilde{S}_N f\|_{L^p} \le C_p \|f\|_{L^p}$$

for all $1 < p < \infty$. (2.2) only holds for $p = 2$ according to Lemma 2.3. The Fejér cutoff defined in (2.9) and the Riesz cutoff defined in (2.10) with $q = 1$ or $q = \infty$ have the desired boundedness properties, but it is not clear how we can split these partial sums into dyadic blocks. Then the choices left are either the square cutoff defined in (2.4) (or the $l^\infty$-cutoff, namely (2.8) with $q = \infty$) or the $l^1$-cutoff, (2.8) with $q = 1$.

We choose the square-cutoff defined in (2.4). We introduce a few notation first. For an integer $j \geq 0$, we set $A_j$ to be the $2^j$-sized block of $d$-dimensional integer lattice points,

$$A_j = \{\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d : |k_m| \leq 2^j,\ m = 1, 2, \ldots, d\}.$$

We define the following localized Fourier projection operators as

$$\begin{aligned} \Delta_0 f(\mathbf{x}) &= \sum_{\mathbf{k} \in A_0} \widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \\ \Delta_j f(\mathbf{x}) &= \sum_{\mathbf{k} \in A_j \setminus A_{j-1}} \widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad j \geq 1,\ j \in \mathbb{N}. \end{aligned} \tag{2.11}$$

For notational convenience, we also write $\Delta_j = 0$ for $j < 0$. With a slight abuse of notation, we set

$$S_j f(\mathbf{x}) = \sum_{m=0}^{j-1} \Delta_m f(\mathbf{x}) = \sum_{\mathbf{k} \in A_{j-1}} \widehat{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.12}$$

In terms of these operators, we can write the Littlewood–Paley decomposition, for any $f \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$,

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \Delta_k f(\mathbf{x}).$$

The following lemma presents useful basic properties of the operators defined above.

**Lemma 2.7.** *Let $j \geq 0$ be an integer. Let $\Delta_j$ and $S_j$ be defined as in* (2.11) *and* (2.12). *Then the following properties hold.*

(a) *If $f \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$, then*

$$\|\Delta_j f\|_{L^p} \leq C\|f\|_{L^p}, \quad \|S_j f\|_{L^p} \leq C\|f\|_{L^p},$$

*where $C$'s are constants depending on $p$ and $d$ only.*

(b) *Let $j \geq 0$ and $k \geq 0$ be integers. Assume $f \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$. Then*

$$\Delta_j \Delta_k f = 0 \quad \textit{if } j \neq k.$$

(c) *Let $j \geq 0$ and $m \geq 0$ be integers. Assume $f, g \in L^p(\mathbb{T}^d)$ with $1 < p \leq \infty$. Then*

$$\Delta_j(S_{m-d} f \Delta_m g) = 0 \quad \textit{if } |m - j| \geq d + 1.$$

**Proof.** (a) Follows directly from Lemma 2.2. (b) Is almost obvious. $\Delta_k f$ contains terms with Fourier modes $\widehat{f}(\mathbf{m})$ for $\mathbf{m} \in A_k \setminus A_{k-1}$. If $j \neq k$, then $\Delta_k f$ does not involve any $\widehat{f}(\mathbf{m})$ for $\mathbf{m} \in A_j \setminus A_{j-1}$ and thus $\Delta_j \Delta_k f = 0$. We now turn to (c). Clearly, $S_{m-d} f \Delta_m g$ contains terms of the form $\widehat{f}(\mathbf{l})\, \widehat{g}(\mathbf{k})\, e^{i(\mathbf{l}+\mathbf{k})\cdot\mathbf{x}}$ with $\mathbf{l}, \mathbf{k}$ satisfying

$$|l_i| \leq 2^{m-d-1} \quad \text{for } i = 1, 2, \ldots, d, \quad \text{and} \quad \mathbf{k} \in A_m \setminus A_{m-1}.$$

$\mathbf{k} \in A_m \backslash A_{m-1}$ implies

$$k_1 + k_2 + \cdots + k_d \geq 2^{m-1} + 1,$$

since at least one of the components is more than $2^{m-1}$ (otherwise $\mathbf{k} \in A_{m-1}$). Therefore,

$$\begin{aligned}|l_1 + k_1| + |l_2 + k_2| + \cdots + |l_d + k_d| &\geq k_1 + k_2 + \cdots + k_d - (l_1 + l_2 + \cdots + l_d)\\ &\geq 2^{m-1} + 1 - d\,2^{m-d-1}.\end{aligned}$$

Since, for any $d \geq 1$, we have $d \leq 2^{d-1}$ and thus

$$\begin{aligned}|l_1 + k_1| + |l_2 + k_2| + \cdots + |l_d + k_d| &\geq 2^{m-1} + 1 - 2^{m-2}\\ &\geq 2^{m-2} + 1 > d\,2^{m-d-1}.\end{aligned}$$

On the other hand, it is clear that

$$|l_1 + k_1| + |l_2 + k_2| + \cdots + |l_d + k_d| \leq d2^{m-d-1} + d2^m < d2^{m+1}.$$

Therefore, $\mathbf{l} + \mathbf{k} \in A_{m+1} \backslash A_{m-d-1}$ and $\Delta_j(S_{m-d} f \Delta_m g) = 0$ if $|m - j| \geq d + 1$. This completes the proof of Lemma 2.7. $\square$

The operators $\Delta_j$ defined for periodic functions share many properties with those for the whole space $\Delta_j$. One crucial property is the following Bernstein type inequalities. The proof of these properties appears to be more difficult than that for the whole space case.

**Proposition 2.8.** *Let $\sigma \geq 0$ and $1 \leq q \leq p \leq \infty$.*

(1) *There exists a constant $C > 0$ such that*

$$\|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T}^d)} \leq C 2^{\sigma j + jd(\frac{1}{q} - \frac{1}{p})} \|\Delta_j f\|_{L^q(\mathbb{T}^d)} \tag{2.13}$$

*and*

$$\|S_j f\|_{L^p(\mathbb{T}^d)} \leq C 2^{jd(\frac{1}{q} - \frac{1}{p})} \|S_j f\|_{L^q(\mathbb{T}^d)}. \tag{2.14}$$

(2) *Let $1 \leq p \leq \infty$. There exist constants $0 < C_1 < C_2$ (depending on $p$) such that, for any integer $j \geq 0$,*

$$C_1 2^{\sigma j} \|\Delta_j f\|_{L^p(\mathbb{T}^d)} \leq \|\Delta_j \Lambda^\sigma f\|_{L^p(\mathbb{T}^d)} \leq C_2 2^{\sigma j} \|\Delta_j f\|_{L^p(\mathbb{T}^d)}. \tag{2.15}$$

**Proof.** To prove (1), we first set $\sigma = 0$ and start with the case when $p = \infty$ and $q = 2$. By Hölder's inequality and Plancherel's identity,

$$\begin{aligned}\|\Delta_j f\|_{L^\infty(\mathbb{T}^d)} &\leq \sum_{\mathbf{k} \in A_j \backslash A_{j-1}} |\widehat{f}(\mathbf{k})|\\ &\leq \left( \sum_{\mathbf{k} \in A_j \backslash A_{j-1}} 1 \right)^{\frac{1}{2}} \left( \sum_{\mathbf{k} \in A_j \backslash A_{j-1}} |\widehat{f}(\mathbf{k})|^2 \right)^{\frac{1}{2}}\\ &\leq C\, 2^{\frac{d}{2} j} \|\Delta_j f\|_{L^2(\mathbb{T}^d)},\end{aligned}$$

which is just (2.13) with $p = \infty$ and $q = 2$. Here we have used the fact that

$$\sum_{\mathbf{k}\in A_j\setminus A_{j-1}} 1 = C(d)2^{dj}$$

for a constant $C(d)$ depending on the dimension $d$. In the case when $p = \infty$ and $1 \le q < 2$, we use the Hausdorff–Young inequality, for $\frac{1}{q} + \frac{1}{r} = 1$, obtain

$$\begin{aligned}
\|\Delta_j f\|_{L^\infty(\mathbb{T}^d)} &\le \sum_{\mathbf{k}\in A_j\setminus A_{j-1}} |\widehat{f}(\mathbf{k})| \\
&\le \left(\sum_{\mathbf{k}\in A_j\setminus A_{j-1}} 1\right)^{\frac{1}{q}} \left(\sum_{\mathbf{k}\in A_j\setminus A_{j-1}} |\widehat{f}(\mathbf{k})|^r\right)^{\frac{1}{r}} \\
&\le C2^{dj\frac{1}{q}} \|\widehat{\Delta_j f}\|_{l^r} \\
&\le C2^{dj\frac{1}{q}} \|\Delta_j f\|_{L^q(\mathbb{T}^d)},
\end{aligned}$$

which is (2.13) with $p = \infty$ and $1 \le q < 2$. Now for $1 \le q \le 2 \le p < \infty$, we have, by an interpolation inequality,

$$\begin{aligned}
\|\Delta_j f\|_{L^p(\mathbb{T}^d)} &\le \|\Delta_j f\|_{L^q(\mathbb{T}^d)}^{\frac{q}{p}} \cdot \|\Delta_j f\|_{L^\infty(\mathbb{T}^d)}^{1-\frac{q}{p}} \\
&\le C\|\Delta_j f\|_{L^q(\mathbb{T}^d)}^{\frac{q}{p}} \cdot 2^{dj\frac{1}{q}(1-\frac{q}{p})} \|\Delta_j f\|_{L^q(\mathbb{T}^d)}^{1-\frac{q}{p}} \\
&= C2^{dj(\frac{1}{q}-\frac{1}{p})} \|\Delta_j f\|_{L^q(\mathbb{T}^d)}.
\end{aligned}$$

We now prove for the case $2 < q < p \le \infty$. We use the duality. Let $\widetilde{p}$ and $\widetilde{q}$ be the dual indices of $p$ and $q$, respectively, namely

$$\frac{1}{p} + \frac{1}{\widetilde{p}} = 1, \quad \frac{1}{q} + \frac{1}{\widetilde{q}} = 1.$$

Since $2 < q \le p \le \infty$, we have $1 \le \widetilde{p} < \widetilde{q} < 2$ and the previous result applies.

$$\begin{aligned}
\|\Delta_j f\|_{L^p} &= \sup_{\|g\|_{L^{\widetilde{p}}}=1} \left| \int_{\mathbb{T}^d} \Delta_j f(\mathbf{x}) \bar{g}(\mathbf{x})\, d\mathbf{x} \right| \\
&= \sup_{\|g\|_{L^{\widetilde{p}}}=1} \int_{\mathbb{T}^d} |\Delta_j f(\mathbf{x}) \Delta_j g(\mathbf{x})| d\mathbf{x} \\
&\le \sup_{\|g\|_{L^{\widetilde{p}}}=1} \|\Delta_j f\|_{L^q} \|\Delta_j g\|_{L^{\widetilde{q}}} \\
&\le \sup_{\|g\|_{L^{\widetilde{p}}}=1} \|\Delta_j f\|_{L^q} 2^{dj(\frac{1}{\widetilde{p}}-\frac{1}{\widetilde{q}})} \|\Delta_j g\|_{L^{\widetilde{p}}} \\
&\le C\|\Delta_j f\|_{L^q} 2^{dj(\frac{1}{q}-\frac{1}{p})},
\end{aligned}$$

where we have used the fact that $\|\Delta_j g\|_{L^{\widetilde{p}}} \le C_p \|g\|_{L^{\widetilde{p}}} \le C_p$. This finishes the proof of (2.13) with $\sigma = 0$. (2.14) can be similarly established. Clearly, (2.13) with

$\sigma > 0$ is a consequence of (2.13) with $\sigma = 0$ and (2). The proof of (2) follows from the proof for the classical Bernstein's inequalities (see, e.g., [16]). The classical Bernstein inequality states that, for any trigonometric polynomial $g_n$ of degree less than or equal to $n$,

$$\|g_n'\|_{L^\infty(\mathbb{T})} \leq Cn\|g\|_{L^\infty(\mathbb{T})}. \tag{2.16}$$

This inequality also holds for $L^1$-norm (see [16, p. 101])

$$\|g_n'\|_{L^1(\mathbb{T})} \leq Cn\|g\|_{L^\infty(\mathbb{T})}. \tag{2.17}$$

The upper bound in (2.15) is obtained by following a similar approach as in the proof of (2.16) and (2.17). To prove the lower bound part, we recall a closely related classical result ([38, p. 55]), which states that, for any trigonometric polynomial of the form

$$g(x) = \sum_{|j|\geq n} a_j e^{ijx}$$

and for any positive integer $m$,

$$n^m \|g\|_{L^p(\mathbb{T})} \leq C_m \|g^{(m)}\|_{L^p(\mathbb{T})},$$

where $g^{(m)}$ denotes the $m$th derivative of $g$. The lower bound part of (2.15) can be shown in a similar fashion. This completes the proof of Proposition 2.8. $\square$

In terms of the operators $\Delta_j$ and $S_j$, we can write a standard product of two periodic functions as a sum of paraproducts, as in the whole space case (see, e.g., [8, 5])

$$fg = T_f g + T_g f + R(f,g),$$

where

$$T_f g = \sum_j S_{j-d} f \Delta_j g, \quad R(f,g) = \sum_j \sum_{k\geq j-1} \Delta_k f \widetilde{\Delta}_k g$$

with $\widetilde{\Delta}_k = \Delta_{k-d} + \Delta_{k-d+1} + \cdots + \Delta_{k+d}$.

