# THE 2D BOUSSINESQ-NAVIER-STOKES EQUATIONS WITH LOGARITHMICALLY SUPERCRITICAL DISSIPATION

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Abstract. This paper studies the global well-posedness of the initial-value problem for the 2D Boussinesq-Navier-Stokes equations with dissipation given by an operator $\mathcal{L}$ that can be defined through both an integral kernel and a Fourier multiplier. When the symbol of $\mathcal{L}$ is represented by $\frac{|\xi|}{a(|\xi|)}$ with $a$ satisfying $\lim_{|\xi|\to\infty} \frac{a(|\xi|)}{|\xi|^\sigma} = 0$ for any $\sigma > 0$, we obtain the global well-posedness. A special consequence is the global well-posedness when the dissipation is logarithmically supercritical.

## 1. Introduction

Attention here is focused on the initial-value problem (IVP) for the Boussinseq-Navier-Stokes equations with dissipation given by a general integral operator,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field denoting the velocity, $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function, $\mathbf{e}_2$ is the unit vector in the $x_2$ direction, and $\mathcal{L}$ is a nonlocal dissipation operator defined by

$$\mathcal{L} f(x) = \text{ p.v. } \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^2} m(|x-y|) dy \tag{1.2}$$

and $m\colon (0,\infty) \to (0,\infty)$ is a smooth, positive, non-increasing function, which obeys

(i) there exists $C_1 > 0$ such that

$$r m(r) \le C_1 \qquad \text{for all } r \le 1;$$

(ii) there exists $C_2 > 0$ such that

$$r|m'(r)| \le C_2 m(r) \qquad \text{for all } r > 0;$$

(iii) there exists $\beta > 0$ such that

$$r^\beta m(r) \text{ is non-increasing.}$$

This type of dissipation operator was introduced by Dabkowski, Kiselev, Silvestre and Vicol when they study the well-posedness of slightly supercritical active scalar equations [13]. As pointed out in [13], $\mathcal{L}$ can be equivalently defined by a Fourier multiplier, namely

$$\widehat{\mathcal{L} f}(\xi) = P(|\xi|) \widehat{f}(\xi) \tag{1.3}$$

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*Date*: September 28, 2012.

2000 *Mathematics Subject Classification.* 35Q35, 35B35, 35B65, 76D03.

*Key words and phrases.* supercritical Boussinesq-Navier-Stokes equations, global regularity.

for $P(|\xi|) = m(\frac{1}{|\xi|})$ when $P(\xi)$ satisfies the following conditions:

(1) $P$ satisfies the doubling condition: for any $\xi \in \mathbb{R}^2$,
$$P(2|\xi|) \le c_D P(|\xi|)$$
with constant $c_D \ge 1$;

(2) $P$ satisfies the Hormander-Mikhlin condition (see [33]): for any $\xi \in \mathbb{R}^2$,
$$|\xi|^{|k|}\, |\partial_\xi^k P(|\xi|)| \le c_H P(|\xi|)$$
for some constant $c_H \ge 1$, and for all multi-indices $k \in \mathbb{Z}^d$ with $|k| \le N$, with $N$ only depending on $c_D$;

(3) $P$ has sub-quadratic growth at $\infty$, i.e.
$$\int_0^1 P(|\xi|^{-1})|\xi| d|\xi| < \infty$$

(4) $P$ satisfies
$$(-\Delta)^2 P(|\xi|) \ge c_H^{-1} P(\xi)|\xi|^{-4}$$
for all $|\xi|$ sufficiently large.

Throughout the rest of this paper we assume that $\mathcal{L}$ satisfies both (1.2) and (1.3) with $P(|\xi|) = m(\frac{1}{|\xi|})$ obeying the conditions stated above. Some examples of $m(r)$ are given below:
$$m(r) = \frac{1}{r^\alpha} \qquad \text{for } r > 0 \text{ and } \alpha \in (0,1], \text{ which yields } \mathcal{L} = \Lambda^\alpha;$$
$$m(r) = \frac{1}{r \log^\gamma(e + 1/r)} \qquad \text{for } r > 0, \gamma \ge 0;$$
$$m(r) = \frac{1}{r \log\log(e^2 + 1/r)} \qquad \text{for } r > 0,$$
where $\Lambda = \sqrt{-\Delta}$ denotes the Zygmund operator and corresponds to the Fourier symbol $|\xi|$ (see, e.g. [33]).

We remark that (1.1) can be reformulated in terms of the vorticity $\omega = \nabla \times u$ as follows:
$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \mathcal{L}\omega = \partial_{x_1}\theta, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta\psi = \omega, \\ \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.4}$$
where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and $\psi$ denotes the stream function. Our main result is a global well-posedness theorem for the IVP (1.1) or (1.4) when $\mathcal{L}$ is slightly supercritical. More precisely, we have the following theorem.

**Theorem 1.1.** *Consider the IVP (1.1) and assume that $\mathcal{L}$ satisfies (1.2) and (1.3) with $P(|\xi|) = m(\frac{1}{|\xi|})$ obeying the aforementioned conditions. We further assume that $a(\xi) = a(|\xi|) \equiv |\xi|/P(|\xi|)$ is positive, non-decreasing and satisfies*
$$\lim_{|\xi|\to\infty} \frac{a(|\xi|)}{|\xi|^\sigma} = 0, \quad \forall\, \sigma > 0. \tag{1.5}$$

*Let $q > 2$ and let the initial data $(u_0, \theta_0)$ be in the class*

$$u_0 \in H^1(\mathbb{R}^2), \quad \omega_0 \in L^q(\mathbb{R}^2) \cap B^0_{\infty,1}(\mathbb{R}^2)\ , \quad \theta_0 \in L^2(\mathbb{R}^2) \cap B^{0,a^2}_{\infty,1}(\mathbb{R}^2),$$

*where $\omega_0 = \nabla \times u_0$ is the initial vorticity. Then (1.1) has a unique global solution $(u, \theta)$ satisfying, for all $t > 0$,*

$$u \in L^\infty_t H^1, \quad \omega \in L^\infty_t L^q \cap L^1_t B^0_{\infty,1}\ , \quad \theta \in L^\infty_t L^2 \cap L^\infty_t B^{0,a^2}_{\infty,1} \cap L^1_t B^{0,a}_{\infty,1}.$$

Here $B^0_{\infty,1}$ denotes an inhomogeneous Besov space, whose precise definition is given in the appendix, and $B^{s,a}_{q,r}$ with $a \geq 0$ being a non-decreasing function is defined through the norm

$$\|f\|_{B^{s,a}_{q,r}} = \|2^{js} a(2^j) \|\Delta_j f\|_{L^q}\|_{l^r} < \infty, \tag{1.6}$$

where $\Delta_j$ denotes the Fourier localization operator, defined in the appendix. A special consequence of Theorem 1.1 is the global existence and uniqueness of classical solutions of (1.1) with logarithmically supercritical dissipation,

$$\widehat{\mathcal{L}u}(\xi) = P(|\xi|)\widehat{u}(\xi) \equiv \frac{|\xi|}{\log^\gamma(e + |\xi|)}\, \widehat{u}(\xi) \quad \text{for any } \gamma \geq 0. \tag{1.7}$$

**Corollary 1.2.** *Consider the IVP (1.1) with $\mathcal{L}$ given by (1.7). Assume that $(u_0, \theta_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 1$. Then IVP (1.1) with $\mathcal{L}$ given by (1.7) has a unique global solution $(u, \theta) \in L^\infty([0, T]; H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2))$ for any $T > 0$.*

We are mainly motivated by very recent progress on the global regularity issue concerning the 2D Boussinesq equations with fractional Laplacian dissipation or with partial dissipation (see, e.g., [1, 2, 5, 6, 8, 11, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 27, 29, 30]). The Boussinesq type equations model geophysical fluids and play a very important role in the study of Raleigh-Bernard convection (see, e.g., [10, 16, 25, 31]). Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the Boussinesq equations retain some key features of the 3D Navier-Stokes and the Euler equations such as the vortex stretching mechanism. As pointed out in [26], the inviscid Boussinesq equations can be identified with the 3D Euler equations for axisymmetric flows. In [20] Hmidi, Keraani and Rousset studied the Boussinseq-Navier-Stokes system with critical dissipation, namely (1.1) with

$$\mathcal{L}u = \Lambda u \quad \text{or} \quad \widehat{\mathcal{L}u}(\xi) = |\xi|\, \widehat{u}(\xi)$$

and obtained the global well-posedness. Our intention here has been to explore how far one can go beyond the critical dissipation and still prove the global regularity. Theorem 1.1 obtains the global well-posedness when the critical dissipation is reduced by a factor weaker than any algebraic power such as any power of a logarithm. This result is compatible with a recent work of Chae and Wu [8], in which they studied a generalized Boussinesq-Navier-Stokes system with a velocity field logarithmically more singular than the one determined by the vorticity through the 2D Biot-Savart law.

