The Zero-Viscosity Limit of the 2D Navier–Stokes Equations

By Jerry L. Bona and Jiahong Wu

It is shown that in a radially symmetric configuration, the zero-viscosity limit of solutions of the Navier–Stokes equations satisfies the associated Euler equations. An ancillary result on continuous dependence of solutions on ν is also established.

1. Introduction

The Navier–Stokes equations for the movement of a viscous, incompressible fluid are considered here. Attention is focussed on the initial- and boundary-value problem

$$\begin{aligned} u_t + u\cdot\nabla u &= -\nabla p + \nu\Delta u, && (x,t)\in\Omega\times\mathbb{R}^+,\\ \nabla\cdot u &= 0, && (x,t)\in\Omega\times\mathbb{R}^+,\\ u(x,0) &= u_0(x), && x\in\Omega,\\ u(x,t) &= 0, && (x,t)\in\partial\Omega\times\mathbb{R}^+, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a $C^{2,\alpha}$ boundary, say, for some $\alpha\in(0,1)$. Here the dependent variables u and p represent the velocity field and pressure, respectively, of a viscous incompressible fluid in a continuum approximation. In this two-dimensional situation, these equations possess

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a globally defined solution $u = u^{\nu}$ corresponding to any given $\nu > 0$ and reasonable assumptions on the initial data u_0. Interest is focussed on the zero-viscosity limit

$$\lim_{\nu \to 0} u^{(\nu)}. \tag{2}$$

This is an old problem that is well understood for reasonably smooth initial data posed on all of $\mathbb{R}^2$ or for the periodic initial-value problem. In these cases, $u^{(\nu)}$ simply converges to $u^{(0)}$ which is the associated solution of the Euler equations ((1) with $\nu = 0$). (For initial data that is less smooth such as a vortex patch with a rough boundary, the issue can be subtle even without boundaries; see Constantin and Wu [1, 2]). The nature of the limit (2) in the presence of the no-slip condition posed on fixed boundaries has remained somewhat elusive. At least a part of the difficulty lies in the change in the boundary condition at $\nu = 0$. Although (1) with $\nu > 0$ features the no-slip condition $u(x, t) = 0$ for $x \in \partial\Omega$, the Euler equations subsist on the weaker condition, namely, $u \cdot n = 0$, where n is the outward normal on $\partial\Omega$. It is our purpose here to cast a little light on the inviscid limit (2) in the context of the problem (1).

We show that at least in very special circumstances, the limit (2) is exactly the solution of the associated Euler equations, including the no-flow condition through the boundary.

Although the principal interest here is the limit (2), we begin our development in Section 2 with a result of continuity of solutions with respect to variations of ν. Section 3 is devoted to the zero-viscosity limit in special circular geometry. Some commentary pertaining to more general geometric configurations is offered in Section 4. The Appendix contains some details pertaining to a Fourier–Bessel expansion.

2. Continuous dependence on viscosity

For $\nu > 0$, let $u^{(\nu)}$ denote the unique solution of (1) corresponding to initial data in the usual space V which is the closure in $H_0^1(\Omega) \times H_0^1(\Omega)$ of those elements ϕ in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ for which $\nabla \cdot \phi = 0$. Here $\mathcal{D}(\Omega)$ is the usual space of C^{∞}-functions whose support is a compact subset of Ω. Similarly, H is the closure in $L^2(\Omega) \times L^2(\Omega)$ of the same linear space of smooth, divergence-free vector fields. It is well known (see, e.g., Temam [3]) that for Ω with a $C^{2,\alpha}$ boundary and for any $T > 0$, the solution u of (1) lies in $L^2(0, T; V)$ and in $C^{1+\alpha}((0, T]; C^{2+\alpha}(\Omega))$.

