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THE COMPLEX KDV EQUATION WITH OR WITHOUT DISSIPATION

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Abstract. Solutions of the complex KdV equation and the complex KdV-Burgers equation are studied theoretically and numerically. Attention is focused on whether their solutions are regular for all time. This is a difficult issue partially because the conservation laws of the KdV equation no longer yield *a priori* bounds for its complex-valued solutions in the L^2-space. The problem is tackled here on several fronts including investigating how the regularity of the real part is related to that of the imaginary part, studying blow-up of series solutions, and assessing the impact of dissipation. Systematic numerical simulations are performed to complement the theoretical results.

1. **Introduction.** In this paper we study solutions of the complex KdV equation

$$u_t + 2u\,u_x + u_{xxx} = 0 \tag{1.1}$$

and of its dissipative counterpart, the complex KdV-Burgers equation

$$u_t + 2u\,u_x + u_{xxx} - \nu u_{xx} = 0, \tag{1.2}$$

where $u = u(x,t)$ is a complex-valued function of the spatial variable x and the temporal variable t, and ν is the diffusion coefficient. Although the real KdV equation is normally the focus of attention, the complex KdV-type equations do arise in physical circumstances. For example, the complex KdV equation with a higher-order correction term models the diffusion-controlled directional crystal growth [8] and the stationary complex KdV equation models small perturbations about a reference velocity in an irrotational flow [12, 13]. In addition, the complex KdV equation also has some remarkable mathematical features. Several authors have previously explored some of the features. In [10], Kruskal represented solitons as the poles of the complex KdV equation and viewed the motion of solitons as the "parade of poles." In [2], Birnir studied a class of elliptic solutions to the complex KdV equation and proved that they blow up in a finite time as a second-order pole. As revealed in the work of Bona and Weissler [5], the complex KdV equation also

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admits solutions with much richer structure than the real KdV equation. Despite these outstanding researches, many fundamental issues concerning the complex KdV equation remain open. Our aim here is to assess whether solutions of the complex KdV equation remain regular for all time and determine how dissipation, in the form of Burgers-type term, impacts the regularity. Results of both theoretical study and numerical computation are presented.

We are mainly concerned with periodic solutions. Both (1.1) and (1.2) are supplemented with the initial condition

$$u(x,0) = u_0(x), \tag{1.3}$$

where u_0 is a reasonably smooth, 2π-periodic complex-valued function. It is not hard to show, by the method of Bourgain ([7]), that both of the initial-value problems (1.1)-(1.3) and (1.2)-(1.3) have at least a unique local (in time) solution ([14]). Whether these local solutions can be extended into global ones depend on *a priori* bounds. Usually conservation laws are the sources of *a priori* bounds, but the complication with the complex KdV equation is that the hierarchy of conservation laws no longer leads to the boundedness of its solutions in the L^2-space. To get around the difficulty, we derive, for any complex-valued solution of the KdV equation, an explicit relationship between the regularity of the real part and that of the imaginary part. In fact, we show *a priori* that the real part is bounded in L^2 or H^k $(k \geq 1)$ if and only if the imaginary part is bounded in the same space. As a consequence, any solution of the complex KdV equation is regular for all time if its real part remains so. This result is proved in Section 2.

Periodic solutions can be represented as Fourier series. In Section 3, we consider solutions of the complex KdV equation that can be written in the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\, e^{ikx}, \tag{1.4}$$

where $a_k(t)$ may be complex-valued. We emphasize that nontrivial real-valued solutions of (1.1) are excluded by the form (1.4). The complex KdV equation is then formally reduced into a system of infinite ordinary differential equations of a_k's. These equations allow us to further represent each a_k as a series in the temporal variable t. That is, u can be written as a double series of the form

$$u(x,t) = \sum_{k=1}^{\infty} \sum_{h=k}^{k^3} \alpha_{k,h}\, e^{iht}\, e^{ikx}. \tag{1.5}$$

Iterative relations for the coefficients $\alpha_{k,h}$'s are derived. These relations, in particular, imply that $\alpha_{k,h} = 0$ for any $k \geq 1$ and $k^3 - 3k^2 + 3k < h < k^3$. In addition, an explicit formula for $\alpha_{k,k}$ is found through these relations. As a consequence, we prove that any solution in the form (1.4) becomes unbounded in $L^2([0,2\pi] \times [0,2\pi])$ in a finite time if $|a_1(0)| \geq 6$, where $a_1(0)$ is the first coefficient in the initial data u_0. This result is sharp in the sense that there exist a series with initial $|a_1(0)| < 6$ that are regular for all time. However, we may not conclude that global in time solutions exist merely under the hypothesis that $|a_1(0)| < 6$.

Aiming at determining the L^2-convergence of the double series (1.5), we computed symbolically through a Maple procedure the expressions of $a_k(t)$'s in terms of the initial coefficients $a_k(0)$'s. Using these formulas, we computed $\|u\|_{L^2([0,2\pi]\times[0,2\pi])}$

for a special class of initial data u_0, namely $a_k(0) = 1/k^\beta$ with $\beta > 1/2$. The numerical results indicate that u is bounded and its L^2-norm has an interesting connection to its initial L^2-norm when $\beta = 1$. We also investigated another interesting and closely related issue: how many $\alpha_{k,h}$ are non-zero for each fixed $k \geq 1$? This appears to be a fairly difficult combinatorial problem. Using a Maple code, we computed the indices (k, h) for which $\alpha_{k,h} \neq 0$ for k up to 50. A careful examination of these indices seems to imply that the number of non-zero $\alpha_{k,h}$'s for extremely large k is about $k^3/6$. More details are discussed in Section 3.

Section 4 is devoted to the complex KdV-Burgers equation. The point is to see how dissipation, in the form of the Burgers-type term, modifies the regularity of solutions to the complex KdV equation. We start with deriving an explicit bound M_0, for which

$$\|u(\cdot, t)\|_{L^2} \leq M_0 \quad \text{for } t < T^*,$$

where T^* is represented in terms of the dissipation coefficient ν and the initial data u_0. Either increasing ν or decreasing $\|u_0\|_{L^2}$ lengthens T^*. It is not known if u remains bounded in L^2 for $t \geq T^*$. However, if the L^2-norm is indeed bounded for all time, then we show that any H^k-norm with $k \geq 1$ is also bounded for all time. This result, in particular, implies that any possible finite-time singularity of the complex KdV equation must develop in the L^2-norm. This result is a little bit surprising. Normally one needs the finiteness of L^∞-norm in order to control the H^k-norms such as in the celebrated result of Beale, Kato and Majda on the 3D Navier-Stokes equations [1].

Section 5 presents the results of systematic numerical simulations. Numerical experiments are carried out in this section to test how the "sizes" of initial data and dissipation impact the regularity of the complex KdV equation. The fully discrete schemes we are using here amount to the Gauss-Legendre methods on the systems of ordinary differential equations that arise from the Galerkin semi-discretization. Adaptive mechanisms are used to adjust the temporal and local spatial grids to retain accuracy in the situation of large dependent variable. More detailed description of the numerical method and its convergence rate is given in Subsection 5.1. Subsection 5.2 reports numerical computations on the complex KdV equation. Attention is focused on how the regularity of its solutions changes with the magnitude of the initial data. In Subsection 5.3, numerical solutions of the complex KdV-Burgers equation are computed. We place a special emphasis on the role of dissipation in the regularity. Solutions corresponding to a range of ν are computed. The results indicate that for any given initial datum u_0 there exists a critical value ν^* such that the solution is smooth for all time if $\nu > \nu^*$, and blows up in a finite time if $\nu < \nu^*$. Efforts are made to pin down the critical ν^*'s. Finally we remark that the numerical experiments are very delicate and time-consuming.

2. **Regularity.** In this section, we prove a general regularity result for solutions of the complex KdV equation

$$u_t + 2u\, u_x + u_{xxx} = 0. \tag{2.1}$$

Roughly speaking, it states that the real part of any solution of (2.1) is as regular as the imaginary part. The precise statement will be presented in Theorem 2.1

below. Throughout this section, we assume that (2.1) is complemented by the initial condition

$$u(x,0) = u_0(x)$$

with u_0 fulfilling required regularity. In addition, the real and imaginary part of a complex-valued solution u are denoted by f and g, respectively.

We now briefly recall the KdV-conversation laws given by Kruskal, Miura, Gardner and Zabusky [11]. Obviously, complex-valued solutions of the KdV equation obey the same laws. For each $k = 1, 2, \cdots$, a sufficiently smooth solution u of the KdV equation satisfies a sequence of identities of the form

$$\frac{\partial}{\partial t} I_k(u) = \frac{\partial}{\partial x} F_k(u), \tag{2.2}$$

where I_k and F_k are polynomials in u and the partial derivatives $\partial_x^j u$, which we write as $u_{(j)}$ for convenience, $j = 1, 2, \cdots$. In more detail, I_k depends on u, $\partial_x u$, $\cdots$, $\partial_x^k u$ and F_k depends on u, $\partial_x u$, $\cdots$, $\partial_x^{k+2} u$. Moreover, suitably normalized, I_k has the form

$$I_k(u) = u_{(k)}^2 + au\, u_{k-1}^2 + \cdots,$$

which is a finite sum of terms of index $k+2$ where the index of a monomial

$$u_{(\beta_1)}^{\alpha_1} \cdots u_{(\beta_r)}^{\alpha_r} \tag{2.3}$$

is

$$\sum_{i=1}^{r} \alpha_i + \frac{1}{2} \sum_{i=1}^{r} \alpha_i\, \beta_i.$$

The fluxes F_k have a similar form except that their general term, which is also of the form (2.3), has index $k+3$.

Integrating both sides of (2.2) with respect to x, there obtains

$$Q_k(u) \equiv \int I_k(u)dx = \int I_k(u_0)dx = Q_k(u_0). \tag{2.4}$$

Here we have written $\int$ for the integral over the spatial domain.