We now explain the main difficulty that one encounters in the study of the global regularity of solutions to (1.1). One key step in proving the global regularity is to establish suitable global *a priori* bounds for the solutions. Clearly, $u$ is bounded *a priori* in $L^2$ and $\theta$ in $L^q$ for any $q \in [2,\infty]$ if they are initially so. To obtain global *a priori* bounds for the Sobolev norms, we make use of the vorticity equation

$$\partial_t \omega + u\cdot\nabla\omega + \mathcal{L}\omega = \partial_{x_1}\theta.$$

But due to the "vortex stretching" term $\partial_{x_1}\theta$, a simple energy estimate will not lead to a global bound for $\|\omega\|_{L^2}$ unless $\mathcal{L}\omega$ is very dissipative. In the case of the critical dissipation $\mathcal{L}\omega = \Lambda\omega$, Hmidi, Kersaani and Rousset [20] were able to overcome this difficulty by considering a new quantity $\omega - \Lambda^{-1}\partial_{x_1}\theta$ to hide $\partial_{x_1}\theta$. Following their idea, we consider the combined quantity

$$G = \omega - \mathcal{R}_a\theta \quad \text{with} \quad \mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}, \tag{1.8}$$

which satisfies

$$\partial_t G + u\cdot\nabla G + \mathcal{L}G = [\mathcal{R}_a, u\cdot\nabla]\theta. \tag{1.9}$$

This equation can be obtained by taking the difference of the equations for $\omega$ and for $\mathcal{R}_a\theta$. Of course, the trade-off is now to deal with the commutator $[\mathcal{R}_a, u\cdot\nabla]\theta$. After obtaining a general bound for this commutator, we are able to prove global *a priori* bounds for $\|G\|_{L^2}$. By fully exploiting the lower bound for the dissipation and suitably controlling the term associated with the commutator, we can further bound $\|G\|_{L^q}$ for $q\in(2,4)$. In order to show a global bound for $\|G\|_{L^q}$ and $\|\omega\|_{L^q}$ with $q\ge 4$, the strategy is first to bound the space-time norm of $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ and consequently $\|G\|_{L^1_t B^{0,a}_{\infty,1}}$. Making use of the relation (1.8) and bounding $\|\theta\|_{L^1_t B^{0,a^2}_{\infty,1}}$ in terms of $\|\nabla u\|_{L^1_t L^\infty}$ algebraically, we establish global bounds for $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ and for $\|\theta\|_{L^1_t B^{0,a^2}_{\infty,1}}$, which, in turn, are sufficient for the global bound $\|\omega\|_{L^q}$ for any $q\ge 2$. These global bounds guarantee a global solution. To show the uniqueness, we consider the difference of two solutions $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$ and show that the difference must vanish by controlling the velocity difference in $B^0_{2,\infty}$ and the difference $\theta^{(2)}-\theta^{(1)}$ in $B^{-1,a}_{2,\infty}$.

The rest of this paper is divided into six sections and one appendix. Section 2 provides several estimates including lower bounds associated with the dissipative operator $\mathcal{L}$ and a commutator estimate. Section 3 proves a global bound for $\|G\|_{L^2}$ and for $\|\omega\|_{B^{0,a-1}_{2,2}}$.

## 2. Preliminary estimates

This section provides several estimates to be used throughout the rest of the paper. First we recall two bounds from [7] for $\|\Delta_j v\|_{L^p}$ and $\|S_N v\|_{L^p}$ when $v$ is related to $\omega$ through

$$v = \mathcal{R}\,Q\,\omega,$$

where $\mathcal{R}$ denotes the standard Riesz transform and $Q$ a very general Fourier multiplier operator (See Condition 1.1 in [7, p.36]). Here $\Delta_j$ denotes the Fourier localization operator and $S_j$ denotes the identity approximation operator (see the appendix for their definitions). Next we derive some pointwise and Lebesgue-normed estimates associated with the dissipative operator $\mathcal{L}$. In addition, a generalized Bernstein type inequality involving $\mathcal{L}$ is also obtained. Finally we prove an estimate for the commutator $[\mathcal{R}_a, u]F$.

**Lemma 2.1.** *Assume that $v$ and $\omega$ are related through*

$$v = \mathcal{R}\,Q\,\omega,$$

*where $\mathcal{R}$ denotes the standard Riesz transform and $Q$ a Fourier multiplier operator satisfying Condition 1.1 in* [7, p.36]*. Then, for any integer $j \geq 0$ and $N \geq 0$,*

$$\begin{aligned} \|S_N v\|_{L^p} &\leq C_p\, Q(C_0 2^N)\, \|S_N \omega\|_{L^p}, \quad 1 < p < \infty, \\ \|\Delta_j v\|_{L^q} &\leq C\, Q(C_0 2^j)\, \|\Delta_j \omega\|_{L^q}, \quad 1 \leq q \leq \infty, \end{aligned}$$

*where $C_p$ is a constant depending on $p$ only, $C_0$ and $C$ are pure constants.*

Throughout the rest of this paper, $\mathcal{L}$ denotes the operator defined by both (1.2) and (1.3). In addition, we recall that

$$a(|\xi|) \equiv \frac{|\xi|}{P(|\xi|)}, \qquad \mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}. \tag{2.1}$$

The first two lemmas provide lower bounds involving $\mathcal{L}$. These bounds are useful when we estimate the $L^p$-norms of the solution. The idea of proving them is similar to [12].

**Lemma 2.2.** *Let $\mathcal{L}$ be the operator defined by (1.2). Then, for $p > 1$,*

$$|f(x)|^{p-2} f(x)(\mathcal{L} f(x)) \geq \frac{1}{p}\mathcal{L}(|f|^p).$$

*Proof.* By (1.2),

$$\mathcal{L} f(x) = \text{ p.v. } \int \frac{f(x) - f(y)}{|x-y|^d} m(|x-y|)dy$$

and thus

$$|f(x)|^{p-2} f(x) \mathcal{L} f(x) = \text{ p.v. } \int \frac{|f(x)|^p - |f(x)|^{p-2} f(x) f(y)}{|x-y|^d} m(|x-y|)dy.$$

By Young's inequality,

$$|f(x)|^{p-2} f(x) f(y) \leq |f(x)|^{p-1} |f(y)| \leq \frac{p-1}{p}|f(x)|^p + \frac{1}{p}|f(y)|^p$$

Therefore,

$$\begin{aligned} &|f(x)|^{p-2} f(x) \mathcal{L} f(x) \\ &\quad \geq \frac{1}{p} \text{ p.v. } \int \frac{p|f(x)|^p - (p-1)|f(x)|^p - |f(y)|^p}{|x-y|^d} m(|x-y|)dy \\ &\quad \geq \frac{1}{p}\mathcal{L}(|f|^p). \end{aligned}$$

This completes the proof of Lemma 2.2. □

**Lemma 2.3.** *Let $\mathcal{L}$ be the operator defined by (1.2). Then, for $p \geq 2$,*

$$\int |f|^{p-2} f(\mathcal{L} f) dx \geq \frac{2}{p} \int \left| \mathcal{L}^{\frac{1}{2}} (|f|^{\frac{p}{2}}) \right|^2 dx.$$

*Proof.* The $p=2$ case is trivial. For $p>2$, let $\beta=\frac{p}{2}-2$. By Lemma 2.2,

$$\begin{aligned}\int |f|^{p-2}f(\mathcal{L}f)dx &= \int |f|^{\frac{p}{2}}|f|^{\beta}f(\mathcal{L}f)dx\\ &\geq \int |f|^{\frac{p}{2}}\frac{2}{p}\,(\mathcal{L}(|f|^{\frac{p}{2}})dx\\ &= \frac{2}{p}\int\left|\mathcal{L}^{\frac12}(|f|^{\frac{p}{2}})\right|^2 dx.\end{aligned}$$

This completes the proof of Lemma 2.3. □

The following lemma is a generalized version of the Bernstein type inequality associated with the operator $\mathcal{L}$.

**Lemma 2.4.** *Let $j\geq 0$ be an integer and $p\in[2,\infty)$. Let $\mathcal{L}$ be defined by (1.2) and (1.3). Then, for any $f\in\mathcal{S}(\mathbb{R}^d)$,*

$$P(2^j)\|\Delta_j f\|^p_{L^p(\mathbb{R}^d)}\leq C\int_{\mathbb{R}^d}|\Delta_j f|^{p-2}\Delta_j f\mathcal{L}\Delta_j f\,dx, \tag{2.2}$$

*where $C$ is a constant depending on $p$ and $d$ only.*

*Proof.* The case when $p=2$ simply follows from Plancherel's theorem. Now we assume $p>2$. The proof modifies the corresponding ones in [9, 17]. Let $N>0$ be an integer to be specified later. Clearly,

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}\leq\|S_N\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}+\|(Id-S_N)\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}\equiv I_1+I_2.$$

By the standard Bernstein inequality (see the appendix), for $s>0$,

$$I_2\leq C2^{-Ns}\||\Delta_j f|^{\frac{p}{2}}\|_{B^{1+s}_{2,2}}.$$

Applying Lemma 3.2 of [9], we have, for $s\in(0,\min(\frac{p}{2}-1,2))$,

$$\||\Delta_j f|^{\frac{p}{2}}\|_{B^{1+s}_{2,2}}\leq C\|\Delta_j f\|^{\frac{p}{2}-1}_{B^0_{p,2}}\|\Delta_j f\|_{B^{1+s}_{p,2}}\leq C2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

Therefore,

$$I_2\leq C2^{-Ns}2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

By Lemma 2.1,

$$I_1=\|S_N\Lambda\mathcal{L}^{-\frac12}\mathcal{L}^{\frac12}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}\leq C\,2^N\,(P(2^N))^{-\frac12}\,\|\mathcal{L}^{\frac12}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Combining the estimates leads to

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}\leq C2^{-Ns}2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p}+C\,2^N(P(2^N))^{-\frac12}\|\mathcal{L}^{\frac12}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

By the generalized Bernstein inequality for $\Lambda$ in [9],

$$2^j\|\Delta_j f\|^{\frac{p}{2}}_{L^p}\leq C\,\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Therefore,

$$2^j\|\Delta_j f\|^{\frac{p}{2}}_{L^p}\leq C2^{-Ns}2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p}+C\,2^N(P(2^N))^{-\frac12}\|\mathcal{L}^{\frac12}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}. \tag{2.3}$$

We now choose $j<N\leq j+N_0$ with $N_0$ independent of $j$ such that

$$C\,2^{-(N-j)s}\leq\frac12.$$

(2.2) then follows from (2.3). This completes the proof of Lemma 2.4. □

To prove the estimates for the commutator $[\mathcal{R}_a, u]F$, we first state a fact given by the following lemma.