We can also define the Besov type space $B^s_{p,q}(\mathbb{T}^d)$ via the operators $\Delta_j$ defined above in the same fashion as in the whole space case. Let $\mathcal{S}$ denote the usual Schwarz class and $\mathcal{S}'$ the distributions.

**Definition 2.9.** Let $f \in \mathcal{S}'$. The Besov space $B^s_{p,q}(\mathbb{T}^d)$ with $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'(\mathbb{T}^d)$ satisfying

$$\|f\|_{B^s_{p,q}(\mathbb{T}^d)} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

We can also define the space-time spaces for periodic functions. This type of functional settings was introduced by Chemin–Lerner for functions defined on the whole space (see, e.g., [5]).

**Definition 2.10.** For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the space-time space $\widetilde{L}_t^r B_{p,q}^s$ is defined through the norm

$$\|f\|_{\widetilde{L}_t^r B_{p,q}^s} \equiv \|2^{js}\|\Delta_j f\|_{L_t^r L^p}\|_{l^q}.$$

These Besov spaces defined above are closely related to some of the standard spaces and share similar properties with their whole space counterparts.

**Lemma 2.11.** *Let $s \in \mathbb{R}$.*

(1) *For $1 \le p \le \infty$ and $q_1 \le q_2$, $B_{p,q_1}^s(\mathbb{T}^d) \subset B_{p,q_2}^s(\mathbb{T}^d)$.*
(2) *$H^s(\mathbb{T}^d)$ can be identified with $B_{2,2}^s(\mathbb{T}^d)$,*

$$H^s(\mathbb{T}^d) \sim B_{2,2}^s(\mathbb{T}^d).$$

**Proof.** (1) Follows directly from the fact that, if $q_1 \le q_2$, then $l^{q_1} \subset l^{q_2}$. (2) Follows from Plancherel's identity. □

## 3. Proof of Theorem 1.1

This section proves Theorem 1.1. A crucial smoothing estimate is obtained using the Littlewood–Paley decomposition and Besov space techniques introduced in the previous section. Naturally the proof is divided into two main parts. The first part is the proof of the global existence of weak solutions of (1.1) with any $\alpha > 0$. This is stated in Proposition 3.2. Since the proof of Proposition 3.2 is lengthy, we leave it to one of the appendices. The second part is the proof of the uniqueness of weak solutions to (1.1) when $\alpha \ge \frac{1}{2} + \frac{d}{4}$. In order to prove the uniqueness, we first prove a major smoothing estimate for the velocity field in Proposition 3.3.

We start with the definition of weak solutions of (1.1) with any $\alpha > 0$.

**Definition 3.1.** Consider (1.1) with $\alpha > 0$, $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$ and $\nabla \cdot \mathbf{u}_0 = 0$. Let $T > 0$ be arbitrarily fixed. A pair $(\mathbf{u}, \theta)$ satisfying

$$\mathbf{u} \in C_w([0,T]; L^2) \cap L^2(0,T; \mathring{H}^\alpha), \quad \nabla \cdot \mathbf{u} = 0,$$
$$\theta \in C_w([0,T]; L^2) \cap L^\infty(0,T; L^2)$$

is weak solution of (1.1) on $[0,T]$ if (a) and (b) below hold:

(a) For any $\phi \in C_0^\infty(\mathbb{T}^d \times [0,T))$ with $\nabla \cdot \phi = 0$ and for any $t \le T$,

$$\begin{aligned} &-\int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \partial_t \phi \, d\mathbf{x} \, d\tau + \int_{\mathbb{T}^d} \mathbf{u}(\mathbf{x},t) \cdot \phi(\mathbf{x},t) d\mathbf{x} - \int_{\mathbb{T}^d} \mathbf{u}_0(\mathbf{x}) \cdot \phi(\mathbf{x},0) d\mathbf{x} \\ &\qquad - \int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla \phi \mathbf{u} d\mathbf{x} d\tau + \int_0^t \int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} \mathbf{u} \cdot (-\Delta)^{\alpha/2} \phi d\mathbf{x} d\tau \\ &= \int_0^t \int_{\mathbb{T}^d} \theta \mathbf{e}_d \cdot \phi \, d\mathbf{x} \, d\tau. \end{aligned}$$

(b) For any $\psi \in C_0^\infty(\mathbb{T}^d \times [0,T))$ and $t \le T$,

$$-\int_0^t \int_{\mathbb{T}^d} \partial_t \psi \theta d\mathbf{x} d\tau + \int_{\mathbb{T}^d} \theta(\mathbf{x},t)\psi(\mathbf{x},t) d\mathbf{x} - \int_{\mathbb{T}^d} \theta_0(\mathbf{x})\psi(\mathbf{x},0) d\mathbf{x}$$
$$= \int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla \psi \theta d\mathbf{x} d\tau + \gamma \int_0^t \int_{\mathbb{T}^d} u_d \psi d\mathbf{x} d\tau.$$

For any $\alpha > 0$ and $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$, (1.1) always has a global weak solution. In the special case when $\theta \equiv 0$, this result assesses the global existence of weak solutions of the generalized Navier–Stokes equations with any $\alpha > 0$ and $\mathbf{u}_0 \in L^2(\mathbb{T}^d)$.

**Proposition 3.2.** *Consider* (1.1) *with* $\alpha > 0$, $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^d)$ *and* $\nabla \cdot \mathbf{u}_0 = 0$. *Let* $T > 0$ *be arbitrarily fixed. Then* (1.1) *has a global weak solution* $(\mathbf{u}, \theta)$ *as given in Definition* 3.1 *satisfying*

$$\|\mathbf{u}(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^2}^2 d\tau \le e^{Ct}(\|\mathbf{u}_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2).$$

The proof of Proposition 3.2 is long and the details will be provided in one of the appendices. Next we establish a smoothing estimate for the weak solution shown in Proposition 3.2.

**Proposition 3.3.** *Let* $d \ge 2$. *Consider* (1.1) *with* $\alpha \ge \frac{1}{2} + \frac{d}{4}$. *Assume* $(\mathbf{u}_0, \theta_0)$ *satisfies*

$$\mathbf{u}_0 \in L^2(\mathbb{T}^d), \quad \nabla \cdot \mathbf{u}_0 = 0, \quad \theta_0 \in L^2(\mathbb{T}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{T}^d).$$

*Let* $(\mathbf{u}, \theta)$ *be the corresponding global weak solution of* (1.1). *Then, for any* $0 < t \le T$,

$$\|\mathbf{u}\|_{\widetilde{L}_t^1 B_{2,2}^{1+\frac{d}{2}}} \le C(t, \|\mathbf{u}_0\|_{L^2}, \|\theta_0\|_{L^2}). \tag{3.1}$$

*As a special consequence,*

$$\sup_{q \ge 2} \int_0^t \frac{\|\nabla \mathbf{u}(\tau)\|_{L^q}}{\sqrt{q}} d\tau \le C(t, \|\mathbf{u}_0\|_{L^2}, \|\theta_0\|_{L^2}). \tag{3.2}$$

Proposition 3.3 is proven via the Littlewood–Paley decomposition and Besov space techniques for periodic functions introduced in the previous section. The proof for the 2D case is partially different from that for the general $d$-D case with $d \ge 3$. We need a lemma for the 2D case.

**Lemma 3.4.** *Assume* $(\mathbf{u}_0, \theta_0) \in L^2(\mathbb{T}^2)$ *with* $\nabla \cdot \mathbf{u}_0 = 0$. *Consider the* 2*D Boussinesq equation in* (1.1) *with* $\alpha \ge 1$. *Let* $(\mathbf{u}, \theta)$ *be the corresponding weak solution. Then* $\mathbf{u}$ *satisfies*

$$\int_0^T \|\mathbf{u}(t)\|_{L^\infty}^2 d\tau \le C(t, \|\mathbf{u}_0\|_{L^2}, \|\theta_0\|_{L^2}). \tag{3.3}$$

**Proof.** It suffices to consider the case when $\alpha = 1$. Let $j \geq 0$ and $\Delta_j$ be as defined in (2.11). Applying $\Delta_j$ to the velocity equation and then dotting with $\Delta_j \mathbf{u}$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta_j \mathbf{u}\|_{L^2}^2 + \nu 2^{2j}\|\Delta_j \mathbf{u}\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^2} \Delta_j \mathbf{u} \cdot \Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\,d\mathbf{x} + \int_{\mathbb{R}^2} \Delta_j \mathbf{u}\cdot\Delta_j(\theta\mathbf{e}_2)\,d\mathbf{x} \\
&\quad \leq \|\Delta_j \mathbf{u}\|_{L^2}\|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2} + \|\Delta_j\mathbf{u}\|_{L^2}\|\Delta_j\theta\|_{L^2}.
\end{aligned}$$

Eliminating $\|\Delta_j \mathbf{u}\|_{L^2}$ from each side and integrating in time yield

$$\begin{aligned}
&\|\Delta_j\mathbf{u}(t)\|_{L^2} + \nu\int_0^t 2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}\,d\tau \\
&\quad \leq \|\Delta_j\mathbf{u}_0\|_{L^2} + \int_0^t \|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}\,d\tau + \int_0^t \|\Delta_j\theta\|_{L^2}\,d\tau.
\end{aligned}$$

Taking the $l^2$-norm and identifying $H^s$ with $B^s_{2,2}$ for $s \geq 0$, we have

$$\begin{aligned}
&\left\|\sup_{0\leq\tau\leq t}\|\Delta_j\mathbf{u}(\tau)\|_{L^2}\right\|_{l^2} + \nu\|\mathbf{u}\|_{\widetilde{L}^1(0,t;H^2)} \\
&\quad \leq \|\mathbf{u}_0\|_{L^2} + \int_0^t \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}\,d\tau + \int_0^t \|\theta(\tau)\|_{L^2}\,d\tau \\
&\quad \leq \|\mathbf{u}_0\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2_tL^2}\|\mathbf{u}\|_{L^2_tL^\infty} + \|\theta\|_{L^1_tL^2}.
\end{aligned} \tag{3.4}$$

By the Littlewood–Paley decomposition and the Bernstein inequalities in Proposition 2.8, we have

$$\begin{aligned}
\int_0^t \|\mathbf{u}(\tau)\|_{L^\infty}^2\,d\tau &\leq \int_0^t \sum_{j=0}^\infty\sum_{k=0}^\infty 2^j 2^k\|\Delta_j\mathbf{u}\|_{L^2}|\Delta_k\mathbf{u}\|_{L^2}\,d\tau \\
&:= H_1 + H_2,
\end{aligned} \tag{3.5}$$

where

$$H_1 = \int_0^t \sum_{|j-k|\leq N}\cdots, \quad H_2 = \int_0^t \sum_{|j-k|>N}\cdots.$$