Theorem 2.1. *Let $k \geq 0$ be an integer and $T > 0$. Let $u_0 \in H^k$ and u be the solution of the complex KdV equation (2.1) with initial data u_0. Denote the real part of u by f and its imaginary part by g. Then for any $t \leq T$,*

$$f(\cdot,t) \text{ is in } H^k \text{ if and only if } g(\cdot,t) \text{ is in } H^k. \tag{2.5}$$

Remark. One interpretation of this theorem is that any solution of the complex KdV equation is regular for all time as long as its real part is so.

Proof. The theorem is proved by induction. With $k = 0$, the real parts of (2.4) gives

$$\int f^2 dx - \int g^2 dx = \mathcal{R}(Q_0(u_0)),$$

where $\mathcal{R}$ stands for the operator that maps a complex number to its real part. Therefore, $f \in L^2$ if and only if $g \in L^2$. For $k = 1$, we use the second conservation law

$$Q_1(u) = \int \left(u_x^2 - \frac{1}{3}u^3\right) dx = Q_1(u_0).$$

In particular, the real parts of both sides are equal, i.e.,

$$\int \left(f_x^2 - g_x^2 - \frac{1}{3} f^3 + f g^2 \right) dx = \mathcal{R}(Q_1(u_0)). \tag{2.6}$$

To prove (2.5) with $k = 1$, we first assume $g \in H^1$. This implies that $g \in L^2$ and thus $f \in L^2$. To show $f_x \in L^2$, we employ the following elementary estimates

$$\int f g^2 dx \le C \|f\|_{L^2} \, \|g\|_{L^2}^{\frac{3}{2}} \, \|g_x\|_{L^2}^{\frac{1}{2}},$$

$$\int f^3 dx \le C \|f\|_{L^\infty} \|f\|_{L^2}^2 \le C \|f\|_{L^2}^{\frac{5}{2}} \, \|f_x\|^{\frac{1}{2}}.$$

Inserting these estimates in (2.6), we obtain, for $\sigma(t) = \|f_x(\cdot, t)\|_{L^2}$,

$$\sigma^2(t) - C \|f\|_{L^2}^{\frac{5}{2}} \, \sigma^{\frac{1}{2}}(t) \le \|g_x\|^2 + C \|f\|_{L^2} \, \|g\|_{L^2}^{\frac{3}{2}} \, \|g_x\|_{L^2}^{\frac{1}{2}} + \mathcal{R}(Q_1(u_0)),$$

where C's denote some pure constants. Applying Lemma 2.1 below, we conclude that $f_x \in L^2$. On the other hand, if $f \in H^1$, then f and g are in L^2. From (2.6), we easily see that g_x is in L^2. This proves (2.5) with $k = 1$.

Now we assume the truth of (2.5) for any $k \le m$ with $m \ge 1$. To show that (2.5) holds for $m + 1$, we invoke the conservation law

$$Q_{m+1}(u) = \int (u_{(m+1)}^2 + au \, u_{(m)}^2 + \cdots) dx = Q_{m+1}(u_0).$$

The above equation can be rearranged into the following form

$$\int (u_{m+1}^2 + au \, u_{(m)}^2) dx = Q_{m+1}(u_0) - \int P(u) dx,$$

where $P(u)$ is a linear combination of the other monomials of index $m+3$. Obviously, the highest order of partial derivatives of u in $P(u)$ is $m - 1$. The equation for the real parts reads

$$\int \left[f_{(m+1)}^2 - g_{(m+1)}^2 + a(f(f_{(m)}^2 - g_{(m)}^2) - 2g f_{(m)} g_{(m)}) \right] dx$$

$$= \int \mathcal{R}(P(u)) dx + \mathcal{R}(Q_{m+1}(u_0)). \tag{2.7}$$

We first assume that $f \in H^{m+1}$. Trivially, f is in H^m and, by the induction hypothesis, g is also in H^m. Since the derivatives of u involved in $P(u)$ have order no more than $m - 1$, the terms on the right of (2.7) are bounded, according to the induction hypothesis. The proof of $g \in H^{m+1}$ is then straightforward. It is quite similar to prove that $g \in H^{m+1}$ implies $f \in H^{m+1}$. This concludes the proof of Theorem 2.1.

Finally, we state the lemma that has been used in the proof of Theorem 2.1.

Lemma 2.1. *Let P, Q and $\beta < 2$ be positive numbers. If $Y \ge 0$ satisfies*

$$Y^2 - P Y^\beta \le Q,$$

then Y is bounded by

$$Y \le \max \left\{ (2P)^{\frac{1}{2-\beta}}, \sqrt{2Q} \right\}.$$

3. **Series solutions.** In this section, we study a special type of solutions to the complex KdV equation

$$u_t + 2uu_x + u_{xxx} = 0. \tag{3.1}$$

More precisely, we seek solutions that can be represented in the form

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) e^{ikx}, \tag{3.2}$$

where the time-dependent coefficients $a_k(t)$ are complex-valued. The initial value is given by

$$u(x,0) = \sum_{k=1}^{\infty} a_k(0) e^{ikx}. \tag{3.3}$$

Formally, the KdV equation (3.1) is reduced to the following infinite system of ordinary differential equations

$$\frac{d}{dt} a_k(t) - ik^3 a_k(t) + ik \sum_{k_1+k_2=k} a_{k_1}(t)\, a_{k_2}(t) = 0, \quad k \geq 1 \tag{3.4}$$

where the indices $k_1 \geq 1$ and $k_2 \geq 1$ are integers. It is clear that each coefficient $a_k(t)$ is determined by the first k equations. For sufficiently regular solutions, solutions of the form (3.2) can be recovered from (3.4). Given initial data $a_k(0)$ for $k \geq 1$, (3.4) is equivalent to the integral equations

$$a_k(t) = a_k(0)\, e^{ik^3 t} - ik \sum_{k_1+k_2=k} e^{ik^3 t} \int_0^t e^{-ik^3\tau}\, a_{k_1}(\tau)\, a_{k_2}(\tau) d\tau, \quad k \geq 1. \tag{3.5}$$

This equation constitutes a base for further analysis on the structure of a_k. In fact, a_k can be written as a summation of terms of the form $\alpha_{k,h} e^{iht}$ with each $\alpha_{k,h}$ satisfying an iterative relation. The details are provided in Subsection 3.1. In Subsection 3.2, we obtain an explicit formula for $\alpha_{k,k}$ and prove a finite-time singularity result for solutions of the form (3.2) corresponding to initial data (3.3) with $|a_1(0)| > 6$. In Subsection 3.3, we discuss the L^2 convergence of solutions (3.2) corresponding to any initial datum in L^2. A special type of data considered here has its coefficients $\{a_k(0)\}$ in l^2.

3.1. **Iterative relations for $\alpha_{k,h}$.** We start with a representation for $a_k(t)$. For notational convenience, we set $I(k) = k^3 - 3k^2 + 3k$.

Proposition 3.1. *For each $k \geq 1$, $a_k(t)$ is of the form*

$$a_k(t) = \sum_{h=k}^{k^3} \alpha_{k,h} e^{iht} \tag{3.6}$$

with real coefficients $\alpha_{k,h}$. In addition,

$$\alpha_{k,h} = 0 \quad \text{for } I(k) < h < k^3. \tag{3.7}$$

As a special consequence, $a_k : \mathbb{R} \to \mathbb{C}$ is a periodic function of period 2π.

Remarks. This proposition indicates that many $\alpha_{k,h}$'s are in fact equal to zero. For example, for $k = 2$, $I(k) = 2$, so $\alpha_{2,3} = \cdots = \alpha_{2,7} = 0$, $\alpha_{2,2} = \frac{1}{3} a_1^2(0)$, $\alpha_{2,8} = a_2(0) - \frac{1}{3} a_1^2(0)$; for $k = 3$, $I(k) = 9$, so $\alpha_{3,10} = \cdots = \alpha_{3,26} = 0$, $\alpha_{3,3} = \frac{1}{12} a_1^3(0)$, $\alpha_{3,9} = \frac{1}{3} a_1(0)\, a_2(0) - \frac{1}{9} a_1^3(0)$ and $\alpha_{3,27} = a_3(0) - \frac{1}{3} a_1(0) a_2(0) + \frac{1}{36} a_1^3(0)$.

Proof. The proof is made by induction on k. For $k = 1$, the formula (3.6) is obvious since (3.4) is reduced to

$$\frac{d}{dt}a_1(t) - ia_1(t) = 0$$

and thus $a_1(t) = a_1(0)e^{it}$. Suppose that (3.6) has been shown to hold through k with $k \geq 1$. Now consider a_{k+1}. According to (3.5),

$$a_{k+1}(t) = a_{k+1}(0)\, e^{i(k+1)^3 t}$$
$$-i(k+1) \sum_{k_1+k_2=k+1} e^{i(k+1)^3 t} \int_0^t e^{-i(k+1)^3 \tau}\, a_{k_1}(\tau)\, a_{k_2}(\tau) d\tau.$$

Using the induction hypothesis, we find that the integral on the right-hand side reduces to a finite summation of terms of the form

$$\frac{k+1}{(k+1)^3 - (h_1+h_2)} \alpha_{k_1,h_1}\, \alpha_{k_2,h_2} \left[e^{i(h_1+h_2)t} - e^{i(k+1)^3 t} \right] \tag{3.8}$$

where $k_1 \geq 1$, $k_2 \geq 1$, $k_1 + k_2 = k+1$, $h_1 \leq k_1^3$ and $h_2 \leq k_2^3$. Since

$$h_1 + h_2 \leq k_1^3 + k_2^3 \leq (k_1+k_2)^3 = (k+1)^3,$$

we conclude that a_{k+1} indeed has the desired representation (3.6). We now show (3.7) by induction. According to (3.8), $h_1 + h_2 < (k+1)^3$ and α_{k+1,h_1+h_2} is a sum of the terms of the form

$$\frac{k+1}{(k+1)^3 - (h_1+h_2)} \alpha_{k_1,h_1}\, \alpha_{k_2,h_2}.$$

If $\alpha_{k_1,h_1} = 0$ for $I(k_1) < h_1 < k_1^3$ and $\alpha_{k_2,h_2} = 0$ for $I(k_2) < h_2 < k_2^3$, then

$$h_1 + h_2 \leq k_1^3 + k_2^3 = (k_1+k_2)^3 - 3k_1k_2(k_1+k_2) = (k+1)^3 - 3k_1k_2(k+1)$$
$$\leq (k+1)^3 - 3k(k+1) = (k+1)^3 - 3(k+1)^2 + 3(k+1) = I(k+1),$$

where we have used $k_1 + k_2 = k$ and $k_1, k_2 \leq k$. In other words, there exits no h_1 and h_2 such that $h_1 + h_2 > I(k+1)$. This completes the proof of (3.7) and thus that of Proposition 3.1.