**Lemma 2.5.** *Consider two different cases: $\delta \in (0,1)$ and $\delta = 1$.*

(1) *Let $\delta \in (0,1)$ and $q \in [1,\infty]$. If $|x|^\delta h \in L^1$, $f \in \mathring{B}^\delta_{q,\infty}$ and $g \in L^\infty$, then*

$$\|h * (fg) - f(h * g)\|_{L^q} \le C \, \||x|^\delta \phi\|_{L^1} \, \|f\|_{\mathring{B}^\delta_{q,\infty}} \|g\|_{L^\infty}, \tag{2.4}$$

*where $C$ is a constant independent of $f, g$ and $h$.*

(2) *Let $\delta = 1$. Let $q \in [1,\infty]$. Let $r_1 \in [1,q]$ and $r_2 \in [1,\infty]$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then*

$$\|h * (fg) - f(h * g)\|_{L^q} \le C \, \||x|h\|_{L^{r_1}} \, \|\nabla f\|_{L^q} \, \|g\|_{L^{r_2}}, \tag{2.5}$$

Here $\mathring{B}^\delta_{q,\infty}$ denotes a homogeneous Besov space, as defined in the appendix. (2.4) is taken from [8] while (2.5) was obtained in [20, p.426]. We also recall the definition of the Besov type norm

$$\|f\|_{B^{s,a}_{q,r}} = \|2^{js} a(2^j) \|\Delta_j f\|_{L^q}\|_{l^r} < \infty, \tag{2.6}$$

as defined in (1.6) or in (A.5) in the appendix. With these notation at our disposal, we are ready to state and prove the commutator estimate.

**Proposition 2.6.** *Let $a$ and $\mathcal{R}_a$ be defined as in (2.1). Assume*

$$p \in [2,\infty), \quad q \in [1,\infty], \quad 0 < s < \delta.$$

*Let $[\mathcal{R}_a, u]F = \mathcal{R}_a(uF) - u\mathcal{R}_a F$ be a standard commutator. Then*

$$\|[\mathcal{R}_a, u]F\|_{B^{s,a}_{p,q}} \le C \,(\|u\|_{\mathring{B}^\delta_{p,\infty}} \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}} + \|u\|_{L^2} \, \|F\|_{L^2}),$$

*where $C$ denotes a constant independent of $a$ and $\mathcal{R}_a$.*

*Proof of Proposition 2.6.* Let $j \ge -1$ be an integer. Using the notion of paraproducts, we decompose $\Delta_j[\mathcal{R}_a, u]F$ into three parts,

$$\Delta_j[\mathcal{R}_a, u]F \;=\; I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(S_{k-1}u \cdot \Delta_k F) - S_{k-1}u \cdot \mathcal{R}_a \Delta_k F), \\ I_2 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(\Delta_k u \cdot S_{k-1}F) - \Delta_k u \cdot \mathcal{R}_a S_{k-1}F), \\ I_3 &= \sum_{k\ge j-1} \Delta_j(\mathcal{R}_a(\Delta_k u \cdot \tilde{\Delta}_k F) - \Delta_k u \mathcal{R}_a \cdot \tilde{\Delta}_k F). \end{aligned}$$

When the operator $\mathcal{R}_a$ acts on a function whose Fourier transform is supported on an annulus, it can be represented as a convolution kernel. Since the Fourier transform of $S_{k-1}u \cdot \Delta_k F$ is supported on an annulus around the radius of $2^k$, we can write

$$h_k \star (S_{k-1}u \cdot \Delta_k F) - S_{k-1}u \cdot (h_k \star \Delta_k F),$$

where $h_k$ is given by the inverse Fourier transform of $i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)$, namely

$$h_k(x) = \Big(i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)\Big)^\vee(x).$$

Here $\widetilde{\Phi}_k(\xi) \in C_0^\infty(\mathbb{R}^2)$, $\widetilde{\Phi}_k(\xi)$ is also supported on an annulus around the radius of $2^k$ and is identically equal to 1 on the support of $S_{k-1}u\cdot\Delta_k F$. Therefore, recalling (2.1), we can write

$$i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi) = i\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(2^{-k}\xi)\,a(|\xi|).$$

Therefore,

$$h_k(x) = 2^{2k}\,h_0(2^k x) * a^\vee(x), \quad h_0(x) = \left(\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(\xi)\right)^\vee.$$

By Lemma 2.5,

$$\begin{aligned}
\|I_1\|_{L^p} &\le C\,\||x|^\delta h_j\|_{L^1}\|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}}\,\|\Delta_j F\|_{L^\infty}\\
&\le C\,2^{-\delta j}\,a(2^j)\,\|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}}\,\|\Delta_j F\|_{L^\infty}.
\end{aligned}$$

$I_2$ in $L^p$ can be estimated as follows.

$$\begin{aligned}
\|I_2\|_{L^p} &\le C\,2^{-\delta j}\,a(2^j)\,\|S_{j-1}F\|_{L^\infty}\|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}\\
&\le C\,2^{-\delta j}\,a(2^j)\sum_{m\le j-1}\|\Delta_m F\|_{L^\infty}\,\|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}\\
&= C\,2^{-sj}a^{-1}(2^j)\sum_{m\le j-1}2^{(s-\delta)(j-m)}\frac{a^2(2^j)}{a^2(2^m)}\,2^{(s-\delta)m}a^2(2^m)\,\|\Delta_m F\|_{L^\infty}\|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}.
\end{aligned}$$

The estimate of $\|I_3\|_{L^p}$ is different. We need to distinguish between low frequency and high frequency terms. For $j = 0, 1$, the terms in $I_3$ with $k = -1, 0, 1$ have Fourier transforms containing the origin in their support and the lower bound part of Bernstein's inequality does not apply. To deal with these low frequency terms, we take advantage of the commutator structure and bound them by Lemma 2.5. The kernel $h$ corresponding to $\mathcal{R}_a$ still satisfies, for any $r_1 \in (1,\infty)$,

$$\||x|\,h\|_{L^{r_1}} \le C.$$

Therefore, by Lemma 2.5 and Bernstein's inequality, for $j = 0, 1$ and $k = -1, 0, 1$,

$$\begin{aligned}
\|\Delta_j(\mathcal{R}_a(\Delta_k u\cdot\tilde{\Delta}_k F) - \Delta_k u\cdot\mathcal{R}_a\tilde{\Delta}_k F)\|_{L^p} &\le C\,\||x|h\|_{L^{r_1}}\,\|\nabla\Delta_k u\|_{L^p}\,\|\Delta_k F\|_{L^{r_2}}\\
&\le C\,\|u\|_{L^2}\,\|F\|_{L^2}.
\end{aligned}$$

where $\frac{1}{r_1}+\frac{1}{r_2}=1$. For the high frequency terms, we do not need the commutator structure. By Lemma 2.1 and Hölder's inequality,

$$\begin{aligned}\|I_{31}\|_{L^p} &\equiv \Big\|\sum_{k\ge j-1}\Delta_j(\mathcal{R}_a(\Delta_k u\cdot\tilde{\Delta}_k F))\Big\|_{L^p} \le \sum_{k\ge j-1} C\,a(2^j)\,\|\Delta_k u\|_{L^p}\,\|\Delta_k F\|_{L^\infty}\\ &\le C\ a(2^j)\sum_{k\ge j-1}2^{-\delta k}\,2^{\delta k}\|\Delta_k u\|_{L^p}\,\|\Delta_k F\|_{L^\infty}\\ &\le C\,2^{-sj}\,a^{-1}(2^j)\,\|u\|_{\mathring{B}^\delta_{p,\infty}}\sum_{k\ge j-1}2^{s(j-k)}\frac{a^2(2^j)}{a^2(2^k)}\,2^{(s-\delta)k}a^2(2^k)\ \|\Delta_k F\|_{L^\infty}.\end{aligned}$$

$I_{32}\equiv\sum_{k\ge j-1}\Delta_k u\cdot\mathcal{R}_a\tilde{\Delta}_k F$ admits the same bound. Therefore, by the definition of the norm in (2.6),

$$\begin{aligned}\|[\mathcal{R}_a,u]F\|_{B^{s,a}_{p,q}} \ \le\ & \Big[\sum_{j\ge-1}2^{qsj}a^q(2^j)\|I_1\|^q_{L^p}\Big]^{\frac1q}+\Big[\sum_{j\ge-1}2^{qsj}a^q(2^j)\|I_2\|^q_{L^p}\Big]^{\frac1q}\\ &+\Big[\sum_{j\ge-1}2^{qsj}a^q(2^j)(\|I_{31}\|^q_{L^p}+\|I_{32}\|^q_{L^p})\Big]^{\frac1q}+C\,\|u\|_{L^2}\,\|F\|_{L^2}.\end{aligned}$$

The first term on the right is clearly bounded by

$$C\,\|u\|_{\mathring{B}^\delta_{p,\infty}}\Big[\sum_{j\ge-1}2^{q(s-\delta)j}a^{2q}(2^j)\|\Delta_j F\|^q_{L^\infty}\Big]^{\frac1q}=C\,\|u\|_{\mathring{B}^\delta_{p,\infty}}\,\|F\|_{B^{s-\delta,\,a^2}_{\infty,q}}.$$

Due to $s<\delta$, (1.5) and a convolution inequality for series,

$$\Big[\sum_{j\ge-1}2^{qsj}a^q(2^j)\|I_2\|^q_{L^p}\Big]^{\frac1q}\le C\,\|u\|_{\mathring{B}^\delta_{p,\infty}}\,\|F\|_{B^{s-\delta,\,a^2}_{\infty,q}}.$$

Thanks to $0<s$, (1.5) and a convolution inequality for series,

$$\Big[\sum_{j\ge-1}2^{qsj}a^q(2^j)\|I_{31}\|^q_{L^p}\Big]^{\frac1q}\le C\,\|u\|_{\mathring{B}^\delta_{p,\infty}}\,\|F\|_{B^{s-\delta,\,a^2}_{\infty,q}}.$$

This completes the proof of Proposition 2.6. □

## 3. Global *a priori* bound for $\|\omega\|_{B^{0,a^{-1}}_{2,2}}$

This section establishes a global *a priori* estimates for $\|G\|_{L^2}$ and consequently for $\|\omega\|_{B^{0,a^{-1}}_{2,2}}$.

**Proposition 3.1.** *Assume that the initial data $(u_0,\theta_0)$ satisfies the conditions in Theorem 1.1. Let $(u,\theta)$ be the corresponding solution and let $\omega=\nabla\times u$ be the vorticity. Let*

$$G=\omega-\mathcal{R}_a\theta,\qquad \mathcal{R}_a=\mathcal{L}^{-1}\partial_{x_1}. \tag{3.1}$$

*Then, for any* $t \geq 0$,

$$\|G\|_{L^2}^2 + \int_0^t \|\mathcal{L}^{\frac{1}{2}} G(\tau)\|_{L^2}^2 \, d\tau \leq B(t)$$

*and consequently*

$$\|\omega(t)\|_{B^{0,\,a-1}_{2,2}} \leq B(t),$$

*where* $B(t)$ *is integrable on any finite-time interval* $[0, T]$.