Due to $|j-k| \leq N$, the summation in $H_1$ includes the diagonal entries $j = k$ and $2N$ sub-diagonal entries. Therefore,

$$\begin{aligned}
H_1 &= \int_0^t \sum_{j=0}^\infty 2^j\|\Delta_j\mathbf{u}\|_{L^2}(2^{j-N}\|\Delta_{j-N}\mathbf{u}\|_{L^2} + 2^{j-N+1}\|\Delta_{j-N+1}\mathbf{u}\|_{L^2} \\
&\quad + \cdots + 2^{j+N}\|\Delta_{j+N}\mathbf{u}\|_{L^2})\,d\tau \\
&\leq \frac{1}{2}\int_0^t \sum_{j=0}^\infty (2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}^2 + 2^{2(j-N)}\|\Delta_{j-N}\mathbf{u}\|_{L^2}^2 + \cdots \\
&\quad + 2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}^2 + 2^{2(j+N)}\|\Delta_{j+N}\mathbf{u}\|_{L^2}^2)\,d\tau
\end{aligned}$$

$$\begin{aligned}&\le CN\int_0^t\sum_{j=0}^{\infty}2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}^2d\tau\\&=CN\|\nabla\mathbf{u}\|_{L_t^2L^2}^2,\end{aligned}\tag{3.6}$$

where we have used Hölder's inequality in the second inequality and $C$ is a pure constant independent of $N$. The summation in $H_2$ contains two identical parts and thus

$$\begin{aligned}H_2&=2\int_0^t\sum_{j-k>N}2^j\|\Delta_j\mathbf{u}\|_{L^2}2^k\|\Delta_k\mathbf{u}\|_{L^2}d\tau\\&=2\int_0^t\sum_{j=N}^{\infty}2^j\|\Delta_j\mathbf{u}\|_{L^2}\sum_{m=0}^{j-N-1}2^m\|\Delta_m\mathbf{u}\|_{L^2}d\tau\\&\le 2^{-N+1}\sum_{j=N}^{\infty}\int_0^t2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}d\tau\sum_{m=0}^{j-N-1}2^{m+N-j}\sup_{0\le\tau\le t}\|\Delta_m\mathbf{u}(\tau)\|_{L^2}\\&\le 2^{-N+1}\left\|\int_0^t2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}\,d\tau\right\|_{l^2}\left\|\sum_{m=0}^{j-N-1}2^{m+N-j}\sup_{0\le\tau\le t}\|\Delta_m u(\tau)\|_{L^2}\right\|_{l^2}\\&\le 2^{-N+1}\|\mathbf{u}\|_{\widetilde{L}^1(0,t;H^2)}\left\|\sup_{0\le\tau\le t}\|\Delta_j\mathbf{u}(\tau)\|_{L^2}\right\|_{l^2},\end{aligned}\tag{3.7}$$

where we have used Young's inequality for sequence convolution. Combining (3.4)–(3.7) yields

$$\begin{aligned}\int_0^t\|\mathbf{u}(\tau)\|_{L^\infty}^2d\tau&\le CN\|\nabla\mathbf{u}\|_{L_t^2L^2}^2\\&\quad+C2^{-N+1}(\|\mathbf{u}_0\|_{L^2}+\|\nabla\mathbf{u}\|_{L_t^2L^2}\|\mathbf{u}\|_{L_t^2L^\infty}+\|\theta\|_{L_t^1L^2})^2\\&\le\frac12\|\mathbf{u}\|_{L_t^2L^\infty}^2+C(t,\|\nabla\mathbf{u}\|_{L_t^2L^2}^2,\|\mathbf{u}_0\|_{L^2},\|\theta_0\|_{L^2}),\end{aligned}\tag{3.8}$$

where we have chosen $N$ such that

$$C2^{-N+1}\|\nabla\mathbf{u}\|_{L_t^2L^2}^2\le\frac12,$$

(3.8) then yields the desired global bound in (3.3). This completes the proof of Lemma 3.4. □

**Proof of Proposition 3.3.** Let $j\in\mathbb{Z}$ and $j\ge 0$. Applying $\Delta_j$ to the first equation of (1.1) yields

$$\partial_t\Delta_j\mathbf{u}+\nu(-\Delta)^\alpha\Delta_j\mathbf{u}=-\Delta_j\nabla P+\Delta_j(\theta\mathbf{e}_d)-\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u}).$$

Dotting with $\Delta_j \mathbf{u}$, integrating by parts and using $\nabla \cdot \mathbf{u} = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j \mathbf{u}\|_{L^2}^2 + \nu 2^{2\alpha j}\|\Delta_j \mathbf{u}\|_{L^2}^2 \le \|\Delta_j \theta\|_{L^2}\|\Delta_j \mathbf{u}\|_{L^2} + I,$$

where

$$I = -\int_{\mathbb{T}^d} \Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta_j \mathbf{u} d\mathbf{x}.$$

We estimate $I$. By the notion of paraproducts provided in Sec. 2,

$$\begin{aligned} I \;=\; & -\sum_{|j-k|\le d}\int_{\mathbb{T}^d} \Delta_j(S_{k-d}\mathbf{u}\cdot\nabla\Delta_k\mathbf{u})\cdot\Delta_j\mathbf{u}d\mathbf{x} \\ & -\sum_{|j-k|\le d}\int_{\mathbb{T}^d} \Delta_j(\Delta_k\mathbf{u}\cdot\nabla S_{k-1}\mathbf{u})\cdot\Delta_j\mathbf{u}d\mathbf{x} \\ & -\sum_{k\ge j-1}\int_{\mathbb{T}^d} \Delta_j(\Delta_k\mathbf{u}\cdot\nabla\widetilde{\Delta}_k\mathbf{u})\cdot\Delta_j\mathbf{u}d\mathbf{x} \\ := {} & I_1 + I_2 + I_3. \end{aligned}$$

By Hölder's inequality, for $d = 2$,

$$\begin{aligned} |I_1| &\le \|\Delta_j\mathbf{u}\|_{L^2}\sum_{|j-k|\le d}\|S_{k-d}\mathbf{u}\|_{L^\infty}\|\nabla\Delta_k\mathbf{u}\|_{L^2} \\ &\le C\|\Delta_j\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^\infty}\sum_{|j-k|\le d}2^k\|\Delta_k\mathbf{u}\|_{L^2}. \end{aligned}$$

The estimate for $d \ge 3$ is slightly different. For $d \ge 3$, by Sobolev's inequality,

$$\begin{aligned} |I_1| &\le \|\Delta_j\mathbf{u}\|_{L^2}\sum_{|j-k|\le d}\|S_{k-d}\mathbf{u}\|_{L^{\frac{4d}{d-2}}}\|\nabla\Delta_k\mathbf{u}\|_{L^{\frac{4d}{d+2}}} \\ &\le \|\Delta_j\mathbf{u}\|_{L^2}\sum_{|j-k|\le d}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}S_{k-d}\mathbf{u}\|_{L^2}\|\Lambda^{\frac{d}{4}-\frac{1}{2}}\nabla\Delta_k\mathbf{u}\|_{L^2} \\ &\le C\|\Delta_j\mathbf{u}\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}\sum_{|j-k|\le d}2^{k(\frac{1}{2}+\frac{d}{4})}\|\Delta_k\mathbf{u}\|_{L^2}. \end{aligned}$$

The estimate of $I_2$ is similar. For $d = 2$,

$$\begin{aligned} |I_2| &\le \|\Delta_j\mathbf{u}\|_{L^2}\sum_{|j-k|\le d}\|\Delta_k\mathbf{u}\|_{L^2}\|\nabla S_{k-d}\mathbf{u}\|_{L^\infty} \\ &\le C\|\Delta_j\mathbf{u}\|_{L^2}\sum_{|j-k|\le d}2^{k-d}\|S_{k-d}\mathbf{u}\|_{L^\infty}\|\Delta_k\mathbf{u}\|_{L^2} \\ &\le C\|\Delta_j\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^\infty}\sum_{|j-k|\le d}2^{k-d}\|\Delta_k\mathbf{u}\|_{L^2}. \end{aligned}$$

For $d \geq 3$, we have

$$
\begin{aligned}
|I_2| &\leq \|\Delta_j \mathbf{u}\|_{L^2} \sum_{|j-k|\leq d} \|\Delta_k \mathbf{u}\|_{L^{\frac{4d}{d-2}}} \|\nabla S_{k-d}\mathbf{u}\|_{L^{\frac{4d}{d+2}}} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} \sum_{|j-k|\leq d} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} S_{k-d}\mathbf{u}\|_{L^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \Delta_k \mathbf{u}\|_{L^2} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \mathbf{u}\|_{L^2} \sum_{|j-k|\leq d} 2^{(\frac{1}{2}+\frac{d}{4})k} \|\Delta_k \mathbf{u}\|_{L^2}.
\end{aligned}
$$

By the fact that $\nabla \cdot u = 0$,

$$
I_3 = -\sum_{k\geq j-1} \int_{\mathbb{T}^d} \Delta_j \nabla \cdot (\Delta_k \mathbf{u} \otimes \widetilde{\Delta}_k \mathbf{u}) \cdot \Delta_j \mathbf{u} d\mathbf{x}.
$$

By Hölder's inequality, for $d = 2$,

$$
\begin{aligned}
|I_3| &\leq C\|\Delta_j \mathbf{u}\|_{L^2} 2^j \sum_{k\geq j-1} \|\Delta_j (\Delta_k \mathbf{u} \otimes \widetilde{\Delta}_k \mathbf{u})\|_{L^2} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} 2^j \sum_{k\geq j-1} \|\Delta_k \mathbf{u}\|_{L^2} \|\widetilde{\Delta}_k \mathbf{u}\|_{L^\infty} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} \|\mathbf{u}\|_{L^\infty} \sum_{k\geq j-1} 2^{j-k} 2^k \|\Delta_k \mathbf{u}\|_{L^2}.
\end{aligned}
$$

For $d \geq 3$, by Proposition 2.8,

$$
\begin{aligned}
|I_3| &\leq C\|\Delta_j \mathbf{u}\|_{L^2} 2^j \sum_{k\geq j-1} \|\Delta_j (\Delta_k \mathbf{u} \otimes \widetilde{\Delta}_k \mathbf{u})\|_{L^2} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} 2^j \sum_{k\geq j-1} \|\Delta_k \mathbf{u}\|_{L^{\frac{4d}{d-2}}} \|\widetilde{\Delta}_k \mathbf{u}\|_{L^{\frac{4d}{d+2}}} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} 2^j \sum_{k\geq j-1} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \Delta_k \mathbf{u}\|_{L^2} \ \|\Lambda^{\frac{d}{4}-\frac{1}{2}} \widetilde{\Delta}_k \mathbf{u}\|_{L^2} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} \sum_{k\geq j-1} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \Delta_k \mathbf{u}\|_{L^2} 2^{j-k} \|\Lambda^{\frac{d}{4}+\frac{1}{2}} \widetilde{\Delta}_k \mathbf{u}\|_{L^2} \\
&\leq C\|\Delta_j \mathbf{u}\|_{L^2} \sum_{k\geq j-1} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \Delta_k \mathbf{u}\|_{L^2} 2^{j-k} \|\Lambda^{\frac{d}{4}+\frac{1}{2}} \mathbf{u}\|_{L^2}.
\end{aligned}
$$

Combining the bounds above yields, for $d = 2$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta_j \mathbf{u}\|_{L^2}^2 + C_0 2^{2\alpha j} \|\Delta_j \mathbf{u}\|_{L^2}^2 \\
&\quad \leq \|\Delta_j \theta\|_{L^2} \|\Delta_j \mathbf{u}\|_{L^2} + C\|\Delta_j \mathbf{u}\|_{L^2} \|\mathbf{u}\|_{L^\infty} \sum_{|j-k|\leq d} 2^k \|\Delta_k \mathbf{u}\|_{L^2} \\
&\qquad + C\|\Delta_j \mathbf{u}\|_{L^2} \|\mathbf{u}\|_{L^\infty} \sum_{k\geq j-1} 2^{j-k} 2^k \|\Delta_k \mathbf{u}\|_{L^2}. \qquad (3.9)
\end{aligned}
$$

For $d \geq 3$,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j \mathbf{u}\|_{L^2}^2 + C_0 2^{2\alpha j}\|\Delta_j \mathbf{u}\|_{L^2}^2$$
$$\leq \|\Delta_j\theta\|_{L^2}\|\Delta_j \mathbf{u}\|_{L^2} + C\|\Delta_j \mathbf{u}\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}\sum_{|j-k|\leq d} 2^{k(\frac{1}{2}+\frac{d}{4})}\|\Delta_k \mathbf{u}\|_{L^2}$$
$$+ C\|\Delta_j \mathbf{u}\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}\sum_{k\geq j-1} 2^{j-k}\|\Lambda^{\frac{d}{4}+\frac{1}{2}}\Delta_k \mathbf{u}\|_{L^2}. \tag{3.10}$$