The coefficients $\alpha_{k,h}$ in (3.6) satisfy an iterative relation, as stated in the following proposition.

Proposition 3.2. *If $k \geq 1$, then*

(a) *for $k \leq h \leq I(k)$,*

$$\alpha_{k,h} = \frac{k}{k^3 - h} \sum_{k_1+k_2=k} \sum_{h_1+h_2=h} \alpha_{k_1,h_1}\, \alpha_{k_2,h_2},$$

where $1 \leq k_1 < k$, $1 \leq k_2 < k$, $k_1 \leq h_1 \leq k_1^3$ and $k_2 \leq h_2 \leq k_2^3$;

(b) *for $h = k^3$,*

$$\alpha_{k,k^3} = a_k(0) - \sum_{k \leq h \leq I(k)} \alpha_{k,h},$$

or more explicitly,

$$\alpha_{k,k^3} = a_k(0) - \sum_{k \leq h \leq I(k)} \frac{k}{k^3 - h} \sum_{k_1+k_2=k} \sum_{h_1+h_2=h} \alpha_{k_1,h_1}\, \alpha_{k_2,h_2},$$

where $1 \leq k_1 < k$, $1 \leq k_2 < k$, $k_1 \leq h_1 \leq k_1^3$ and $k_2 \leq h_2 \leq k_2^3$.

Proof of Proposition 3.2. Inserting (3.6) in (3.5) and comparing the coefficients of e^{iht}, we obtain for $k \le h < k^3$

$$\alpha_{k,h} = \frac{k}{k^3 - h} \sum_{k_1+k_2=k} \sum_{h_1+h_2=h} \alpha_{k_1,h_1}\, \alpha_{k_2,h_2}, \tag{3.9}$$

which, in particular, implies (a). To establish (b), it suffices to let $t = 0$ in (3.6).

3.2. **Explicit formula for $\alpha_{k,k}$ and a finite-time singularity result.** The iterative relationship in Proposition 3.2 allows us to derive an explicit formula for $\alpha_{k,k}$ with $k \ge 1$.

Proposition 3.3. *If $a_1(0) = a$, then for all $k \ge 1$*

$$\alpha_{k,k} = 6k \left(\frac{a}{6}\right)^k. \tag{3.10}$$

Remark. If the family of functions $a_k(t)$ is a solution to the system (3.5), then the value of $a_1(0)$ alone determines all coefficients $\alpha_{k,k}$ in the expansion (3.6). This also indicates that not all $\alpha_{k,h}$ are equal to zero when $a_k(0) = 0$.

Proof. By (b) of Proposition 3.2, for $k \ge 1$,

$$\alpha_{k,k} = \frac{1}{k^2 - 1} \sum_{k_1+k_2=k} \sum_{h_1+h_2=k} \alpha_{k_1,h_1}\, \alpha_{k_2,h_2}.$$

Since $h_1 \ge k_1$ and $h_2 \ge k_2$, the terms on the right must be of the form $\alpha_{k_1,k_1}\, \alpha_{k_2,k_2}$. That is,

$$\alpha_{k,k} = \frac{1}{k^2 - 1} \sum_{k_1=1}^{k-1} \alpha_{k_1,k_1}\, \alpha_{k-k_1,k-k_1}.$$

We now prove (3.10) by induction. Clearly, (3.10) holds for $k = 1$. Assume that it holds through k for $k \ge 1$. Then

$$\begin{aligned} \alpha_{k+1,k+1} &= \frac{1}{(k+1)^2 - 1} \sum_{k_1=1}^{k} \alpha_{k_1,k_1}\, \alpha_{k+1-k_1,k+1-k_1} \\ &= \frac{36}{(k+1)^2 - 1} \left(\frac{a}{6}\right)^{k+1} \sum_{k_1=1}^{k} k_1\,(k+1-k_1) \end{aligned}$$

Using the following basic identity

$$\sum_{k_1=1}^{k} k_1\,(k+1-k_1) = \frac{1}{6}(k^3 + 3k^2 + 2k) = \frac{1}{6}(k+1)\left[(k+1)^2 - 1\right],$$

we obtain that $\alpha_{k+1,k+1} = 6(k+1)(a/6)^{k+1}$. This completes the proof.

Consider a regular solution $u(x,t)$ of the complex KdV equation (3.1) in the form (3.2). Proposition 3.2 gives the form of the solution $a_k(t)$ in the formula (3.6). Let $a_1(0) = a$. Since the functions $e^{iht}\, e^{ikx}$ are orthogonal in $L^2([0,2\pi] \times [0,2\pi])$, we can easily calculate the norm of u in this space. We obtain

$$\|u\|_{L^2([0,2\pi]\times[0,2\pi])} = 4\pi^2 \sum_{k,h} |\alpha_{k,h}|^2 \ge \sum_{k\ge1} |\alpha_{k,k}|^2 = \sum_{k\ge1} 36k^2 |a/6|^{2k}.$$

Therefore, we have the following theorem.

Theorem 3.1. *There is no regular, global in time solution of the complex KdV equation (3.1) of the form (3.2) with* $|a_1(0)| \geq 6$.

Remark. This theorem can also be obtained as a special consequence of Theorem 3.1 in [5].

Remark. This result is sharp in the sense that (3.1) does have global in time solution of the form (3.2) if $a = a_1(0)$ and $a < 6$. In fact, if $a_k = 6k(a/6)^k$, then

$$\sum_{k=1}^{\infty} a_k e^{ikt} e^{ikx} = \frac{a\, e^{i(x+t)}}{(1-(a/6)e^{i(x+t)})^2}$$

is an associated traveling solution of the complex KdV equation.

The question that is not answered here is: does any solution of the form (3.2) with $|a_1(0)| < 6$ exist for all time? In the next subsection, we will have a discussion on the L^2 convergence for the case when $a_k(0) = 1/k^\beta$.

3.3. **A discussion on the L^2-convergence.** Assume that u is a solution of the complex KdV equation (3.1) and u is of the form (3.2). A very interesting and challenging issue is whether $\|u\|_{L^2([0,2\pi]\times[0,2\pi])}$ is bounded if $|a_1(0)| < 6$. In this subsection, we present some evidence we have on its boundedness for a special type of initial data and defer the results of a systematic numerical computation to Section 5. Since $u_0 \in L^2([0,2\pi])$ is equivalent to $\{a_k(0)\} \in l^2$, we focus our attention on

$$a_k(0) = \frac{1}{k^\beta}. \tag{3.11}$$

for $\beta > \frac{1}{2}$. Our test for the finiteness of $\|u\|_{L^2([0,2\pi]\times[0,2\pi])}$ is carried out in two stages. First, we obtain an explicit dependence of each $\alpha_{k,h}$ on $a_k(0)'s$ by performing a symbolic computation using Maple codes. In fact, $\alpha_{k,h}$ only involves terms of the form

$$C(h, m_1, m_2, \cdots, m_k)\, a_1^{m_1}(0)\, a_2^{m_2}(0) \cdots a_k^{m_k}(0)$$

with $\sum_{j=1}^{k} j \cdot m_j = k$, where $C(h, m_1, m_2, \cdots, m_k)$ is a pure number with dependence on $h, m_1, m_2, \cdots$, and m_k. For large k and $h < k^3$, $C(h, m_1, m_2, \cdots, m_k)$ is found to be very small and this is probably due to the small factor $k/(k^3 - h)$ in the iterative relations stated in Proposition 3.1. Here are two examples of $\alpha_{k,h}$:

$$\alpha_{5,11} = -\frac{25}{2916}\, a_1^5(0) + \frac{25}{972}\, a_1^3(0)\, a_2(0),$$

$$\begin{aligned}\alpha_{7,133} = &-\frac{1}{2099520}\, a_1^7(0) + \frac{7}{233280}\, a_1^5(0)\, a_2(0) - \frac{7}{19440}\, a_1^4(0)\, a_3(0) \\ &+\frac{7}{19440}\, a_1^3(0)\, a_2^2(0) + \frac{1}{270}\, a_1^3(0)\, a_4(0) - \frac{1}{45} a_1^2(0)\, a_5(0) + \frac{23}{6480}\, a_1^2(0)\, a_2(0)\, a_3(0) \\ &+\frac{1}{90}\, a_1(0)\, a_2(0)\, a_4(0) + \frac{1}{1215}\, a_1(0)\, a_2^3(0) - \frac{1}{135}\, a_2^2(0)\, a_3(0) + \frac{1}{15}\, a_2(0)\, a_5(0).\end{aligned}$$

In the second stage, we substitute (3.11) into the symbolic formulas and investigate the convergence of the following sequence

$$s_N = \sum_{k=1}^{N} \sum_{k \leq h \leq k^3} |\alpha_{k,h}|^2.$$

If s_N converges, then it converges to $\|u\|^2_{L^2([0,2\pi]\times[0,2\pi])}$. We computed s_N for $\beta = 1$ and $N = 10, 15, 20, 25, 30$. The results for s_N and its initial counterpart

$$s_N(0) = \sum_{k=1}^{N} |a_k(0)|^2$$

are recorded in the following table.

Table 1. Partial sum s_N for L^2-norm of u with $a_k(0) = 1/k$

$\beta = 1$	N=10	N=15	N=20	N=25	N=30
s_N	1.29954	1.32432	1.33790	1.34645	1.35232
$s_N(0)$	1.54977	1.58044	1.59616	1.60572	1.61215
$s_N(0) - s_N$	0.2503	0.25612	0.25826	0.25927	0.25983

This table appears to indicate that

$$s_N \approx s_N(0) - 0.26,$$

which implies that s_N converges $\frac{\pi^2}{6} - 0.26$ as $N \to \infty$ since $s_N(0) \to \frac{\pi^2}{6}$. Therefore, this computation suggests that $u \in L^2([0, 2\pi] \times [0, 2\pi])$ for $a_k(0) = 1/k$. We remark that the summation of s_N with $N = 30$ involves several thousands of non-zero $\alpha_{k,h}$'s. The $\alpha_{k,h}$'s are computed symbolically using MAPLE and the process becomes extremely slow when N is getting larger. It would be more convincing if the values of s_N for very large N are available. Similar computations involving other values of $\beta > 1/2$ appear to lead to the same conclusion.