*Proof.* Trivially $u$ and $\theta$ obey the following global *a priori* bounds

$$\|\theta(t)\|_{L^2\cap L^\infty} \leq \|\theta_0\|_{L^2\cap L^\infty}, \quad \|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}. \tag{3.2}$$

It is easy to check that $G$ satisfies

$$\partial_t G + u\cdot\nabla G + \mathcal{L}G = [\mathcal{R}_a, u\cdot\nabla]\theta. \tag{3.3}$$

Taking the inner product with $G$ leads to

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + \int G\,\mathcal{L}G\,dx = \int G\nabla\cdot[\mathcal{R}_a, u]\theta\,dx. \tag{3.4}$$

By the Hölder inequality and the boundedness of Riesz transforms on $L^2$,

$$\left|\int G\nabla\cdot[\mathcal{R}_a, u]\theta dx\right| \leq \|\mathcal{L}^{\frac{1}{2}}G\|_{L^2}\,\|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2}.$$

Inserting this estimate in (3.4) and applying Young's inequality, we obtain

$$\frac{d}{dt}\|G\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}}G\|_{L^2}^2 \leq \|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2}^2. \tag{3.5}$$

By the definition of the norm in (2.6), $\|\mathcal{L}^{-\frac{1}{2}}\Lambda f\|_2 \leq \|f\|_{B^{\frac{1}{2},\,\frac{a}{2}}_{2,2}}$. Applying Proposition 2.6 with $\delta > \frac{1}{2}$ and $p = q = 2$, we obtain

$$\|[\mathcal{R}_a, u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \leq C\|u\|_{B^{\delta}_{2,\infty}}\,\|\theta\|_{B^{\frac{1}{2}-\delta,\,\frac{a^2}{4}}_{\infty,2}} + C\,\|u\|_{L^2}\,\|\theta\|_{L^2}.$$

Since $u = \nabla^\perp\Delta^{-1}\omega$,

$$\begin{aligned}\|u\|_{B^{\delta}_{2,\infty}} &= \sup_{j\geq -1} 2^{\delta j}\,\|\Delta_j u\|_{L^2} \leq \|\Delta_{-1}u\|_{L^2} + \sup_{j\geq 0} 2^{\delta j}\,\|\Delta_j\nabla^\perp\Delta^{-1}\omega\|_{L^2}\\ &\leq \|u\|_{L^2} + \sup_{j\geq 0} 2^{(\delta-1)j}\|\Delta_j\omega\|_{L^2} \leq \|u\|_{L^2} + \|\omega\|_{B^{0,a-1}_{2,2}}.\end{aligned}$$

For $\delta > \frac{1}{2}$, $\|\theta\|_{B^{\frac{1}{2}-\delta,\,\frac{a^2}{4}}_{\infty,2}} \leq \|\theta\|_{L^\infty}$. Therefore,

$$\|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2} \leq \|[\mathcal{R}_a, u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \leq C\,\|u\|_{L^2}\,\|\theta\|_{L^2\cap L^\infty} + \|\omega\|_{B^{0,a-1}_{2,2}}\|\theta\|_{L^\infty}. \tag{3.6}$$

We can bound the $\|\omega\|_{B^{0,a-1}_{2,2}}$ by

$$\|\omega\|_{B^{0,a-1}_{2,2}} \leq \|G\|_{B^{0,a-1}_{2,2}} + \|\mathcal{R}_a\theta\|_{B^{0,a-1}_{2,2}} \leq \|G\|_2 + \|\theta\|_2. \tag{3.7}$$

Since $\|u\|_{L^2}$ and $\|\theta\|_{L^2\cap L^\infty}$ are bounded by (3.2), we combine (3.5), (3.6) and (3.7) to obtain the desired result. This completes the proof of Proposition 3.1. □

## 4. Global *a priori* bound for $\|G\|_{L^q}$ with $q \in (2,4)$

This section establishes a global *a priori* bounds for $\|\omega\|_{L^q}$ with $q \in (2,4)$.

**Proposition 4.1.** *Assume that the initial data $(u_0, \theta_0)$ satisfies the conditions stated in Theorem 1.1. Let $(u, \theta)$ be the corresponding solution and $G$ be defined as in (3.1). Then, for any $q \in (2,4)$, $G$ obeys the global bound, for any $T > 0$ and $t \le T$,*

$$\|G(t)\|_{L^q}^q + C\int_0^t \int \left|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\right|^2 dxdt + C\int_0^t \|G\|^q_{L^{\frac{2q}{1+\epsilon}}}\, d\tau \le B(t), \tag{4.1}$$

*where $C$ is a constant depending on $q$ only and $B(t)$ is integrable on any finite time interval. A special consequence is that, for any small $\epsilon > 0$,*

$$\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}} \le B(t). \tag{4.2}$$

*Proof.* Multiplying (3.3) by $G|G|^{q-2}$ and integrating with respect to $x$, we obtain

$$\frac{1}{q}\frac{d}{dt}\|G\|_{L^q}^q + \int G|G|^{q-2}\mathcal{L}G dx = -\int G|G|^{q-2}\nabla\cdot[\mathcal{R}_a, u]\theta\, dx.$$

By Lemma 2.3,

$$\int G|G|^{q-2}\mathcal{L}G dx \ge C\int |\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})|^2\, dx.$$

Set $\epsilon > 0$ to be small, say, for $q \in (2,4)$,

$$(1+\epsilon)\left(1-\frac{2}{q}\right) < \frac{1}{2}.$$

Thanks to the condition in (1.5) and by a Sobolev embedding,

$$\begin{aligned}
\|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 &= \sum_{j\ge -1}\|\Delta_j \mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= \sum_{j\ge -1} 2^j a^{-1}(2^j)\|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\ge C\sum_{j\ge -1} 2^{(1-\epsilon)j}\|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= C\,\|\Lambda^{\frac{1}{2}-\frac{\epsilon}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\ge C\,\|G\|^q_{L^{\frac{2q}{1+\epsilon}}}.
\end{aligned}$$

For $q \in (2,4)$, we choose $s > 0$ such that

$$s > \epsilon, \qquad s + (1+\epsilon)\left(1-\frac{2}{q}\right) = \frac{1}{2} - \epsilon.$$

By Hölder's inequality,

$$\left|\int G|G|^{q-2}\nabla\cdot[\mathcal{R}_a, u]\theta\right| \le \|G|G|^{q-2}\|_{\mathring{H}^s}\|[\mathcal{R}_a, u]\theta\|_{\mathring{H}^{1-s}}.$$

By Lemma 4.2 below,

$$\|G|G|^{q-2}\|_{\mathring{H}^s} \le C\,\|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\,\|G\|_{\mathring{H}^{s+(1+\epsilon)(1-\frac{2}{q})}} = C\,\|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\,\|G\|_{\mathring{H}^{\frac{1}{2}-\epsilon}}.$$

In addition, due to the condition in (1.5),

$$\|G\|^2_{\mathring{H}^{\frac12-\epsilon}} = \sum_{j\ge -1} 2^{j-2\epsilon j}\|\Delta_j G\|^2_{L^2} \le \sum_{j\ge -1} 2^j a^{-2}(2^j)\|\Delta_j G\|^2_{L^2} \le \|\mathcal{L}^{\frac12}(G)\|^2_{L^2}.$$

By Proposition 2.6, recalling $s>\epsilon$ and $u=\nabla^\perp\Delta^{-1}\omega$,

$$\begin{aligned}\|[\mathcal{R}_a,u]\theta\|_{\mathring{H}^{1-s}} &\le C\,\|u\|_{\mathring{B}^{1-s+\epsilon}_{2,\infty}}\|\theta\|_{B^{-\epsilon,1}_{\infty,2}} + C\,\|u\|_{L^2}\,\|\theta\|_{L^2}\\ &\le C\,\|\omega\|_{B^{0,\frac1a}_{2,2}}\|\theta\|_{L^\infty} + C\,\|u\|_{L^2}\|\theta\|_{L^2}.\end{aligned}$$

Putting the estimates together, we obtain

$$\begin{aligned}&\frac1q\frac{d}{dt}\|G\|^q_{L^q} + C\int |\mathcal{L}^{\frac12}(|G|^{\frac q2})|^2\,dx + C\,\|G\|^q_{L^{\frac{2q}{1+\epsilon}}}\\ &\le C\,\|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\|\mathcal{L}^{\frac12}(G)\|_{L^2}\left(\|\omega\|_{B^{0,\frac1a}_{2,2}}\|\theta\|_{L^\infty} + C\,\|u\|_{L^2}\|\theta\|_{L^2}\right).\end{aligned}$$

Applying Young's inequality to the right-hand side, noticing that $q\in(2,4)$ and resorting to the bounds in Proposition 3.1, we obtain (4.1). (4.2) follows from the inequality

$$\|\omega\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{B^{-\epsilon}_{q,\infty}} + \|\mathcal{R}_a\theta\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{L^q} + \|\theta\|_{L^q}.$$

This completes the proof of Proposition 4.1. □

We have used the following lemma in the proof of Proposition 4.1.

