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BEHAVIOR OF SOLUTIONS OF 2D QUASI-GEOSTROPHIC EQUATIONS*

PETER CONSTANTIN† AND JIAHONG WU‡

Abstract. We study solutions to the 2D quasi-geostrophic (QGS) equation

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^{\alpha}\theta = f$$

and prove global existence and uniqueness of smooth solutions if $\alpha \in (\frac{1}{2}, 1]$; weak solutions also exist globally but are proven to be unique only in the class of strong solutions. Detailed aspects of large time approximation by the linear QGS equation are obtained.

Key words. quasi-geostrophic equation, existence, uniqueness, large time approximation

AMS subject classifications. 76U05, 35Q35

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1. Introduction.

This paper is concerned with the 2D surface quasi-geostrophic (QGS) equation

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^{\alpha}\theta = f,$$

where $\alpha \in [0, 1]$, $\kappa > 0$, and $\theta = \theta(x, t)$ is a real scalar function of two space variables x and a time variable t. The velocity $u = (u_1, u_2)$ is incompressible and determined from θ by a stream function ψ:

$$(u_1, u_2) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right), \tag{1.1}$$

and the stream function ψ satisfies

$$(-\Delta)^{\frac{1}{2}}\psi = -\theta. \tag{1.2}$$

The nonlocal operator $(-\Delta)^{\beta}$ ($\beta \geq 0$) is defined through the Fourier transform

$$\widehat{(-\Delta)^{\beta}} f(\xi) = |\xi|^{2\beta} \widehat{f}(\xi),$$

where $\widehat{f}$ is the Fourier transform of f [11]. For notational convenience, we write Λ for $(-\Delta)^{\frac{1}{2}}$.

The variable θ in the 2D QGS equation represents the potential temperature, u is the fluid velocity, and the stream function ψ can be identified with the pressure. When the fractional power $\alpha = 1/2$, the equation, derived from the more general quasi-geostrophic models (see pages 345–368 and 653–670 of [7]), describes the evolution

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†Department of Mathematics, The University of Chicago, Chicago, IL 60637 (const@cs.uchicago.edu). The work of this author was partially supported by NSF/DMS grant 9207080.

‡School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540. Current address: Department of Mathematics, University of Texas, Austin, TX 78712-1082 (jiahong@math.utexas.edu). The work of this author while at the Institute for Advanced Study was partially supported by NSF/DMS grant 9304580.

of the temperature on the 2D boundary of a rapidly rotating half-space with small Rossby and Ekman numbers. Dimensionally, the 2D QGS equation with $\alpha = 1/2$ is the analogue of the 3D Navier–Stokes equations. The general fractional power α is considered here in order to observe the minimal power of Laplacian necessary in the analysis and thus make a comparison with the 3D Navier–Stokes equations [3], [6].

Recently, this equation has been intensively investigated because of both its mathematical importance and its potential for applications in meteorology and oceanography [4], [7], [5]. Mathematically, the behavior of solutions to the 2D QGS equation is strikingly similar to that of the potentially singular solutions to the 3D hydrodynamics equations. Despite exhibiting a number of similar features, the 2D QGS equation is considerably simpler than the 3D Euler or Navier–Stokes equations.

The smooth solution of the QGS equation is unique but, if $\kappa = 0$, it is known to exist only for a finite time [4]. On the other hand, weak solutions are global but their uniqueness is unknown [8]. Whether the smooth solution develops singularity in finite time and whether weak solutions are unique are fundamental mathematical issues related to the QGS equation. We show in section 2 that the solution remains smooth for all time for $\alpha \in (\frac{1}{2}, 1]$ and any weak solution must coincide with a more regular solution as long as such a strong solution exists.

Large time behavior of weak solutions is investigated in sections 3 and 4. In section 3, the L^2 decay rate of order $t^{-\frac{1}{2\alpha}}$ is obtained. For generic initial data, this rate is optimal. The solution θ of the nonlinear equation may be approximated by the solution Θ of the linear equation with a higher-order correction. An explicit form for the higher-order correction is attempted in section 4. A rate of order $t^{\frac{1}{2}-\frac{1}{\alpha}}$ is first obtained without any smoothness assumption. With the assumption that

$$\|\Lambda^{2-2\alpha+\delta}\theta(\cdot,t)\|_{L^2} \quad \sim \quad t^{-\epsilon},$$

the ratio and the difference are shown to behave as follows:

$$\frac{\|\theta(\cdot,t)\|_{L^2}}{\|\Theta(\cdot,t)\|_{L^2}} \sim 1 + O(t^{-\min\{\frac{1}{2\alpha},\epsilon\}}), \quad \|\theta(\cdot,t) - \Theta(\cdot,t)\|_{L^2} \sim t^{-\frac{1}{2\alpha}-\min\{\frac{1}{2\alpha},\epsilon\}},$$

which imply that the effect of the nonlinearity is felt only in the higher-order correction.