Another interesting and related issue is exactly how many $\alpha_{k,h}$'s are non-zero for each $k \geq 1$. According to Proposition 3.2, $\alpha_{k,h} = 0$ for $k^3 - 3k^2 + 2k < h < k^3$. Results from running a simple Maple procedure indicate that there are far few non-zero terms than the predicted $k^3 - 3k^2 + 3k + 1$. The following table presents the actual number of non-zero $\alpha_{k,h}$'s for $k = 10, 20, 30, 40, 50$. We have used Non(k) to denote the number of non-zero $\alpha_{k,h}$'s for $k \geq 1$.

Table 2. Actual number of non-zero $\alpha_{k,h}$'s vs. $k^3 - 3k^2 + 3k + 1$

	k=10	k=20	k=30	k=40	k=50
Non(k)	36	278	1046	2710	5759
$k^3 - 3k^2 + 3k + 1$	731	6861	24391	59321	117651

The exact number of non-zero $\alpha_{k,h}$'s for k up to 50 can be fitted with the cubic curve $0.06169x^3 - 0.7834x^2$. It is also interesting to observe that the difference between h_1 and h_2 for two adjacent non-zero α_{k,h_1} and α_{k,h_2} is mostly 6. This leads us to conjecture that the number of non-zero $\alpha_{k,h}$'s is $k^3/6 + O(k^2)$ for very large k.

4. **The complex KdV-Burgers equation.** In this section, we investigate whether the dissipative effects, in the form of the Burgers-type term, can overcome the nonlinear and dispersive effects and lead to global regularity. More precisely, attention is paid to the regularity of solutions to the initial-value problem (IVP)

$$\begin{cases} u_t + 2u\, u_x - \nu u_{xx} + u_{xxx} = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

where $\nu > 0$ and $u_0(x)$ is a reasonably smooth, 2π-periodic complex-valued function on $\mathbb{R}$. For notational convenience, we will use $\mathbb{T}$ to denote the interval of one period $[0, 2\pi]$.

It is straightforward to show that the IVP (4.1) has unique solutions corresponding to reasonably smooth initial data, at least locally in time [14]. A local (in time) solution has a global extension if it is bounded *a priori* in certain norm. This section is divided into two subsections. The first subsection derives an *a priori* bound for $\|u(\cdot,t)\|_{L^2}$ valid for t in an explicit finite-time interval. In the second subsection, we bound the Sobolev norm $\|u(\cdot,t)\|_{H^k}$ $(k \geq 1)$ in terms of $\|u(\cdot,t)\|_{L^2}$. As a consequence, we conclude that there is no finite-time blow-up in the H^k-norm of any solution u to the IVP (4.1) unless its L^2-norm blows up in a finite-time.

4.1. **A finite-time bound for the L^2-norm.** We show in this subsection that any solution u of the IVP (4.1) with $u_0 \in L^2$ is bounded, at least for t in a finite-time interval.

Theorem 4.1. *Let $\nu > 0$. Assume that $u_0 \in L^2$ and u is the corresponding solution of the IVP (4.1). Then there exists a $T^* > 0$ such that $\|u(\cdot,t)\|_{L^2}$ is bounded for any $t \in [0, T^*)$. In fact, u satisfies*

$$\|u(\cdot,t)\|_{L^2} \leq M_0(t) \equiv \frac{\|u_0\|_{L^2}}{(1 - 216\nu^{-3}\|u_0\|_{L^2}^4\, t)^{1/4}}$$

and

$$\int_0^t \|u_x(\cdot,\tau)\|_{L^2}^2 d\tau \leq N_0(t) \equiv \nu^{-1}\, \|u_0\|_{L^2}^2 + 108\nu^{-4}\, t\, M_0^6(t)$$

for

$$t < T^* = \frac{\nu^3}{216\, \|u_0\|_{L^2}^4}. \tag{4.2}$$

Remark. We do not know at this moment if $\|u(\cdot,t)\|_{L^2}$ remains bounded for $t \geq T^*$, but increasing the viscosity ν or decreasing $\|u_0\|_{L^2}$ lengthens T^*, according to (4.2).

Proof of Theorem 4.1. Multiplying the first equation in (4.1) by $\bar{u}$ (the conjugate of u) and its conjugate by u, adding the results and then integrating over $\mathbb{T}$, we obtain

$$\frac{d}{dt}\int |u|^2 dx + 2\nu \int |u_x|^2 dx = -4\int \mathcal{R}(u_x)\, |u|^2 dx. \tag{4.3}$$

Applying the basic inequality

$$\|u(\cdot,t)\|_{L^\infty} \leq \sqrt{2}\, \|u(\cdot,t)\|_{L^2}^{\frac{1}{2}}\, \|u_x(\cdot,t)\|_{L^2}^{\frac{1}{2}}$$

and Young's inequality to the term on the right-hand side of (4.3) (denoted by K), we obtain

$$|K| \leq 4\sqrt{2}\, \|u(\cdot,t)\|_{L^2}^{\frac{3}{2}}\, \|u_x(\cdot,t)\|_{L^2}^{\frac{3}{2}} \leq \nu \|u_x(\cdot,t)\|_{L^2}^2 + 108\nu^{-3}\|u(\cdot,t)\|_{L^2}^6.$$

Inserting the above estimate in (4.3),

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2 + \nu\|u_x(\cdot,t)\|_{L^2}^2 \leq 108\nu^{-3}\|u(\cdot,t)\|_{L^2}^6. \tag{4.4}$$

If we let $Y(t) = \|u(\cdot,t)\|_{L^2}$, then the above differential inequality implies

$$\frac{d}{dt} Y(t) \le \frac{54}{\nu^3}\, Y^5(t).$$

It then follows that

$$\|u(\cdot,t)\|_{L^2} \le M_0(t) \equiv \frac{\|u_0\|_{L^2}}{(1 - 216\nu^{-3}\|u_0\|_{L^2}^4\, t)^{1/4}},$$

which is valid for

$$t < T^* = \frac{\nu^3}{216\, \|u_0\|_{L^2}^4}.$$

Integrating (4.4) over $[0,t]$ with $t < T^*$, we obtain

$$\int_0^t \|u_x(\cdot,\tau)\|_{L^2}^2 d\tau \le N_0(t) \equiv \nu^{-1}\, \|u_0\|_{L^2}^2 + 108\nu^{-4}\, t\, M_0^6(t).$$

This completes the proof of Theorem 4.1.

4.2. **Bounding $\|u(\cdot,t)\|_{H^k}$ by $\|u(\cdot,t)\|_{L^2}$.** Using an induction argument, we show that any H^k-norm of u with $k \ge 1$ can be bounded in terms of its L^2-norm.

Theorem 4.2. *Let $\nu > 0$, $k \ge 1$ and $T > 0$. Let $u_0 \in H^k$ and u be the corresponding solution of the IVP (4.1). Assume the L^2-norm of u is bounded over $[0,T]$, i.e., for some M_0 depending only on T*

$$\|u(\cdot,t)\|_{L^2} \le M_0, \quad \textit{for } t \le T. \tag{4.5}$$

Then, there exists a C_k depending only on ν, T, M_0 and $\|u_0\|_{H^k}$ such that

$$\|u(\cdot,t)\|_{H^k} \le C_k, \quad \textit{for } t \le T. \tag{4.6}$$

Remark. It can be seen from the previous theorem that the assumption (4.5) on the L^2-norm implies

$$\int_0^t \|u_x(\cdot,\tau)\|_{L^2}^2 d\tau \le N_0, \quad \text{for } t \le T,$$

where N_0 depends only on ν, T, M_0 and $\|u_{0x}\|_{L^2}$.

Proof. We start with an estimate for $\|u_x(\cdot,t)\|_{L^2}$. Taking ∂_x of (4.1), we have

$$u_{xt} + 2u_x^2 + 2u\, u_{xx} - \nu\, u_{xxx} + u_{xxxx} = 0.$$

From this equation, we obtain

$$\frac{d}{dt}\int |u_x|^2 dx + 2\nu \int |u_{xx}|^2 dx = -4\int \mathcal{R}(u_x)|u_x|^2 dx - 4\int \mathcal{R}(u\,\bar{u}_x\, u_{xx}) dx. \tag{4.7}$$

Inserting the following estimates in (4.7)

$$4\int \mathcal{R}(u_x)|u_x|^2 dx \le 4\sqrt{2}\, \|u_x\|_{L^2}^{\frac{5}{2}}\, \|u_{xx}\|_{L^2}^{\frac{1}{2}} \le \frac{\nu}{2}\, \|u_{xx}\|_{L^2}^2 + 3\left(\frac{8}{\nu}\right)^{\frac{1}{3}} \|u_x\|_{L^2}^{\frac{10}{3}},$$

$$4\int \mathcal{R}(u\,\bar{u}_x\, u_{xx}) dx \le 4\sqrt{2}\, \|u\|_{L^2}^{\frac{1}{2}}\, \|u_x\|_{L^2}^{\frac{3}{2}}\, \|u_{xx}\|_{L^2} \le \frac{\nu}{2}\, \|u_{xx}\|_{L^2}^2 + \frac{16}{\nu}\, \|u\|_{L^2}\, \|u_x\|_{L^2}^3,$$

we obtain

$$\frac{d}{dt}\|u_x\|_{L^2}^2 + \nu\|u_{xx}(\cdot,t)\|_{L^2}^2 \le 3\left(\frac{8}{\nu}\right)^{\frac{1}{3}} \|u_x\|_{L^2}^{\frac{10}{3}} + \frac{16}{\nu}\, \|u\|_{L^2}\, \|u_x\|_{L^2}^3. \tag{4.8}$$

