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High Reynolds number and high Weissenberg number Oldroyd-B model with dissipation

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To Matthias Hieber, with friendship, respect and admiration

Abstract. We give a small data global well-posedness result for an incompressible Oldroyd-B model with wave-number dissipation in the equation of stress tensor. The result is uniform in solvent Reynolds numbers and requires only fractional wave number-dependent dissipation $(-\Delta)^{\beta}$, $\beta \geq \frac{1}{2}$ in the added stress.

1. Introduction

A class of models of complex fluids is based on an equation for a solvent coupled with a kinetic description of particles suspended in it. In the case of dilute suspensions weakly confined by a Hookean spring potential, a rigorously established exact closure for the moments in the kinetic equation of this Navier–Stokes–Fokker–Planck system yields the Oldroyd-B system [21]. After non-dimensionalization, the coupled Oldroyd-B system is

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \frac{1}{Re}\Delta u = K\nabla \cdot \sigma, \\ \partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^* - \frac{1}{We}(\sigma - \mathbb{I}), \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where σ is the conformation tensor, $\sigma = \mathbb{E}(m \otimes m)$ with m the end-to-end vector in $\mathbb{R}^d$ and $\mathbb{E}$ the average with respect to the local distribution, u is the solvent velocity, p is the pressure, Re is the Reynolds number of the solvent, We is the Weissenberg number, $K = \frac{1}{\gamma Re We}$ and γ is the ratio of solvent viscosity to polymeric viscosity. In the limit of zero Reynolds number, the system (1.1) reduces further and it becomes a nonlinear evolution for σ

$$\partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^* - \frac{1}{We}(\sigma - \mathbb{I}) \tag{1.2}$$

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where u is obtained from σ by solving the Stokes system

$$-\Delta u + \nabla p = \frac{1}{\gamma We} \nabla \cdot \sigma, \qquad \nabla \cdot u = 0. \tag{1.3}$$

The system (1.2) with (1.3) is an example of an equation which might develop finite-time singularities for large data, even in $\mathbb{R}^2$. The forcing in the right-hand side of (1.3) or in the right-hand side of the momentum equation of (1.1) depends only on the added stress

$$\tau = \sigma - \mathbb{I}, \tag{1.4}$$

because any multiple of the identity matrix added to σ is balanced by a pressure, even if the factor is a function of space and time. For small added stress, it is known [7] that the system (1.2), (1.3) has global solutions. The problem of global existence of smooth solutions for large data is open and challenging. The large Weissenberg number problem is challenging both numerically and analytically. If we replace the damping term by a wave number-dependent dissipative term, we obtain an equation for the conformation stress

$$\partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^* - \eta P(D)(\sigma - \mathbb{I}) \tag{1.5}$$

with $P(D)$ being a dissipative differential operator and η a positive number. If a small diffusive term ($P(D) = -\Delta$ in (1.5)) is added to the equation for σ coupled with (1.3), then global existence of smooth solutions with arbitrary data has been established [8] in $d = 2$. For the small data problem, one can discuss a less stringent wave-number dependence and allow the solvent Reynolds number to be arbitrarily large.

In this paper, we consider an Oldroyd-B model

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nabla \cdot \tau, \quad x \in \mathbb{R}^d, \ \ t > 0, \\ \partial_t \tau + u \cdot \nabla \tau + \eta(-\Delta)^\beta \tau + Q(\tau, \nabla u) = D(u), \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x), \ \ \tau(0, x) = \tau_0(x), \end{cases} \tag{1.6}$$

where $0 \leqslant \beta \leqslant 1$ and $\eta > 0$ are real parameters, $u = u(x, t)$ represents the velocity field of the fluid, $p = p(x, t)$ the pressure and $\tau = \tau(x, t)$ the non-Newtonian added stress tensor (see (1.4)) (a d-by-d symmetric matrix). Here, $D(u)$ is the symmetric part of the velocity gradient

$$D(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^\top\big)$$

and the bilinear term Q is taken to be

$$Q(\tau, \nabla u) = \tau W(u) - W(u)\tau - b\big(D(u)\tau + \tau D(u)\big)$$

with $b \in [-1, 1]$ a constant and $W(u)$ the skew-symmetric part of the ∇u:

$$W(u) = \frac{1}{2}\big(\nabla u - (\nabla u)^\top\big).$$

The fractional Laplacian operator $(-\Delta)^\gamma$ is defined through the Fourier transform:

$$\widehat{(-\Delta)^\gamma f}(\xi) = |\xi|^{2\gamma} \widehat{f}(\xi).$$

For notational convenience, we also write $\Lambda = (-\Delta)^{\frac{1}{2}}$ denoting the Zygmund operator. Background information on the Oldroyd-B model can be found in many references (see, e.g., [2,26]).

Our main result is the small data global well-posedness of (1.6) with any $\frac{1}{2} \leqslant \beta \leqslant 1$. There is no damping mechanism in the equation of τ in (1.6): Strictly speaking, the Weissenberg number is infinite, but wave number-dependent dissipation is added. Whether or not (1.6) with $0 \leq \beta < \frac{1}{2}$ possesses small data global well-posedness remains an open problem.

Theorem 1.1. *Consider* (1.6) *with* $\frac{1}{2} \leqslant \beta \leqslant 1$. *Let* $d = 2, 3$ *and* $s > 1 + \frac{d}{2}$. *Assume* $(u_0, \tau_0) \in H^s(\mathbb{R}^d)$, $\nabla \cdot u_0 = 0$, *and* τ_0 *is symmetric. Then, there exists a small constant* $\varepsilon > 0$ *such that if*

$$\|u_0\|_{H^s} + \|\tau_0\|_{H^s} \leqslant \varepsilon,$$

then (1.6) *has a unique global solution* (u, b) *satisfying, for some constant* $C > 0$ *and all* $t > 0$,

$$\|u\|_{H^s} + \|\tau\|_{H^s} \leqslant C\varepsilon.$$

The requirement that $\beta \geq \frac{1}{2}$ appears to be sharp. When $\beta < \frac{1}{2}$, even the local well-posedness problem is open and the main difficulty is how to provide a suitable upper bound for the nonlinear term Q, due to the lack of sufficient dissipation.

The small data global well-posedness for an Oldroyd-B model without dissipation in the velocity equation has previously been examined by T. Elgindi and F. Rousset in the 2D case [11] and by T. Elgindi and J. Liu for the 3D case [12]. They focus on the following Oldroyd-B model without velocity dissipation:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \nabla \cdot \tau, \quad x \in \mathbb{R}^d, \ t > 0, \\ \tau_t + u \cdot \nabla \tau + Q(\tau, \nabla u) - \eta \Delta \tau + a\tau = D(u), \\ \nabla \cdot u = 0, \end{cases} \tag{1.7}$$

where $a > 0$ is a parameter. The small data global well-posedness result in [11] is for (1.7) with $d = 2$ and $a > 0$. The damping term plays a crucial role in the proof of their result and cannot be removed. It was used to form a damping term in the equation of a combined quantity. [12] examined (1.7) with $d = 3$ and $a > 0$ and obtained the small data global well-posedness for any sufficiently small data $(u_0, \tau_0) \in H^3$. The damping term $a\tau$ in (1.7) is also necessary for their result.