We conclude this introduction by mentioning the global existence result of weak solutions obtained in [8]. When not specified, the spatial domain can be either the whole $\mathbb{R}^2$ or the 2D torus $\mathbb{T}^2$.

PROPOSITION 1.1. *Let $T > 0$ be arbitrary. Then for every $\theta_0 \in L^2$ and $f \in L^2([0,T]; H^{-\alpha})$, there exists a weak solution of*

$$\partial_t\theta + u \cdot \nabla\theta + \kappa\Lambda^{2\alpha}\theta = f, \tag{1.3}$$

$$\theta\,|_{t=0} = \theta_0 \tag{1.4}$$

which satisfies

$$\theta \in L^\infty([0,T]; L^2) \cap L^2([0,T]; H^\alpha).$$

2. Global smooth solution and uniqueness. It is shown here that weak solutions of the QGS equation are globally smooth for $\alpha \in (\frac{1}{2}, 1]$ and "strong" solutions are unique. The spatial domain here is the 2D torus $\mathbb{T}^2$.

THEOREM 2.1. *Let* $\alpha \in (\frac{1}{2}, 1]$, $\beta \geq 0$, *and* $\beta + 2\alpha > 2$. *If* $\theta_0 \in H^\beta(\mathbb{T}^2)$ *and if, for* $T > 0$,

$$f \in L^2([0,T]; H^{\beta-\alpha}), \qquad \int_0^T \|f(\tau)\|_{L^q} d\tau < \infty,$$

where $q = \infty$ *for* $\beta \geq 1$ *and* $q = 2/(1-\beta)$ *for* $\beta < 1$, *then the solution* θ *of* (1.3) *and* (1.4) *obeys for all* $t \leq T$

$$\|\Lambda^\beta \theta(t)\|_{L^2} \leq C, \tag{2.1}$$

where C *is constant depending only on* T, $\|\theta_0\|_{H^\beta}$, $\|f\|_{L^2([0,T];H^{\beta-\alpha})}$, *and* $\int_0^T \|f(\tau)\|_{L^q} d\tau$.

Proof. We sketch the proof. Taking the scalar product of (1.3) with $\Lambda^{2\beta}\theta$

$$\frac{1}{2}\frac{d}{dt}\int |\Lambda^\beta \theta|^2 + \kappa \int |\Lambda^{\alpha+\beta}\theta|^2 = -\int (u\cdot\nabla\theta)\Lambda^{2\beta}\theta + \int \Lambda^{2\beta}\theta f$$

and using the estimates

$$\left|\int \Lambda^{2\beta}\theta f\right| \leq \frac{\kappa}{4}\|\Lambda^{\alpha+\beta}\theta\|_{L^2}^2 + \frac{1}{\kappa}\|\Lambda^{\beta-\alpha} f\|_{L^2}^2, \tag{2.2}$$

$$\left|\int (u\cdot\nabla\theta)\Lambda^{2\beta}\theta\right| \leq \frac{\kappa}{4}\|\theta\|_{H^{\alpha+\beta}}^2 + C(\kappa,\theta_0,f)\|\theta\|_{H^\beta}^2, \tag{2.3}$$

where $C(\kappa,\theta_0,f)$ is constant, we obtain (2.1) after applying Gronwall's inequality. The estimate (2.3) is obtained by using the calculus inequality (see page 61 of [8] and inequality (3.1.59) on page 74 of [12])

$$\|\Lambda^s(gh)\|_{L^2} \leq C(\|g\|_{L^q}\|\Lambda^s h\|_{L^p} + \|h\|_{L^q}\|\Lambda^s g\|_{L^p})$$

with $1/p + 1/q = 1/2$, $g = u$, $h = \theta$, $s = \beta + 1 - \alpha$, and the maximum principle

$$\|\theta\|_{L^q} \leq \|\theta_0\|_{L^q} + \int_0^t \|f(\tau)\|_{L^q} d\tau. \qquad \square$$

Although weak solutions may not be unique, there is at most one solution in the class of "strong" solutions.

THEOREM 2.2. *Assume that* $\alpha \in (\frac{1}{2}, 1]$, $T > 0$, p *and* q *satisfy*

$$p \geq 1, \quad q > 0, \qquad \frac{1}{p} + \frac{\alpha}{q} = \alpha - \frac{1}{2}; \tag{2.4}$$

then there is at most one solution θ *of the QGS equation with initial data* $\theta_0 \in L^2$ *such that*

$$\theta \in L^\infty([0,T]; L^2) \cap L^2([0,T]; H^\alpha), \tag{2.5}$$

$$\theta \in L^q([0,T]; L^p). \tag{2.6}$$

We make two remarks.