To sharpen the zero-viscosity issue (2), it is interesting to know that if initial data $u_0 \in V$ is fixed and the solution viewed as a mapping $\nu \mapsto u^{(\nu)}$, then this mapping is C^1 and thus we may think of $\{u^{(\nu)}\}_{\nu > 0}$ as a curve in function space. Our main aim may be seen as finding the $\nu = 0$ end of this curve.

PROPOSITION 1. *Let* $\nu_0 > 0$ *and* $u_0 \in V$. *If* $u^{(\nu_0)}(u_0)$ *is the corresponding solution of the initial-boundary-value problem (1) with* $\nu = \nu_0$ *and initial data* u_0, *then for any fixed* $T > 0$, *as* $\nu \to \nu_0$ *in* $\mathbb{R}^+$ *and* $v \to u_0 \in V$,

$$u^{(\nu)}(v) \to u^{(\nu_0)}(u_0)$$

in $L^2(0, T; V) \cap C^{1+\alpha}((0, T]; C^{2+\alpha}(\Omega))$.

To prove Proposition 1, the following version of the Implicit-Function Theorem [4] is used.

THEOREM 1. *Let* E, F, *and* G *be three Banach spaces*, g *a continuously differentiable mapping of an open subset* W *of* $E \times F$ *into* G. *Let* (x_0, y_0) *be a point of* W *such that* $g(x_0, y_0) = 0$ *and suppose the partial Fréchet-derivative* $D_2 g(x_0, y_0)$ *to be a linear homeomorphism from* F *onto* G. *Then there exists an open neighborhood* U_0 *of* x_0 *such that for every open connected neighborhood* U *of* x_0 *contained in* U_0, *there is a unique continuous mapping* u *of* U *into* F *such that* $u(x_0) = y_0$, $(x, u(x)) \in W$, *and* $g(x, u(x)) = 0$ *for any* $x \in U$. *Furthermore*, u *is continuously differentiable in* U *and its derivative is given by*

$$u'(x) = -(D_2 g(x, u(x)))^{-1} D_1 g(x, u(x)).$$

Proof of Proposition 1: This is a straightforward application of Theorem 1. To apply the Implicit-Function Theorem, let $E = \mathbb{R} \times V$,

$$F = C^{1+\alpha}((0, T]; C^{2+2\alpha}(\Omega)) \cap L^2(0, T; V)$$

and

$$G = C^{\alpha}((0, T]; C^{2\alpha}(\Omega)) \cap L^2(0, T; H),$$

where $\alpha \in (0, 1/2)$. As has been the common practice since Leray's pioneering work, the system (1) is written in the form

$$\begin{aligned} w_t + \nu A(w) + B(w, w) &= 0 \qquad \text{in } \Omega \times [0, T], \\ w(x, 0) &= w_0(x) \qquad \text{for } x \in \Omega, \end{aligned}$$

(see Temam [3]). Here, $w_0 = P u_0$ is the projection of u_0 onto the divergence-free vector fields, $Au = P(-\Delta u)$ and $B(u, v) = P(u \cdot \nabla u)$.

Let g be the operator from $E \times F \to G$ given by

$$g(\nu, u) = u_t + \nu A u + B(u, u).$$

We show that g satisfies the hypotheses of Theorem 1. The mapping g is clearly continuously differentiable. When the initial data u_0 lies in V, the solution $u^{(\nu_0)}$ of the initial-boundary-value problem with $\nu = \nu_0$ lies in F and

$g(\nu_0, u^{(\nu_0)}) = 0$. It is easy to check that

$$D_2 g(\nu_0, u_0)v = v_t + \nu_0 A v + B\big(u^{(\nu_0)}, v\big) + B\big(v, u^{(\nu_0)}\big).$$