The velocity equation in (1.6) is a forced Euler equation. As it is known, the H^s-norm of a solution of the Euler equation may grow in time, even perhaps at a double exponential rate (see, e.g., [10,20,35]). The Oldroyd-B system discussed has a dissipative structure, and a main reason why Theorem 1.1 holds is a key observation on the linearized system of (1.6). Clearly, any solution (u, τ) of (1.6) also solves

$$\begin{cases} \partial_t u + \mathbb{P}(u \cdot \nabla u) = \mathbb{P}\nabla \cdot \tau, \quad x \in \mathbb{R}^d, \quad t > 0, \\ \partial_t \mathbb{P}\nabla \cdot \tau + \mathbb{P}\nabla \cdot (u \cdot \nabla\tau) + \eta(-\Delta)^\beta \mathbb{P}\nabla \cdot \tau + \mathbb{P}\nabla \cdot Q(\tau, \nabla u) = \frac{1}{2}\Delta u, \\ \nabla \cdot u = 0, \end{cases} \tag{1.8}$$

where $\mathbb{P}$ denotes the Leray projection onto divergence-free vector fields. The corresponding linearized system is given by

$$\begin{cases} \partial_t u = \mathbb{P}\nabla \cdot \tau, \\ \partial_t \mathbb{P}\nabla \cdot \tau + \eta(-\Delta)^\beta \mathbb{P}\nabla \cdot \tau = \frac{1}{2}\Delta u, \\ \nabla \cdot u = 0, \end{cases}$$

which can be easily reduced to a system of decoupled wave-type equations

$$\begin{cases} \partial_{tt} u + \eta(-\Delta)^\beta \partial_t u - \frac{1}{2}\Delta u = 0, \\ \partial_{tt}(\mathbb{P}\nabla \cdot \tau) + \eta(-\Delta)^\beta \partial_t(\mathbb{P}\nabla \cdot \tau) - \frac{1}{2}\Delta(\mathbb{P}\nabla \cdot \tau) = 0, \\ \nabla \cdot u = 0. \end{cases} \tag{1.9}$$

The structure in (1.9) reveals that there are both dissipative and dispersive effects on u in (1.6). We remark that the Oldroyd-B model with only velocity dissipation shares a similar structure and has been shown by Yi Zhu to possess a unique global small solution [33]. In order to prove the existence part of Theorem 1.1, we construct a suitable Lyapunov functional that incorporates these effects. We set the Lyapunov functional to be

$$L(t) = \|u(t)\|^2_{H^s(\mathbb{R}^d)} + \|\tau(t)\|^2_{H^s(\mathbb{R}^d)} + 2k(u(t), \nabla \cdot \tau(t))_{H^{s-\beta}(\mathbb{R}^d)},$$

where $(f, g)_{H^\sigma(\mathbb{R}^d)}$ denotes the inner product in $H^\sigma(\mathbb{R}^d)$. When the parameter $k > 0$ is sufficiently small and when $\frac{1}{2} \le \beta \le 1$, we are able to show that, for any $t \ge 0$,

$$\begin{aligned} E(t) := &\|u(t)\|^2_{H^s(\mathbb{R}^d)} + \|\tau(t)\|^2_{H^s(\mathbb{R}^d)} \\ &+2\int_0^t \left(\eta\|\Lambda^\beta \tau(t')\|^2_{H^s} + \frac{k}{2}\|\nabla u(t')\|^2_{H^{s-\beta}}\right) dt' \end{aligned} \tag{1.10}$$

obeys

$$E(t) \le E(0) + C\, E^{\frac{3}{2}}(t). \tag{1.11}$$

A bootstrap argument applied to (1.11) implies that if $E(0)$ is sufficiently small, namely

$$E(0) \leq \varepsilon$$

for some suitable $\varepsilon > 0$, then $E(t)$ is bounded uniformly for all time $t > 0$, or

$$E(t) \leq C\,\varepsilon,$$

which allows us to establish the global existence of solutions to (1.6). In order to prove the uniqueness, we distinguish between two cases: $\beta = 1$ and $\frac{1}{2} \leq \beta < 1$. When $\beta = 1$, the term $Q(\tau, \nabla u)$ can be bounded directly. When $\frac{1}{2} \leq \beta < 1$, one needs to make use of the wave structure to generate a dissipative term in the velocity field in order to deduct a suitable bound for $Q(\tau, \nabla u)$.

The second part of this paper rigorously assesses that the Oldroyd-B system in (1.6) is the vanishing viscosity limit of the Oldroyd-B system with kinematic dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p + \nu(-\Delta)^{\alpha} u = \nabla \cdot \tau, \quad x \in \mathbb{R}^d, \quad t > 0, \\ \partial_t \tau + u \cdot \nabla \tau + \eta(-\Delta)^{\beta} \tau + Q(\tau, \nabla u) = D(u), \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x), \quad \tau(0, x) = \tau_0(x), \end{cases} \tag{1.12}$$

where $\nu > 0$, $\eta > 0$, $0 \leq \alpha \leq 1$ and $\frac{1}{2} \leq \beta \leq 1$. First of all, (1.12) always possesses a unique global solution when the initial data are sufficiently small.

Theorem 1.2. *Consider* (1.12) *with*

$$\nu > 0, \quad \eta > 0, \quad \frac{1}{2} \leq \beta \leq 1 \quad \textit{and} \quad 0 \leq \alpha \leq \min\{1, 3\beta - 1\}.$$

Assume $(u_0, \tau_0) \in H^s(\mathbb{R}^d)$ *with* $s > 1 + \frac{d}{2}$. *There exists small number* $\varepsilon > 0$ *(independent of* ν*) such that if*

$$\|(u_0, \tau_0)\|_{H^s} \leq \varepsilon,$$

then (1.12) *has a unique global solution* $(u^{(\nu)}, \tau^{(\nu)})$ *satisfying*

$$\begin{aligned} &u^{(\nu)} \in C([0, \infty); H^s) \cap L^2(0, \infty; H^{s+\alpha}) \cap L^2(0, \infty; H^{s+1-\beta}); \\ &\tau^{(\nu)} \in C([0, \infty); H^s) \cap L^2(0, \infty; H^{s+\beta}). \end{aligned}$$

In addition, $(u^{(\nu)}, \tau^{(\nu)})$ *admits the following bound that is uniform in time and in* ν*:*

$$\|(u^{(\nu)}(t), \tau^{(\nu)}(t))\|_{H^s} \leq C\,\varepsilon, \tag{1.13}$$

where C *is independent of* t *and* ν*.*

In particular, Theorem 1.2 holds for the case when $\alpha = 1$ and $\beta = 1$, namely the standard Laplacian case. We emphasize that ε in Theorem 1.2 is independent of ν. In addition, the fact that the bound for the solution $(u^{(\nu)}, \tau^{(\nu)})$ in H^s is uniform in terms of ν plays a crucial role in the proof of the following vanishing viscosity limit. As $\nu \to 0$, (1.12) converges to (1.6) in the sense as stated in the following theorem.

Theorem 1.3. *Assume*

$$\nu > 0, \quad \eta > 0, \quad \frac{1}{2} \le \beta \le 1 \quad \text{and} \quad 0 \le \alpha \le \min\{1, 3\beta - 1\}.$$

Let $(u_0, \tau_0) \in H^s(\mathbb{R}^d)$ *with* $s > 1 + \frac{d}{2}$ *and* $s \ge 2\alpha + 2\beta - 1$. *Assume that the norm of* $(u_0, \tau_0) \in H^s$ *is sufficiently small, namely*

$$\|(u_0, \tau_0)\|_{H^s} \le \varepsilon$$

such that (1.6) *and* (1.12) *each have a unique global solution. Let* (u, τ) *and* $(u^{(\nu)}, \tau^{(\nu)})$ *be the solutions of* (1.6) *and* (1.12), *respectively. Then,*

$$\|(u^{(\nu)}(t), \tau^{(\nu)}(t)) - (u(t), \tau(t))\|_{L^2} \le C\,\nu, \tag{1.14}$$

where C *may depend on* t *and the initial data but is independent of* ν.

The parameter C in the inviscid limit estimate (1.14) may depend on time. It appears difficult to make C uniformly independent of time. As we can see from the proof of this theorem, one reason is the lack of the uniform time integrability on $\|\Lambda^{2\alpha} u(t)\|_{L^2}^2$ for α in the range specified here.

We remark that small data global solutions of (1.6) in critical homogeneous Besov spaces have also been obtained [28]. Due to its special features, the Oldroyd-B model has recently attracted considerable interests from the community of mathematical fluids. A rich array of results have been established on the well-posedness and closely related problems. Interested readers can consult some of the references listed here, see, e.g., [1,3–6,8,9,11–19,22–25,27–34]. This list is by no means exhaustive.

We finally mention that there are many other interesting problems on (1.6) to be examined. One problem is the precise large-time behavior of the solutions obtained in this paper. Due to the lack of dissipation in the velocity equation, this is not a trivial problem. It is difficult to show that the solution itself decays to zero just using the energy estimates presented in this paper. Even in the case of the linear heat equation, considerations of energy alone are not sufficient and one has to use the solution representation $u(t) = e^{\nu t \Delta} u_0$ to show that $\|u(t)\|_{L^2(\mathbb{R}^d)}$ decays to zero when $u_0 \in L^2(\mathbb{R}^d)$. If we impose more stringent condition on the initial data, a different method might lead to the convergence to zero in the Oldroyd-B model studied here.

2. Proof of Theorem 1.1

This section proves Theorem 1.1.

Proof. The proof is naturally divided into two parts. The first part is for the existence, while the second part is for the uniqueness.