Remark 2.3. It is clear from the proof given below that we can assume that only one of the two solutions is "strong," i.e., in the class (2.5), (2.6), the other being only a weak solution.

Remark 2.4. By taking $\alpha = 1$, (2.6) with (2.4) reduces exactly to the regularity assumptions in obtaining uniqueness for weak solutions to the 3D Navier–Stokes equations (cf. Temam [13, p. 299]). Theorem 2.2 is a sort of generalization in the sense that it holds for a range of $\alpha \in (\frac{1}{2}, 1]$.

Proof of Theorem 2.2. The difference $\theta = \theta_A - \theta_B$ of two solutions θ_A and θ_B satisfies

$$\partial_t \theta + u \cdot \nabla \theta_A + u_B \cdot \nabla \theta + \kappa \Lambda^{2\alpha} \theta = 0 \tag{2.7}$$

in which $u = u_A - u_B$ with u_A and u_B being the velocities corresponding to θ_A and θ_B. We take the scalar product of (2.7) with $\psi = -\Lambda^{-1}\theta$ and use

$$\int_{\mathbb{T}^2} \psi u \cdot \nabla \theta_A = 0,$$

$$\left| \int_{\mathbb{T}^2} \theta u_B \cdot \nabla \psi \right| \leq \kappa \|\psi\|^2_{H^{\alpha+\frac{1}{2}}} + C(\kappa) \|\theta_B\|^{\frac{1}{1-\beta}}_{L^p} \|\psi\|^2_{H^{\frac{1}{2}}},$$

where $\beta = \frac{1}{\alpha}\left(\frac{1}{2} + \frac{1}{p}\right)$ and $C(\kappa) = C\kappa^{-\frac{\beta}{1-\beta}}$ (see page 32 of [8]). It then follows that

$$\frac{d}{dt}\|\psi\|^2_{H^{\frac{1}{2}}} \leq C(\kappa)\|\theta_B\|^{\frac{1}{1-\beta}}_{L^p} \|\psi\|^2_{H^{\frac{1}{2}}},$$

which implies that $\psi = 0$ and thus $\theta = 0$. $\square$

3. Large time behavior. The large time behavior of weak solutions is investigated in this section. We adapt well-known ideas of Amick, Bona, and Schonbek [1] and Schonbek [9], [10].

We first analyze the case when the force $f = 0$ and the result can be stated as follows.

THEOREM 3.1. *Let $\alpha \in (0, 1]$ and $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then there exists a weak solution θ of the 2D QGS equation*

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda^{2\alpha}\theta = 0, \qquad \theta\,|_{t=0} = \theta_0 \tag{3.1}$$

such that

$$\|\theta(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{2\alpha}}, \tag{3.2}$$

where C is a constant depending on L^1 and L^2 norms of θ_0.

Remark 3.2. For generic initial data, the rate obtained in Theorem 3.1 is optimal, as implied by Theorem 4.6 of section 4.

The proof of Theorem 3.1 consists of two major steps. The first step is a formal argument to show that (3.2) holds for smooth solutions. In the second step the formal argument is applied to a sequence of "retarded mollifications" [2] and we obtain Theorem 3.1 after passing to the limit. We will need a simple estimate.

LEMMA 3.3. *Assume that $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then θ satisfies the a priori estimate*

$$|\widehat{\theta}(\xi, t)| \leq \|\theta_0\|_{L^1} + |\xi| \int_0^t \|\theta(\tau)\|_{L^2}^2 d\tau.$$

Proof. We have from (3.1)

$$\partial_t \widehat{\theta} + |\xi|^{2\alpha} \widehat{\theta} = -\widehat{u \cdot \nabla \theta}. \tag{$*$}$$

Since $\nabla \cdot u = 0$,

$$|\widehat{u \cdot \nabla \theta}| \leq |\xi| \|\theta(t)\|_{L^2}^2.$$

After integrating $(*)$, we obtain

$$|\widehat{\theta}(\xi, t)| \leq |\widehat{\theta_0}(\xi)| + |\xi| \int_0^t \|\theta(\tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^1} + |\xi| \|\theta_0\|_{L^2}^2 t. \qquad \square$$

Proof of Theorem 3.1. Taking the scalar product of (3.1) with θ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\theta|^2 + \int_{\mathbb{R}^n} (\Lambda^\alpha \theta)^2 = 0.$$

Using Plancherel's theorem,

$$\frac{d}{dt} \int_{\mathbb{R}^2} |\widehat{\theta}|^2 + 2 \int_{\mathbb{R}^2} |\xi|^{2\alpha} |\widehat{\theta}|^2 = 0.$$