**Lemma 4.2.** *Let $q\in(2,\infty)$, $s\in(0,1)$, $0<\epsilon(q-2)\le 2$ and $f\in L^{\frac{2q}{1+\epsilon}}\cap \mathring{H}^{s+(1-\frac2q)(1+\epsilon)}$. Then*

$$\||f|^{q-2}f\|_{\mathring{H}^s} \le C\,\|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\|f\|_{\mathring{B}^s_{\frac{2q}{2-\epsilon(q-2)},2}} \le C\,\|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}}\|f\|_{\mathring{H}^{s+(1-\frac2q)(1+\epsilon)}}. \tag{4.3}$$

*Proof.* This proof modifies that of [20]. Identifying $\mathring{H}^s$ with $\mathring{B}^s_{2,2}$ and by the definition of $\mathring{B}^s_{2,2}$, we have

$$\||f|^{q-2}f\|^2_{\mathring{H}^s} = \int \frac{\||f|^{q-2}f(x+y)-|f|^{q-2}f(x)\|^2_{L^2}}{|y|^{2+2s}}\,dy.$$

Thanks to the inequality

$$\big||f|^{q-2}f(x+y)-|f|^{q-2}f(x)\big| \le C\,\left(|f|^{q-2}(x+y)+|f|^{q-2}(x)\right)|f(x+y)-f(x)|,$$

we have, by Hölder's inequality

$$\||f|^{q-2}f(x+y)-|f|^{q-2}f(x)\|^2_{L^2} \le C\,\|f\|^{2(q-2)}_{L^{\frac{2q}{1+\epsilon}}}\|f(x+y)-f(x)\|^2_{L^\rho},$$

where

$$\rho = \frac{2q}{2-\epsilon(q-2)}.$$

Therefore,

$$\||f|^{q-2}f\|^2_{\mathring{H}^s} \le C\,\|f\|^{2(q-2)}_{L^{\frac{2q}{1+\epsilon}}}\|f\|^2_{\mathring{B}^s_{\rho,2}}.$$

Further applying the Besov embedding inequality

$$\|f\|_{\mathring{B}^s_{\rho,2}} \le C\,\|f\|_{\mathring{H}^{s+1-\frac2\rho}},$$

we obtain (4.3) and this completes the proof of Lemma 4.2. □

## 5. Global *a priori* bound for $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ with $q \in [2,4)$

This section proves a global *a priori* bound for $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ with $q \in (2,4)$. This bound serves as an important step towards a global bound for $\|\omega\|_{L^q}$ with general $q \in [2,\infty)$.

**Proposition 5.1.** *Assume that the initial data $(u_0, \theta_0)$ satisfies the conditions stated in Theorem 1.1. Let*

$$r \in [1,\infty], \quad s \in [0,1), \quad q \in (2,4).$$

*Then, for any $t > 0$, $G$ obeys the following global bound*

$$\|G\|_{\widetilde{L}^r_t B^s_{q,1}} \le B(t), \tag{5.1}$$

*where $B$ is integrable on any finite-time interval.*

*Proof.* Let $j \ge -1$ be an integer. Applying $\Delta_j$ to (3.3) yields

$$\partial_t \Delta_j G + \mathcal{L}\Delta_j G = -\Delta_j(u\cdot\nabla G) - \Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta.$$

Taking the inner product with $\Delta_j G|\Delta_j G|^{q-2}$, we have

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|^q_{L^q} + \int \Delta_j G|\Delta_j G|^{q-2}\mathcal{L}\Delta_j G = J_1 + J_2, \tag{5.2}$$

where

$$J_1 = -\int \Delta_j(u\cdot\nabla G)\,\Delta_j G|\Delta_j G|^{q-2}, \tag{5.3}$$

$$J_2 = -\int \Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta\ \Delta_j G|\Delta_j G|^{q-2}.$$

According to Lemma 2.4, for $j \ge 0$, the dissipation part can be bounded below by

$$\int \Delta_j G|\Delta_j G|^{q-2}\mathcal{L}\Delta_j G \ge C P(2^j)\|\Delta_j G\|^q_{L^q}. \tag{5.4}$$

By Lemma 5.2 below, $J_1$ can be bounded by

$$\begin{aligned} \|J_1\|_{L^q} \le\ & C\, 2^{j(\epsilon+\frac{2}{q})}\, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} \\ & + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]\, \|\Delta_j G\|^{q-1}_{L^q}, \end{aligned} \tag{5.5}$$

where we have taken $\epsilon$ to be small positive number, especially

$$s - 1 + 3\epsilon < 0.$$

To bound $J_2$, we first apply Hölder's inequality and then employ similar estimates as in the proof of Proposition 2.6 to obtain

$$\begin{aligned} |J_2| \le\ & \|\Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta\|_{L^q}\|\Delta_j G\|^{q-1}_{L^q} \\ \le\ & C\left(2^{j\epsilon} a(2^j)\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\|\theta\|_{L^\infty} + \|u\|_{L^2}\|\theta\|_{L^2}\right)\, \|\Delta_j G\|^{q-1}_{L^q}. \end{aligned} \tag{5.6}$$

Inserting (5.4), (5.5) and (5.6) in (5.2) and writing the bound for $\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}}$ by $B(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\Delta_j G\|_{L^q} + C2^j\, a^{-1}(2^j)\,\|\Delta_j G\|_{L^q} &\le C2^{\epsilon j}\, a(2^j)\, B(t)\\ &+ C2^{j(\epsilon+\frac{2}{q})} B(t)\Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q}\\ &\qquad + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big].\end{aligned}$$

Due to (1.5), $a(2^j) \le 2^{\epsilon j}$. Integrating in time yields

$$\begin{aligned}\|\Delta_j G(t)\|_{L^q} &\le e^{-C\,2^{(1-\epsilon)j}t}\|\Delta_j G(0)\|_{L^q} + C\,2^{-j(1-3\epsilon)}\,B(t)\\ &+ C\,2^{j(\epsilon+\frac{2}{q})} B(t)\int_0^t e^{-C\,2^{(1-\epsilon)j}(t-\tau)} L(\tau)\,d\tau,\end{aligned}$$

where, for notational convenience, we have written

$$L(t) = \Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big].$$

Taking the $L^r$ norm in time and applying Young's inequality for convolution lead to

$$\begin{aligned}\|\Delta_j G\|_{L^r_t L^q} &\le C\,2^{-\frac{1}{r}(1-\epsilon)j}\,\|\Delta_j G(0)\|_{L^q} + C\,2^{-j(1-3\epsilon)}\,\widetilde{B}(t)\\ &+ C\,2^{j(-1+2\epsilon+\frac{2}{q})}\widetilde{B}(t)\,\|L\|_{L^r}.\end{aligned}$$

Multiplying by $2^{js}$, summing over $j \ge -1$ and noticing $s - 1 + 3\epsilon < 0$, we obtain

$$\|G\|_{\widetilde{L}^r_t B^s_{q,1}} \le C\,\|G(0)\|_{B^{s-1/r(1-\epsilon)}_{q,1}} + C\,\widetilde{B}(t) + K_1 + K_2 + K_3, \tag{5.7}$$

where

$$\begin{aligned}K_1 &= C\sum_{j\ge -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\,\widetilde{B}(t)\,2^{js}\|\Delta_j G\|_{L^r_t L^q},\\ K_2 &= C\sum_{j\ge -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\,\widetilde{B}(t)\,2^{js}\sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^r_t L^q},\\ K_3 &= C\sum_{j\ge -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\,\widetilde{B}(t)\,2^{js}\sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^r_t L^q}.\end{aligned}$$

Since $-1 + 2\epsilon + \frac{2}{q} < 0$, we can choose an integer $N > 0$ such that

$$C\,2^{N(-1+2\epsilon+\frac{2}{q})}\widetilde{B}(t) \le \frac{1}{8}.$$

The sums in $K_1$, $K_2$ and $K_3$ can then be split into two parts: $j \le N$ and $j > N$. Since $\|G\|_{L^q}$ is bounded, the sum for the first part is bounded by $C\,\widetilde{B}(t)2^{sN}$. The second part of the sum over $j > N$ is bounded by $\frac{1}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$. Therefore,

$$K_1, K_2, K_3 \le C\,\widetilde{B}(t)2^{sN} + \frac{3}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}.$$

Combining these bounds with (5.7) yields the desired estimates. This completes the proof of Proposition 5.1. □

We now provide the details leading to (5.5). They bear some similarities as those in [8], but they are provided here for the sake of completeness.

**Lemma 5.2.** *Let $J_1$ be defined as in (5.3). Then we have the following bound*

$$\begin{aligned}\|J_1\|_{L^q} \;&\le\; C\,2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} \\ &\quad + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]\,\|\Delta_j G\|_{L^q}^{q-1}.\end{aligned}$$

*Proof.* Using the notion of paraproducts, we write

$$\Delta_j(u\cdot\nabla G) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15},$$

where

$$\begin{aligned}J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k G,\\ J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k G,\\ J_{13} &= S_j u\cdot\nabla\Delta_j G,\\ J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\ J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k G).\end{aligned}$$

Since $\nabla\cdot u = 0$, we have

$$\int J_{13}|\Delta_j G|^{q-2}\Delta_j G\,dx = 0.$$

By Hölder's inequality,

$$\left|\int J_{11}|\Delta_j G|^{q-2}\Delta_j G\right| \le \|J_{11}\|_{L^q}\|\Delta_j G\|_{L^q}^{q-1}.$$

We write the commutator in terms of the integral,

$$J_{11} = \int \Phi_j(x-y)\,(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k G(y)\,dy,$$

where $\Phi_j$ is the kernel of the operator $\Delta_j$ and more details can be found in the Appendix. As in the proof of Lemma 3.3, we have, for any $0<\epsilon<1$,

$$\|J_{11}\|_{L^q} \le \||x|^{1-\epsilon}\Psi_j(x)\|_{L^1}\,\|S_{j-1}u\|_{\mathring{B}^{1-\epsilon}_{q,\infty}}\,\|\nabla\Delta_j G\|_{L^\infty}.$$

By the definition of $\Phi_j$ and Bernstein's inequality (see the Appendix), we have

$$\begin{aligned}\|J_{11}\|_{L^q} \;&\le\; C\,2^{j(\epsilon+\frac{2}{q})}\,\||x|^{1-\epsilon}\Psi_0(x)\|_{L^1}\,\|S_{j-1}\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\|\Delta_j G\|_{L^\infty}\\ &\le\; C\,2^{j(\epsilon+\frac{2}{q})}\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\,\|\Delta_j G\|_{L^q}.\end{aligned}$$

Again, by Bernstein's inequality,

$$\begin{aligned}\|J_{12}\|_{L^q} \;&\le\; C\|\Delta_j u\|_{L^q}\|\nabla\Delta_j G\|_{L^\infty}\\ &\le\; C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\,\|\Delta_j G\|_{L^q};\end{aligned}$$

$$\begin{aligned}\|J_{14}\|_{L^q} &\le C\|\Delta_j u\|_{L^q}\|\nabla S_{j-1}G\|_{L^\infty}\\ &\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q};\end{aligned}$$

$$\begin{aligned}\|J_{15}\|_{L^q} &\le C2^{j(\epsilon+\frac{2}{q})}\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Lambda^{1-\epsilon}\Delta_k u\|_{L^q}\|\Delta_k G\|_{L^q}\\ &\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}.\end{aligned}$$

Combining the estimates above yields

$$\begin{aligned}\|J_1\|_{L^q} \le\ & C\,2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\Big[\|\Delta_j G\|_{L^q}+\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q}\\ &+\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]\,\|\Delta_j G\|_{L^q}^{q-1}.\end{aligned}$$

This completes the proof of Lemma 5.2. □

## 6. Global *a priori* bounds for $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ and $\|\omega\|_{L^q}$ for any $q \ge 2$

This section shows that, if the initial data $\omega_0$ is in $L^q$, then the solution $\omega$ is also *a priori* in $L^q$ at any time. This is established by first proving the time integrability $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$. More precisely, we have the following theorem.