To prove the global existence of solutions, it suffices to establish the energy inequality in (1.11) with $E(t)$ being defined in (1.10). The proof of (1.11) is via energy estimates. We need to separate the homogeneous part of the H^s-norm from the inhomogeneous part. Due to the equivalence of the norm $\|f\|_{H^s}$ with $\|f\|_{L^2} + \|\Lambda^s f\|_{L^2}$, we combine the L^2-part with the homogeneous $\dot{H}^s$-part. Dotting (1.6) by (u, τ) in L^2, integrating by parts and making use of $\nabla \cdot u = 0$, we find

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|\tau\|_{L^2}^2) + \eta\|\Lambda^\beta \tau\|_{L^2}^2 = -(Q(\tau, \nabla u), \tau), \tag{2.1}$$

where (f, g) denotes the inner product in $L^2(\mathbb{R}^2)$, and we used

$$\int_{\mathbb{R}^2} (u \cdot (\nabla \cdot \tau) + D(u) \cdot \tau)\,\mathrm{d}x = 0.$$

Applying Λ^s to (1.6) and dotting by $(\Lambda^s u, \Lambda^s \tau)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s \tau\|_{L^2}^2) + \eta\|\Lambda^{s+\beta}\tau\|_{L^2}^2 \\
&\quad = -(\Lambda^s(u \cdot \nabla u), \Lambda^s u) - (\Lambda^s(u \cdot \nabla \tau), \Lambda^s \tau) - (\Lambda^s Q(\tau, \nabla u), \Lambda^s \tau),
\end{aligned} \tag{2.2}$$

where we used

$$\int_{\mathbb{R}^2} (\Lambda^s u \cdot (\Lambda^s \nabla \cdot \tau) + \Lambda^s D(u) \cdot \Lambda^s \tau)\mathrm{d}x = 0.$$

We now make use of (1.8) to generate a dissipative term on the velocity field u. It is not difficult to check that

$$\begin{aligned}
&\frac{d}{dt}(u, \nabla \cdot \tau) + \frac{1}{2}\|\nabla u\|_{L^2}^2 - \|\mathbb{P}\nabla \cdot \tau\|_{L^2}^2 \\
&\quad = -((u \cdot \nabla u), \mathbb{P}\nabla \cdot \tau) - (\mathbb{P}\nabla \cdot (u \cdot \nabla \tau), u) - (\mathbb{P}\nabla \cdot Q(\tau, \nabla u), u) \\
&\qquad - \eta((-\Delta)^\beta \mathbb{P}\nabla \cdot \tau, u).
\end{aligned} \tag{2.3}$$

A similar equality also holds for the $\dot{H}^{s-\beta}$ inner product:

$$\begin{aligned}
&\frac{d}{dt}(\Lambda^{s-\beta} u, \Lambda^{s-\beta}\nabla \cdot \tau) + \frac{1}{2}\|\Lambda^{s-\beta}\nabla u\|_{L^2}^2 - \|\Lambda^{s-\beta}\mathbb{P}\nabla \cdot \tau\|_{L^2}^2 \\
&\quad = -(\Lambda^{s-\beta}(u \cdot \nabla u), \Lambda^{s-\beta}\mathbb{P}\nabla \cdot \tau) - (\Lambda^{s-\beta}\mathbb{P}\nabla \cdot (u \cdot \nabla \tau), \Lambda^{s-\beta} u) \\
&\qquad - (\Lambda^{s-\beta}\mathbb{P}\nabla \cdot Q(\tau, \nabla u), \Lambda^{s-\beta} u) - \eta(\Lambda^{s-\beta}(-\Delta)^\beta \mathbb{P}\nabla \cdot \tau, \Lambda^{s-\beta} u).
\end{aligned} \tag{2.4}$$

For a constant $k > 0$, (2.1)+(2.2)+ k(2.3)+ k(2.4) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|_{H^s}^2 + \|\tau\|_{H^s}^2 + 2k(u, \nabla \cdot \tau)_{H^{s-\beta}}) + \eta\|\Lambda^\beta \tau\|_{H^s}^2 \\
&\qquad + \frac{k}{2}\|\nabla u\|_{H^{s-\beta}}^2 - k\|\mathbb{P}\nabla \cdot \tau\|_{H^{s-\beta}}^2 = \sum_{i=1}^{7} I_i,
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
I_1 &= -k((u\cdot\nabla u), \mathbb{P}\nabla\cdot\tau)_{H^{s-\beta}},\\
I_2 &= -k(\mathbb{P}\nabla\cdot(u\cdot\nabla\tau), u)_{H^{s-\beta}},\\
I_3 &= -k(\mathbb{P}\nabla\cdot Q(\tau,\nabla u), u)_{H^{s-\beta}},\\
I_4 &= -k\eta(\Lambda^{2\beta}\mathbb{P}\nabla\cdot\tau, u)_{H^{s-\beta}},\\
I_5 &= -(\Lambda^s(u\cdot\nabla u), \Lambda^s u),\\
I_6 &= -(\Lambda^s(u\cdot\nabla\tau), \Lambda^s\tau),\\
I_7 &= -(Q(\tau,\nabla u),\tau)_{H^s}.
\end{aligned}$$

Now, we estimate I_1 through I_7. We use the simple facts that $\mathbb{P}u = u$ if u is divergence free, $\mathbb{P}$ is bounded by 1 on $H^s(\mathbb{R}^d)$ and $(\mathbb{P}f, g) = (f, \mathbb{P}g)$. Thanks to $s > 1 + \frac{d}{2}$, $\frac{1}{2} \leqslant \beta \leqslant 1$ and $\nabla\cdot u = 0$, we have

$$\begin{aligned}
|I_1| &\lesssim \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\tau\|_{L^2} + \|\Lambda^{s-\beta+1}\tau\|_{L^2}\|u\|_{L^\infty}\|\Lambda^{s-\beta+1}u\|_{L^2}\\
&\lesssim \|u\|_{H^s}\|\nabla u\|_{H^{s-\beta}}\|\Lambda^\beta\tau\|_{H^s}.
\end{aligned}$$

Due to $\frac{1}{2} \leqslant \beta \leqslant 1$ and $\nabla\cdot u = 0$, we have, by integration by parts,

$$\begin{aligned}
|I_2| &\lesssim \|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\tau\|_{L^2}\\
&\quad + \|\Lambda^{s-\beta+1}u\|_{L^2}(\|\Lambda^{s-\beta+1}u\|_{L^2}\|\tau\|_{L^\infty} + \|u\|_{L^\infty}\|\Lambda^{s-\beta+1}\tau\|_{L^2})\\
&\lesssim \|u\|_{H^s}\|\nabla u\|_{H^{s-\beta}}\|\Lambda^\beta\tau\|_{H^s} + \|\nabla u\|_{H^{s-\beta}}^2\|\tau\|_{H^s}.
\end{aligned}$$

Due to $s > 1 + \frac{d}{2}$ and $\frac{1}{2} \leqslant \beta \leqslant 1$, we have, by integration by parts,

$$\begin{aligned}
|I_3| &\lesssim \|\nabla u\|_{L^2}^2\|\tau\|_{L^\infty} + \|\Lambda^{s-\beta+1}u\|_{L^2}(\|\Lambda^{s-\beta+1}u\|_{L^2}\|\tau\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|\Lambda^{s-\beta}\tau\|_{L^2})\\
&\lesssim \|\tau\|_{H^s}\|\nabla u\|_{H^{s-\beta}}^2 + \|u\|_{H^s}\|\nabla u\|_{H^{s-\beta}}\|\Lambda^\beta\tau\|_{H^s}.
\end{aligned}$$

I_4 is bounded by

$$|I_4| \leqslant k\eta\|\Lambda^\beta\tau\|_{H^s}\|\nabla u\|_{H^{s-\beta}} \leqslant \frac{\eta}{4}\|\Lambda^\beta\tau\|_{H^s}^2 + k^2\eta\|\nabla u\|_{H^{s-\beta}}^2.$$