Eliminating $\|\Delta_j \mathbf{u}\|_{L^2}$ from each side of (3.9) and (3.10) and integrating in time yield, for $d = 2$,

$$\|\Delta_j \mathbf{u}(t)\|_{L^2} + C_0 2^{2\alpha j}\int_0^t \|\Delta_j \mathbf{u}(\tau)\|_{L^2}\,d\tau$$
$$\leq \|\Delta_j \mathbf{u}_0\|_{L^2} + \int_0^t \|\Delta_j\theta(\tau)\|_{L^2}\,d\tau + C\int_0^t \|\mathbf{u}\|_{L^\infty}\sum_{|j-k|\leq d} 2^k\|\Delta_k \mathbf{u}\|_{L^2}\,d\tau$$
$$+ C\int_0^t \|\mathbf{u}\|_{L^\infty}\sum_{k\geq j-1} 2^{j-k}2^k\|\Delta_k \mathbf{u}\|_{L^2}\,d\tau. \tag{3.11}$$

For $d \geq 3$,

$$\|\Delta_j \mathbf{u}(t)\|_{L^2} + C_0 2^{2\alpha j}\int_0^t \|\Delta_j \mathbf{u}(\tau)\|_{L^2}\,d\tau$$
$$\leq \|\Delta_j \mathbf{u}_0\|_{L^2} + \int_0^t \|\Delta_j\theta(\tau)\|_{L^2}\,d\tau$$
$$+ C\int_0^t \|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}\sum_{|j-k|\leq d} 2^{k(\frac{1}{2}+\frac{d}{4})}\|\Delta_k \mathbf{u}\|_{L^2}\,d\tau$$
$$+ C\int_0^t \|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}\sum_{k\geq j-1} 2^{j-k}\|\Lambda^{\frac{d}{4}+\frac{1}{2}}\Delta_k \mathbf{u}\|_{L^2}\,d\tau. \tag{3.12}$$

Taking the $l^2$-norm of the sequence in (3.11) and identifying $B^0_{2,2}$ with $L^2$, we obtain, after recalling the bound for $\|u\|_{L^2_t L^\infty}$ in Lemma 3.4, for $d = 2$,

$$\|\mathbf{u}(t)\|_{L^2} + C_0\left\|2^{2\alpha j}\int_0^t \|\Delta_j \mathbf{u}(\tau)\|_{L^2}\,d\tau\right\|_{l^2}$$
$$\leq 2\|\mathbf{u}_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2}\,d\tau + C\int_0^t \|\mathbf{u}(t)\|_{L^\infty}\|\nabla \mathbf{u}(\tau)\|_{L^2}\,d\tau$$
$$+ C\int_0^t \|\mathbf{u}(t)\|_{L^\infty}\left\|\sum_{k\geq j-1} 2^{j-k}2^k\|\Delta_k \mathbf{u}(\tau)\|_{L^2}\right\|_{l^2}\,d\tau$$
$$\leq 2\|\mathbf{u}_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2}\,d\tau + C\|\nabla \mathbf{u}\|_{L^2_t L^2}\|\mathbf{u}\|_{L^2_t L^\infty} < \infty,$$

where we have used the bound, by Young's inequality for sequence convolution,

$$\left\| \sum_{k\geq j-1} 2^{j-k}2^k \|\Delta_k \mathbf{u}(\tau)\|_{L^2} \right\|_{l^2} \leq C\|2^k\|\Delta_k \mathbf{u}(\tau)\|_{L^2}\|_{l^2} = C\|\nabla \mathbf{u}(\tau)\|_{L^2}.$$

We thus have obtained (3.1) for the case $d = 2$. For $d \geq 3$, we have, by taking the $l^2$-norm of the sequence in (3.12)

$$\begin{aligned}
&\|\mathbf{u}(t)\|_{L^2} + C_0 \left\| 2^{2\alpha j} \int_0^t \|\Delta_j \mathbf{u}(\tau)\|_{L^2} d\tau \right\|_{l^2} \\
&\quad \leq 2\|\mathbf{u}_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2} d\tau + C\int_0^t \|2^{(\frac{1}{2}+\frac{d}{4})j}\|\Delta_j \mathbf{u}\|_{L^2}\|_{l^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2} d\tau \\
&\qquad + C\int_0^t \left\| \sum_{k\geq j-1} 2^{j-k}2^{(\frac{d}{4}+\frac{1}{2})k}\|\Delta_k \mathbf{u}(\tau)\|_{L^2} \right\|_{l^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2} d\tau \\
&\quad \leq 2\|\mathbf{u}_0\|_{L^2} + \int_0^t \|\theta(\tau)\|_{L^2} d\tau + C\int_0^t \|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}(\tau)\|_{L^2}^2 d\tau,
\end{aligned}$$

which is the desired global bound in (3.1). Next we show (3.1) implies (3.2). By Proposition 2.8,

$$\|\nabla \mathbf{u}\|_{L^q(\mathbb{T}^d)} \leq \sum_{j=0}^{\infty} \|\nabla \Delta_j \mathbf{u}\|_{L^q(\mathbb{T}^d)} \leq C\sum_{j=0}^{\infty} 2^j 2^{dj(\frac{1}{2}-\frac{1}{q})}\|\Delta_j \mathbf{u}\|_{L^2(\mathbb{T}^d)},$$

where $C$ is a constant independent of $q$. By Hölder's inequality for sequences,

$$\begin{aligned}
\int_0^t \|\nabla \mathbf{u}\|_{L^q} dt &\leq \sum_{j=0}^{\infty} 2^{-\frac{d}{q}j} \int_0^t 2^{(1+\frac{d}{2})j}\|\Delta_j \mathbf{u}\|_{L^2} d\tau \\
&\leq \left(\sum_{j=0}^{\infty} 2^{-\frac{2d}{p}j}\right)^{\frac{1}{2}} \left\| \int_0^t 2^{(1+\frac{d}{2})j}\|\Delta_j \mathbf{u}\|_{L^2} d\tau \right\|_{l^2}.
\end{aligned}$$

Since

$$\left(\sum_{j=0}^{\infty} 2^{-\frac{2d}{q}j}\right)^{\frac{1}{2}} \leq C\left(\int_0^{\infty} 2^{-\frac{2d}{q}r} dr\right)^{\frac{1}{2}} = C\sqrt{q},$$

(3.1) then implies

$$\int_0^t \|\nabla \mathbf{u}\|_{L^p} dt \leq C\sqrt{q}\|\mathbf{u}\|_{\widetilde{L}_t^1 B_{2,2}^{(1+\frac{d}{2})j}} \leq C\sqrt{q},$$

where $C$ depends on $T$, $\|\mathbf{u}_0\|_{L^2}$ and $\|\theta_0\|_{L^2}$ only. We thus have shown (3.2). This completes the proof of Proposition 3.3. □

We now prove Theorem 1.1.

**Proof.** Due to Propositions 3.2 and 3.3, it suffices to show the uniqueness of the weak solutions of (1.1). Suppose (1.1) has two weak solutions $(\mathbf{u}^{(1)}, \theta^{(1)})$ and $(\mathbf{u}^{(2)}, \theta^{(2)})$ with the same initial data $(\mathbf{u}_0, \theta_0)$. We show that $(\mathbf{u}^{(1)}, \theta^{(1)})$ and $(\mathbf{u}^{(2)}, \theta^{(2)})$ must coincide. To do so, we consider the difference $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ with

$$\widetilde{\mathbf{u}} := \mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \quad \widetilde{\theta} := \theta^{(1)} - \theta^{(2)}.$$

Let $P^{(1)}$ and $P^{(2)}$ be the corresponding pressure terms and $\widetilde{P} := P^{(1)} - P^{(2)}$. In addition, we introduce the lower regularity quantities $h^{(1)}$ and $h^{(2)}$ satisfying

$$-\Delta h^{(1)} = \theta^{(1)}, \quad -\Delta h^{(2)} = \theta^{(2)}$$

and set

$$\widetilde{h} = h^{(1)} - h^{(2)}.$$

It follows from (1.1) that $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ satisfies

$$\begin{cases} \partial_t \widetilde{\mathbf{u}} + \mathbf{u}^{(1)} \cdot \nabla \widetilde{\mathbf{u}} + \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} + \nu(-\Delta)^\alpha \widetilde{\mathbf{u}} + \nabla \widetilde{P} = \widetilde{\theta} \mathbf{e}_d, \\ \partial_t \widetilde{\theta} + \mathbf{u}^{(1)} \cdot \nabla \widetilde{\theta} + \widetilde{\mathbf{u}} \cdot \nabla \theta^{(2)} = \gamma \widetilde{u}_d, \\ \nabla \cdot \widetilde{\mathbf{u}} = 0, \\ (\widetilde{\mathbf{u}}, \widetilde{\theta})|_{t=0} = 0. \end{cases} \tag{3.13}$$

Dotting the first equation of (3.13) by $\widetilde{\mathbf{u}}$ and integrating by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \nu\|\Lambda^\alpha \widetilde{\mathbf{u}}\|_{L^2}^2 &= -\int \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u}^{(2)} \cdot \widetilde{\mathbf{u}} d\mathbf{x} + \int \widetilde{\theta} \cdot (\mathbf{e}_d \cdot \widetilde{\mathbf{u}}) d\mathbf{x} \\ &:= K_1 + K_2, \end{aligned} \tag{3.14}$$

where we have invoked the fact that, for $\alpha \geq \frac{1}{2} + \frac{d}{4}$,

$$\int_{\mathbb{T}^d} \mathbf{u}^{(1)} \cdot \nabla \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} \, d\mathbf{x} = 0,$$

due to $\nabla \cdot \mathbf{u}^{(1)} = 0$, $\nabla \cdot \widetilde{\mathbf{u}} = 0$ and

$$\int_0^T \int_{\mathbb{T}^d} |\mathbf{u}^{(1)} \cdot \nabla \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}}| d\mathbf{x} dt \leq \int_0^T \|\mathbf{u}^{(1)}(\tau)\|_{L^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \widetilde{\mathbf{u}}(\tau)\|_{L^2}^2 \, d\tau < \infty.$$

By Hölder's and Sobolev's inequalities, for $d = 2$,

$$\begin{aligned} |K_1| &\leq \|\widetilde{\mathbf{u}}\|_{L^4}^2 \|\nabla \mathbf{u}^{(2)}\|_{L^2} \\ &\leq \|\widetilde{\mathbf{u}}\|_{L^2} \|\nabla \widetilde{\mathbf{u}}\|_{L^2} \|\nabla \mathbf{u}^{(2)}\|_{L^2} \\ &\leq \frac{\nu}{16} \|\nabla \widetilde{\mathbf{u}}\|_{L^2}^2 + C\|\nabla \mathbf{u}^{(2)}\|_{L^2}^2 \|\widetilde{\mathbf{u}}\|_{L^2}^2. \end{aligned} \tag{3.15}$$

For $d \geq 3$,

$$\begin{aligned} |K_1| &\leq \|\widetilde{\mathbf{u}}\|_{L^2} \|\nabla \mathbf{u}^{(2)}\|_{\frac{4d}{d+2}} \|\widetilde{\mathbf{u}}\|_{L^{\frac{4d}{d-2}}} \\ &\leq C\|\widetilde{\mathbf{u}}\|_{L^2} \, \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \mathbf{u}^{(2)}\|_{L^2} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \widetilde{\mathbf{u}}\|_{L^2} \\ &\leq \frac{\nu}{16} \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \widetilde{\mathbf{u}}\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}+\frac{d}{4}} \mathbf{u}^{(2)}\|_{L^2}^2 \|\widetilde{\mathbf{u}}\|_{L^2}^2. \end{aligned} \tag{3.16}$$

By integration by parts and an interpolation inequality,

$$
\begin{aligned}
|K_2| &= \left|\int_{\mathbb{T}^d} (-\Delta\widetilde{h})(\mathbf{e}_d\cdot\widetilde{\mathbf{u}})d\mathbf{x}\right| \\
&\le \|\nabla\widetilde{h}\|_{L^2}\|\nabla\widetilde{\mathbf{u}}\|_{L^2} \\
&\le C\|\nabla\widetilde{h}\|_{L^2}\|\widetilde{\mathbf{u}}\|_{L^2}^{\frac{d-2}{d+2}}\|\Lambda^{\frac{d+2}{4}}\widetilde{\mathbf{u}}\|_{L^2}^{\frac{4}{d+2}} \\
&\le C\|\nabla\widetilde{h}\|_{L^2}(\|\widetilde{\mathbf{u}}\|_{L^2}+\|\Lambda^{\frac{d+2}{4}}\widetilde{\mathbf{u}}\|_{L^2}) \\
&\le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}^2 + C\,\|\nabla\widetilde{h}\|_{L^2}(\|\widetilde{\mathbf{u}}\|_{L^2}+\|\nabla\widetilde{h}\|_{L^2}).
\end{aligned}\tag{3.17}
$$