In fact, the bounded linear operator $D_2 g$ is a homeomorphism from F to G. To verify this property, it suffices to prove the existence and uniqueness of the initial-value problem,

$$v_t + \nu_0 A v + B\big(u^{(\nu_0)}, v\big) + B\big(v, u^{(\nu_0)}\big) = f,$$
$$v(0) = v_0,$$

where $f \in G$ is arbitrary and $v_0 \in V$. The existence follows from linear theory ([5]). One can also devise a Galerkin approximation. The uniqueness is obtained through a simple energy estimate. In fact, if v and w are two solutions, then

$$(v - w)_t - \nu_0 \Delta(v - w) + u^{(\nu_0)} \cdot \nabla(v - w) + (v - w) \cdot \nabla u^{(\nu_0)} = 0.$$

Multiply the above equation by $v - w$ and integrate the result over Ω. After integrating by parts and using the fact that $\nabla \cdot (v - w) = 0$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (v(x) - w(x))^2\,dx &+ \nu_0 \int_\Omega (\nabla(v(x) - w(x))^2\,dx \\ &\le \left\|\nabla u^{(\nu_0)}\right\|_{L^\infty} \int_\Omega (u(x) - v(x))^2\,dx, \end{aligned}$$

which implies that $v = w$.

3. Vorticity with radial symmetry

The zero-viscosity limit (2) is addressed by way of the vorticity equations

$$\omega_t + u \cdot \nabla\omega = \nu\Delta\omega, \quad (x, t) \in \Omega \times \mathbb{R}^+, \tag{3}$$

$$\omega(x, 0) = \omega_0(x), \quad x \in \Omega, \tag{4}$$

where $\omega = \omega^{(\nu)} = \nabla \times u$ is the vorticity corresponding to the flow defined by (1). The velocity u may be recovered from the vorticity by the usual recipe of first solving for the stream function ψ via

$$\Delta\psi = \omega \quad \text{in } \Omega \times \mathbb{R}^+, \tag{5}$$

and then obtaining the velocity field as $u = (\partial_{x_2}\psi, -\partial_{x_1}\psi)$. An advantage of the vorticity formulation is that the pressure does not intrude explicitly and the divergence-free condition is automatically satisfied. This could be helpful because obtaining bounds on the pressure tends to be challenging (see, e.g., [6]).

A drawback is that the no-slip boundary condition is not imposed in an explicit way on the variable ω.

There is a particular situation where the boundary condition on the vorticity can be made explicit, albeit global, namely, in the case of radial symmetry. It is in this special circumstance that the aforementioned zero-viscosity limit results are established.

THEOREM 2. *Consider Navier–Stokes flows in the unit disc D whose initial vorticity has radial symmetry; namely,*

$$\tilde{\omega}_0(x) = \omega_0(|x|). \tag{6}$$

Suppose ω_0 to be continuous on $[0, 1)$, *integrable over* $[0, 1]$ *and that*

$$\int_0^1 r\omega_0(r)\, dr = 0.$$

Let u_0 be the corresponding initial velocity field and let $u^{(\nu)}$ be the solution of the initial-boundary-value problem (1) *corresponding to* $\nu > 0$. *Let $u^{(0)}$ be the solution of* (1) *with* $\nu = 0$ *and the no-flow boundary condition*

$$u \cdot n = 0$$

in place of the no-slip condition. Let $\omega^{(\nu)}$ and $\omega^{(0)}$ be the vorticities corresponding to the velocity fields $u^{(\nu)}$ and $u^{(0)}$, respectively. Then for any fixed $T > 0$ *and all* $t \in [0, T]$, *the differences*

$$u^{(\nu)}(\cdot, t) - u^{(0)}(\cdot, t) \quad \textit{and} \quad \omega^{(\nu)}(\cdot, t) - \omega^{(0)}(\cdot, t)$$

converge to zero as $\nu \to 0$, *in $L^2(D)$ and uniformly on compact subsets of D.*