By $\nabla\cdot u = 0$, $\frac{1}{2} \leqslant \beta \leqslant 1$ and $s > 1 + \frac{d}{2}$, we obtain

$$\begin{aligned}
|I_5| &= \left|\int(\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u)\Lambda^s u\,\mathrm{d}x\right|\\
&\lesssim \|\Lambda^s u\|_{L^2}^2\|\nabla u\|_{L^\infty} \lesssim \|u\|_{H^s}\|\nabla u\|_{H^{s-\beta}}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|I_6| &= \left|\int(\Lambda^s(u\cdot\nabla\tau) - u\cdot\nabla\Lambda^s\tau)\Lambda^s\tau\,\mathrm{d}x\right|\\
&\lesssim \|\Lambda^s\tau\|_{L^2}(\|\nabla u\|_{L^\infty}\|\Lambda^s\tau\|_{L^2} + \|\Lambda^s u\|_{L^2}\|\nabla\tau\|_{L^\infty})
\end{aligned}$$

$$\lesssim \|u\|_{H^s}\|\Lambda^\beta\tau\|_{H^s}^2.$$

Thanks to $\frac{1}{2} \leqslant \beta \leqslant 1$, $s > 1 + \frac{d}{2}$ and $d = 2, 3$, we have

$$\begin{aligned}
|I_7| &= |(Q(\tau,\nabla u),\tau) + (\Lambda^{s-\beta}Q(\tau,\nabla u),\Lambda^{s+\beta}\tau)| \\
&\lesssim \|\nabla u\|_{L^2}\|\tau\|_{L^4}^2 + \|\Lambda^{s+\beta}\tau\|_{L^2}(\|\Lambda^{s-\beta}\nabla u\|_{L^2}\|\tau\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|\Lambda^{s-\beta}\tau\|_{L^2}) \\
&\lesssim \|\nabla u\|_{L^2}\|\tau\|_{L^2}^{2(1-\frac{d}{4})}\|\nabla\tau\|_{L^2}^{\frac{d}{2}} + \|\tau\|_{H^s}\|\Lambda^\beta\tau\|_{H^s}\|\nabla u\|_{H^{s-\beta}} + \|\Lambda^\beta\tau\|_{H^s}^2\|u\|_{H^s} \\
&\lesssim \|\tau\|_{H^s}\|\Lambda^\beta\tau\|_{H^s}\|\nabla u\|_{H^{s-\beta}} + \|\Lambda^\beta\tau\|_{H^s}^2\|u\|_{H^s}.
\end{aligned}$$

In addition, due to $\frac{1}{2} \leqslant \beta \leqslant 1$,

$$k\|\mathbb{P}\nabla\cdot\tau\|_{H^{s-\beta}}^2 \leqslant k\|\Lambda^\beta\tau\|_{H^s}^2.$$

Inserting the estimates for I_1 through I_7 into (2.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|\tau\|_{H^s}^2 + 2k(u,\nabla\cdot\tau)_{H^{s-\beta}}\right) \\
&\quad + \left(\frac{3}{4}\eta - k\right)\|\Lambda^\beta\tau\|_{H^s}^2 + \left(\frac{k}{2} - k^2\eta\right)\|\nabla u\|_{H^{s-\beta}}^2 \\
&\lesssim \|u\|_{H^s}\|\nabla u\|_{H^{s-\beta}}\|\Lambda^\beta\tau\|_{H^s} + \|\nabla u\|_{H^{s-\beta}}^2\|\tau\|_{H^s} + \|u\|_{H^s}\|\nabla u\|_{H^{s-\beta}}^2 \\
&\quad + \|u\|_{H^s}\|\Lambda^\beta\tau\|_{H^s}^2 + \|\tau\|_{H^s}\|\Lambda^\beta\tau\|_{H^s}\|\nabla u\|_{H^{s-\beta}} \\
&\lesssim (\|u\|_{H^s} + \|\tau\|_{H^s})(\|\Lambda^\beta\tau\|_{H^s}^2 + \|\nabla u\|_{H^{s-\beta}}^2).
\end{aligned} \tag{2.6}$$

By moving Λ^s on u, and in view of $\frac{1}{2} \leqslant \beta \leqslant 1$, importantly, we have

$$\begin{aligned}
|2k(u,\nabla\cdot\tau)_{H^{s-\beta}}| &\leqslant 2k\|u\|_{H^s}\|\tau\|_{H^{s+1-2\beta}} \\
&\leqslant 2c_3k\|u\|_{H^s}\|\tau\|_{H^s} \\
&\leqslant \frac{1}{2}\|u\|_{H^s}^2 + 2c_3^2k^2\|\tau\|_{H^s}^2.
\end{aligned} \tag{2.7}$$

Choosing k small enough and integrating (2.6) in time and using (2.7), we have

$$\begin{aligned}
&\sup_t \|u\|_{H^s}^2 + \sup_t \|\tau\|_{H^s}^2 + 2\int_0^t (\eta\|\Lambda^\beta\tau\|_{H^s}^2 + \frac{k}{2}\|\nabla u\|_{H^{s-\beta}}^2)\mathrm{d}t' \\
&\quad \lesssim \|u_0\|_{H^s}^2 + \|\tau_0\|_{H^s}^2 + (\sup_t \|u\|_{H^s} + \sup_t \|\tau\|_{H^s})\int_0^t (\|\Lambda^\beta\tau\|_{H^s}^2 + \|\nabla u\|_{H^{s-\beta}}^2)\mathrm{d}t'.
\end{aligned}$$

Thus, we have established (1.11). This concludes the proof for the existence part.

We now prove the uniqueness. The term $Q(\tau,\nabla u)$ requires special attention. We split the consideration into two cases: $\beta = 1$ and $\frac{1}{2} \leq \beta < 1$. The uniqueness for the case when $\beta = 1$ is direct, but the case when $\frac{1}{2} \leq \beta < 1$ is difficult and has to be dealt with by constructing suitable energy functional.

Case 1: $\beta = 1$. Assume (u_1, τ_1) and (u_2, τ_2) are two solutions of (1.6) with the same initial data. Denote $\delta u = u_1 - u_2$, $\delta\tau = \tau_1 - \tau_2$. Then, $(\delta u, \delta\tau)$ satisfies

$$\begin{cases} \partial_t \delta u = \nabla \cdot \delta\tau - u_1 \cdot \nabla \delta u - \delta u \cdot \nabla u_2 - \nabla \delta P, \\ \partial_t \delta\tau + u_1 \cdot \nabla \delta\tau - \eta \Delta \delta\tau = D(\delta u) - \delta u \cdot \nabla \tau_2 - Q(\tau_1, \nabla \delta u) - Q(\delta\tau, \nabla u_2), \\ \nabla \cdot \delta u = 0, \\ \delta u(x, 0) = 0; \ \delta\tau(x, 0) = 0, \end{cases} \tag{2.8}$$

where δP is the corresponding pressure difference. Taking the L^2 inner product of (2.8) with $(\delta u, \delta\tau)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\delta u\|_{L^2}^2 + \|\delta\tau\|_{L^2}^2) + \eta\|\nabla\delta\tau\|_{L^2}^2 \\ &\quad = -\int \delta u \cdot \nabla u_2 \cdot \delta u \mathrm{d}x - \int \delta u \cdot \nabla\tau_2 \cdot \delta\tau \mathrm{d}x - \int Q(\tau_1, \nabla\delta u) \cdot \delta\tau \mathrm{d}x \\ &\qquad - \int Q(\delta\tau, \nabla u_2) \cdot \delta\tau \mathrm{d}x + \int (\delta u \cdot (\nabla \cdot \delta\tau) + D(\delta u) \cdot \delta\tau) \mathrm{d}x \\ &\quad \leqslant \|\nabla u_2\|_{L^\infty}\|\delta u\|_{L^2}^2 + \|\nabla\tau_2\|_{L^\infty}\|\delta u\|_{L^2}\|\delta\tau\|_{L^2} \\ &\qquad + c\,(\|\tau_1\|_{L^\infty}\|\nabla\delta\tau\|_{L^2} + \|\nabla\tau_1\|_{L^\infty}\|\delta\tau\|_{L^2})\|\delta u\|_{L^2} + \|\nabla u_2\|_{L^\infty}\|\delta\tau\|_{L^2}^2 \\ &\quad \leqslant c\,(\|\nabla u_2\|_{L^\infty} + \|\nabla\tau_2\|_{L^\infty} + \|\nabla\tau_1\|_{L^\infty} + \|\tau_1\|_{L^\infty}^2)(\|\delta u\|_{L^2}^2 + \|\delta\tau\|_{L^2}^2) + \frac{\eta}{2}\|\nabla\delta\tau\|_{L^2}^2, \end{aligned}$$

where we have used the fact that

$$\int (\delta u \cdot (\nabla \cdot \delta\tau) + D(\delta u) \cdot \delta\tau) \mathrm{d}x = 0.$$

It then follows from Gronwall's inequality that $\delta u = \delta\tau = 0$.