In particular, it implies that

$$\frac{d}{dt}\|u_x\|_{L^2} \le \frac{3}{2}\left(\frac{8}{\nu}\right)^{\frac{1}{3}} \|u_x\|_{L^2}^{\frac{7}{3}} + \frac{8}{\nu}\,\|u\|_{L^2}\,\|u_x\|_{L^2}^2.$$

Integrating with respect to t, we have

$$\|u_x(\cdot,t)\|_{L^2} \le \|u_{0x}\|_{L^2}$$

$$+\frac{3}{2}\left(\frac{8}{\nu}\right)^{\frac{1}{3}} \int_0^t \|u_x(\cdot,\tau)\|_{L^2}^{\frac{7}{3}}\,d\tau + \frac{8}{\nu}\int_0^t \|u(\cdot,\tau)\|_{L^2}\,\|u_x(\cdot,\tau)\|_{L^2}^2\,d\tau \qquad (4.9)$$

Using the assumption on the finiteness of L^2-norm of u, i.e.

$$\|u(\cdot,t)\|_{L^2} \le M_0 \quad \text{and} \quad \int_0^t \|u_x(\cdot,\tau)\|_{L^2}^2 d\tau \le N_0 \quad \text{for } t \le T,$$

we obtain, after taking the maximum of (4.9) for $t \in [0,T]$,

$$Z \le \|u_{0x}\|_{L^2} + \frac{3}{2}\left(\frac{8}{\nu}\right)^{\frac{1}{3}} N_0 Z^{\frac{1}{3}} + \frac{\nu}{8} M_0\, N_0,$$

where

$$Z = \max_{0\le t\le T} \|u_x(\cdot,t)\|_{L^2}.$$

We then apply Lemma 2.1 to this inequality and obtain

$$Z = \max_{0\le t\le T} \|u_x(\cdot,t)\|_{L^2} \le M_1$$

with

$$M_1 = \max\left\{6\sqrt{\frac{6}{\nu}}\,N_0^{\frac{3}{2}}, 2\|u_{0x}\|_{L^2} + \frac{\nu}{4}\,M_0\,N_0\right\}.$$

Integrating (4.8) over $[0,t]$, we obtain

$$\int_0^t \|u_{xx}(\cdot,\tau)\|_{L^2}^2 d\tau \le N_1$$

with

$$N_1 = \nu^{-1}\,\|u_{0x}\|_{L^2}^2 + 6\nu^{-4/3} N_0\, M_1^{4/3}(t) + 16\,\nu^{-2} M_0\, N_0\, M_1.$$

This completes the proof for the case $k = 1$.

We now turn to the case $k = 2$ and start with the basic identity

$$\frac{d}{dt}\int |u_{xx}|^2 dx + 2\nu \int |u_{xxx}|^2 dx = -12\int \mathcal{R}(u_x)|u_{xx}|^2 dx - 4\int \mathcal{R}(u\bar{u}_{xx}\,u_{xxx})dx.$$

Using the following estimates for the terms on the right hand side

$$-12\int \mathcal{R}(u_x)|u_{xx}|^2 dx \le 12\sqrt{2}\|u_x\|_{L^2}^{\frac{1}{2}}\,\|u_{xx}\|_{L^2}^{\frac{5}{2}}$$

and

$$-4\int \mathcal{R}(u\bar{u}_{xx}\,u_{xxx})dx \le 4\sqrt{2}\|u\|_{L^2}^{\frac{1}{2}}\,\|u_x\|_{L^2}^{\frac{1}{2}}\,\|u_{xx}\|_{L^2}\,\|u_{xxx}\|_{L^2}$$

$$\le \nu\|u_{xxx}\|_{L^2}^2 + \frac{8}{\nu}\|u\|_{L^2}\,\|u_x\|_{L^2}\|u_{xx}\|_{L^2}^2,$$

we have

$$\frac{d}{dt}\int |u_{xx}|^2 dx + \nu\int |u_{xxx}|^2 dx \le 12\sqrt{2}\|u_x\|_{L^2}^{\frac{1}{2}}\,\|u_{xx}\|_{L^2}^{\frac{5}{2}} + \frac{8}{\nu}\|u\|_{L^2}\,\|u_x\|_{L^2}\|u_{xx}\|_{L^2}^2.$$

This, in particular, implies

$$\frac{d}{dt}\|u_{xx}\|_{L^2} \le 6\sqrt{2}\|u_x\|_{L^2}^{\frac{1}{2}}\|u_{xx}\|_{L^2}^{\frac{3}{2}} + \frac{4}{\nu}\|u\|_{L^2}\|u_x\|_{L^2}\,\|u_{xx}\|_{L^2}. \tag{4.10}$$

Applying Young's inequality to the right hand side of (4.10), we obtain

$$\frac{d}{dt}\|u_{xx}\|_{L^2} \le \|u_{xx}\|_{L^2}^2 + 18\,\|u_x\|_{L^2}^2 + \left(\frac{4}{\nu}\right)^2 \|u\|_{L^2}^2\,\|u_x\|_{L^2}^2.$$

Integrating with respect to t gives

$$\|u_{xx}(\cdot,t)\|_{L^2} \le M_2 \equiv \|u_{0xx}\|_{L^2} + N_1 + 18N_0 + \left(\frac{4}{\nu}\right)^2 M_0 N_0.$$

This completes the proof for the case $k = 2$.

The general case is proved by induction on k. Assume that (4.6) is valid for all $k \le m$ with $m \ge 2$. That is, for $k = 1, 2, \cdots, m$, there exist two numbers M_k and N_k satisfying

$$\|u_{(k)}(\cdot,t)\|_{L^2} \le M_k \quad \text{and} \quad \int_0^t \|u_{(k+1)}(\cdot,\tau)\|_{L^2}^2 d\tau \le N_k, \quad t \le T.$$

We show that (4.6) also holds for $k = m+1$. Multiplying the KdV-Burgers equation by $(-1)^{m+1}\bar{u}_{(2m+2)}$, its conjugate by $(-1)^{m+1}u_{(2m+2)}$, adding the results and then integrating over $\mathbb{T}$, we obtain

$$\frac{d}{dt}\int |u_{(m+1)}|^2 dx + 2\nu \int |u_{(m+2)}|^2 dx = -4(m+2)\int \mathcal{R}(u_x)|u_{(m+1)}|^2 dx$$
$$-4\int \mathcal{R}(u\bar{u}_{(m+1)}\,u_{(m+2)})\,dx - 2\sum_{j=2}^{m} \binom{m+2}{j} \int \mathcal{R}(u_{(j)}u_{(m+2-j)}\bar{u}_{m+1})dx.$$

For notational convenience, we label the terms on the right hand side as I, II and III. Using the induction hypothesis, we can bound them as follows.

$$\begin{aligned}
|I| &\le 4\sqrt{2}(m+1)\|u_x\|_{L^2}^{\frac{1}{2}}\,\|u_{xx}\|_{L^2}^{\frac{1}{2}}\,\|u_{(m+1)}\|_{L^2}^2 \\
&\le 4\sqrt{2}(m+1)M_1^{\frac{1}{2}}\,M_2^{\frac{1}{2}}\,\|u_{(m+1)}\|_{L^2}^2 \qquad , \\
|II| &\le 4\sqrt{2}\|u\|_{L^2}^{\frac{1}{2}}\,\|u_x\|_{L^2}^{\frac{1}{2}}\,\|u_{(m+1)}\|_{L^2}\,\|u_{(m+2)}\|_{L^2} \\
&\le \nu\|u_{(m+2)}\|_{L^2}^2 + \frac{8}{\nu}\|u\|_{L^2}\|u_x\|_{L^2}\|u_{(m+1)}\|_{L^2}^2 \\
&\le \nu\|u_{(m+2)}\|_{L^2}^2 + \frac{8}{\nu}M_0\,M_1\,\|u_{(m+1)}\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
|III| &\le \left(2\sum_{j=2}^{m}\binom{m+2}{j}\|u_{(j)}\,u_{(m+2-j)}\|_{L^2}\right)\|u_{(m+1)}\|_{L^2} \\
&\le C(M_2, M_3, \cdots, M_k)\,\|u_{(m+1)}\|_{L^2} \\
&\le \frac{1}{4}\,C(M_2, M_3, \cdots, M_k) + \|u_{(m+1)}\|_{L^2}^2,
\end{aligned}$$

where C is a constant depending on $M_2, M_3, \cdots, M_m$ only. Collecting these estimates, we obtain

$$\frac{d}{dt}\int |u_{(m+1)}|^2 dx + \nu\int |u_{(m+2)}|^2 dx$$
$$\le C(\nu, M_0, M_1, M_2)\,\|u_{(m+1)}\|_{L^2}^2 + C(M_2, M_3, \cdots, M_m),$$

where C's are constants depending on the specified quantities. Integrating with respect to t completes the proof. This also concludes the proof of Theorem 4.2.

5. **Numerical results.** In this section, we present the results of systematic numerical computations for solutions of the complex KdV equation and the complex KdV-Burgers equation. The numerical scheme is a fully discrete Galerkin method, but we have also used the spectral method for the purpose of comparison. The results from both methods are extremely close, so we will focus on the Galerkin method and omit details concerning the spectral method. This section is divided into three subsections. Subsection 5.1 introduces the numerical method and explains the convergence rates. Subsection 5.2 details the numerical tests on the complex KdV equation. We computed its solutions corresponding to the initial data of the form $ae^{2\pi ix}$ for a ranging from 2 to 5. The results suggest that the solutions blow up for $a > 4.5$ and are bounded for $a \leq 4$. Subsection 5.3 presents the results of numerical experiments on the complex KdV-Burgers equation with $u(x,0) = ae^{2\pi ix}$ for $a = 6$ and a range of ν. These computations indicate that the solutions of the complex KdV-Burgers equation are bounded for relatively large ν. Throughout this section, the spatial period 2π will be normalized to 1. First, we briefly review the numerical method.