Dotting the second equation in (3.13) with $\widetilde{h}$ yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{h}\|_{L^2}^2 &= \int_{\mathbb{T}^d}\mathbf{u}^{(1)}\cdot\nabla\widetilde{\theta}\,\widetilde{h}d\mathbf{x} + \int_{\mathbb{T}^d}\widetilde{\mathbf{u}}\cdot\nabla\theta^{(2)}\widetilde{h}d\mathbf{x} + \gamma\int \widetilde{u}_d\widetilde{h}d\mathbf{x} \\
&:= K_3+K_4+K_5.
\end{aligned}\tag{3.18}
$$

We estimate $K_4$ first. The case with $d=2$ is treated differently from $d\ge 3$. For $d=2$, by Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
|K_4| &\le \|\theta^{(2)}\|_{L^2}\|\widetilde{\mathbf{u}}\|_{L^{2p}}\|\nabla\widetilde{h}\|_{L^q} \\
&\le C\sqrt{p}\|\widetilde{\mathbf{u}}\|_{L^2}^{1/p}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^{1-1/p}\|\theta_0\|_{L^2}\|\nabla\widetilde{h}\|_{L^2}^{1-\frac{1}{p}}\|\Delta\widetilde{h}\|_{L^2}^{\frac{1}{p}} \\
&\le C\sqrt{p}(\|\widetilde{\mathbf{u}}\|_{L^2}+\|\nabla\widetilde{\mathbf{u}}\|_{L^2})\|\nabla\widetilde{h}\|_{L^2}^{1-\frac{1}{p}}\|\widetilde{\theta}\|_{L^2}^{\frac{1}{p}} \\
&\le \frac{\nu}{16}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2+\|\widetilde{\mathbf{u}}\|_{L^2}^2+Cp\,M^{\frac{2}{p}}\|\nabla\widetilde{h}\|_{L^2}^{2(1-\frac{1}{p})},
\end{aligned}\tag{3.19}
$$

where $1<p,q<\infty$ satisfy

$$\frac{1}{p}+\frac{2}{q}=1$$

and we have used the fact that

$$\|\Delta\widetilde{h}\|_{L^2}\le\|\theta^{(1)}\|_{L^2}+\|\theta^{(2)}\|_{L^2}\le C(T,\|\mathbf{u}_0\|_{L^2},\|\theta_0\|_{L^2}):=\sqrt{M}.$$

For $d\ge 3$, by integration by parts, Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
|K_4| &= \int_{\mathbb{T}^d}|\theta^{(2)}\widetilde{\mathbf{u}}\cdot\nabla\widetilde{h}|\,d\mathbf{x} \\
&\le \|\theta^{(2)}\|_{L^{\frac{4d}{d+2}}}\|\nabla\widetilde{h}\|_{L^2}\|\widetilde{\mathbf{u}}\|_{L^{\frac{4d}{d-2}}} \\
&\le \|\theta_0\|_{L^{\frac{4d}{d+2}}}\|\nabla\widetilde{h}\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2} \\
&\le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}^2+C\|\theta_0\|_{L^{\frac{4d}{d+2}}}^2\|\nabla\widetilde{h}\|_{L^2}^2.
\end{aligned}\tag{3.20}
$$

Recalling $\widetilde{\theta} = -\Delta \widetilde{h}$ and integrating by parts, we have

$$\begin{aligned}
K_3 &= -\int_{\mathbb{T}^d} \mathbf{u}^{(1)} \cdot \nabla \Delta \widetilde{h}\, \widetilde{h} d\mathbf{x} \\
&= \int_{\mathbb{T}^d} \partial_{x_k} u_j^{(1)} \partial_{x_j} \partial_{x_k} \widetilde{h}\, \widetilde{h} d\mathbf{x} + \int_{\mathbb{T}^d} u_j^{(1)} \partial_{x_j} \partial_{x_k} \widetilde{h} \partial_{x_k} \widetilde{h} d\mathbf{x} \\
&= -\int_{\mathbb{T}^d} \partial_{x_k} u_j^{(1)} \partial_{x_k} \widetilde{h}\, \partial_{x_j} \widetilde{h} d\mathbf{x},
\end{aligned}$$

where the repeated indices are summed and we have used $\nabla \cdot \mathbf{u}^{(1)} = 0$. By Hölder's inequality, for $p > \frac{d}{2}$ and $\frac{1}{p} + \frac{2}{q} = 1$,

$$\begin{aligned}
|K_3| &\le C \|\nabla \mathbf{u}^{(1)}\|_{L^p} \|\nabla \widetilde{h}\|_{L^q}^2 \\
&\le C \|\nabla \mathbf{u}^{(1)}\|_{L^p} \|\nabla \widetilde{h}\|_{L^2}^{2-\frac{d}{p}} \|\widetilde{\theta}\|_{L^2}^{\frac{d}{p}} \\
&\le C \|\nabla \mathbf{u}^{(1)}\|_{L^p} M^{\frac{d}{2p}} \|\nabla \widetilde{h}\|_{L^2}^{2-\frac{d}{p}}. \qquad (3.21)
\end{aligned}$$

Clearly, $K_5$ can be similarly estimated as $K_3$ and the bound is the same. Adding (3.14) and (3.18) and collecting the estimates in (3.15)–(3.17), (3.19)–(3.21), we find that, for $\delta > 0$,

$$G_\delta(t) := \|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\nabla \widetilde{h}(t)\|_{L^2}^2 + \delta$$

obeys the differential inequality, when $d = 2$,

$$\frac{d}{dt} G_\delta(t) \le C \left(1 + \|\Lambda \mathbf{u}^{(2)}\|_{L^2}^2\right) G_\delta(t) + C \left(1 + \frac{\|\nabla \mathbf{u}^{(1)}\|_{L^p}}{p}\right) p M^{\frac{1}{p}} G_\delta(t)^{1-\frac{1}{p}} \qquad (3.22)$$

and, for $d \ge 3$,

$$\frac{d}{dt} G_\delta(t) \le C \left(1 + \|\Lambda^{\frac{1}{2}+\frac{d}{4}} \mathbf{u}^{(2)}\|_{L^2}^2\right) G_\delta(t) + C \frac{\|\nabla \mathbf{u}^{(1)}\|_{L^p}}{p} p M^{\frac{d}{2p}} G_\delta(t)^{1-\frac{d}{2p}}. \qquad (3.23)$$

Optimizing the quantities $p\, M^{\frac{1}{p}}\, G_\delta(t)^{1-\frac{1}{p}}$ and $p\, M^{\frac{d}{2p}}\, G_\delta(t)^{1-\frac{d}{2p}}$ with respect to $p$, we obtain

$$\begin{aligned}
p M^{\frac{1}{p}} G_\delta(t)^{1-\frac{1}{p}} &\le e G_\delta (\ln M - \ln G_\delta), \\
p M^{\frac{d}{2p}} G_\delta(t)^{1-\frac{d}{2p}} &\le \frac{d}{2} e G_\delta\, (\ln M - \ln G_\delta).
\end{aligned}$$

Incorporating these bounds in (3.22) and (3.23), we find that both (3.22) and (3.23) are reduced to the following form:

$$G_\delta(t) \le G_\delta(0) + C \int_0^t \gamma(s) \phi(G_\delta(s))\, ds,$$

where

$$\gamma(t) = C + C\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}^{(2)}\|_{L^2}^2 + C\,\frac{\|\nabla\mathbf{u}^{(1)}\|_{L^p}}{p}, \quad \phi(r) = r + r(\ln M - \ln r).$$

It follows from Proposition 3.1 that

$$\int_0^T \gamma(t)\,d\tau < \infty.$$

Let

$$\Omega(x) = \int_x^1 \frac{dr}{\phi(r)} = \int_x^1 \frac{dr}{r + r(\ln M - \ln r)}$$
$$= \ln(1 + \ln M - \ln x) - \ln(1 + \ln M).$$

It then follows from Lemma B.6 that

$$-\Omega(G_\delta(t)) + \Omega(G_\delta(0)) \le \int_0^t \gamma(s)ds.$$

Therefore,

$$-\ln(1 + \ln M - \ln G_\delta(t)) + \ln(1 + \ln M - \ln G_\delta(0)) \le \int_0^t \gamma(s)ds.$$

Therefore, for $\widetilde{C}(t) = \int_0^t \gamma(s)ds$,

$$G_\delta(t) \le (eM)^{1-e^{-\widetilde{C}(t)}}\, G_\delta(0)^{e^{-\widetilde{C}(t)}}.$$

Letting $\delta \to 0$ and noting that $G_0(0) = 0$, we obtain

$$G_0(t) := \|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\nabla\widetilde{h}(t)\|_{L^2}^2 = 0.$$

This completes the proof of Theorem 1.1. □

## 4. Proof of Theorem 1.3

This section provides the proof of Theorem 1.3.

**Proof.** Let $(\mathbf{u}, \theta)$ and $(\mathbf{u}^{(\eta)}, \theta^{(\eta)})$ be the weak solutions of (1.1) and (1.4), respectively. Then the difference $(\widetilde{\mathbf{u}}, \widetilde{\theta})$ with

$$\widetilde{\mathbf{u}} = \mathbf{u}^{(\eta)} - \mathbf{u}, \qquad \widetilde{\theta} = \theta^{(\eta)} - \theta$$

satisfies

$$\begin{cases} \partial_t\widetilde{\mathbf{u}} + \mathbf{u}^{(\eta)}\cdot\nabla\widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}\cdot\nabla\mathbf{u} + \nu(-\Delta)^\alpha\widetilde{\mathbf{u}} + \nabla\widetilde{P} = \widetilde{\theta}\mathbf{e}_d, \\ \partial_t\widetilde{\theta} + \mathbf{u}^{(\eta)}\cdot\nabla\widetilde{\theta} + \widetilde{\mathbf{u}}\cdot\nabla\theta = \eta\Delta\widetilde{\theta} + \eta\Delta\theta + \gamma\widetilde{u}_d, \\ \nabla\cdot\widetilde{\mathbf{u}} = 0, \\ (\widetilde{\mathbf{u}}, \widetilde{\theta})|_{t=0} = (\widetilde{\mathbf{u}}_0, \widetilde{\theta}_0), \end{cases} \tag{4.1}$$

where $\widetilde{P} := P^{(\eta)} - P$ with $P^{(\eta)}$ and $P$ being the corresponding pressure terms of (1.1) and (1.4), respectively. We introduce the lower regularity quantities $h^{(\eta)}$

and $h$ satisfying

$$-\Delta h^{(\eta)} = \theta^{(\eta)}, \quad -\Delta h = \theta$$

and set

$$\widetilde{h} = h^{(\eta)} - h.$$

Dotting the first equation of (4.1) by $\widetilde{\mathbf{u}}$ and integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \nu\|\Lambda^\alpha \widetilde{\mathbf{u}}\|_{L^2}^2 &= -\int \widetilde{\mathbf{u}} \cdot \nabla \mathbf{u} \cdot \widetilde{\mathbf{u}} d\mathbf{x} + \int \widetilde{\theta} \cdot (\mathbf{e}_d \cdot \widetilde{\mathbf{u}}) d\mathbf{x} \\ &:= L_1 + L_2. \end{aligned} \tag{4.2}$$

The two terms on the right of (4.2) can be bounded similarly as in the proof of Theorem 1.1 and we have

$$|L_1| \le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}^2 \|\widetilde{\mathbf{u}}\|_{L^2}^2$$

and

$$\begin{aligned}|L_2| &\le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\widetilde{h}\|_{L^2}(\|\widetilde{\mathbf{u}}\|_{L^2} + \|\nabla\widetilde{h}\|_{L^2}) \\ &\le \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}^2 + C(\|\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\nabla\widetilde{h}\|_{L^2}^2).\end{aligned}$$

Dotting the second equation in (4.1) with $\widetilde{h}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{h}\|_{L^2}^2 + \eta\|\Delta\widetilde{h}\|_{L^2}^2 = L_3 + L_4 + L_5 + L_6, \tag{4.3}$$

where

$$\begin{aligned} L_3 &:= \int_{\mathbb{T}^d} \mathbf{u}^{(\eta)} \cdot \nabla\widetilde{\theta}\, \widetilde{h} d\mathbf{x}, \\ L_4 &:= \int_{\mathbb{T}^d} \widetilde{\mathbf{u}} \cdot \nabla\theta\widetilde{h} d\mathbf{x}, \\ L_5 &:= -\eta \int_{\mathbb{T}^d} \Delta\theta\widetilde{h} d\mathbf{x}, \\ L_6 &:= -\gamma \int_{\mathbb{T}^d} \widetilde{u}_d\widetilde{h} d\mathbf{x}. \end{aligned}$$