Case 2: $\frac{1}{2} \leqslant \beta < 1$. Assume (u_1, τ_1) and (u_2, τ_2) are two solutions of (1.6) with the same initial data. Denote $\delta u = u_1 - u_2$, $\delta\tau = \tau_1 - \tau_2$. Then, $(\delta u, \delta\tau)$ satisfies

$$\begin{cases} \partial_t \delta u = \nabla \cdot \delta\tau - u_1 \cdot \nabla \delta u - \delta u \cdot \nabla u_2 + \nabla \delta P, \\ \partial_t \delta\tau + u_1 \cdot \nabla \delta\tau + \eta \Lambda^{2\beta} \delta\tau = D(\delta u) - \delta u \cdot \nabla \tau_2 - Q(\tau_1, \nabla \delta u) - Q(\delta\tau, \nabla u_2), \\ \nabla \cdot \delta u = 0, \\ \delta u(x, 0) = 0; \ \delta\tau(x, 0) = 0. \end{cases} \tag{2.9}$$

Dotting (2.9) by $(\delta u, \delta\tau)$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\delta u\|_{L^2}^2 + \|\delta\tau\|_{L^2}^2) + \eta\|\Lambda^\beta\delta\tau\|_{L^2}^2 \\ &\quad = -(\delta u \cdot \nabla u_2, \delta u) - (\delta u \cdot \nabla\tau_2, \delta\tau) - (Q(\tau_1, \nabla\delta u), \delta\tau) - (Q(\delta\tau, \nabla u_2), \delta\tau). \end{aligned} \tag{2.10}$$

Applying Λ^β to (2.9) and then dotting by $(\Lambda^\beta \delta u, \Lambda^\beta \delta\tau)$ lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^\beta \delta u\|_{L^2}^2 + \|\Lambda^\beta \delta\tau\|_{L^2}^2) + \eta\|\Lambda^{2\beta}\delta\tau\|_{L^2}^2 \\
&\quad = -(\Lambda^\beta(u_1 \cdot \nabla\delta u), \Lambda^\beta \delta u) - (\Lambda^\beta(\delta u \cdot \nabla u_2), \Lambda^\beta \delta u) - (\Lambda^\beta(u_1 \cdot \nabla\delta\tau), \Lambda^\beta\delta\tau) \\
&\qquad -(\Lambda^\beta(\delta u \cdot \nabla\tau_2), \Lambda^\beta\delta\tau) - (\Lambda^\beta Q(\tau_1, \nabla\delta u), \Lambda^\beta\delta\tau) - (\Lambda^\beta Q(\delta\tau, \nabla u_2), \Lambda^\beta\delta\tau).
\end{aligned} \tag{2.11}$$

Applying $\mathbb{P}\nabla\cdot$ to the second equation of (2.9), we have

$$\begin{aligned}
&\partial_t \mathbb{P}\nabla\cdot\delta\tau + \mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau) + \eta\Lambda^{2\beta}\mathbb{P}\nabla\cdot\delta\tau \\
&\quad = \frac{1}{2}\Delta\delta u - \mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2) - \mathbb{P}\nabla\cdot Q(\tau_1, \nabla\delta u) - \mathbb{P}\nabla\cdot Q(\delta\tau, \nabla u_2).
\end{aligned} \tag{2.12}$$

Taking the L^2 inner product of the first equation of (2.9) with $\mathbb{P}\nabla\cdot\delta\tau$ and the L^2 inner product of (2.12) with δu separately, we have

$$\begin{aligned}
&\frac{d}{dt}(\delta u, \nabla\cdot\delta\tau) + \frac{1}{2}\|\nabla\delta u\|_{L^2}^2 - \|\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}^2 \\
&\quad = -((u_1\cdot\nabla\delta u), \mathbb{P}\nabla\cdot\delta\tau) - ((\delta u\cdot\nabla u_2), \mathbb{P}\nabla\cdot\delta\tau) - (\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau), \delta u) \\
&\qquad -(\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2), \delta u) - (\mathbb{P}\nabla\cdot Q(\tau_1, \nabla\delta u), \delta u) - (\mathbb{P}\nabla\cdot Q(\delta\tau, \nabla u_2), \delta u) \\
&\qquad -\eta(\Lambda^{2\beta}\mathbb{P}\nabla\cdot\delta\tau, \delta u).
\end{aligned} \tag{2.13}$$

For a positive constant k_1 to be determined later, (2.10)+(2.11)+k_1(2.13) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\delta u\|_{H^\beta}^2 + \|\delta\tau\|_{H^\beta}^2 + 2k_1(\delta u, \nabla\cdot\delta\tau)) + \eta\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 \\
&\quad +\frac{k_1}{2}\|\nabla\delta u\|_{L^2}^2 - k_1\|\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}^2 = \sum_{i=1}^{7} I_i',
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
I_1' &= -k_1((u_1\cdot\nabla\delta u), \mathbb{P}\nabla\cdot\delta\tau) - k_1((\delta u\cdot\nabla u_2), \mathbb{P}\nabla\cdot\delta\tau), \\
I_2' &= -k_1(\mathbb{P}\nabla\cdot(u_1\cdot\nabla\delta\tau), \delta u) - k_1(\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau_2), \delta u), \\
I_3' &= -k_1(\mathbb{P}\nabla\cdot Q(\tau_1, \nabla\delta u), \delta u) - k_1(\mathbb{P}\nabla\cdot Q(\delta\tau, \nabla u_2), \delta u), \\
I_4' &= -k_1\eta(\Lambda^{2\beta}\mathbb{P}\nabla\cdot\delta\tau, \delta u), \\
I_5' &= -(\Lambda^\beta(u_1\cdot\nabla\delta u), \Lambda^\beta\delta u) - ((\delta u\cdot\nabla u_2), \delta u)_{H^\beta}, \\
I_6' &= -(\Lambda^\beta(u_1\cdot\nabla\delta\tau), \Lambda^\beta\delta\tau) - ((\delta u\cdot\nabla\tau_2), \delta\tau)_{H^\beta}, \\
I_7' &= -(Q(\tau_1, \nabla\delta u), \delta\tau)_{H^\beta} - (Q(\delta\tau, \nabla u_2), \delta\tau)_{H^\beta}.
\end{aligned}$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}
|I_1'| &\lesssim \|u_1\|_{L^\infty}\|\nabla\delta u\|_{L^2}\|\nabla\delta\tau\|_{L^2} + \|\nabla u_2\|_{L^\infty}\|\delta u\|_{L^2}\|\nabla\delta\tau\|_{L^2},\\
|I_2'| &\lesssim \|u_1\|_{L^\infty}\|\nabla\delta u\|_{L^2}\|\nabla\delta\tau\|_{L^2} + \|\nabla\tau_2\|_{L^\infty}\|\nabla\delta u\|_{L^2}\|\delta u\|_{L^2},\\
|I_3'| &\lesssim \|\tau_1\|_{L^\infty}\|\nabla\delta u\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty}\|\nabla\delta u\|_{L^2}\|\delta\tau\|_{L^2},\\
|I_4'| &\leqslant \frac{\eta}{4}\|\Lambda^{2\beta}\delta\tau\|_{L^2}^2 + k_1^2\eta\|\nabla\delta u\|_{L^2}^2.
\end{aligned}$$

Since $\nabla\cdot u_1 = 0$, I_5' can be written as

$$I_5' = -(\Lambda^\beta(u_1\cdot\nabla\delta u) - u_1\cdot\nabla\Lambda^\beta\delta u, \Lambda^\beta\delta u) - ((\delta u\cdot\nabla u_2),\delta u)_{H^\beta}.$$