For the second term

$$\int_{\mathbb{R}^2} |\xi|^{2\alpha} |\widehat{\theta}|^2 \geq \int_{B(t)^c} |\xi|^{2\alpha} |\widehat{\theta}|^2 \geq g^{2\alpha}(t) \int_{B(t)^c} |\widehat{\theta}|^2$$

$$= g^{2\alpha}(t) \int_{\mathbb{R}^2} |\widehat{\theta}|^2 - g^{2\alpha}(t) \int_{B(t)} |\widehat{\theta}|^2,$$

where $g \in C([0, \infty); \mathbb{R}^+)$ remains to be determined and $B(t)^c$ is the complement of $B(t)$ with

$$B(t) = \left\{ \xi \in \mathbb{R}^2 : \quad |\xi| < g(t) \right\}.$$

By Lemma 3.3, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} |\widehat{\theta}|^2 + 2 g^{2\alpha}(t) \int_{\mathbb{R}^2} |\widehat{\theta}|^2$$

$$\leq 2\pi g^{2\alpha}(t) \int_0^{g(t)} \left[\|\theta_0\|_{L^1} + r \int_0^t \|\theta(\tau)\|_{L^2}^2 d\tau \right]^2 r dr. \tag{3.3}$$

By integrating (3.3), we have

$$e^{2\int_0^t g^{2\alpha}(\tau)d\tau} \int_{\mathbb{R}^2} |\widehat{\theta}|^2 \le \|\theta_0\|_{L^2}^2$$

$$+ \int_0^t e^{2\int_0^s g^{2\alpha}(\tau)d\tau} \left[C_1 g^{2\alpha+2}(s) + C_2 s g^{2\alpha+4}(s) \int_0^s \|\theta(\tau)\|_{L^2}^4 d\tau \right] ds, \tag{3.4}$$

where $C_1 = 2\pi\|\theta_0\|_{L^1}^2$ and $C_2 = \pi$.

To obtain a basic estimate, we take $g^{2\alpha}(t) = \left(\frac{1}{2} + \frac{1}{2\alpha}\right)[(e+t)\ln(e+t)]^{-1}$ and thus $e^{2\int_0^t g^{2\alpha}(\tau)d\tau} = [\ln(e+t)]^{(1+\frac{1}{\alpha})}$. We then obtain from (3.4)

$$\|\theta\|_{L^2}^2 \le C[\ln(e+t)]^{-1-\frac{1}{\alpha}}.$$

To obtain the sharp decay result, we take $g^{2\alpha}(t) = \frac{1}{2\alpha(t+1)}$ and thus $e^{2\int_0^t g^{2\alpha}(\tau)d\tau} = (1+t)^{\frac{1}{\alpha}}$. From (3.4),

$$\|\theta(t)\|_{L^2}^2 \le C(t+1)^{-\frac{1}{\alpha}} + C(t+1)^{(1-\frac{2}{\alpha})} \int_0^t \|\theta(s)\|_{L^2}^2 [\ln(e+s)]^{-1-\frac{1}{\alpha}} ds.$$

Using Gronwall's inequality and the fact that $\alpha \le 1$,

$$\|\theta(t)\|_{L^2}^2 \le C(1+t)^{-\frac{1}{\alpha}}, \tag{3.5}$$

where the constant C depends on the L^1 and L^2 norms of θ_0. We note here that (3.5) is actually obtained by first taking $g^{2\alpha}(t) = \left(\frac{1}{2\alpha} - \epsilon\right)\frac{1}{1+t}$ and then passing to the limit as $\epsilon \to 0$. This completes the formal argument step.

Next we construct a sequence of retarded mollifications θ_n and carry over the formal arguments to θ_n. We will present here only the main ideas. We approximate the QGS equation by a sequence of equations

$$\partial_t \theta_n + u_n \cdot \nabla \theta_n + \Lambda^{2\alpha}\theta_n = 0, \tag{3.6}$$

where $\delta_n \to 0$ and $u_n = S_{\delta_n}(\theta_n)$ is obtained from θ_n by

$$S_{\delta_n}(\theta_n) = \int_0^\infty \phi(\tau)\mathcal{R}^\perp \theta_n(t - \delta_n \tau)d\tau.$$

We denote here $\mathcal{R}^\perp = (-\partial_{x_2}\Lambda, \partial_{x_1}\Lambda)$ as the Riesz transform. The smooth function ϕ is nonnegative with compact support in $[1,2]$ and $\int_0^\infty \phi(t)dt = 1$. For each n, (3.6) is a linear equation since the values of $u_n(t)$ depend only on the values of θ_n in $[t - 2\delta_n, t - \delta_n]$.

Without giving details, we point out that θ_n converges to a weak solution θ strongly in L^2 for almost every t. Hence

$$\|\theta(t)\|_{L^2} \le \|\theta_n(t) - \theta(t)\|_{L^2} + \|\theta_n(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2\alpha}},$$

where C is a constant depending only on the L^1 and L^2 norms of θ_0. This completes the proof of Theorem 3.1. □

We now consider the case when the force f is not zero.