**Proposition 6.1.** *Assume that the initial data $(u_0, \theta_0)$ satisfies the conditions as stated in Theorem 1.1. Then we have the following global a priori bounds. For any $T > 0$ and $t \le T$,*

$$\|\omega(t)\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T), \quad \|\theta(t)\|_{B^{0,a^2}_{\infty,1}} \le C(T), \quad \|\omega(t)\|_{L^q} \le C(T),$$

*where $C(T)$ are constants depending on $T$ and the initial norms only.*

In order to prove this proposition, we need the following fact.

**Lemma 6.2.** *Let $T > 0$ and let $u$ be a divergence-free smooth vector field satisfying*

$$\int_0^T \|\nabla u\|_{L^\infty}\,dt < \infty.$$

*Assume that $\theta$ solves*

$$\partial_t\theta + u\cdot\nabla\theta = f.$$

*Let $a : (0,\infty) \to (0,\infty)$ be an nondecreasing and radially symmetric function satisfying (1.5). Let $\rho \in [1,\infty]$. For any $t > 0$,*

$$\|\theta\|_{B^{0,a}_{\rho,1}} \le (\|\theta_0\|_{B^{0,a}_{\rho,1}} + \|f\|_{L^1_t B^{0,a}_{\rho,1}})\left(1+\int_0^t \|\nabla u\|_{L^\infty}dt\right).$$

This lemma can be proven in a similar fashion as that of Lemma 4.5 in [8]. A crucial assumption is that $a$ satisfies (1.5).

*Proof of Proposition 6.1.* We first explains that (5.1) in Proposition 5.1 implies that, for $t \le T$,

$$\|G\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T).$$

In fact, if we choose $s \in [0, 1)$ satisfying $s > \frac{2}{q}$ for $q \in (2, 4)$ and set $\epsilon > 0$ satisfying $\epsilon + \frac{2}{q} - s < 0$, then

$$\begin{aligned} \|G\|_{B^{0,a}_{\infty,1}} &\equiv \sum_{j\ge -1} a(2^j)\|\Delta_j G\|_{L^\infty} \le \sum_{j\ge -1} a(2^j) 2^{\frac{2}{q}j} \|\Delta_j G\|_{L^q} \\ &\le \sum_{j\ge -1} a(2^j)2^{-\epsilon j}\, 2^{j(\epsilon+\frac{2}{q}-s)} 2^{js}\|\Delta_j G\|_{L^q} \le C\,\|G\|_{B^s_{q,1}}, \end{aligned}$$

where we have used the fact that $a(2^j)2^{-\epsilon j} \le C$ for $C$ independent of $j$. Furthermore,

$$\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \le \|G\|_{L^1_t B^{0,a}_{\infty,1}} + \|\mathcal{R}_a\theta\|_{L^1_t B^{0,a}_{\infty,1}}.$$

By the definition of the norm in $B^{0,a}_{\infty,1}$ and recalling that $\mathcal{R}_a\theta$ is defined by the multiplier $a(|\xi|)\frac{i\xi_1}{|\xi|}$, we have

$$\begin{aligned} \|\mathcal{R}_a\theta\|_{B^{0,a}_{\infty,1}} &= a(2^{-1})\,\|\Delta_{-1}\mathcal{R}_a\theta\|_{L^\infty} + \sum_{j\ge 0} a(2^j)\,\|\Delta_j \mathcal{R}_a\theta\|_{L^\infty} \\ &\le C\,\|\theta_0\|_{L^2} + \sum_{j\ge 0} a^2(2^j)\,\|\Delta_j\theta\|_{L^\infty} \\ &\le C\,\|\theta_0\|_{L^2} + \|\theta\|_{B^{0,a^2}_{\infty,1}}. \end{aligned}$$

By Lemma 6.2,

$$\begin{aligned} \|\theta\|_{B^{0,a^2}_{\infty,1}} &\le C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \left(1 + \int_0^t \|\nabla u\|_{L^\infty} dt\right) \\ &\le C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \left(1 + \|u\|_{L^1_t L^2} + \|\omega\|_{L^1_t B^0_{\infty,1}}\right) \\ &\le C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \left(1 + \|u\|_{L^1_t L^2} + \|\omega\|_{L^1_t B^{0,a}_{\infty,1}}\right). \end{aligned} \tag{6.1}$$

Therefore,

$$\begin{aligned} \|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \le\ & \|G\|_{L^1_t B^{0,a}_{\infty,1}} + C\left(\|\theta_0\|_{L^2} + \|\theta_0\|_{B^{0,a^2}_{\infty,1}}\right) t \\ & + C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \int_0^t \|u\|_{L^1_\tau L^2}\, d\tau + C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \int_0^t \|\omega\|_{L^1_\tau B^{0,a}_{\infty,1}}\, d\tau. \end{aligned}$$

By Gronwall's inequality, $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T)$, which, in turn, implies that, by (6.1),

$$\|\theta(t)\|_{B^{0,a^2}_{\infty,1}} \le C(T).$$

Now we prove the bound for $\|\omega\|_{L^q}$. From the equations of $G$ and $\mathcal{R}_a\theta$,

$$\begin{aligned}\|\omega\|_{L^q} &\le \|G\|_{L^q} + \|\mathcal{R}_a\theta\|_{L^q}\\ &\le \|G_0\|_{L^q} + \|\mathcal{R}_a\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{L^q}\, d\tau\\ &\le \|G_0\|_{L^q} + \|\mathcal{R}_a\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}}\, d\tau.\end{aligned}$$

Following the steps as in the proof of Proposition 2.6, we can show that

$$\|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}} \le C\|\omega\|_{L^q}\,\|\theta\|_{B^{0,a}_{\infty,1}} + C\,\|\theta_0\|_{L^2}\,\|u\|_{L^2}.$$

Gronwall's inequality and the bound $\|\theta\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T)$ then imply the bound for $\|\omega\|_{L^q}$. This completes the proof of Proposition 6.1. □

## 7. Uniqueness and Proof of Theorem 1.1

This section proves the uniqueness of solutions in the class stated in Theorem 1.1 and sketches the proof of Theorem 1.1. First we state and prove the uniqueness theorem.

**Theorem 7.1.** *Assume that the initial data $(u_0, \theta_0)$ satisfies the conditions stated in Theorem 1.1. Then, the solutions $(u, \theta)$ in the class*

$$u \in L^\infty([0,T]; H^1), \quad \omega \in L^\infty([0,T]; L^q)\cap L^1_T B^{0,a}_{\infty,1}, \quad \theta \in L^\infty([0,T], L^2\cap B^{0,a}_{\infty,1}) \tag{7.1}$$

*must be unique.*

*Proof.* Assume that $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ are two solutions in the class (7.1). Let $p^{(1)}$ and $p^{(2)}$ be the associated pressure. The differences

$$u = u^{(2)} - u^{(1)}, \qquad p = p^{(2)} - p^{(1)}, \qquad \theta = \theta^{(2)} - \theta^{(1)}$$

satisfy

$$\begin{cases}\partial_t u + u^{(1)}\cdot\nabla u + u\cdot\nabla u^{(2)} + \mathcal{L}u = -\nabla p + \theta\mathbf{e}_2,\\ \partial_t\theta + u^{(1)}\cdot\nabla\theta + u\cdot\nabla\theta^{(2)} = 0.\end{cases}$$

By Lemmas 7.2 and 7.3 below, we have the following estimates

$$\begin{aligned}\|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\,\|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}}\\ &\quad + C\int_0^t \|u(\tau)\|_{L^2}\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}})\, d\tau\end{aligned}$$

and

$$\begin{aligned}\|\theta(t)\|_{B^{-1,a}_{2,\infty}} &\le \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C\int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\, d\tau\\ &\quad + C\int_0^t \|u(\tau)\|_{L^2}\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}\, d\tau.\end{aligned}$$

In addition, we bound $\|u\|_{L^2}$ by the following interpolation inequality

$$\|u\|_{L^2} \le C\,\|u\|_{B^0_{2,\infty}}\,\log\left(1 + \frac{\|u\|_{H^1}}{\|u\|_{B^0_{2,\infty}}}\right)$$

together with $\|u\|_{H^1} \le \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}$. These inequalities allow us to conclude that

$$Y(t) \equiv \|u(t)\|_{B^0_{2,\infty}} + \|\theta(t)\|_{B^{-1,a}_{2,\infty}}$$

obeys

$$Y(t) \le 2\,Y(0) + C \int_0^t D_1(\tau) Y(\tau)\, \log\left(1 + D_2(\tau)/Y(\tau)\right)\, d\tau, \tag{7.2}$$

where

$$\begin{aligned} D_1 &= \|\theta^{(2)}\|_{B^{0,a}_{\infty,1}} + \|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}, \\ D_2 &= \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}. \end{aligned}$$

Applying Osgood's inequality to (7.2) and noticing that $Y(0) = 0$, we conclude that $Y(t) = 0$. This completes the proof of Theorem 7.1. □

We now state and prove two estimates used in the proof of Theorem 7.1.