By a standard commutator estimate,

$$\begin{aligned}
|I_5'| \lesssim\ & \|\nabla u_1\|_{L^{\frac{d}{2-2\beta}}}\|\Lambda^\beta\delta u\|^2_{L^{\frac{2d}{d-2+2\beta}}} + \|\Lambda^\beta u_1\|_{L^{\frac{d}{1-\beta}}}\|\nabla\delta u\|_{L^2}\|\Lambda^\beta\delta u\|_{L^{\frac{2d}{d-2+2\beta}}}\\
&+\|\nabla u_2\|_{L^\infty}\|\delta u\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty}\|\Lambda^\beta\delta u\|_{L^2}^2 + \|\delta u\|_{L^{\frac{2d}{d-2\beta}}}\|\Lambda^\beta\nabla u_2\|_{L^{\frac{d}{\beta}}}\|\Lambda^\beta\delta u\|_{L^2}\\
\lesssim\ & (\|\nabla u_1\|_{L^{\frac{d}{2-2\beta}}} + \|\Lambda^\beta u_1\|_{L^{\frac{d}{1-\beta}}})\|\nabla\delta u\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty}\|\delta u\|_{L^2}^2\\
&+(\|\nabla u_2\|_{L^\infty} + \|\Lambda^\beta\nabla u_2\|_{L^{\frac{d}{\beta}}})\|\Lambda^\beta\delta u\|_{L^2}^2.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
|I_6'| \lesssim\ & \|u_1\|_{L^\infty}\|\nabla\delta\tau\|_{L^2}\|\Lambda^{2\beta}\delta\tau\|_{L^2}\\
&+\|\nabla\tau_2\|_{L^\infty}\|\delta u\|_{L^2}(\|\delta\tau\|_{L^2} + \|\Lambda^{2\beta}\delta\tau\|_{L^2}),\\
|I_7'| \lesssim\ & \|\tau_1\|_{L^\infty}\|\nabla\delta u\|_{L^2}(\|\delta\tau\|_{L^2} + \|\Lambda^{2\beta}\delta\tau\|_{L^2})\\
&+\|\nabla u_2\|_{L^\infty}\|\delta\tau\|_{L^2}(\|\delta\tau\|_{L^2} + \|\Lambda^{2\beta}\delta\tau\|_{L^2}).
\end{aligned}$$

We insert the estimates above for I_1' through I_7' in (2.14). If the initial data are small enough, namely

$$\|u_0\|_{H^s} + \|\tau_0\|_{H^s} \leqslant \varepsilon$$

for sufficiently small $\varepsilon > 0$, we can choose k_1 and t small enough to obtain the desired uniqueness. This completes the proof of Theorem 1.1. □

3. Proof of Theorems 1.2 and 1.3

This section proves Theorems 1.2 and 1.3 .

Proof of Theorem 1.2. The proof of Theorem 1.2 is very close to that for Theorem 1.1. We shall omit most of the details but to point out the differences. The differences are due to the extra term $\nu(-\Delta)^\alpha u^{(\nu)}$. (2.5) would now contain two extra terms and is given by

$$\frac{1}{2}\frac{d}{dt}(\|u^{(\nu)}\|_{H^s}^2 + \|\tau^{(\nu)}\|_{H^s}^2 + 2k(u^{(\nu)}, \nabla\cdot\tau^{(\nu)})_{H^{s-\beta}}) + \eta\|\Lambda^\beta\tau^{(\nu)}\|_{H^s}^2 + \nu\|\Lambda^\alpha u^{(\nu)}\|_{H^s}^2$$
$$+\frac{k}{2}\|\nabla u^{(\nu)}\|_{H^{s-\beta}}^2 - k\|\mathbb{P}\nabla\cdot\tau^{(\nu)}\|_{H^{s-\beta}}^2 = \sum_{i=1}^{8} I_i,$$

where I_1 through I_7 is the same as before, and I_8 is given by

$$I_8 = \nu\, k\, ((-\Delta)^\alpha u^{(\nu)}, \nabla\cdot\tau^{(\nu)})_{H^{s-\beta}}.$$

The estimates for I_1 through I_7 are the same as before, and I_8 can be bounded by

$$|I_8| \le \nu k\|\Lambda^{2\alpha-3\beta+1}u^{(\nu)}\|_{H^s}\,\|\Lambda^\beta\tau^{(\nu)}\|_{H^s}.$$

When $\alpha \le \min\{1, 3\beta-1\}$, we have $2\alpha-3\beta+1 \le \alpha$ and

$$|I_8| \le \frac{\nu}{2}\|\Lambda^\alpha u^{(\nu)}\|_{H^s}^2 + \frac{\nu k^2}{2}\|\Lambda^\beta\tau^{(\nu)}\|_{H^s}^2.$$

The rest of the proof is almost identical to that for Theorem 1.1. The crucial fact is that the bound for $(u^{(\nu)}, \tau^{(\nu)})$ in H^s obtained from this process is uniform in ν. We omit further details. □

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. We distinguish between two cases: Case I: $\beta = 1$ and Case II: $\frac{1}{2} \le \beta < 1$. The first case is relatively easy, while the second case is more delicate. The fact that the bound for the solution $(u^{(\nu)}, \tau^{(\nu)})$ in H^s is uniform in terms of ν plays a crucial role in the proof.

Case I: $\beta = 1$. The difference $(\delta u, \delta\tau)$ with

$$\delta u = u^{(\nu)} - u, \quad \delta\tau = \tau^{(\nu)} - \tau$$

satisfies

$$\begin{cases} \partial_t\delta u + u^{(\nu)}\cdot\nabla\delta u + \nu\,(-\Delta)^\alpha\delta u = -\nu\,(-\Delta)^\alpha u + \nabla\cdot\delta\tau - \delta u\cdot\nabla u - \nabla\delta P, \\ \partial_t\delta\tau + u^{(\nu)}\cdot\nabla\delta\tau - \eta\Delta\delta\tau = D(\delta u) - \delta u\cdot\nabla\tau - Q(\tau, \nabla\delta u) - Q(\delta\tau, \nabla u^{(\nu)}), \\ \nabla\cdot\delta u = 0, \\ \delta u(x,0) = 0;\ \ \delta\tau(x,0) = 0, \end{cases} \tag{3.1}$$

where δP is the corresponding pressure difference. Taking the L^2 inner product of (3.1) with $(\delta u, \delta\tau)$, we have

$$\frac{1}{2}\frac{d}{dt}(\|\delta u\|_{L^2}^2 + \|\delta\tau\|_{L^2}^2) + \nu\|\Lambda^\alpha\delta u\|_{L^2}^2 + \eta\|\nabla\delta\tau\|_{L^2}^2$$
$$= -\nu\int(-\Delta)^\alpha u\cdot\delta u\,dx - \int\delta u\cdot\nabla u\cdot\delta u\,\mathrm{d}x - \int\delta u\cdot\nabla\tau\cdot\delta\tau\,\mathrm{d}x$$

$$
\begin{aligned}
&-\int Q(\tau, \nabla\delta u)\cdot\delta\tau\,\mathrm{d}x - \int Q(\delta\tau, \nabla u^{(\nu)})\cdot\delta\tau\,\mathrm{d}x\\
&\leqslant \nu\|u\|_{H^{2\alpha}}\|\delta u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\delta u\|_{L^2}^2 + \|\nabla\tau\|_{L^\infty}\|\delta u\|_{L^2}\|\delta\tau\|_{L^2}\\
&\quad + c\,(\|\tau\|_{L^\infty}\|\nabla\delta\tau\|_{L^2} + \|\nabla\tau\|_{L^\infty}\|\delta\tau\|_{L^2})\|\delta u\|_{L^2} + \|\nabla u^{(\nu)}\|_{L^\infty}\|\delta\tau\|_{L^2}^2\\
&\leqslant \nu^2\|u\|_{H^{2\alpha}}^2 + \frac{\eta}{2}\|\nabla\delta\tau\|_{L^2}^2\\
&\quad + C\,(1 + \|u\|_{H^s} + \|\tau\|_{H^s} + \|u^{(\nu)}\|_{H^s} + \|\tau\|_{H^{s-1}}^2)(\|\delta u\|_{L^2}^2 + \|\delta\tau\|_{L^2}^2).
\end{aligned}
$$

Here, we have used the fact that

$$\int(\delta u\cdot(\nabla\cdot\delta\tau) + D(\delta u)\cdot\delta\tau)\mathrm{d}x = 0.$$

(1.14) then follows from Gronwall's inequality and the uniform bound (in ν) for $\|\tau^{(\nu)}\|_{H^s}$.