As in the proof of Theorem 1.1, $L_3$ admits the following bound,

$$|L_3| \le C\|\nabla\mathbf{u}^{(\eta)}\|_{L^p} M^{\frac{d}{2p}} \|\nabla\widetilde{h}\|_{L^2}^{2-\frac{d}{p}},$$

where $\frac{1}{p} + \frac{2}{q} = 1$ and $p > \frac{d}{2}$. $L_4$ can also be similarly bounded as $K_4$ in the proof of Theorem 1.1. For $d = 2$,

$$|L_4| \le \frac{\nu}{16}\|\nabla\widetilde{\mathbf{u}}\|_{L^2}^2 + \|\widetilde{\mathbf{u}}\|_{L^2}^2 + CpM^{\frac{1}{p}}\|\nabla\widetilde{h}\|_{L^2}^{2(1-\frac{1}{p})}$$

and, for $d \geq 3$,

$$|L_4| \leq \frac{\nu}{16}\|\Lambda^{\frac{1}{2}+\frac{d}{4}}\widetilde{\mathbf{u}}\|_{L^2}^2 + C\|\theta_0\|^2_{L^{\frac{4d}{d+2}}}\|\nabla\widetilde{h}\|_{L^2}^2.$$

By integration by parts and Hölder's inequality,

$$|L_5| \leq \eta\|\theta\|_{L^2}\|\Delta\widetilde{h}\|_{L^2} \leq \frac{\eta}{2}\|\Delta\widetilde{h}\|_{L^2}^2 + \frac{\eta}{2}\|\theta\|_{L^2}^2.$$

The bound for $L_6$ is the same as that for $L_3$. Adding (4.2) and (4.3) and incorporating the bounds for $L_1$ through $L_5$, we find, for $\delta > 0$,

$$E_\delta(t) := \|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\nabla\widetilde{h}(t)\|_{L^2}^2 + \delta$$

satisfies, for $d = 2$,

$$\frac{d}{dt}E_\delta(t) \leq \frac{\eta}{2}\|\theta\|_{L^2}^2 + C\left(1 + \|\Lambda\mathbf{u}\|_{L^2}^2\right)E_\delta(t) + C\left(1 + \frac{\|\nabla\mathbf{u}^{(\eta)}\|_{L^p}}{p}\right)pM^{\frac{1}{p}}E_\delta(t)^{1-\frac{1}{p}}$$

and, for $d \geq 3$,

$$\frac{d}{dt}E_\delta(t) \leq \frac{\eta}{2}\|\theta\|_{L^2}^2 + C\left(1 + \|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}^2\right)E_\delta(t) + C\frac{\|\nabla\mathbf{u}^{(\eta)}\|_{L^p}}{p}pM^{\frac{d}{2p}}E_\delta(t)^{1-\frac{d}{2p}}.$$

By following a similar procedure as in the proof of Theorem 1.1 and applying Lemma B.6, we obtain

$$E_\delta(t) \leq (eM)^{1-e^{-\widetilde{C}(t)}}(E_\delta(0) + \eta t\|\theta\|_{L^2}^2)^{e^{-\widetilde{C}(t)}} \tag{4.4}$$

where $\widetilde{C}(t)$ is the uniform bound (independent of $\eta$)

$$\widetilde{C}(t) = C\int_0^t\left(1 + \|\Lambda^{\frac{1}{2}+\frac{d}{4}}\mathbf{u}\|_{L^2}^2 + \frac{\|\nabla\mathbf{u}^{(\eta)}\|_{L^p}}{p}\right)d\tau < \infty.$$

Even though $\mathbf{u}^{(\eta)}$ is the solution of (1.4), the bound

$$\sup_{p\geq 2}\int_0^t\frac{\|\nabla\mathbf{u}^{(\eta)}\|_{L^p}}{p}\,d\tau < \infty$$

is uniform in $\eta$ since it only depends on $\|\theta^{(\eta)}\|_{L^2}$. Letting $\delta \to 0$ in (4.4) yields

$$\|\widetilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\nabla\widetilde{h}(t)\|_{L^2}^2 \leq (eM)^{1-e^{-\widetilde{C}(t)}}(\|\widetilde{\mathbf{u}}_0\|_{L^2}^2 + \|\nabla\widetilde{h}_0\|_{L^2}^2 + \eta t\|\theta\|_{L^2}^2)^{e^{-\widetilde{C}(t)}},$$

which is the desired bound (1.5). This completes the proof of Theorem 1.1. $\square$

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## Appendix A. Global Existence of Weak Solutions

This appendix provides the proof of Proposition 3.2. For readers' convenience, we first list several simple facts to be used in the proof. The first two lemmas are Picard's existence and extension results (see, e.g., [47]).

**Lemma A.1 (Picard Existence and Uniqueness Theorem).** *Let $E$ be a Banach space. Let $O \subseteq E$ be an open subset. Let $F: O \to E$ be a locally Lipschitz map. More precisely, for any $y \in O$, there is a neighborhood of $y$, denoted by $U(y)$ and $L = L(y, U)$ such that*

$$\|F(y) - F(z)\|_E \leq L\|y - z\|_E, \quad \forall\, z \in U(y).$$

*Then, for any $y_0 \in O$, the ODE*

$$\begin{cases} \dfrac{dy}{dt} = F(y), \\ \\ y|_{t=0} = y_0 \in O \end{cases}$$

*has a unique local solution, namely, there is $T > 0$ and a unique solution $y = y(t)$ satisfying $y \in C^1(0, T; O)$.*

**Lemma A.2 (Picard Extension Theorem).** *Assume the conditions in Lemma* A.1 *hold. Let $y = y(t)$ be the local solution. Then either $y(t)$ is global in time, namely, $T = \infty$, or for a finite $T_0 > 0$, $\lim_{t \to T_0} y(t) \notin O$.*

In addition, we will also need the following Lions–Aubin compactness Lemma.

**Lemma A.3 (Lions–Aubin compactness lemma).** *Let $X_1 \hookrightarrow X_2 \hookrightarrow X_3$ be three Banach spaces with the first embedding being compact and the second being continuous. Let $T > 0$. For $1 \leq p, q \leq +\infty$, let*

$$W = \{\mathbf{u} \in L^p(0, T; X_1), \partial_t \mathbf{u} \in L^q(0, T; X_3)\}.$$

*Then,*

(i) *If $p < +\infty$, then the embedding of $W$ into $L^p(0, T; X_2)$ is compact;*
(ii) *If $p = +\infty$ and $q > 1$, then the embedding of $W$ into $C(0, T; X_2)$ is compact.*

Lemma A.3 states that any bounded sequence in $W$ has a convergent subsequence in $L^p(0, T; X_2)$ when $p < \infty$.

We are now ready to prove Proposition 3.2.

**Proof of Proposition 3.2.** The proof is divided into several steps. The first step is to show the global existence of smooth solutions to a sequence of approximate systems. The second is to establish uniform bounds for this sequence of solutions and extract a strongly convergent subsequence. The third is to verify that the limit of the subsequence is actually the weak solution.

**Step I:** The global existence of smooth solutions to an approximate system. Let $n \in \mathbb{N}$. Set

$$L_n^2(\mathbb{T}^d) = \left\{ g \in L^2(\mathbb{T}^d),\ g(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d, |\mathbf{k}|\le 2^n} \widehat{g}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \right\}.$$

We seek a solution $(\mathbf{u}^{(n)}, \theta^{(n)}) \in L_n^2$ satisfying

$$\begin{cases} \partial_t \mathbf{u}^{(n)} + \mathbb{P}S_n(\mathbf{u}^{(n)} \cdot \nabla \mathbf{u}^{(n)}) + \nu(-\Delta)^\alpha \mathbf{u}^{(n)} = \mathbb{P}S_n(\theta^{(n)}\mathbf{e}_d), \\ \partial_t \theta^{(n)} + S_n(\mathbf{u}^{(n)} \cdot \nabla \theta^{(n)}) = \gamma S_n u_d^{(n)}, \\ \nabla \cdot \mathbf{u}^{(n)} = 0, \\ \mathbf{u}^{(n)}(x,0) = S_n \mathbf{u}_0, \quad \theta^{(n)}(x,0) = S_n\theta_0, \end{cases} \tag{A.1}$$

where $\mathbb{P}$ denotes the standard projection operator onto divergence-free vector fields and $S_n$ is the identity approximation operator as defined in (2.12). We remark that functions in $L_n^2(\mathbb{T}^d)$ are smooth. In fact,

$$L_n^2 \subseteq \bigcap_{m=0}^{\infty} \dot{H}^m.$$

We use the Picard theorem to show that (A.1) has a unique global solution in $L_n^2$. To this end, we first apply Lemma A.1 to show (A.1) has a local-in-time solution. We can rewrite (A.1) as

$$\frac{dY}{dt} = F(Y),$$

with

$$\begin{aligned} Y &= (\mathbf{u}^{(n)}, \theta^{(n)})^T, \quad F(Y) = (F_1(Y), F_2(Y))^T \\ F_1(Y) &= -\mathbb{P}S_n(\mathbf{u}^{(n)} \cdot \nabla \mathbf{u}^{(n)}) - \nu(-\Delta)^\alpha \mathbf{u}^{(n)} + \mathbb{P}S_n(\theta^{(n)}\mathbf{e}_d), \\ F_2(Y) &= -S_n(\mathbf{u}^{(n)} \cdot \nabla \theta^{(n)}) + \gamma S_n u_d^{(n)}. \end{aligned}$$

We set $E = L_n^2$ and $O = E$. We verify that $F : E \to E$ is locally Lipschitz. Assume $Y \in L_n^2$ and show $F(Y) \in L_n^2$. It suffices to show that $F(Y) \in L^2$. In fact,

$$\begin{aligned} \|F_1(Y)\|_{L^2} &\le \|\mathbf{u}^{(n)} \cdot \nabla \mathbf{u}^{(n)}\|_{L^2} + \|\nu(-\Delta)^\alpha \mathbf{u}^{(n)}\|_{L^2} + \|\theta^{(n)}\|_{L^2} \\ &\le \|\mathbf{u}^{(n)}\|_{L^4} \|\nabla \mathbf{u}^{(n)}\|_{L^4} + \nu\|\mathbf{u}^{(n)}\|_{\dot{H}^{2\alpha}} + \|\theta^{(n)}\|_{L^2} \\ &\le \|\mathbf{u}^{(n)}\|_{\dot{H}^{\frac{d}{4}}} \|\mathbf{u}^{(n)}\|_{\dot{H}^{1+\frac{d}{4}}} + \nu\|\mathbf{u}^{(n)}\|_{\dot{H}^{2\alpha}} + \|\theta^{(n)}\|_{L^2} \\ &\le (2^n)^{2(1+\frac{d}{4})} \|\mathbf{u}^{(n)}\|_{L^2}^2 + \nu(2^n)^{2\alpha}\|\mathbf{u}^{(n)}\|_{L^2} + \|\theta^{(n)}\|_{L^2}. \end{aligned}$$

That is $F_1(Y) \in L^2(\mathbb{T}^d)$. Similarly, $F_2(Y) \in L^2(\mathbb{T}^d)$. Next we show $F(Y)$ is locally Lipschitiz. Let $Y = (\mathbf{u}^{(n)}, \theta^{(n)})^T \in L_n^2$ and $Z = (\mathbf{v}^{(n)}, \rho^{(n)})^T \in L_n^2$ and consider

$$
\begin{aligned}
&\|F_2(Y) - F_2(Z)\|_{L^2} \\
&\quad = \|-S_n(\mathbf{u}^{(n)} \cdot \nabla\theta^{(n)}) + S_n(\mathbf{v}^{(n)} \cdot \nabla\rho^{(n)})\|_{L^2} + |\gamma|\, \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2} \\
&\quad = \|-S_n((\mathbf{u}^{(n)} - \mathbf{v}^{(n)}) \cdot \nabla\theta^{(n)}) - S_n(\mathbf{v}^{(n)} \cdot \nabla(\theta^{(n)} - \rho^{(n)}))\|_{L^2} \\
&\qquad + |\gamma|\, \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2} \\
&\quad \le \|(\mathbf{u}^{(n)} - \mathbf{v}^{(n)}) \cdot \nabla\theta^{(n)}\|_{L^2} + \|\mathbf{v}^{(n)} \cdot \nabla(\theta^{(n)} - \rho^{(n)})\|_{L^2} + |\gamma|\, \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2} \\
&\quad \le \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2}\|\nabla\theta^{(n)}\|_{L^\infty} + \|\mathbf{v}^{(n)}\|_{L^\infty}\|\nabla(\theta^{(n)} - \rho^{(n)})\|_{L^2} \\
&\qquad + |\gamma|\, \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2} \\
&\quad \le \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2}\|\theta^{(n)}\|_{\dot{H}^{1+\frac{d}{2}+\epsilon}} + \|\mathbf{v}^{(n)}\|_{\dot{H}^{\frac{d}{2}+\epsilon}}\|\theta^{(n)} - \rho^{(n)}\|_{\dot{H}^1} \\
&\qquad + |\gamma|\, \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2} \\
&\quad \le (2^n)^{1+\frac{d}{2}+\epsilon}\|\theta^{(n)}\|_{L^2}\|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2} + (2^n)^{1+\frac{d}{2}+\epsilon}\|\mathbf{v}^{(n)}\|_{L^2}\|\theta^{(n)} - \rho^{(n)}\|_{L^2} \\
&\qquad + |\gamma|\, \|\mathbf{u}^{(n)} - \mathbf{v}^{(n)}\|_{L^2} \\
&\quad \le L\|Y - Z\|_{L^2},
\end{aligned}
$$

where $\epsilon > 0$ is a small parameter and $L = (2^n)^{1+\frac{d}{2}+\epsilon}(\|Y\|_{L^2} + r) + |\gamma|\, \|Y\|_{L^2}$ for $\|Z - Y\| \le r$. Therefore $F_2(Y)$ is locally Lipschitz. Similarly, $F_1(Y)$ is locally Lipschitz. Lemma A.1 implies (A.1) has a unique local-in-time solution in $L^2_n$.