THEOREM 3.4. *Let* $\alpha \in (0,1]$ *and* $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. *Assume that* $f \in L^1([0,\infty); L^2)$, *satisfying*

$$\|f(\cdot,t)\|_{L^2} \le C(1+t)^{-\frac{1}{\alpha}-1}, \qquad |\widehat{f(\xi,t)}| \le C|\xi|^\alpha \tag{3.7}$$

for some constant C. *Then there is a weak solution of the QGS equation*

$$\partial_t \theta + u \cdot \theta + \Lambda^{2\alpha}\theta = f$$

such that

$$\|\theta(\cdot,t)\|_{L^2} \le C(1+t)^{-\frac{1}{2\alpha}}. \tag{3.8}$$

Proof. The arguments of Theorem 3.1 work here and we will point out only the difference. It is easy to see that

$$\|\theta(t)\|_{L^2} \le \|\theta_0\|_{L^2} + \int_0^t \|f(\tau)\|_{L^2} d\tau \le C$$

by energy estimates. When the force f is present, the estimate for $\widehat{\theta}$ is given by

$$|\widehat{\theta}(\xi,t)| \le e^{-|\xi|^{2\alpha}t}|\widehat{\theta_0}| + \int_0^t e^{-|\xi|^{2\alpha}(t-\tau)} \left[\widehat{f} + |\xi|\|\theta\|_{L^2}^2\right] d\tau.$$

Then the procedures of the proof of Theorem 3.1 can be repeated and the assumptions (3.7) are sufficient in establishing (3.8). ∎

4. Large time approximation. In this section we intend to understand the higher-order correction in the large time approximation of the solution θ to the nonlinear equation by the solution Θ to the linear equation. The approach is to study the difference and the ratio

$$\|\theta(\cdot,t) - \Theta(\cdot,t)\|_{L^2}, \qquad \frac{\|\theta(\cdot,t)\|_{L^2}}{\|\Theta(\cdot,t)\|_{L^2}}.$$

We start with some estimates for the linear equation. The solution of the linear equation on $\mathbb{R}^n$

$$\partial_t \theta + \Lambda^{2\alpha}\theta = 0, \qquad \theta\,|_{t=0} = \theta_0 \tag{4.1}$$

is given by

$$\Theta(t) = k_t^\alpha * \theta_0, \tag{4.2}$$

where the kernel k_t^α is defined by its Fourier transform

$$\widehat{k}_t^\alpha(\xi) = e^{-|\xi|^{2\alpha}t}. \tag{4.3}$$

PROPOSITION 4.1. *Assume that* $\alpha > 0$ *and the initial data* $\theta_0 \in L^1(\mathbb{R}^n)$. *Then we have*

$$\lim_{t\to\infty} t^{\frac{n}{2\alpha}} \|\Theta(\cdot,t)\|_{L^2}^2 = A(n,\alpha)\left[\int_{\mathbb{R}^n} \theta_0(x)dx\right]^2, \tag{4.4}$$

$$\lim_{t\to\infty} t^{\frac{n+2}{2\alpha}} \|\nabla\Theta(\cdot,t)\|_{L^2}^2 = B(n,\alpha)\left[\int_{\mathbb{R}^n} \theta_0(x)dx\right]^2, \tag{4.5}$$

where the constants $A(n,\alpha) = \int_{\mathbb{R}^n} e^{-2|\eta|^2} d\eta$ *and* $B(n,\alpha) = \int_{\mathbb{R}^n} |\eta|^2 e^{-2|\eta|^2} d\eta$.

Especially for $n = 2$, *the* L^2 *decay rates of* Θ *and* $\nabla\Theta$ *are* $t^{-\frac{1}{2\alpha}}$ *and* $t^{-\frac{1}{\alpha}}$, *respectively.*

Proof. We first prove (4.4). By Plancherel's theorem,

$$\lim_{t\to\infty} t^{\frac{n}{2\alpha}} \|\Theta(\cdot,t)\|_{L^2}^2 = \lim_{t\to\infty} t^{\frac{n}{2\alpha}} \|\widehat{\Theta}(\cdot,t)\|_{L^2}^2$$

$$= \lim_{t\to\infty} t^{\frac{n}{2\alpha}} \int_{\mathbb{R}^n} e^{-2|\xi|^{2\alpha}t} |\widehat{\theta}_0|^2(\xi) d\xi = \lim_{t\to\infty} \int_{\mathbb{R}^n} e^{-2|\eta|^2} |\widehat{\theta}_0|^2(\eta t^{-\frac{1}{2\alpha}}) d\eta.$$

Since for any $t \in [0,\infty)$

$$\int_{\mathbb{R}^n} e^{-2|\eta|^2} |\widehat{\theta}_0|^2(\eta t^{-\frac{1}{2\alpha}}) d\eta$$

$$\le \|\widehat{\theta}_0\|_{L^\infty}^2 \int_{\mathbb{R}^n} e^{-2|\eta|^2} d\eta \le A(n,\alpha) \|\theta_0\|_{L^1}^2,$$

we can apply the dominated convergence theorem, which leads to (4.4).