Proof: The proof of Theorem 2 requires some preliminary manipulations. First, note that the flow corresponding via (3) and (4) to $\tilde{\omega}_0$ remains radially symmetric for all $t > 0$, because of the uniqueness of the initial-value problem coupled with the fact that the equations of motion are invariant under rotations. The stream function ψ is likewise radially symmetric because the Laplacian is rotationally invariant; thus, ψ is given as the solution of the ordinary differential equation

$$\psi''(r, t) + \frac{1}{r}\psi'(r, t) = \Delta\psi = \omega(r, t). \tag{7}$$

In this special case, the velocity field u turns out to be

$$u(x, t) = (-\partial_{x_2}\psi, \partial_{x_1}\psi) = \left(-\frac{x_2}{r}, \frac{x_1}{r}\right)\psi'(r, t) \tag{8}$$

where $'$ denotes differentiation with respect to r and ψ satisfies (7). Rewriting (7) in the form

$$\frac{d}{dr}(r\psi'(r,t)) = r\omega^{(\nu)}(r,t),$$

it follows readily that

$$\psi'(r,t) = \frac{1}{r}\int_0^r \rho\omega^{(\nu)}(\rho,t)\,d\rho.$$

Formula (8) and the last result yield

$$u(x,t) = \frac{1}{r^2}\int_0^r \rho\omega^{(\nu)}(\rho,t)\,d\rho \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}. \tag{9}$$

Then, the no-slip boundary condition $u(x,t) = 0$ imposed when $|x| = 1$ becomes simply

$$\int_0^1 r\omega^{(\nu)}(r,t)\,dr = 0. \tag{10}$$

Furthermore, it is easy to see from the representation (9) for u that

$$u \cdot \nabla\omega^{(\nu)} = 0,$$

and thus the vorticity equation (3) reduces to the linear heat equation

$$\omega_t^{(\nu)} = \nu\Delta\omega^{(\nu)}. \tag{11}$$

In terms of vorticity, the Euler equations become simply (11) with $\nu = 0$, and thus express the time-independence of the vorticity

$$\omega^{(0)}(x,t) = \tilde{\omega}_0(x) = \omega_0(r).$$

In consequence of these ruminations, the zero-viscosity limit boils down to comparing the solution $\omega^{(\nu)}$ of (11) with the initial data $\tilde{\omega}_0$ in the limit as ν tends to zero.

Attention is now turned to solving the initial- and boundary-value problem (11), (10), and (4) for the vorticity. Assuming a solution in the form

$$\omega(r,t) = Q(r)P(t), \tag{12}$$

equation (11) gives

$$P'(t)Q(r) = \nu P(t)(Q''(r) + \frac{1}{r}Q'(r)).$$

Thus there must be a constant λ such that

$$\frac{P'(t)}{P(t)} = \frac{\nu\big(Q''(r) + \frac{1}{r}Q'(r)\big)}{Q(r)} = -\lambda^2,$$

or what is the same

$$P'(t) + \lambda^2 P(t) = 0, \tag{13}$$

$$Q''(r) + \frac{1}{r}Q'(r) + \frac{\lambda^2}{\nu}Q(r) = 0. \tag{14}$$

The solution of equation (14) bounded near $r = 0$ can be expressed in terms of Bessel functions, namely,

$$Q(r) = CJ_0\left(\frac{\lambda}{\sqrt{\nu}}r\right),$$

where C is a constant and J_0 is the Bessel function of the first kind of order *0* (see Watson [7]). The solution of (13) is

$$P(t) = P(0)e^{-\lambda^2 t}.$$

Hence, solutions of equation (11) in the separated form (12) are

$$Ae^{-\lambda^2 t}J_0\left(\frac{\lambda}{\sqrt{\nu}}r\right),$$

where $A = CP(0)$ is a constant. To satisfy the no-slip boundary (10), one must require

$$\int_0^1 rJ_0\left(\frac{\lambda}{\sqrt{\nu}}r\right)dr = 0. \tag{15}$$

To solve (15), write $y = \frac{\lambda}{\sqrt{\nu}}r$ and use the identity

$$\frac{d}{dy}yJ_1(y) = yJ_0(y)$$

to reach the formula

$$0 = \int_0^1 rJ_0\left(\frac{\lambda}{\sqrt{\nu}}r\right)dr = \frac{\nu}{\lambda^2}\int_0^{\frac{\lambda}{\sqrt{\nu}}} yJ_0(y)\,dy = \frac{\sqrt{\nu}}{\lambda}J_1\left(\frac{\lambda}{\sqrt{\nu}}\right).$$