Next we use the Picard Extension Theorem, Lemma A.2 to show that the solution is global in time. It suffices to show that for any $t \le T$, $\|(\mathbf{u}^{(n)}, \theta^{(n)})\|_{L^2} < +\infty$. This is done by the energy method. Dotting (A.1) by $(\mathbf{u}^{(n)}, \theta^{(n)})$ yields

$$
\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}^{(n)}\|^2_{L^2} + \|\theta^{(n)}\|^2_{L^2}) + \nu\|\Lambda^\alpha \mathbf{u}^{(n)}\|^2_{L^2} := M_1 + M_2 + M_3 + M_4,
$$

where

$$
\begin{aligned}
M_1 &= -\int_{\mathbb{T}^d} \mathbb{P}S_n(\mathbf{u}^{(n)} \cdot \nabla\mathbf{u}^{(n)}) \cdot \mathbf{u}^{(n)} d\mathbf{x}, \\
M_2 &= \int_{\mathbb{T}^d} \mathbb{P}S_n(\theta^{(n)}\mathbf{e}_d) \cdot \mathbf{u}^{(n)} d\mathbf{x}, \\
M_3 &= -\int_{\mathbb{T}^d} S_n(\mathbf{u}^{(n)} \cdot \nabla\theta^{(n)}) \cdot \theta^{(n)} d\mathbf{x}, \\
M_4 &= \gamma \int_{\mathbb{T}^d} S_n u_d^{(n)} \theta^{(n)} d\mathbf{x}.
\end{aligned}
$$

We note that

$$
\begin{aligned}
M_1 &= -\int \mathbb{P}S_n(\mathbf{u}^{(n)} \cdot \nabla\mathbf{u}^{(n)}) \cdot \mathbf{u}^{(n)} d\mathbf{x} = -\int S_n(\mathbf{u}^{(n)} \cdot \nabla\mathbf{u}^{(n)}) \cdot \mathbb{P}\mathbf{u}^{(n)} d\mathbf{x} \\
&= -\int S_n(\mathbf{u}^{(n)} \cdot \nabla\mathbf{u}^{(n)}) \cdot \mathbf{u}^{(n)} d\mathbf{x} = -\int (\mathbf{u}^{(n)} \cdot \nabla\mathbf{u}^{(n)}) \cdot \mathbf{u}^{(n)} d\mathbf{x} = 0.
\end{aligned}
$$

Similarly, $M_3 = 0$. Clearly, $|M_2|, |M_4| \le C\|\mathbf{u}^{(n)}\|_{L^2}\|\theta^{(n)}\|_{L^2}$. Therefore,

$$\frac{d}{dt}(\|\mathbf{u}^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2) + 2\nu\|\Lambda^\alpha \mathbf{u}^{(n)}\|_{L^2}^2 \le C\,\|\mathbf{u}^{(n)}\|_{L^2}\|\theta^{(n)}\|_{L^2}.$$

Gronwall's inequality then implies that

$$\|\mathbf{u}^{(n)}(t)\|_{L^2}^2 + 2\nu\int_0^t \|\Lambda^\alpha \mathbf{u}^{(n)}\|_{L^2}^2 d\tau \le e^{Ct}(\|\mathbf{u}_0\|_{L^2} + \|\theta_0\|_{L^2})^2.$$

Therefore, $(\mathbf{u}^{(n)}, \theta^{(n)}) \in L_n^2$ for all time $t \le T$. Then Lemma A.2 allows us to conclude that $(\mathbf{u}^{(n)}, \theta^{(n)})$ is global in time.

**Step 2.** Extraction of a strongly convergent subsequence.

This step extracts a subsequence of $\mathbf{u}^{(n)}$ that converges strongly in $L^2(0,T;L^2(\mathbb{T}^d))$ using the Lions–Aubin lemma. In order to use the Lions–Aubin lemma, we show that

$$\partial_t \mathbf{u}^{(n)} \in L^2(0,T;H^{-s}), \tag{A.2}$$

where $s = \max\{\alpha, 1 + \frac{d}{2} - \alpha\}$. Let $\phi \in H^s$. We take the $L^2$-inner product of $\phi$ and the velocity equation in (A.1) leads to

$$\int_{\mathbb{T}^d} \phi\cdot\partial_t \mathbf{u}^{(n)}\ d\mathbf{x} := Q_1 + Q_2 + Q_3,$$

with

$$Q_1 = -\int \phi\cdot\mathbb{P}S_n(\mathbf{u}^{(n)}\cdot\nabla\mathbf{u}^{(n)})d\mathbf{x},$$

$$Q_2 = -\nu\int \phi\cdot(-\Delta)^\alpha \mathbf{u}^{(n)} d\mathbf{x},$$

$$Q_3 = \int \phi\cdot\mathbb{P}S_n(\theta^{(n)}\mathbf{e}_d)d\mathbf{x}.$$

Integrating by parts, and applying Hölder's and Sobolev's inequalities yield

$$\begin{aligned}
|Q_1| &\le \|\mathbf{u}^{(n)}\|_{L^{\frac{2d}{d-\alpha}}}^2 \|\nabla\mathbb{P}S_n\phi\|_{L^{\frac{d}{\alpha}}} \\
&\le C\|\mathbf{u}^{(n)}\|_{L^2}^{\frac12}\|\Lambda^\alpha\mathbf{u}^{(n)}\|_{L^2}^{\frac12}\,\|\mathbb{P}S_n\phi\|_{\dot H^{1+\frac{d}{2}-\alpha}} \\
&\le C\|\mathbf{u}^{(n)}\|_{L^2}^{\frac12}\|\Lambda^\alpha\mathbf{u}^{(n)}\|_{L^2}^{\frac12}\|\phi\|_{\dot H^{1+\frac{d}{2}-\alpha}}.
\end{aligned}$$

Using integration by parts and Hölder's inequality, we have

$$|Q_2| \le \nu\|\Lambda^\alpha\phi\|_{L^2}\|\Lambda^\alpha\mathbf{u}^{(n)}\|_{L^2} \le \nu\|\phi\|_{\dot H^s}\|\Lambda^\alpha\mathbf{u}^{(n)}\|_{L^2}.$$

Clearly,

$$|Q_3| \le \|\phi\|_{\dot H^s}\|\theta^{(n)}\|_{L^2}.$$

Therefore,

$$\left|\int \phi\cdot\partial_t\mathbf{u}^{(n)} d\mathbf{x}\right| \le C\|\phi\|_{\dot H^s}(\|\Lambda^\alpha\mathbf{u}^{(n)}\|_{L^2}(1 + \|\mathbf{u}^{(n)}\|_{L^2}) + \|\theta^{(n)}\|_{L^2}).$$

That is,

$$\|\partial_t \mathbf{u}^{(n)}\|_{H^{-s}} \le C(\|\Lambda^\alpha \mathbf{u}^{(n)}\|_{L^2}(1+\|\mathbf{u}^{(n)}\|_{L^2}) + \|\theta^{(n)}\|_{L^2}).$$

Squaring and integrating in time yields

$$\begin{aligned}
&\int_0^T \|\partial_t \mathbf{u}^{(n)}\|_{H^{-s}}^2 dt \\
&\quad \le C\int_0^T (1+\|\mathbf{u}^{(n)}\|_{L^2})^2 \|\Lambda^\alpha \mathbf{u}^{(n)}\|_{L^2}^2 dt + C\int_0^T \|\theta^{(n)}\|_{L^2}^2 dt \\
&\qquad + C\int_0^T (1+\|\mathbf{u}^{(n)}\|_{L^2})\|\Lambda^\alpha \mathbf{u}^{(n)}\|_{L^2}\|\theta^{(n)}\|_{L^2}^2 dt \\
&\quad \le C\sup_{0\le t\le T}(1+\|\mathbf{u}^{(n)}\|_{L^2}^2)\int_0^T \|\Lambda^\alpha \mathbf{u}^{(n)}\|_{L^2}^2\ dt + CT\sup_{0\le t\le T}\|\theta^{(n)}\|_{L^2} \\
&\qquad + C\left(T\sup_{0\le t\le T}\|\theta^{(n)}\|_{L^2}\right)\cdot\left(\sup_{0\le t\le T}(1+\|\mathbf{u}^{(n)}\|_{L^2})\right)\int_0^T \|\Lambda^\alpha \mathbf{u}^{(n)}\|_{L^2}^2 dt \\
&\quad < +\infty.
\end{aligned}$$

Thus we have obtained (A.2). Since we have

$$\mathbf{u}^{(n)} \in L^2(0,T;\dot{H}^\alpha(\mathbb{T}^d)), \quad \partial_t \mathbf{u}^{(n)} \in L^2(0,T;\dot{H}^{-s}(\mathbb{T}^d)),$$

and the facts that $H^\alpha(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d)$ is compact and $L^2(\mathbb{T}^d) \hookrightarrow H^{-(1+d/2-\alpha)}(\mathbb{T}^d)$ is continuous, we can apply the Lions–Aubin Lemma to conclude that $\mathbf{u}^{(n)}$ has a convergent subsequence in $L^2(0,T;L^2(\mathbb{T}^d))$. Let $\mathbf{u}$ be the limit of $\mathbf{u}^{(n)}$ and $\theta$ be the weak limit of $\theta^{(n)}$. Clearly,

$$\theta \in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad \mathbf{u} \in L^\infty(0,T;L^2(\mathbb{T}^d)) \cap L^2(0,T;H^\alpha(\mathbb{T}^d)).$$

**Step 3.** Passing to the limit.

This step shows that $(\mathbf{u},\theta)$ obtained in the previous step is a weak solution of (1.1). It is easy to see from (A.1) that, for any $\phi \in C_0^\infty(\mathbb{T}^d \times [0,T))$ with $\nabla\cdot\phi = 0$, and for any $\psi \in C_0^\infty(\mathbb{T}^d \times [0,T))$ and for any $t \le T$,

$$\begin{aligned}
&-\int_0^t\int_{\mathbb{T}^d} \mathbf{u}^{(n)}\cdot\partial_t\phi d\mathbf{x}d\tau + \int_{\mathbb{T}^d} \mathbf{u}^{(n)}(\mathbf{x},t)\cdot\phi(\mathbf{x},t)d\mathbf{x} - \int_{\mathbb{T}^d} \mathbf{u}_0^{(n)}(\mathbf{x})\cdot\phi(\mathbf{x},0)d\mathbf{x} \\
&\qquad -\int_0^t\int_{\mathbb{T}^d} \mathbf{u}^{(n)}\cdot\nabla(S_n\phi)\mathbf{u}^{(n)} d\mathbf{x}d\tau\ + \nu\int_0^t\int_{\mathbb{T}^d} \Lambda^\alpha \mathbf{u}^{(n)}\cdot\Lambda^\alpha\phi d\mathbf{x}dt \\
&\quad = \int_0^t\int_{\mathbb{T}^d} \theta^{(n)}\mathbf{e}_d\cdot S_n\phi d\mathbf{x}d\tau, \\
&\qquad -\int_0^t\int_{\mathbb{T}^d} \partial_t\psi\theta^{(n)} d\mathbf{x}d\tau + \int_{\mathbb{T}^d} \theta^{(n)}\psi(\mathbf{x},t)d\mathbf{x} - \int_{\mathbb{T}^d} \theta_0^{(n)}(\mathbf{x})\psi(\mathbf{x},0)d\mathbf{x} \\
&\quad = \int_0^t\int_{\mathbb{T}^d} \mathbf{u}^{(n)}\cdot\nabla(S_n\psi)\theta^{(n)} d\mathbf{x}d\tau.
\end{aligned}$$