5.1. **Numerical method.** The numerical scheme is one of the fully discrete Galerkin methods and has been developed by Bona, Dougalis, Karakashian and McKinney [4], and Bona and Yuan [14]. It consists of two major parts. The first part involves a standard semi-discretization in a space of the N-dimensional vector space of 1-periodic smooth splines with uniform mesh length $h = 1/N$. The second part approximates the systems of ordinary differential equations resulted from the first part by the q-stage Gauss-Legendre family, a class of implicit Runge-Kutta methods of collocation type. These methods possess good accuracy and stability properties.

The convergence theory for the scheme can be found in [4] and [6]. Throughout this paper, we use 2-stage Gauss-Legendre method and cubic splines. It can be shown (see [4] and [6]) that, provided u is smooth enough in a time interval $[0,T]$ and the temporal step size and the mesh length satisfy some condition, there is a unique numerical solution U^n satisfying

$$\max_{0\leq n\leq T/k} \|U^n - u(\cdot, nk)\|_{L^2} \leq C(k^4 + h^r) \quad \text{for } nk \leq T$$

where r is the order of the splines used in the spatial discretization (i.e., r =the degree of the spline +1). We checked the convergence rate on an explicit example. More precisely, we compared the analytic traveling-wave solution

$$u(x,t) = \frac{ae^{2\pi i(x+t)}}{\left(1 - \frac{a}{6}\, e^{2\pi i(x+t)}\right)^2}$$

of the complex KdV equation

$$u_t + 2u\, u_x + \frac{1}{4\pi^2}\, u_{xxx} = 0$$

with its solution computed using the scheme outlined above. In our tests, we set $a = 1.0$ and used cubic splines (i.e., r=4). Spatial error and temporal error are analyzed in Table 3 and Table 4, respectively. First, we fixed the temporal step

$k = 0.0001$ and varied the mesh length h from 1/96 to 1/768. The results of the corresponding errors

$$E(t) = \|U^n - u(\cdot,t)\|_{L^2}, \quad n = t/k$$

for $t = 0.1, 0.5$, and 1.0 are presented in Table 3. It is clear that the convergence rate is $O(h^4)$. In Table 4, we fixed $h = 1/480$ and varied k from 1/640 to 1/1440. The corresponding errors $E(t)$ at $t = 0.1, 0.5$, and 1.0 are recorded in Table 4, which shows that the convergence rate is $O(k^4)$.

Table 3. $k^{-1} = 10000$ and $E(t)$ at $t = 0.1$, 0.5, and 1.0

h^{-1}	k^{-1}	E(0.1)	rate	E(0.5)	rate	E(1.0)	rate
96	10000	1.739(-7)	4.03	1.739(-7)	4.03	1.739(-7)	4.03
144	10000	3.398(-8)	4.01	3.398(-8)	4.01	3.398(-8)	4.01
192	10000	1.071(-8)	4.01	1.071(-8)	4.01	1.071(-8)	4.01
256	10000	3.381(-9)	4.00	3.381(-9)	4.00	3.381(-9)	4.00
320	10000	1.384(-9)	4.00	1.384(-9)	4.00	1.384 (-9)	3.99
512	10000	2.109(-10)	4.00	2.112(-10)	3.94	2.120(-10)	3.78
768	10000	4.168(-11)		4.268(-11)		4.572(-11)	

Table 4. $h = 1/480$ and $E(t)$ at $t = 0.1$, 0.5, and 1.0

k^{-1}	E(0.1)	rate	E(0.5)	rate	E(1.0)	rate
640	2.2460(-9)	4.17	2.2393(-9)	4.18	2.4531(-9)	4.41
720	1.3740(-9)	4.15	1.3694(-9)	3.97	1.4587(-9)	4.3
960	4.1665(-10)	4.01	4.3752(-10)	4.1	4.2230(-10)	3.97
1440	8.207(-11)		8.2986(-11)		8.45816(-11)	

5.2. **Solutions of the complex KdV equation with $|a_1(0)| < 6$.** As we have mentioned above, the spatial period 2π will be normalized to 1 in this subsection. The aim here is to determine whether solutions of the complex KdV equation remains finite for all time if $|a_1(0)| < 6$. We recall that $a_k(0)$ stands for the coefficient of the kth-term in the series representation of the initial data u_0. In subsection 3.2, we concluded that there is no regular, global in time solution of the complex KdV equation in the form (3.2) if $|a_1(0)| \geq 6$, but it is not clear if the solution with $|a_1(0)| < 6$ is regular for all time.

In this subsection, we compute numerical solutions of the IVP

$$\begin{cases} u_t + 2u\, u_x + \frac{1}{4\pi^2} u_{xxx} = 0, & x \in [0,1], \\ u(x,0) = a\, e^{2\pi i x}, & x \in [0,1] \end{cases} \tag{5.1}$$

with $a < 6$. Notice that solutions of (5.1) are not traveling waves. Even though $a_k(0) = 0$ for $k \geq 2$, the corresponding coefficient $a_k(t)$ of the series solution is not zero. Our numerical experiments involve a's ranging from 2 to 5. For each a, we plot both the real part (denoted f) and the imaginary part (denoted g) of the corresponding solution at various times.

We start with an explanation for the formation of the initial peaks and sharpness in the graphs of f and g. We consider the real part f and the analysis for the imaginary part is similar. According to (5.1), f satisfies the equation

$$\begin{cases} f_t + 2f\, f_x - 2g\, g_x + \frac{1}{4\pi^2} f_{xxx} = 0, & x \in [0,1], \\ f(x,0) = a\, \cos(2\pi x), & x \in [0,1]. \end{cases}$$

When t is near 0, $f(x,t)$ is close to $a\, \cos(2\pi x)$ and $g(x,t)$ is close to $a\, \sin(2\pi x)$ and the evolution of f is roughly governed by

$$f_t = -2\pi a\, \sin(2\pi\, x)\Big(4\, a\, \cos(2\pi x) - 1\Big).$$

When $a = 4$, $f_t > 0$ and f increases for $x \in (0, 0.24) \cup (0.5, 0.74)$, but $f_t < 0$ and decreases for $x \in (0.24, 0.5) \cup (0.74, 1)$. That is, the sharpness forms around $x = 0.24$ and a peak appears in the interval $(0, 0.24)$. This explains the behavior of f plotted in Figure 1. The initial behavior of g is plotted in the second box of Figure 1. A similar analysis can be employed to explain plots corresponding to other a's.

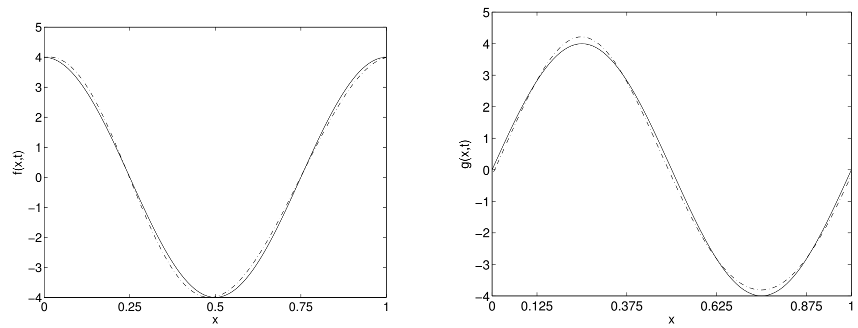

FIGURE 1. Graphs of f and g at $t = 0$ (solid curves) and $t = 0.001$ (dashed curves) with $a = 4, \nu = 0$, $h = 0.01$ and $k = 0.0001$

Numerical solutions of (5.1) with a ranging from 2 to 5 are computed. One of our goals has been to find the critical amplitude a^* such that solutions of (5.1) with $a > a^*$ will blow up, and exist globally if $a < a^*$. We started with $a = 2$. In Figure 2, the graphs of f (solid curves) and g (dashed curves) are plotted. We clearly observe that they are regular for all time. Furthermore, f and g are seen to be periodic in the temporal variable t with period 1. This is consistent with the theoretical result of Proposition 3.1.

In Figure 3, we plot f and g associated with $a = 5$ at times t=0.008, 0.016, 0.024, and 0.032. The plots clearly show that both f and g quickly lose their shape and their peaks become unbounded. By $t^* = 0.032$, the numerical stability sets in and the numerical solution blows up. To reinforce the fact that the solution does become infinite by $t^* = 0.032$, we refined both spatial and temporal scales. In Figure 4, the plots depict f and g after the refinement. They are almost the same plots as in Figure 3 before the solution blows up.

We then computed numerical solutions of (5.1) with $a = 4$ and $a = 4.5$. The plots in Figure 5 strongly indicate, for $a = 4.5$, that both f and g blows up. When t is near 0.06, the instability again sets in and f and g become unbounded. For $a = 4$, we plotted in Figure 6 the graphs of f and g for t from 0.08 to 1 (a whole

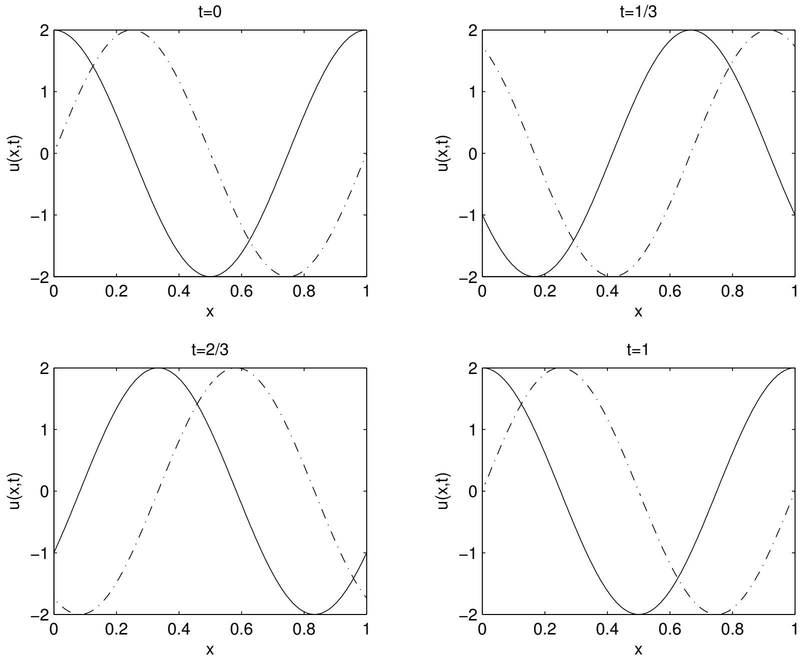

FIGURE 2. $a = 2, \nu = 0$, $h = 0.002$ and $k = 0.0001$.