Case 2: $\frac{1}{2}\leqslant\beta<1$. The difference $(\delta u, \delta\tau)$ satisfies

$$
\begin{cases}
\partial_t\delta u + u^{(\nu)}\cdot\nabla\delta u + \nu\,(-\Delta)^\alpha\delta u = -\nu\,(-\Delta)^\alpha u + \nabla\cdot\delta\tau - \delta u\cdot\nabla u - \nabla\delta P,\\
\partial_t\delta\tau + u^{(\nu)}\cdot\nabla\delta\tau + \eta(-\Delta)^\beta\delta\tau = D(\delta u) - \delta u\cdot\nabla\tau - Q(\tau, \nabla\delta u) - Q(\delta\tau, \nabla u^{(\nu)}),\\
\nabla\cdot\delta u = 0,\\
\delta u(x,0) = 0;\ \ \delta\tau(x,0) = 0.
\end{cases}
\tag{3.2}
$$

Dotting (3.2) by $(\delta u, \delta\tau)$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\delta u\|_{L^2}^2 + \|\delta\tau\|_{L^2}^2) + \eta\|\Lambda^\beta\delta\tau\|_{L^2}^2 + \nu\|\Lambda^\alpha\delta u\|_{L^2}^2\\
&\quad = -\nu(\Lambda^{2\alpha}u, \delta u) - (\delta u\cdot\nabla u, \delta u) - (\delta u\cdot\nabla\tau, \delta\tau)\\
&\qquad -(Q(\tau, \nabla\delta u), \delta\tau) - (Q(\delta\tau, \nabla u^{(\nu)}), \delta\tau).
\end{aligned}
$$

Applying Λ^β to (3.2) and then dotting by $(\Lambda^\beta\delta u, \Lambda^\beta\delta\tau)$ lead to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^\beta\delta u\|_{L^2}^2 + \|\Lambda^\beta\delta\tau\|_{L^2}^2) + \eta\|\Lambda^{2\beta}\delta\tau\|_{L^2}^2 + \nu\|\Lambda^{\alpha+\beta}\delta u\|_{L^2}^2\\
&\quad = -\nu(\Lambda^{2\alpha+\beta}u, \Lambda^\beta\delta u) - (\Lambda^\beta(u^{(\nu)}\cdot\nabla\delta u), \Lambda^\beta\delta u) - (\Lambda^\beta(\delta u\cdot\nabla u), \Lambda^\beta\delta u)\\
&\qquad -(\Lambda^\beta(u^{(\nu)}\cdot\nabla\delta\tau), \Lambda^\beta\delta\tau) - (\Lambda^\beta(\delta u\cdot\nabla\tau), \Lambda^\beta\delta\tau) - (\Lambda^\beta Q(\tau, \nabla\delta u), \Lambda^\beta\delta\tau)\\
&\qquad -(\Lambda^\beta Q(\delta\tau, \nabla u^{(\nu)}), \Lambda^\beta\delta\tau).
\end{aligned}
\tag{3.3}
$$

Applying $\mathbb{P}\nabla\cdot$ to the second equation of (3.2), we have

$$
\begin{aligned}
&\partial_t\mathbb{P}\nabla\cdot\delta\tau + \mathbb{P}\nabla\cdot(u^{(\nu)}\cdot\nabla\delta\tau) + \eta\Lambda^{2\beta}\mathbb{P}\nabla\cdot\delta\tau\\
&\quad = \frac{1}{2}\Delta\delta u - \mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau) - \mathbb{P}\nabla\cdot Q(\tau, \nabla\delta u) - \mathbb{P}\nabla\cdot Q(\delta\tau, \nabla u^{(\nu)}).
\end{aligned}
\tag{3.4}
$$

Taking the L^2 inner product of the first equation of (3.2) with $\mathbb{P}\nabla\cdot\delta\tau$ and the L^2 inner product of (3.4) with δu, we have

$$\begin{aligned}
&\frac{d}{dt}(\delta u, \nabla\cdot\delta\tau) + \frac{1}{2}\|\nabla\delta u\|_{L^2}^2 - \|\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}^2 \\
&= -\nu((-\Delta)^\alpha u^{(\nu)}, \nabla\cdot\delta\tau) - ((u^{(\nu)}\cdot\nabla\delta u), \mathbb{P}\nabla\cdot\delta\tau) - ((\delta u\cdot\nabla u), \mathbb{P}\nabla\cdot\delta\tau) \\
&\quad -(\mathbb{P}\nabla\cdot(u^{(\nu)}\cdot\nabla\delta\tau), \delta u) - (\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau), \delta u) - (\mathbb{P}\nabla\cdot Q(\tau, \nabla\delta u), \delta u) \\
&\quad -(\mathbb{P}\nabla\cdot Q(\delta\tau, \nabla u^{(\nu)}), \delta u) - \eta(\Lambda^{2\beta}\mathbb{P}\nabla\cdot\delta\tau, \delta u).
\end{aligned} \tag{3.5}$$

We choose a positive constant k_3 satisfying, for a suitable constant $C > 0$,

$$0 < k_3 \le C\ \min\{1, \eta\}.$$

Then, (3.3)+k_3(3.5) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\delta u\|_{H^\beta}^2 + \|\delta\tau\|_{H^\beta}^2 + 2k_3(\delta u, \nabla\cdot\delta\tau)) + \eta\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 \\
&\quad +\nu\|\Lambda^\alpha\delta u\|_{H^\beta}^2 + \frac{k_3}{2}\|\nabla\delta u\|_{L^2}^2 - k_3\|\mathbb{P}\nabla\cdot\delta\tau\|_{L^2}^2 = \sum_{i=1}^{20} K_i,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
&K_1 = -\nu(\Lambda^{2\alpha+\beta}u, \Lambda^\beta\delta u), \qquad K_2 = -(\Lambda^\beta(u^{(\nu)}\cdot\nabla\delta u), \Lambda^\beta\delta u), \\
&K_3 = -(\Lambda^\beta(\delta u\cdot\nabla u), \Lambda^\beta\delta u), \qquad K_4 = -(\Lambda^\beta(u^{(\nu)}\cdot\nabla\delta\tau), \Lambda^\beta\delta\tau), \\
&K_5 = -(\Lambda^\beta(\delta u\cdot\nabla\tau), \Lambda^\beta\delta\tau), \qquad K_6 = -(\Lambda^\beta Q(\tau, \nabla\delta u), \Lambda^\beta\delta\tau), \\
&K_7 = -(\Lambda^\beta Q(\delta\tau, \nabla u^{(\nu)}), \Lambda^\beta\delta\tau), \qquad K_8 = -k_3\,\nu((-\Delta)^\alpha u^{(\nu)}, \nabla\cdot\delta\tau), \\
&K_9 = -k_3((u^{(\nu)}\cdot\nabla\delta u), \mathbb{P}\nabla\cdot\delta\tau), \qquad K_{10} = -k_3((\delta u\cdot\nabla u), \mathbb{P}\nabla\cdot\delta\tau), \\
&K_{11} = -k_3(\mathbb{P}\nabla\cdot(u^{(\nu)}\cdot\nabla\delta\tau), \delta u), \qquad K_{12} = -k_3(\mathbb{P}\nabla\cdot(\delta u\cdot\nabla\tau), \delta u), \\
&K_{13} = -k_3(\mathbb{P}\nabla\cdot Q(\tau, \nabla\delta u), \delta u), \qquad K_{14} = -k_3(\mathbb{P}\nabla\cdot Q(\delta\tau, \nabla u^{(\nu)}), \delta u), \\
&K_{15} = -k_3\eta(\Lambda^{2\beta}\mathbb{P}\nabla\cdot\delta\tau, \delta u), \qquad K_{16} = -\nu(\Lambda^{2\alpha}u, \delta u), \\
&K_{17} = -(\delta u\cdot\nabla u, \delta u), \qquad K_{18} = -(\delta u\cdot\nabla\tau, \delta\tau), \\
&K_{19} = -(Q(\tau, \nabla\delta u), \delta\tau), \qquad K_{20} - (Q(\delta\tau, \nabla u^{(\nu)}), \delta\tau).
\end{aligned}$$

The terms above can be bounded as follows. All the constants in the estimates are independent of ν. By Hölder's inequality,

$$|K_1| \le \nu^2\|\Lambda^{2\alpha+\beta}u\|_{L^2}^2 + C\,\|\delta u\|_{H^\beta}^2.$$

Due to $\nabla\cdot u^{(\nu)} = 0$ and by a standard commutator estimate,

$$\begin{aligned}
|K_2| &\le C\,\|u^{(\nu)}\|_{H^s}\,\|\delta u\|_{H^\beta}^2 + C\,\|u^{(\nu)}\|_{H^s}\,\|\nabla\delta u\|_{L^2}\|\Lambda^\beta\delta u\|_{L^2} \\
&\le \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2 + C\,(1 + k_3^{-1}\|u^{(\nu)}\|_{H^s})\,\|u^{(\nu)}\|_{H^s}\,\|\delta u\|_{H^\beta}^2.
\end{aligned}$$