The task is then to verify that, as $n \to \infty$, the terms above converge to the corresponding terms in the definition of the weak solution given in Definition 3.1. We need the strong convergence $\mathbf{u}^{(n)} \to \mathbf{u}$ in $L^2(0,T;L^2)$. It suffices to consider the convergence of the nonlinear terms. Let

$$A := -\int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla(S_n\phi)\mathbf{u}d\mathbf{x}d\tau,$$

$$A^{(n)} := -\int_0^t \int_{\mathbb{T}^d} \mathbf{u}^{(n)} \cdot \nabla(S_n\phi)\mathbf{u}^{(n)}d\mathbf{x}d\tau$$

and consider the difference

$$\begin{aligned}
A^{(n)} - A &= -\int_0^t \int_{\mathbb{T}^d} (\mathbf{u}^{(n)} - \mathbf{u}) \cdot \nabla(S_n\phi)\mathbf{u}^{(n)}d\mathbf{x}d\tau \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla(S_n\phi - \phi)\mathbf{u}^{(n)}d\mathbf{x}d\tau \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla\phi \cdot (\mathbf{u}^{(n)} - \mathbf{u})d\mathbf{x}d\tau \\
&:= R_1 + R_2 + R_3.
\end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}
|R_1| &\le \|\mathbf{u}^{(n)} - \mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])}\|\nabla S_n\phi\|_{L^\infty(\mathbb{T}^d\times[0,T])}\|\mathbf{u}^{(n)}\|_{L^2(\mathbb{T}^d\times[0,T])} \\
&\le C\|\mathbf{u}^{(n)} - \mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])}\|\phi\|_{H^{2+\frac{d}{2}}}\|\mathbf{u}_0\|_{L^2(\mathbb{T}^d\times[0,T])} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|R_2| &\le \|\mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])}\|\nabla(S_n\phi - \phi)\|_{L^\infty(\mathbb{T}^d\times[0,T])}\|\mathbf{u}^{(n)}\|_{L^2(\mathbb{T}^d\times[0,T])} \\
&\le C\|\mathbf{u}_0\|_{L^2}\|S_n\phi - \phi\|_{H^{2+\frac{d}{2}}}\|\mathbf{u}_0\|_{L^2} \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

and, as $n \to \infty$,

$$|R_3| \le \|\mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])}\|\nabla\phi\|_{L^\infty(\mathbb{T}^d\times[0,T])}\|\mathbf{u}^{(n)} - \mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])} \to 0.$$

Therefore $|A^{(n)} - A| \to 0$ as $n \to \infty$. The convergence of the other nonlinear term is slightly different. We do not have strong convergence in $\theta^{(n)}$. Define

$$B := -\int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla(S_n\psi)\theta d\mathbf{x}d\tau,$$

$$B^{(n)} := -\int_0^t \int_{\mathbb{T}^d} \mathbf{u}^{(n)} \cdot \nabla(S_n\psi)\theta^{(n)}d\mathbf{x}d\tau$$

and consider the difference

$$
\begin{aligned}
B^{(n)} - B &= -\int_0^t \int_{\mathbb{T}^d} (\mathbf{u}^{(n)} - \mathbf{u}) \cdot \nabla (S_n\psi)\theta^{(n)} d\mathbf{x}d\tau \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla (S_n\psi - \psi)\theta^{(n)} d\mathbf{x}d\tau \\
&\quad + \int_0^t \int_{\mathbb{T}^d} \mathbf{u} \cdot \nabla \psi \cdot (\theta^{(n)} - \theta) d\mathbf{x}d\tau \\
&= W_1 + W_2 + W_3.
\end{aligned}
$$

Using Hölder's inequality, we have

$$
\begin{aligned}
|W_1| &\le \|\mathbf{u}^{(n)} - \mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])} \|\nabla S_n\psi\|_{L^\infty(\mathbb{T}^d\times[0,T])} \|\theta^{(n)}\|_{L^2(\mathbb{T}^d\times[0,T])} \\
&\le C\|\mathbf{u}^{(n)} - \mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])} \|\psi\|_{H^{2+\frac{d}{2}}} \|\theta_0\|_{L^2(\mathbb{T}^d\times[0,T])} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|W_2| &\le \|\mathbf{u}\|_{L^2(\mathbb{T}^d\times[0,T])} \|\nabla (S_n\psi - \psi)\|_{L^\infty(\mathbb{T}^d\times[0,T])} \|\theta^{(n)}\|_{L^2(\mathbb{T}^d\times[0,T])} \\
&\le C\|\mathbf{u}_0\|_{L^2} \|S_n\psi - \psi\|_{H^{2+\frac{d}{2}}} \|\theta_0\|_{L^2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

$W_3$ is estimated differently from $R_3$ since we do not have strong convergence in $\theta^{(n)}$. Since $L^2$ functions can be approximated by smooth functions with compact support, $\mathbf{u} \cdot \nabla\psi$ can be treated as a test function. Since $\theta^{(n)}$ converges weakly to $\theta$, we have

$$
W_3 \to 0 \quad \text{as } n \to \infty.
$$

Hence $|B^{(n)} - B| \to 0$ as $n \to \infty$. Therefore, $(\mathbf{u}, \theta)$ is indeed a weak solution. This completes the proof of Proposition 3.2. □

## Appendix B. The Littlewood–Paley Decomposition in $\mathbb{R}^d$, Besov Spaces and Related Facts

This appendix provides the definitions of the Littlewood–Paley decomposition in $\mathbb{R}^d$, functional settings associated with the Besov spaces and related facts. The reason that this is provided here is to make a comparsion with the Littlewood–Paley decomposition in $\mathbb{T}^d$ and the associated tools developed in Sec. 2. In addition, an Osgood type inequality used in the previous sections is also stated here for the convenience of readers. More details can be found in several books and many papers (see, e.g., [5, 6, 48, 51, 55]).

To introduce the Besov spaces, we start with a few notation. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{\phi \in \mathcal{S} \colon \int_{\mathbb{R}^d} \phi(\mathbf{x})\mathbf{x}^\gamma d\mathbf{x} = 0, |\gamma| = 0, 1, 2, \ldots\right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ denotes the space of multinomials. For each $j \in \mathbb{Z}$, we write

$$A_j = \{\xi \in \mathbb{R}^d \colon 2^{j-1} \leq |\xi| < 2^{j+1}\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^d\backslash\{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d\backslash\{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \quad f \in \mathcal{S}_0' \tag{B.1}$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f = 2^{jd} \int \Phi_0(2^j(x-y))f(y)dy, \quad j \in \mathbb{Z}.$$

The homogeneous Littlewood–Paley decomposition (B.1) can then be written as

$$f = \sum_{j=-\infty}^{\infty} \mathring{\Delta}_j f, \quad f \in \mathcal{S}_0'.$$

**Definition B.1.** For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{B.2}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \ldots. \end{cases} \tag{B.3}$$

The inhomogeneous Littlewood–Paley decomposition (B.2) can then be written as

$$f = \sum_{j=-1}^{\infty} \Delta_j f, \quad f \in \mathcal{S}'.$$

**Definition B.2.** The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \le p, q \le \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left( \int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left( \int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way. We will also use the space-time spaces introduced by Chemin–Lerner (see, e.g., [5]).

**Definition B.3.** For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the space-time spaces $\widetilde{L}_t^r \mathring{B}_{p,q}^s$ and $\widetilde{L}_t^r B_{p,q}^s$ are defined though the norms

$$\|f\|_{\widetilde{L}_t^r \mathring{B}_{p,q}^s} \equiv \|2^{js} \|\mathring{\Delta}_j f\|_{L_t^r L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}_t^r B_{p,q}^s} \equiv \|2^{js} \|\Delta_j f\|_{L_t^r L^p}\|_{l^q}.$$

Here $L_t^r$ is the abbreviation for $L^r(0,t)$. These spaces are related to the classical space-time spaces $L_t^r \mathring{B}_{p,q}^s$ and $L_t^r B_{p,q}^s$ via the Minkowski inequality, if $r \ge q$,

$$\widetilde{L}_t^r \mathring{B}_{p,q}^s \subseteq L_t^r \mathring{B}_{p,q}^s, \quad \widetilde{L}_t^r B_{p,q}^s \subseteq L_t^r B_{p,q}^s$$

and, if $r < q$,

$$\widetilde{L}_t^r \mathring{B}_{p,q}^s \supset L_t^r \mathring{B}_{p,q}^s, \quad \widetilde{L}_t^r B_{p,q}^s \supset L_t^r B_{p,q}^s.$$

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

**Proposition B.4.** *For any $s \in \mathbb{R}$,*

$$\mathring{H}^s \sim \mathring{B}_{2,2}^s, \quad H^s \sim B_{2,2}^s.$$

*For any $s \in \mathbb{R}$ and $1 < q < \infty$,*

$$\mathring{B}_{q,\min\{q,2\}}^s \hookrightarrow \mathring{W}_q^s \hookrightarrow \mathring{B}_{q,\max\{q,2\}}^s.$$

*In particular, $\mathring{B}_{q,\min\{q,2\}}^0 \hookrightarrow L^q \hookrightarrow \mathring{B}_{q,\max\{q,2\}}^0$.*

Besides the Fourier localization operators $\Delta_j$, the partial sum $S_j$ is also a useful notation. For an integer $j$,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where $\Delta_k$ is given by (B.3). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius $2^j$ and

$$S_j f(x) = 2^{dj} \Psi(2^j x) * f(x) = 2^{dj} \int \Psi(2^j(x-y)) f(y) dy.$$

The operators $\Delta_j$ and $S_j$ defined above satisfy the following properties:

$$\Delta_j \Delta_k f = 0 \quad \text{if } |k-j| \ge 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |k-j| \ge 3.$$

Bernstein's inequalities is a useful tool on Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

**Proposition B.5.** *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

(1) *If $f$ satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \le K 2^j\},$$

*for some integer $j$ and a constant $K > 0$, then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) *If $f$ satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : K_1 2^j \le |\xi| \le K_2 2^j\}$$

*for some integer $j$ and constants $0 < K_1 \le K_2$, then*

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

*where $C_1$ and $C_2$ are constants depending on $\alpha, p$ and $q$ only.*

We shall also use Bony's notion of paraproducts to decompose a product into three parts

$$fg = T_f g + T_g f + R(f,g),$$

where

$$T_f g = \sum_j S_{j-1} f \Delta_j g,$$

$$R(f,g) = \sum_j \sum_{k \ge j-1} \Delta_k f \widetilde{\Delta}_k g$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. Finally, we state an Osgood type inequality to be used in the subsequent sections (see, e.g., [5]).

**Lemma B.6.** *Let $a > 0$ and $0 \le t_0 < T$. Let $\rho$ be a measurable function from $[t_0, T]$ to $[0, a]$. Let $\gamma(t) > 0$ be a locally integrable function on $[t_0, T]$. Let $\phi \ge 0$ be a continuous and non-decreasing function on $[0, a]$. Assume that $\rho$ satisfies, for some constant $c$*

$$\rho(t) \le c + \int_{t_0}^t \gamma(s)\phi(\rho(s))ds \quad \text{for a.e. } t \in [t_0, T].$$

*Then, if $c > 0$, we have, for a.e. $t \in [t_0, T]$,*

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \le \int_{t_0}^t \gamma(\tau)d\tau,$$

*where*

$$\mathcal{M}(x) = \int_x^a \frac{dr}{\phi(r)}.$$

*If $c = 0$ and*

$$\int_0^a \frac{dr}{\phi(r)} = \infty,$$

*then $\rho(t) = 0$ a.e. $t \in [t_0, T]$.*

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