The proof of (4.5) is similar to that of (4.4). We have

$$\lim_{t\to\infty} t^{\frac{n+2}{\alpha}} \|\nabla\Theta(\cdot,t)\|_{L^2}^2 = \lim_{t\to\infty} t^{\frac{n+2}{\alpha}} \int_{\mathbb{R}^n} |\xi|^2 e^{-2|\xi|^{2\alpha}t} |\widehat{\theta}_0|^2(\xi) d\xi$$

$$= \lim_{t\to\infty} \int_{\mathbb{R}^n} |\eta|^2 e^{-2|\eta|^2} |\widehat{\theta}_0|^2(\eta t^{-\frac{1}{2\alpha}}) d\eta = B(n,\alpha) \left[\int_{\mathbb{R}^n} \theta_0(x) dx\right]^2. \qquad \square$$

PROPOSITION 4.2. *Let* $\alpha \in (0,1]$ *and* $\theta_0 \in L^2(\mathbb{R}^2)$. *Then the solution* Θ *of* (4.1) *satisfies*

$$\|\nabla\Theta(t)\|_{L^\infty(\mathbb{R}^2)} \le C t^{-\frac{1}{\alpha}},$$

where the constant C *depends only on the* L^2 *norm of* θ_0.

Proof. We have by (4.2) and (4.3)

$$\|\nabla\Theta\|_{L^\infty} \le \int_{\mathbb{R}^2} |\xi||\widehat{\Theta}(\xi)| d\xi = \int_{\mathbb{R}^2} |\xi| e^{-|\xi|^{2\alpha}t} |\widehat{\theta}_0(\xi)| d\xi$$

$$\le \|\theta_0\|_{L^2} \left(\int_{\mathbb{R}^2} |\xi|^2 e^{-2|\xi|^{2\alpha}t} d\xi\right)^{\frac{1}{2}} \le C \left(\int_0^\infty r^3 e^{-2r^{2\alpha}t} dr\right)^{\frac{1}{2}} \le C t^{-\frac{1}{\alpha}},$$

where the constant C depends only on the L^2 norm of θ_0. $\square$

THEOREM 4.3. *Let* $\alpha \in (0,1]$ *and* $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. *Then the difference* $\theta - \Theta$ *between a weak solution* θ *of the QGS equation and the solution* Θ *of the linear QGS equation with the data* θ_0 *satisfies*

$$\|\theta(t) - \Theta(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{\frac{1}{2}-\frac{1}{\alpha}},$$

where the constant C *depends only on the* L^1 *and* L^2 *norms of* θ_0.

Remark 4.4. By comparing the rates in Theorems 3.1 and 4.3, we see that $\theta - \Theta$ decays faster than θ does for large time for $\alpha < 1$.

Proof. We will present only a formal argument. The justification process can be done similarly as in the proof of Theorem 3.1. The difference $w = \theta - \Theta$ satisfies

$$\partial_t w + \Lambda^{2\alpha} w = -u \cdot \nabla\theta. \tag{4.6}$$

Taking the scalar product of (4.6) with w and using the fact that

$$\int_{\mathbb{R}^2} (u \cdot \nabla\theta)\theta dx = 0,$$

we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} |w|^2 + 2\int_{\mathbb{R}^2} |\Lambda^\alpha w|^2 = \int_{\mathbb{R}^2} \Theta(u \cdot \nabla\theta)dx.$$

Using the results of Proposition 4.2 and Theorem 3.1, we bound the right-hand term by

$$\left|\int_{\mathbb{R}^2} \Theta(u \cdot \nabla\theta)dx\right| \le \|\nabla\Theta\|_{L^\infty}\|\theta\|_{L^2}^2 \le C(1+t)^{-\frac{2}{\alpha}}.$$

As in the proof of Theorem 3.1,

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\widehat{w}|^2 + 2g^{2\alpha}(t)\int_{\mathbb{R}^2} |\widehat{w}|^2 \le 2g^{2\alpha}(t)\int_{|\xi|\le g(t)} |\widehat{w}|^2 + C(1+t)^{-\frac{2}{\alpha}}, \tag{4.7}$$

where $g(t)$ remains to be decided.