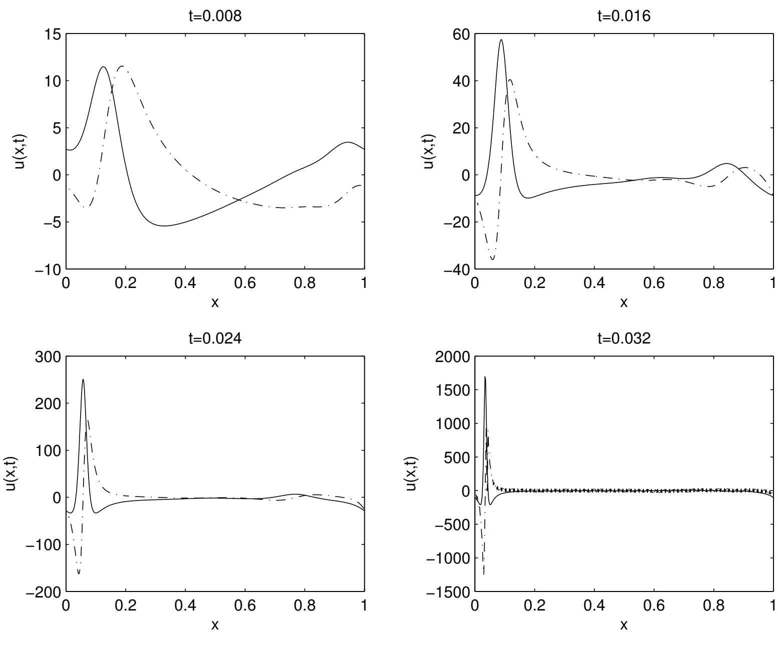

FIGURE 3. $a = 5, \nu = 0$, $h = 0.002$ and $k = 10^{-6}$.

temporal period). The difference is that there is no numerical instability and f and g remains bounded for the whole temporal period.

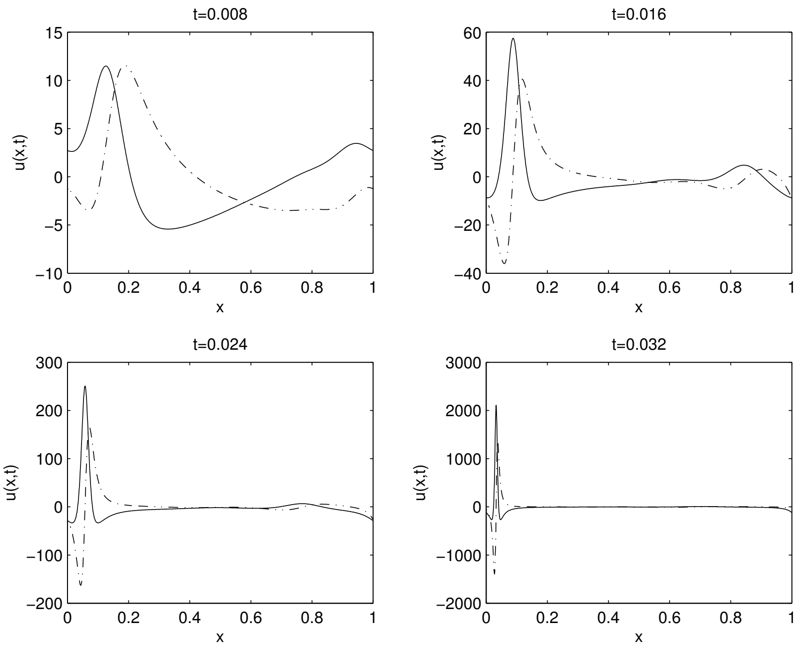

FIGURE 4. $a = 5, \nu = 0$, $h = 0.001$ and $k = 10^{-7}$.

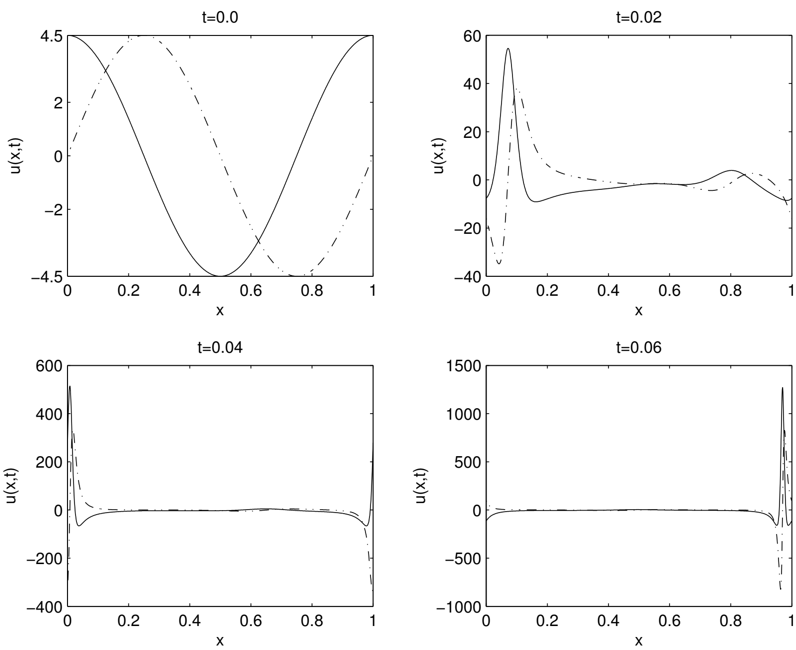

FIGURE 5. $a = 4.5, \nu = 0$, $h = 0.001$ and $k = 10^{-7}$.

5.3. **Numerical results for the complex KdV-Burgers equation.** In this subsection, we compute solutions of the IVP for the KdV-Burgers equation

$$\begin{cases} u_t + 2u\, u_x + \frac{1}{4\pi^2} u_{xxx} - \nu u_{xx} = 0, & x \in [0,1], \\ u(x,0) = a\, e^{2\pi i x}, & x \in [0,1] \end{cases} \tag{5.2}$$

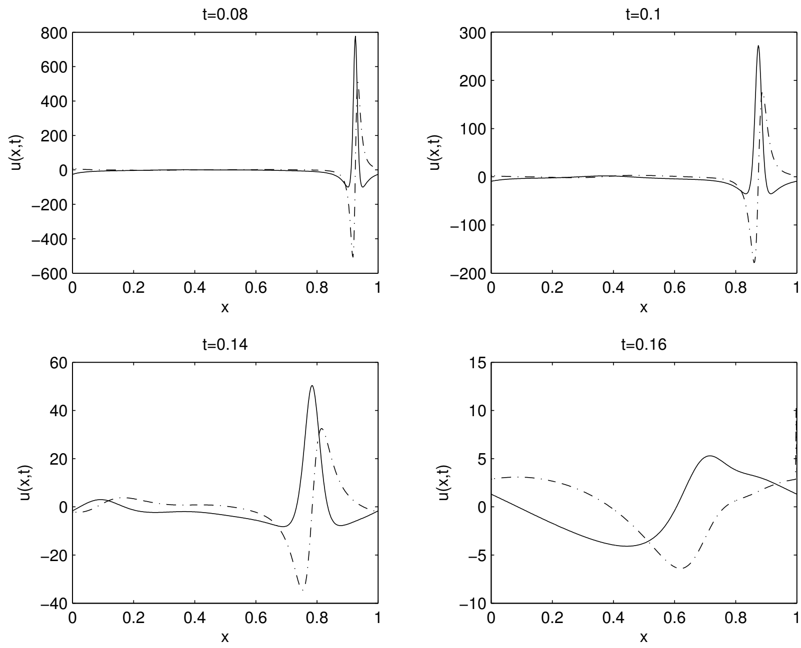

FIGURE 6. $a = 4, \nu = 0$, $h = 0.002$ and $k = 10^{-6}$.

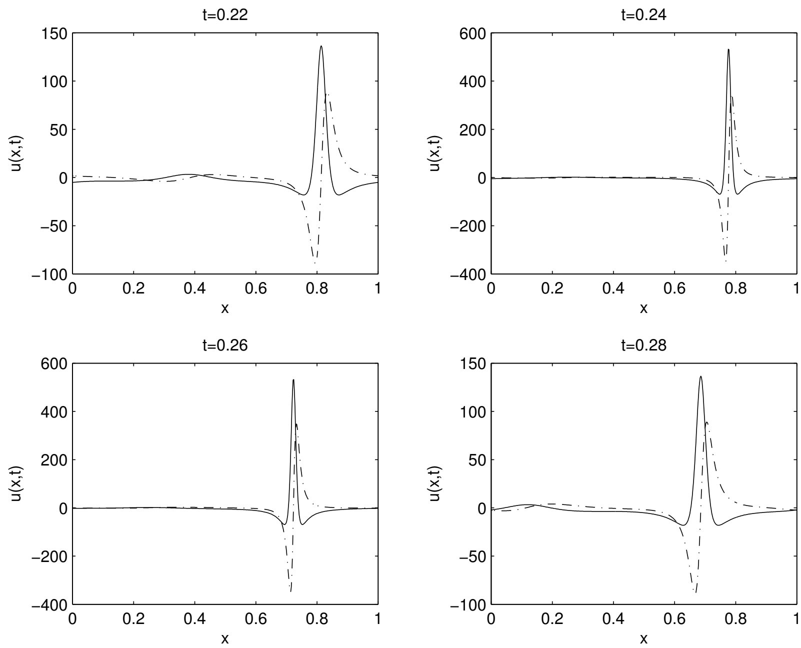

FIGURE 6. Continued. $a = 4, \nu = 0$, $h = 0.002$ and $k = 10^{-6}$.

for $a = 6$ and a range of $\nu > 0$. The goal here is to investigate numerically how the regularity of solutions to (5.2) is modified by the the Burgers-type dissipation.