Clearly, for q_1 and q_2 satisfying $\frac{1}{q_1} = \frac{1}{2} - \frac{\beta}{d}$ and $\frac{1}{q_2} = \frac{1}{2} - \frac{1}{q_1}$,

$$\begin{aligned}|K_3| &\le C\,\|u\|_{H^s}\,\|\delta u\|_{H^\beta}^2 + \|\delta u\|_{L^{q_1}}\,\|\Lambda^\beta \nabla u\|_{L^{q_2}}\,\|\Lambda^\beta \delta u\|_{L^2}\\ &\le C\,\|u\|_{H^s}\,\|\delta u\|_{H^\beta}^2.\end{aligned}$$

By a commutator estimate,

$$\begin{aligned}|K_4| &\le C\,\|\Lambda^\beta u^{(\nu)}\|_{L^\infty}\,\|\nabla\delta\tau\|_{L^2}\,\|\Lambda^\beta\delta\tau\|_{L^2} + C\,\|\nabla u^{(\nu)}\|_{L^\infty}\|\Lambda^\beta\delta\tau\|_{L^2}^2\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + C\,(\eta^{-1}\|u^{(\nu)}\|_{H^s}^2 + \|u^{(\nu)}\|_{H^s})\|\delta\tau\|_{H^\beta}^2.\end{aligned}$$

K_5 can be similarly bounded as K_3:

$$|K_5| \le C\,\|\tau\|_{H^s}\,(\|\delta u\|_{H^\beta}^2 + \|\delta\tau\|_{H^\beta}^2).$$

By Hölder's inequality,

$$\begin{aligned}|K_6| &\le \|\Lambda^{2\beta}\delta\tau\|_{L^2}\,\|\tau\|_{L^\infty}\,\|\nabla\delta u\|_{L^2}\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + C\,\eta^{-1}\|\tau\|_{H^{s-1}}^2\,\|\nabla\delta u\|_{L^2}^2\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2,\end{aligned}$$

where we have used the smallness of the solution

$$C\,\eta^{-1}\|\tau\|_{H^{s-1}}^2 \le C\,\eta^{-1}\,\varepsilon^2 \le \frac{k_3}{16}.$$

By Hölder's inequality,

$$\begin{aligned}|K_7| &\le \|\Lambda^{2\beta}\delta\tau\|_{L^2}\,\|\nabla u^{(\nu)}\|_{L^\infty}\,\|\delta\tau\|_{L^2}\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + C\,\eta^{-1}\|u^{(\nu)}\|_{H^s}^2\,\|\delta\tau\|_{L^2}^2.\end{aligned}$$

Clearly,

$$\begin{aligned}|K_8| &\le k_3\nu\|\Lambda^{2\alpha+1-\beta}u^{(\nu)}\|_{L^2}\,\|\Lambda^\beta\delta\tau\|_{L^2}\\ &\le \nu^2\|\Lambda^{2\alpha+1-\beta}u^{(\nu)}\|_{L^2}^2 + C\,k_3^2\,\|\delta\tau\|_{H^\beta}^2.\end{aligned}$$

K_9 can be similarly handled as K_6:

$$\begin{aligned}|K_9| &\le k_3\,\|u^{(\nu)}\|_{L^\infty}\,\|\nabla\delta u\|_{L^2}\,\|\nabla\delta\tau\|_{L^2}\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + C\,\eta^{-1}k_3^2\,\|u^{(\nu)}\|_{H^{s-1}}^2\,\|\nabla\delta u\|_{L^2}^2\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2,\end{aligned}$$

where we have used the smallness of the solution

$$C\,k_3\eta^{-1}\|u^{(\nu)}\|_{H^{s-1}}^2 \le C\,k_3\eta^{-1}\,\varepsilon^2 \le \frac{1}{16}.$$

We emphasize that $\|u^{(\nu)}\|_{H^s} \le C\,\varepsilon$ with ε independent of ν, as stated in Theorem 1.2. For $\beta \ge \frac{1}{2}$,

$$\begin{aligned}|K_{10}| &\le k_3\,\|\delta u\|_{L^2}\,\|\nabla u\|_{L^\infty}\,\|\nabla\delta\tau\|_{L^2}\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + C\,k_3^2\,\|u\|_{H^s}^2\,\|\delta u\|_{L^2}^2.\end{aligned}$$

K_{11} admits the same bound as K_9:

$$\begin{aligned}|K_{11}| &\le k_3\,\|u^{(\nu)}\|_{L^\infty}\,\|\nabla\delta u\|_{L^2}\,\|\nabla\delta\tau\|_{L^2}\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2.\end{aligned}$$

K_{12} can be bounded directly:

$$\begin{aligned}|K_{12}| &\le k_3\,\|\delta u\|_{L^2}\,\|\nabla\tau\|_{L^\infty}\,\|\nabla\delta u\|_{L^2}\\ &\le \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2 + C\,k_3\|\tau\|_{H^s}^2\,\|\delta u\|_{L^2}^2.\end{aligned}$$

We use the smallness of the solution to bound K_{13}:

$$|K_{13}| \le C\,k_3\,\|\tau\|_{L^\infty}\,\|\nabla\delta u\|_{L^2}^2 \le \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2,$$

where we have used

$$C\,\|\tau\|_{L^\infty} \le C\,\|\tau\|_{H^{s-1}} \le C\,\varepsilon \le \frac{1}{16}.$$

K_{14} is bounded similarly as K_{12}:

$$\begin{aligned}|K_{14}| &\le k_3\,\|\nabla\delta u\|_{L^2}\,\|\delta\tau\|_{L^2}\,\|\nabla u^{(\nu)}\|_{L^\infty}\\ &\le \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2 + C\,k_3\|u^{(\nu)}\|_{H^s}^2\,\|\delta\tau\|_{L^2}^2.\\ |K_{15}| &\le k_3\eta\,\|\Lambda^{2\beta}\delta\tau\|_{L^2}\,\|\nabla\delta u\|_{L^2}\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + C\,k_3^2\eta^{-1}\|\nabla\delta u\|_{L^2}^2\\ &\le \frac{\eta}{16}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2.\end{aligned}$$

In addition, it is easy to obtain the following estimates:

$$\begin{aligned}|K_{16}| &\le \nu^2\|\Lambda^{2\alpha}u\|_{L^2}^2 + C\,\|\delta u\|_{L^2}^2,\\ |K_{17}| &\le C\,\|u\|_{H^s}\,\|\delta u\|_{L^2}^2,\\ |K_{18}| &\le C\,\|\tau\|_{H^s}\,(\|\delta u\|_{L^2}^2 + \|\delta\tau\|_{L^2}^2),\\ |K_{19}| &\le C\|\tau\|_{H^s}^2\|\delta\tau\|_{L^2}^2 + \frac{k_3}{16}\|\nabla\delta u\|_{L^2}^2,\\ |K_{20}| &\le C\|u^{(\nu)}\|_{H^s}\|\delta\tau\|_{L^2}^2.\end{aligned}$$

Inserting the bounds for K_1 through K_{20} above in (3.6), we find

$$\begin{aligned}
&\frac{d}{dt}(\|\delta u\|_{H^\beta}^2 + \|\delta\tau\|_{H^\beta}^2 + 2k_3(\delta u, \nabla\cdot\delta\tau))\\
&\quad +2\nu\|\Lambda^\alpha\delta u\|_{H^\beta}^2 + \frac{\eta}{4}\|\Lambda^\beta\delta\tau\|_{H^\beta}^2 + \frac{k_3}{4}\|\nabla\delta u\|_{L^2}^2\\
&\le C(1+\|u\|_{H^s}^2 + \|u^{(\nu)}\|_{H^s}^2 + \|\tau\|_{H^s}^2)(\|\delta u\|_{H^\beta}^2 + \|\delta\tau\|_{H^\beta}^2)\\
&\quad +C\,\nu^2(\|u\|_{H^s}^2 + \|u^{(\nu)}\|_{H^s}^2).
\end{aligned}$$

Choosing $k_3 \le \frac{1}{2}$, applying Gronwall's inequality and using the fact that $\|u^{(\nu)}\|_{H^s}$ is bounded uniformly in ν (see (1.13)), we obtain (1.14). This completes the proof of Theorem 1.3. □

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