We need an estimate of $\widehat{w}$, which can be obtained by taking the Fourier transform of (4.6) and proceeding as in Lemma 3.3. By Theorem 3.1 and noticing $\alpha \le 1$,

$$|w(\xi,t)| \le |\xi|\int_0^t \|\theta(\tau)\|_{L^2}^2 d\tau \le |\xi|\int_0^t (1+\tau)^{-\frac{1}{\alpha}} d\tau \le C|\xi|.$$

Taking $g^{2\alpha} = \frac{\beta}{2(1+t)}$, we obtain, by integrating (4.7),

$$(1+t)^\beta \int_{\mathbb{R}^2} |\widehat{w}|^2 \le C\left[\int_0^t (1+\tau)^{\beta-\frac{2}{\alpha}} d\tau + \int_0^t (1+\tau)^\beta g^4(\tau) d\tau\right].$$

Therefore,

$$\|w\|_{L^2}^2 \le C(1+t)^{1-\frac{2}{\alpha}}.$$

This completes the proof of Theorem 4.3. $\square$

We can consider lower bounds for the decay of θ with the aid of Theorem 4.3. It is easy to see that Θ can decay exponentially fast. For example, if $\widehat{\theta}_0 = 0$ for $|\xi| \le \gamma$, then

$$\|\Theta(t)\|_{L^2}^2 = \int e^{-2|\xi|^{2\alpha}t}|\widehat{\theta}_0(\xi)|^2 \le \|\theta_0\|_{L^2}^2 e^{-2\gamma^2 t}.$$

However, for those θ_0 satisfying

$$|\widehat{\theta}_0(\xi)| \ge \lambda \qquad \text{for} \qquad |\xi| \le \gamma, \tag{4.8}$$

we have the following.

PROPOSITION 4.5. *Let $\alpha \in (0,1]$ and $\theta_0 \in L^2(\mathbb{R}^2)$ satisfy (4.8). Then if Θ is a solution of the linear QGS equation,*

$$\|\Theta(t)\|_{L^2(\mathbb{R}^2)} \geq C(1+t)^{-\frac{1}{2\alpha}},$$

where C is a constant depending only on λ, γ, and the L^2 norm of θ_0.

As a corollary of Theorem 4.3 and Proposition 4.5, we have the following.

THEOREM 4.6. *Let $\alpha \in (0,1]$ and $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ satisfy (4.8). Then for a weak solution θ of the QGS equation with data θ_0,*

$$\|\theta(\cdot,t)\|_{L^2(\mathbb{R}^2)} \geq C(1+t)^{-\frac{1}{2\alpha}},$$

where C depends on λ, γ, and the L^1 and L^2 norms of θ_0.

The following theorem reveals more detailed aspects of the higher-order correction.

THEOREM 4.7. *Let $\alpha \in (\frac{1}{2}, 1]$ and $\delta > 0$ such that $2\alpha - 1 - \delta \geq 0$. Assume that θ is a weak solution of the 2D QGS equation*

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda^{2\alpha} \theta = 0$$

with initial data $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and that satisfies

$$\|\Lambda^{2-2\alpha+\delta} \theta(\cdot,t)\|_{L^2} \leq C t^{-\epsilon} \tag{4.9}$$

for some constant C and $\epsilon > 0$. Let Θ be the solution of the linear equation with the same initial data θ_0. Then

$$\frac{\|\theta(\cdot,t)\|_{L^2}}{\|\Theta(\cdot,t)\|_{L^2}} = 1 + O(t^{-\min\{\frac{1}{2\alpha}, \epsilon\}}), \tag{4.10}$$

$$t^{\frac{1}{2\alpha} + \min\{\frac{1}{2\alpha}, \epsilon\}} \|\theta(\cdot,t) - \Theta(\cdot,t)\|_{L^2} = O(1). \tag{4.11}$$

Proof. By taking the Fourier transform of the equation for θ

$$\partial_t \theta + \Lambda^{2\alpha} \theta = -u \cdot \nabla \theta,$$

we obtain

$$\widehat{\theta}(\xi,t) = e^{-|\xi|^{2\alpha} t} \widehat{\theta}_0(\xi) - \int_0^t e^{-|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla \theta}(\xi,s) ds.$$

Then the ratio

$$\frac{\|\theta(\cdot,t)\|_{L^2}^2}{\|\Theta(\cdot,t)\|_{L^2}^2} = \frac{\|\widehat{\theta}(\cdot,t)\|_{L^2}^2}{\|\widehat{\Theta}(\cdot,t)\|_{L^2}^2} = 1 + 2\mathcal{J}(t) + \mathcal{J}^2(t),$$

where $\mathcal{J}$ is given by

$$\mathcal{J}(t) = \frac{\int_{\mathbb{R}^2} \left| \int_0^t e^{-|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla \theta}(\xi,s) ds \right|^2 d\xi}{\|\Theta\|_{L^2}^2}.$$