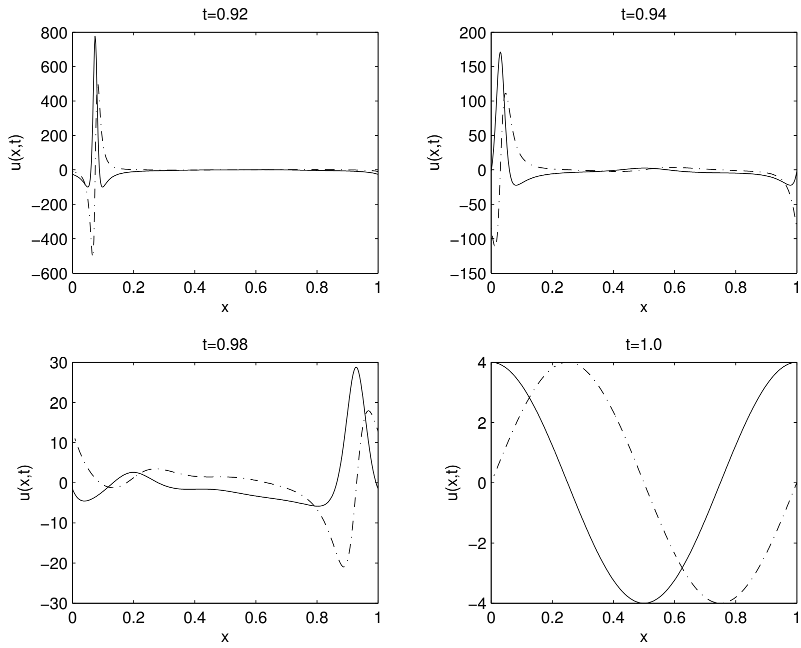

FIGURE 6. Continued. $a = 4, \nu = 0$, $h = 0.002$ and $k = 10^{-6}$

The theoretical results in Section 4 assess that the L^2-norm of any solution is finite, at least for $t \in [0, T^*)$ with T^* depending on ν and the L^2-norm of u_0. Furthermore, any H^k-norm is bounded as long as the L^2-norm is bounded. Therefore, our attention in this subsection will be focused on any possible singularity in u itself.

First, we set $a = 6$ and $\nu = 0.5$ and compute solutions of (5.2). The graphs of f and g are plotted in Figure 7. As we have seen in Subsection 3.2, the solution of the corresponding non-dissipative equation becomes infinite since $|a_1(0)| = a \geq 6$. What we see here in Figure 7 is different. Both the peaks of f and g are bounded and there is no indication of finite-time singularity. Thus, for large ν, solutions of the IVP (5.2) are regular and global in time.

The next two sets of experiments are designed to find the critical ν^* such that solutions of (5.2) with $a = 6$ will form singularity in a finite time if $\nu < \nu^*$, and will exist globally in time if $\nu > \nu^*$. We recorded two nearby values $\nu = 0.09$ and $\nu = 0.095$. Solutions of (5.2) blow up if $\nu \leq 0.09$ and exist globally if $\nu \geq 0.095$. The graphs of f and g corresponding to $\nu = 0.09$ are plotted in Figure 8. As the graphs show, both the peaks of f and g increase very rapidly and they reach 4300 and 3900, respectively when $t = 0.04$. They then become unbounded. In Figure 9, we plot the graphs of the solution of (5.2) with $\nu = 0.095$. Although the peaks of f and g still reach around 2500 and 1900, respectively, they start to decrease at about $t = 0.05$. When $t = 0.1$, the peaks reduce to below 110 and 75, respectively. This is a strong indication that solutions of (5.2) with $\nu \geq 0.095$ are global.

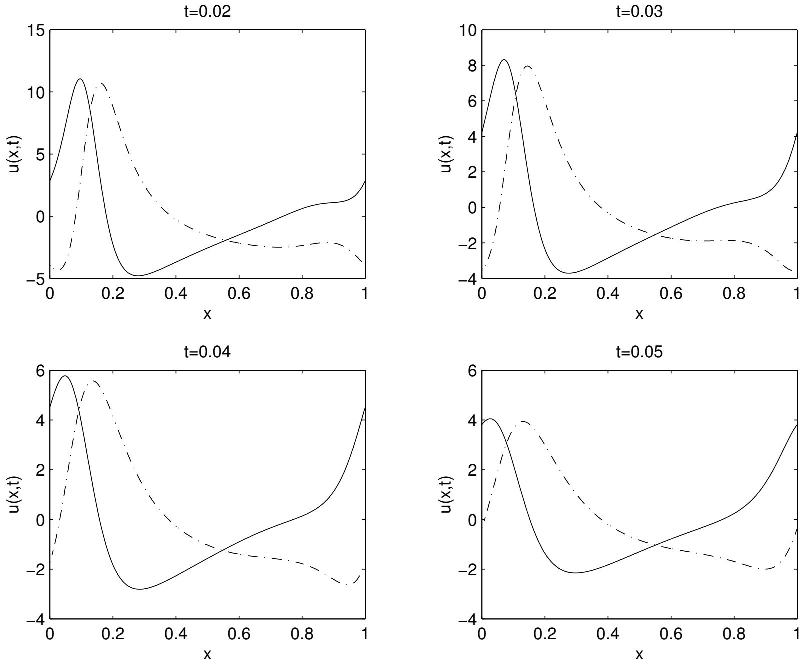

FIGURE 7. $a = 6$, $\nu = 0.5$, $h = 0.002$ and $k = 10^{-7}$.

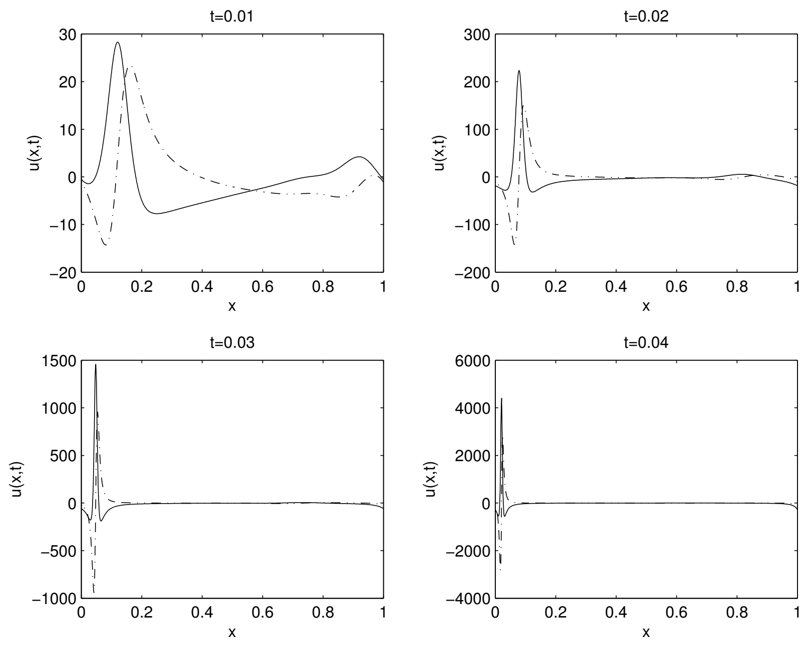

FIGURE 8. $a = 6, \nu = 0.09$, $h = 0.001$ and $k = 10^{-7}$.

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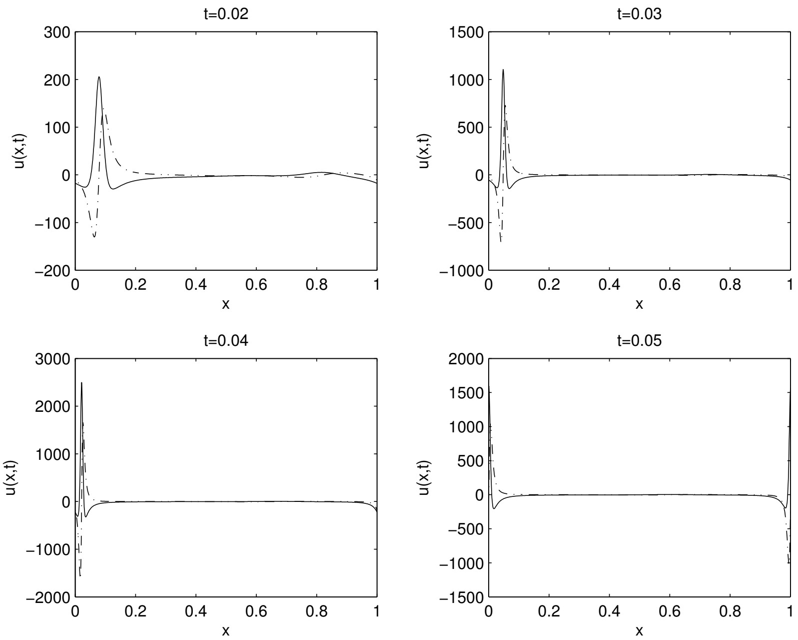

FIGURE 9. $a = 6, \nu = 0.095$, $h = 0.001$ and $k = 10^{-7}$

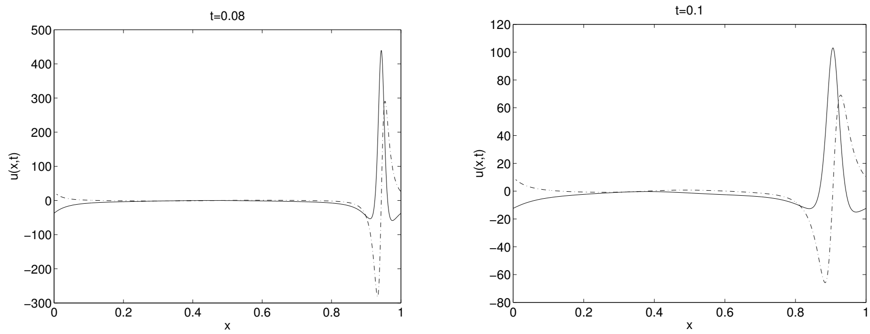

FIGURE 9. Continued. $a = 6, \nu = 0.095$, $h = 0.001$ and $k = 10^{-7}$.

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