**Lemma 7.2.** *Assume that $u^{(1)}$, $u^{(2)}$, $u$, $p$ and $\theta$ are defined as in the proof of Theorem 7.1 and satisfy*

$$\partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2. \tag{7.3}$$

*Then we have the a priori bound*

$$\begin{aligned} \|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\,\|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}} \\ &+ C \int_0^t \|u(\tau)\|_{L^2} \left(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}\right) d\tau. \end{aligned} \tag{7.4}$$

*Proof of Lemma 7.2.* Let $j \ge -1$ be an integer. Applying $\Delta_j$ to (7.3) and taking the inner product with $\Delta_j u$, we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2}^2 + \|\mathcal{L}^{\frac12}\Delta_j u\|_{L^2}^2 = J_1 + J_2 + J_3, \tag{7.5}$$

where

$$\begin{aligned} J_1 &= -\int \Delta_j u\, \Delta_j(u^{(1)} \cdot \nabla u)\, dx, \\ J_2 &= -\int \Delta_j u\, \Delta_j(u \cdot \nabla u^{(2)})\, dx, \\ J_3 &= \int \Delta_j u\, \Delta_j(\theta \mathbf{e}_2)\, dx. \end{aligned}$$

By Plancherel's theorem,

$$\|\mathcal{L}^{\frac12}\Delta_j u\|_{L^2}^2 \ge C\, 2^j a^{-1}(2^j)\, \|\Delta_j u\|_{L^2}^2,$$

where $C = 0$ in the case of $j = -1$ and $C > 0$ for $j \ge 0$. The estimate for $J_3$ is easy and we have, by Hölder's inequality,

$$|J_3| \le \|\Delta_j u\|_{L^2}\, \|\Delta_j \theta\|_{L^2} \le 2^j a^{-1}(2^j)\, \|\Delta_j u\|_{L^2}\, \|\theta\|_{B^{-1,a}_{2,\infty}}.$$

To estimate $J_1$, we need to use a commutator structure to shift one derivative to $u^{(1)}$. For this purpose, we write

$$\Delta_j(u^{(1)} \cdot \nabla u) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15}, \tag{7.6}$$

where

$$\begin{aligned} J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u^{(1)}\cdot\nabla]\Delta_k u,\\ J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u^{(1)} - S_j u^{(1)})\cdot\nabla\Delta_j\Delta_k u,\\ J_{13} &= S_j u^{(1)}\cdot\nabla\Delta_j u,\\ J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u^{(1)}\cdot\nabla S_{k-1}u),\\ J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u^{(1)}\cdot\nabla\widetilde{\Delta}_k u). \end{aligned}$$

Since $\nabla\cdot u^{(1)} = 0$, we have

$$\int J_{13}\,\Delta_j u\,dx = 0.$$

$J_{11}$, $J_{12}$, $J_{14}$ and $J_{15}$ can be bounded in a similar fashion as in the proof of Lemma 5.2 and we have

$$\begin{aligned} &\|J_{11}\|_{L^2},\ \|J_{12}\|_{L^2} \le C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\|\Delta_j u\|_{L^2},\\ &\|J_{14}\|_{L^2} \le C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\le j-1} 2^{m-j}\|\Delta_m u\|_{L^2},\\ &\|J_{15}\|_{L^2} \le C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k}\|\Delta_k u\|_{L^2}. \end{aligned}$$

To estimate $J_2$, we write

$$\Delta_j(u\cdot\nabla u^{(2)}) = J_{21} + J_{22} + J_{23}, \tag{7.7}$$

where

$$\begin{aligned} J_{21} &= \sum_{|j-k|\le 2} \Delta_j(S_{k-1}u\cdot\nabla\Delta_k u^{(2)}),\\ J_{22} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}u^{(2)}),\\ J_{23} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k u^{(2)}). \end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned} &\|J_{21}\|_{L^2} \le C\,\|u\|_{L^2}\,\|\nabla\Delta_j u^{(2)}\|_{L^\infty},\\ &\|J_{22}\|_{L^2} \le C\,\|\Delta_j u\|_{L^2}(\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}),\\ &\|J_{23}\|_{L^2} \le C\,(\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k}\|\Delta_k u\|_{L^2}. \end{aligned}$$

Inserting the estimates above in (7.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2} + C\,2^j a^{-1}(2^j)\,\|\Delta_j u\|_{L^2} \le C\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}} + K(t), \tag{7.8}$$

where

$$\begin{aligned} K(t) &= C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}})\|\Delta_j u\|_{L^2} \\ &\quad + C\,\|u\|_{L^2}\,\|\nabla\Delta_j u^{(2)}\|_{L^\infty} + (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\le j-1} 2^{m-j}\|\Delta_m u\|_{L^2} \\ &\quad + C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k}\|\Delta_k u\|_{L^2}. \end{aligned}$$

Integrating (7.8) in time and taking $\sup_{j\ge -1}$, we obtain (7.4). This completes the proof of Lemma 7.2. □

**Lemma 7.3.** *Assume that $\theta$, $u^{(1)}$, $u$ and $\theta^{(2)}$ are defined as in the proof of Theorem 7.1 and satisfy*

$$\partial_t\theta + u^{(1)}\cdot\nabla\theta + u\cdot\nabla\theta^{(2)} = 0. \tag{7.9}$$

*Then we have the a priori bound*

$$\begin{aligned} \|\theta(t)\|_{B^{-1,a}_{2,\infty}} &\le \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C\int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\,d\tau \\ &\quad + C\int_0^t \|u(\tau)\|_{L^2}\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}\,d\tau. \end{aligned} \tag{7.10}$$

*Proof of Lemma 7.3.* Let $j \ge -1$ be an integer. Applying $\Delta_j$ to (7.9) and taking the inner product with $\Delta_j\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta\|^2_{L^2} = K_1 + K_2, \tag{7.11}$$

where

$$\begin{aligned} K_1 &= -\int \Delta_j\theta\,\Delta_j(u^{(1)}\cdot\nabla\theta)\,dx, \\ K_2 &= -\int \Delta_j\theta\,\Delta_j(u\cdot\nabla\theta^{(2)})\,dx. \end{aligned}$$

To estimate $K_1$, we decompose $\Delta_j(u^{(1)}\cdot\nabla\theta)$ as in (7.6) and estimate each component in a similar fashion to obtain

$$\begin{aligned} |K_1| &\le C\,\|\Delta_j\theta\|^2_{L^2}\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \\ &\quad + C\,\|\Delta_j\theta\|_{L^2}\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}). \end{aligned}$$

To estimate $K_2$, we decompose $\Delta_j(u\cdot\nabla\theta^{(2)})$ as in (7.7) and bound the components in a similar fashion to have

$$|K_2| \le C\,\|\Delta_j\theta\|_{L^2}\,\|u\|_{L^2}2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}.$$

Combining these estimates, we find

$$\begin{aligned} \frac{d}{dt}\|\Delta_j\theta\|_{L^2} &\le C\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \\ &\quad + C\,\|u\|_{L^2}2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}. \end{aligned}$$

Integrating in time, multiplying by $2^{-j}a(2^j)$ and taking $\sup_{j\geq -1}$, we obtain (7.10). This completes the proof of Lemma 7.3. □

We now sketch the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Thanks to Theorem 7.1, it suffices to establish the existence of solutions. The first step is to obtain a local (in time) solution and then extend it into a global solution through the global *a priori* bounds obtained in the previous section. The local solution can be constructed through the method of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)}, \theta^{(n)})\}$ solving

$$\begin{cases} \omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0, \\ \partial_t\omega^{(n+1)} + u^{(n)} \cdot \nabla\omega^{(n+1)} + \mathcal{L}\omega^{(n+1)} = \partial_{x_1}\theta^{(n+1)}, \\ \partial_t\theta^{(n+1)} + u^{(n)} \cdot \nabla\theta^{(n+1)} = 0, \\ \omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x). \end{cases} \tag{7.12}$$

To show that $\{(\omega^{(n)}, \theta^{(n)})\}$ converges to a solution of (1.4), it suffices to prove that $\{(\omega^{(n)}, \theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0, T_1]$ over which $\{(\omega^{(n)}, \theta^{(n)})\}$ are bounded uniformly in terms of $n$. More precisely, we show that
$$\|\omega^{(n)}\|_{L_t^\infty(L^2\cap L^q)\cap L_t^1 B^{0,a}_{\infty,1}} \leq C(T_1), \quad \|\theta^{(n)}\|_{L_t^\infty(L^2\cap B^{0,a^2}_{\infty,1})\cap L_t^1 B^{0,a}_{\infty,1}} \leq C(T_1),$$
where $C(T_1)$ is a constant independent of $n$.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ is a Cauchy sequence in $L_t^\infty B^{-1}_{\infty,1}$ and $\theta^{(n+1)} - \theta^{(n)}$ is Cauchy in $L_t^1 B^{-1,a}_{\infty,1}$, namely
$$\|\omega^{(n+1)} - \omega^{(n)}\|_{L_t^\infty B^{-1}_{\infty,1}} \leq C(T_2)\, 2^{-n}, \quad \|\theta^{(n+1)} - \theta^{(n)}\|_{L_t^1 B^{-1,a}_{\infty,1}} \leq C(T_2)\, 2^{-n}$$
for any $t \in [0, T_2]$, where $C(T_2)$ is independent of $n$.

If the properties stated in (1) and (2) hold, then there exists $(\omega, \theta)$ satisfying

$$\omega \in L_t^\infty(L^2 \cap L^q) \cap L_t^1 B^{0,a}_{\infty,1}, \quad \theta \in L_t^\infty(L^2 \cap B^{0,a^2}_{\infty,1}) \cap L_t^1 B^{0,a}_{\infty,1},$$
$$\omega^{(n)} \to \omega \quad \text{in} \quad L_t^\infty B^{-1}_{\infty,1}, \qquad \theta^{(n)} \to \theta \quad \text{in} \quad L_t^1 B^{-1,a}_{\infty,1}$$

for any $t \leq \min\{T_1, T_2\}$. It is then easy to show that $(\omega, \theta)$ solves (1.4) and we thus obtain a local solution and the global bounds in the previous sections allow us to extend it into a global solution. It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in the previous sections (Section 3 through Section 6) while Property (2) can be checked as in the proof of Theorem 7.1. We thus omit further details. This completes the proof of Theorem 1.1. □

## Appendix A. Functional spaces and Osgood inequality

This appendix provides the definitions of some of the functional spaces and related facts used in the previous sections. In addition, the Osgood inequality used in the proof of Theorem 7.1 is also provided here for the convenience of readers. Materials presented in this appendix can be found in several books and many papers (see, e.g., [3, 4, 28, 32, 34]).