In consequence, λ must be such that

$$J_1\left(\frac{\lambda}{\sqrt{\nu}}\right) = 0, \tag{16}$$

so $\frac{\lambda}{\sqrt{\nu}}$ has to be a zero of J_1. Let $j_1 < j_2 < j_3 \cdots$ connote the sequence of zeros of J_1. Thus (16) is valid only for the discrete set of modes

$$\lambda_k = j_k\sqrt{\nu}$$

and solutions of (10) and (11) in the separated form (12) are

$$A_k e^{-\nu j_k^2 t} J_0(j_k r)$$

where A_k is a constant. Because the equations are linear, one expects the complete set of solutions to be given by the series

$$\sum_{k=1}^{\infty} A_k e^{-\nu j_k^2 t} J_0(j_k r)\,. \tag{17}$$

Appeal is now made to the theory of Dini expansions that generalize the usual Fourier–Bessel expansions arising in heat conduction and elsewhere. For this theory, we refer to the classical treatise of Watson [7]. For the reader's convenience, a brief précis of the elements of the theory that come to the fore here may be found in the Appendix. In modern terminology, the theory is naturally set in $L^2(0,1; r\,dr)$. For any h in this latter class, define

$$A_l = \frac{2}{J_0(j_l)^2}\int_0^1 h(r) J_0(j_l r) r\,dr. \tag{18}$$

Then,

$$h(r) = 2\int_0^1 r h(r)\,dr + \sum_{k=1}^{\infty} A_k J_0(j_k r), \tag{19}$$

with the convergence taking place in $L^2(0,1; r\,dr)$ and, in case h is continuous on [0, 1), is uniform on compact subsets of [0, 1).

Applying (19) to $\omega^{(\nu)}(r,t)$, imposing the boundary condition (10) and the initial condition $\omega^{(\nu)}(r,0) = \omega_0(r)$, $r \in [0,1)$, and using the orthogonality relations

$$\frac{2}{J_0(j_k)^2}\int_0^1 J_0(j_k r) J_0(j_l r) r\,dr = \delta_{k,l},$$

the Kronecker delta, leads exactly to (17) where

$$A_k = \frac{2}{J_0(j_k)^2}\int_0^1 \omega_0(r) J_0(j_k r) r\,dr.$$

It is now demonstrated that the conditions that $\omega_0(r)$ is continuous in [0,1) and

$$\int_0^1 r\omega_0(r)\,dr = 0$$

are sufficient to infer the zero viscosity limit

$$\omega^{\nu}(r,t) \to \omega_0(r), \quad \text{as } \nu \to 0 \tag{20}$$

for $r \in (0, 1)$ and uniformly for $t \in [0, T]$ for any fixed $T > 0$. In fact, given an arbitrary positive number ϵ and a compact set $K \subset [0, 1)$, there is an N such that for $n \geq N$,

$$\left| \sum_{k=n+1}^{\infty} A_k e^{-\nu j_k^2 t} J_0(j_k r) \right| < \frac{\epsilon}{4}$$

for $r \in K$ and $0 \leq t \leq T$. This follows from local Lipschitz properties of the heat flow. Once n is fixed, there exists a positive number ν_0 such that

$$\left| \sum_{k=1}^{n} \left(1 - e^{-\nu j_k^2 t}\right) A_k J_0(j_k r) \right| < \frac{\epsilon}{2}$$

whenever $0 < \nu \leq \nu_0$ and $0 \leq t \leq T$. For $r \in K$ and $t \in [0, T]$, it follows that

$$\begin{aligned} \left|\omega^{(\nu)}(r, t) - \omega^{(0)}(r)\right| &= \left| \sum_{k=1}^{\infty} A_k e^{-\nu j_k^2 t} J_0(j_k r) - \sum_{k=1}^{\infty} A_k J_0(j_k r) \right| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