To prove (4.10), it suffices to show that

$$\mathcal{J}(t) = O(t^{-\min\{\frac{1}{2\alpha}, \epsilon\}}).$$

The difference $w = \theta - \Theta$ satisfies

$$\partial_t w + \Lambda^{2\alpha} w = -u \cdot \nabla \theta.$$

Since $w(x,0) = \theta(x,0) - \Theta(x,0) = 0$,

$$\widehat{w}(\xi,t) = -\int_0^t e^{-|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla\theta}(\xi,s)ds,$$

$$\|w(\cdot,t)\|_{L^2}^2 = \|\widehat{w}(\cdot,t)\|_{L^2}^2 = \int_{\mathbb{R}^2} \left| \int_0^t e^{-|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla\theta}(\xi,s)ds \right|^2 d\xi.$$

Thus, to prove (4.10) and (4.11), we need only to estimate the integral

$$I \equiv \int_{\mathbb{R}^2} \left| \int_0^t e^{-|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla\theta}(\xi,s)ds \right|^2 d\xi.$$

To this end, we divide the integral I into the following two parts:

$$II = \int_{\mathbb{R}^2} \left| \int_0^{t/2} \cdots \right|^2 d\xi \quad \text{and} \quad III = \int_{\mathbb{R}^2} \left| \int_{t/2}^t \cdots \right|^2 d\xi.$$

Since $\nabla \cdot u = 0$, $\widehat{u \cdot \nabla\theta} = i\xi \cdot \widehat{u\theta}$ and we obtain, by setting $\eta = t^{\frac{1}{2\alpha}}\xi$,

$$II = t^{-\frac{2}{\alpha}} \int_{\mathbb{R}^2} e^{-2|\eta|^{2\alpha}} \left| \int_0^{t/2} e^{\frac{s}{t}|\eta|^2} \eta \cdot \widehat{u\theta}(\eta t^{-\frac{1}{2\alpha}}, s)ds \right|^2 d\eta.$$

Since

$$\|\widehat{u\theta}(\cdot,s)\|_{L^\infty} \le \|u\theta(\cdot,s)\|_{L^1} \le \|u(\cdot,s)\|_{L^2}\|\theta(\cdot,s)\|_{L^2} \le C\|\theta(\cdot,s)\|_{L^2}^2,$$

we have the bound

$$II \le Ct^{-\frac{2}{\alpha}} \int_{\mathbb{R}^2} |\eta|^2 e^{-\frac{7}{4}|\eta|^2} \left[\int_0^\infty \|\theta(\cdot,s)\|_{L^2}^2 ds \right]^2 d\eta$$

$$\le Ct^{-\frac{2}{\alpha}} \left(\int_{\mathbb{R}^2} |\eta|^2 e^{-\frac{7}{4}|\eta|^2} d\eta \right) \|\theta\|_{L^2([0,\infty);L^2)}^4.$$

The estimate of III seems tricky. Intuitively, the idea is to split the whole derivative ∇ into two fractional parts $\Lambda^{2\alpha-1-\delta}$ and $\Lambda^{2-2\alpha+\delta}$:

$$III = \int_{\mathbb{R}^2} \left| \int_{t/2}^t e^{-|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla\theta}(\xi,s)ds \right|^2 d\xi$$

$$\le \int_{\mathbb{R}^2} e^{-2|\xi|^{2\alpha}t} |\xi|^{2(2\alpha-1-\delta)} \sup_{t/2 \le s \le t} |\widehat{u \cdot \Lambda^{2-2\alpha+\delta}\theta}|^2(\xi,s) \left[\int_{t/2}^t e^{s|\xi|^{2\alpha}} ds \right]^2 d\xi.$$

Using the assumption (4.9), we obtain

$$|\widehat{u \cdot \Lambda^{2-2\alpha+\delta}\theta}|(\xi, s) \le \|\widehat{u \cdot \Lambda^{2-2\alpha+\delta}\theta}(\cdot, s)\|_{L^\infty}$$

$$\le \|(u \cdot \Lambda^{2-2\alpha+\delta}\theta)(\cdot, s)\|_{L^1} \le C\|u(\cdot, s)\|_{L^2}\|\Lambda^{2-2\alpha+\delta}\theta(\cdot, s)\|_{L^2} \le Cs^{-\frac{1}{2\alpha}-\epsilon},$$

where C is a constant. Therefore

$$III \le Ct^{-\frac{1}{\alpha}-2\epsilon}\int_{\mathbb{R}^2}|\xi|^{-2-\delta}(1-e^{-\frac{1}{2}|\xi|^{2\alpha}t})^2 d\xi \le Ct^{-\frac{1}{\alpha}-2\epsilon}.$$

Combining the estimates for II and III, we conclude that

$$I \equiv \int_{\mathbb{R}^2}\left|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}\widehat{u \cdot \nabla\theta}(\xi, s)ds\right|^2 d\xi \le Ct^{-\frac{1}{\alpha}-\min\{\frac{1}{\alpha},2\epsilon\}},$$

and (4.10), (4.11) are therefore established. ∎

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