We start with several notation. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma \, dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials.

To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{ \xi \in \mathbb{R}^d :\ 2^{j-1} \le |\xi| < 2^{j+1} \right\}. \tag{A.1}$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 & , \quad \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 & , \quad \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}. \tag{A.2}$$

**Definition A.1.** *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{A.3}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A.4}$$

**Definition A.2.** *The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \le p, q \le \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left( \int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left( \int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt \right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way. We have also used the following generalized version of Besov spaces.

**Definition A.3.** *Let $a(x) = a(|x|) : (0, \infty) \to (0, \infty)$ be a non-decreasing function satisfying (1.5), namely*

$$\lim_{|x| \to \infty} \frac{a(x)}{|x|^\sigma} = 0, \quad \forall \sigma > 0.$$

*For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the generalized Besov spaces $\mathring{B}^{s,a}_{p,q}$ and $B^{s,a}_{p,q}$ are defined through the norms*

$$\|f\|_{\mathring{B}^{s,a}_{p,q}} \equiv \|2^{js} a(2^j) \, \|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty,$$

$$\|f\|_{B^{s,a}_{p,q}} \equiv \|2^{js} a(2^j) \, \|\Delta_j f\|_{L^p}\|_{l^q} < \infty. \tag{A.5}$$

We have also used the space-time spaces defined below.

**Definition A.4.** *For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the space-time spaces $\widetilde{L}^r_t \mathring{B}^s_{p,q}$ and $\widetilde{L}^r_t B^s_{p,q}$ are defined through the norms*

$$\|f\|_{\widetilde{L}^r_t \mathring{B}^s_{p,q}} \equiv \|2^{js} \|\mathring{\Delta}_j f\|_{L^r_t L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^r_t L^p}\|_{l^q}.$$

*$\widetilde{L}^r_t \mathring{B}^{s,a}_{p,q}$ and $\widetilde{L}^r_t B^{s,a}_{p,q}$ are similarly defined.*

These spaces are related to the classical space-time spaces $L^r_t \mathring{B}^s_{p,q}$, $L^r_t B^{s,\gamma}_{p,q}$, $L^r_t \mathring{B}^{s,a}_{p,q}$ and $L^r_t B^{s,a}_{p,q}$ via the Minkowski inequality.

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

**Proposition A.5.** *For any $s \in \mathbb{R}$,*

$$\mathring{H}^s \sim \mathring{B}^s_{2,2}, \quad H^s \sim B^s_{2,2}.$$

*For any $s \in \mathbb{R}$ and $1 < q < \infty$,*

$$\mathring{B}^s_{q,\min\{q,2\}} \hookrightarrow \mathring{W}^s_q \hookrightarrow \mathring{B}^s_{q,\max\{q,2\}}.$$

*In particular, $\mathring{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \mathring{B}^0_{q,\max\{q,2\}}$.*

For notational convenience, we write $\Delta_j$ for $\mathring{\Delta}_j$. There will be no confusion if we keep in mind that $\Delta_j$'s associated with the homogeneous Besov spaces is defined in (A.2) while those associated with the inhomogeneous Besov spaces are defined in (A.4). Besides the Fourier localization operators $\Delta_j$, the partial sum $S_j$ is also a useful notation. For an integer $j$,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where $\Delta_k$ is given by (A.4). For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius $2^j$.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

**Proposition A.6.** *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

1) *If $f$ satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \le K 2^j\},$$

*for some integer $j$ and a constant $K > 0$, then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_1\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If $f$ satisfies*

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \le |\xi| \le K_2 2^j\}$$

*for some integer $j$ and constants $0 < K_1 \le K_2$, then*

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^d)} \le \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \le C_2\, 2^{2\alpha j + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)},$$

*where $C_1$ and $C_2$ are constants depending on $\alpha, p$ and $q$ only.*

Finally we recall the Osgood inequality.

**Proposition A.7.** *Let $\alpha(t) > 0$ be a locally integrable function. Assume $\omega(t) \geq 0$ satisfies*

$$\int_0^\infty \frac{1}{\omega(r)} dr = \infty.$$

*Suppose that $\rho(t) > 0$ satisfies*

$$\rho(t) \leq a + \int_{t_0}^t \alpha(s)\omega(\rho(s))ds$$

*for some constant $a \geq 0$. Then if $a = 0$, then $\rho \equiv 0$; if $a > 0$, then*

$$-\Omega(\rho(t)) + \Omega(a) \leq \int_{t_0}^t \alpha(\tau)d\tau,$$

*where*

$$\Omega(x) = \int_x^1 \frac{dr}{\omega(r)}.$$


## Acknowledgements

This work was partially supported by NSF grants DMS0907913 and DMS1209153. The authors thank Professor Dongho Chae for discussions.


## References


[1] D. Adhikari, C. Cao and J. Wu, The 2D Boussinesq equations with vertical viscosity and vertical diffusivity, *J. Differential Equations* **249** (2010), 1078–1088.

[2] D. Adhikari, C. Cao and J. Wu, Global regularity results for the 2D Boussinesq equations with vertical dissipation, *J. Differential Equations* **251** (2011), 1637-1655.

[3] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, 2011.

[4] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[5] C. Cao and J. Wu, Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation, arXiv:1108.2678v1 [math.AP] 12 Aug 2011.

[6] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, *Advances in Math.* **203** (2006), 497-513.

[7] D. Chae, P. Constantin and J. Wu, Inviscid models generalizing the 2D Euler and the surface quasi-geostrophic equations, *Arch. Ration. Mech. Anal.* **202** (2011), 35-62.

[8] D. Chae and J. Wu, The 2D Boussinesq equations with logarithmically supercritical velocities, *Advances in Math.* **230** (2012), 1618-1645.

[9] Q. Chen, C. Miao and Z. Zhang, A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation, *Commun. Math. Phys.* **271** (2007), 821–838.

[10] P. Constantin and C.R. Doering, Infinite Prandtl number convection, *J. Statistical Physics* **94** (1999), 159-172.

[11] P. Constantin and V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, arXiv:1110.0179v1 [math.AP] 2 Oct 2011.

[12] A. Córdoba and D. Córdoba, A maximum princple applied to quasi-geostroohhic equations; *Comm. Math. Phys.* **249**(2004), 511-528.

[13] M. Dabkowski, A. Kiselev, L. Silvestre and V. Vicol, Global well-posedness of slightly supercritical active scalar equations, arXiv:1203.6302v1 [math.AP] Mar 28 2012.

[14] R. Danchin and M. Paicu, Global existence results for the anisotropic Boussinesq system in dimension two, *Math. Models Methods Appl. Sci.* **21** (2011), 421–457.

[15] W. E and C. Shu, Samll-scale structures in Boussinesq convection, *Phys. Fluids* **6** (1994), 49-58.

[16] A.E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press (London), 1982.

[17] T. Hmidi, On a maximum principle and its application to the logarithmically critical Boussinesq system, *Anal. Partial Differential Equations* **4** (2011), 247–284.

[18] T. Hmidi and S. Keraani, On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity, *Adv. Differential Equations* **12** (2007), 461–480.

[19] T. Hmidi and S. Keraani, On the global well-posedness of the Boussinesq system with zero viscosity, *Indiana Univ. Math. J.* **58** (2009), 1591–1618.

[20] T. Hmidi, S. Keraani and F. Rousset, Global well-posedness for a Boussinesq-Navier-Stokes system with critical dissipation, *J. Differential Equations* **249** (2010), 2147-2174.

[21] T. Hmidi, S. Keraani and F. Rousset, Global well-posedness for Euler-Boussinesq system with critical dissipation, *Comm. Partial Differential Equations* **36** (2011), 420-445.

[22] T. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, *Discrete and Cont. Dyn. Syst.* **12** (2005), 1-12.

[23] D. KC, D. Regmi, L. Tao and J. Wu, The 2D Euler-Boussinesq equations with a logarithemically supercritical velocity, preprint.

[24] A. Larios, E. Lunasin and E.S. Titi, Global well-posedness for the 2D Boussinesq system without heat diffusion and with either anisotropic viscosity or inviscid Voigt-$a$ regularization, arXiv:1010.5024v1 [math.AP] 25 Oct 2010.

[25] A.J. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lecture Notes in Mathematics **9**, AMS/CIMS, 2003.

[26] A.J. Majda and A.L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, 2001.

[27] C. Miao and L. Xue, On the global well-posedness of a class of Boussinesq- Navier-Stokes systems, *NoDEA Nonlinear Differential Equations Appl.* **18** (2011), 707-735.

[28] C. Miao, J. Wu and Z. Zhang, *Littlewood-Paley Theory and its Applications in Partial Differential Equations of Fluid Dynamics*, Science Press, Beijing, China, 2012 (in Chinese).

[29] H.K. Moffatt, Some remarks on topological fluid mechanics, in: R.L. Ricca (Ed.), *An Introduction to the Geometry and Topology of Fluid Flows*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001, pp. 3-10.

[30] K. Ohkitani, Comparison between the Boussinesq and coupled Euler equations in two dimensions. Tosio Kato's method and principle for evolution equations in mathematical physics (Sapporo, 2001). Surikaisekikenkyusho Kokyuroku No. 1234 (2001), 127-145.

[31] J. Pedlosky, *Geophysical Fluid Dyanmics*, Springer-Verlag, New York, 1987.

[32] T. Runst and W. Sickel, *Sobolev Spaces of fractional order, Nemytskij operators and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, New York, 1996.

[33] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Unviersity Press, Princeton, NJ, 1970.

[34] H. Triebel, *Theory of Function Spaces II*, Birkhauser Verlag, 1992.

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