The pointwise convergence of $\omega^{(\nu)}$ to $\omega^{(0)}$ for $x \in D$ immediately implies that of the velocity field. In fact, according to (9),

$$u^{(\nu)}(x, t) - u^{(0)}(x, t) = \frac{1}{r^2} \int_0^r \rho \left[\omega^{(\nu)}(\rho, t) - \omega^{(0)}(\rho, t)\right] d\rho \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

It follows from the Dominated Convergence Theorem that for any $t \in [0, T]$, $u^{(\nu)}(x, t) \to u^{(0)}(x, t)$ as $\nu \to 0$ for any $x \in D$. The convergence of $u^{(\nu)}$ to $u^{(0)}$ is uniform on compact subsets of D and on $[0, T]$.

It is now proposed that the pointwise convergence implies convergence of both the vorticity and velocity field in L^2, which is to say

$$\omega^{(\nu)}(\cdot, t) \to \omega^{(0)}(\cdot, t) \quad \text{and} \quad u^{(\nu)}(\cdot, t) \to u^{(0)}(\cdot, t) \quad \text{in } L^2(D)$$

as $\nu \to 0$, uniformly for $t \in [0, T]$ for any fixed $T > 0$. For smooth initial vorticity ω_0, the vorticities $\omega^{(\nu)}$ and $\omega^{(0)}$ will remain smooth and, in particular, in L^2. Therefore

$$\left\|\omega^{(\nu)}(\cdot, t) - \omega^{(0)}(\cdot, t)\right\|_{L^2}^2 = \int_D \left|\omega^{(\nu)}(x, t) - \omega^{(0)}(x, t)\right|^2 dx$$

approaches 0 as $\nu \to 0$ by the Dominated Convergence Theorem and the pointwise convergence of $\omega^{(\nu)}(x, t)$ to $\omega^{(0)}(x, t)$ for any $x \in D$.

We know from (9) that

$$\left\|u^{(\nu)}(\cdot, t) - u^{(0)}(\cdot, t)\right\|_{L^2}^2 = 2\pi \int_0^1 \left[\int_0^r \rho \left(\omega^{(\nu)}(\rho, t) - \omega^{(0)}(\rho, t)\right) d\rho\right]^2 \frac{dr}{r}.$$

Remark. The Dominated Convergence Theorem applied to the above formula ensures the L^2-convergence of $u^{(\nu)}(\cdot, t)$ to $u^{(0)}(\cdot, t)$ follows from the pointwise convergence of $\omega^{(\nu)}(\cdot, t)$ to $\omega^{(0)}(\cdot, t)$.

Remark. (i) If ω_0 is taken to be in $L^2(0, 1)$, but is not continuous, similar but weakened results may be deduced by the same methods.

(ii) Notice that Prandtl's boundary-layer equation coincides with the Navier–Stokes equations near the boundary ∂D.

4. General vorticity

The zero dissipation limit problem in the unit disk with a general initial vorticity appears to be more difficult and we do not currently know how to resolve it. Nevertheless, we present here some primitive thoughts about the general case. Attention is still given to the vorticity equation for the reasons mentioned previously.

1. It may be helpful to rewrite the equations in polar coordinates because of the special geometry of the unit disk. In terms of vorticity, the two-dimensional Navier–Stokes equations may be written as

$$\omega_t^{(\nu)} + U^{(\nu)}\omega_r^{(\nu)} + V^{(\nu)}\frac{\omega_\theta^{(\nu)}}{r} = \nu\left(\omega_{rr}^{(\nu)} + \frac{1}{r}\omega_r^{(\nu)} + \frac{1}{r^2}\omega_{\theta\theta}^{(\nu)}\right)$$

where $U^{(\nu)}$ and $V^{(\nu)}$ are

$$U^{(\nu)} = u^{(\nu)}\cos\theta + v^{(\nu)}\sin\theta, \quad V^{(\nu)} = -u^{(\nu)}\sin\theta + v^{(\nu)}\cos\theta.$$

On the other hand, the Euler equations in polar coordinates are

$$\omega_t^{(0)} + U^{(0)}\omega_r^{(0)} + V^{(0)}\frac{\omega_\theta^{(0)}}{r} = 0,$$

with $U^{(0)}$ and $V^{(0)}$ being similarly defined.

A drawback of this approach is that the energy methods tend to be complicated in the polar coordinates.

2. To resolve the zero-viscosity limit problem using vorticity equations, a determination of the boundary conditions for $\omega^{(\nu)}$ and $\omega^{(0)}$ is needed that corresponds to the natural conditions

$$u^{(\nu)} = 0, \quad u^{(0)} \cdot r = 0, \quad \text{on } \partial D,$$

on the velocity fields.

We temporarily drop the upper indices (ν) and (0) because the following argument applies equally to both the Navier–Stokes and the Euler equations.

As before, the velocity field (u, v) is defined through the stream function ψ by

$$(u, v) = (-\psi_{x_2}, \psi_{x_1})$$

and thus the no-slip boundary condition $(u, v) = 0$ on $S = \partial D$ can be expressed as

$$\phi_r(r, \theta) = 0, \qquad \phi_\theta(r, \theta) = 0$$

as $r \to 1$. Furthermore, the stream function ψ satisfies the Poisson equation

$$\Delta \psi = \omega \quad \text{on } D. \tag{21}$$

Thus ψ solves the Neumann problem which is (21) coupled with

$$\psi_r = 0, \quad \text{on } S \tag{22}$$

and must also satisfy the extra condition $\phi_\theta = 0$ on S. An obvious condition for solvability of this problem is

$$\int_D \omega \, dx = 0. \tag{23}$$

If $\omega \in L^2$, the Neumann problem (21) and (22) can be solved by expansion in spherical harmonics and Bessel functions. More precisely, if

$$\omega(x) = \sum a_k^{lm} F_k^{lm}(x),$$

then

$$\psi(x) = \sum a_k^{lm} \left(\lambda_k^l\right)^{-2} F_k^{lm}(x). \tag{24}$$

Details are given in the Appendix. It can be shown that the series (24) converges in $H^1(D)$. Furthermore, if $\omega \in C^\alpha(D)$ where $\alpha > 0$, then $\phi \in C^2(\bar{D})$, so ϕ is a classical solution of the Neumann problem.

Appendix

The validity of (19) with the A_l determined by (18) is established here. In this, we simply follow the lead of Watson [7, ch. 18] in his analysis of Dini expansions.

Consider the expansion of a real-valued function f defined on [0, 1] in the form

$$f(x) = \sum_{m=1}^{\infty} b_m J_\mu(\lambda_m x) \tag{A.1}$$

where J_μ is the Bessel function of the first kind of order μ, and $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ are the complete set of positive zeros of the function

$$z^{-\mu}\{zJ'_\mu(z) + HJ_\mu(z)\} \tag{A.2}$$

when $\mu \geq -1/2$ and H are given constants. The coefficients in the expansion (A.1) are formally determined by the formula

$$\left\{\left(\lambda_m^2 - \mu^2\right)J_\nu^2(\lambda_m) + \lambda_m^2 J_\mu'^2(\lambda_m)\right\} b_m = 2\lambda_m^2 \int_0^1 tf(t)J_\mu(\lambda_m t)\,dt.$$

This type of expansion, first investigated by Dini [7, p. 10], will be called a Dini expansion. The following convergence theorem is proved by means of the Hankel–Schläfli technique in [7, p. 601].

Theorem 3. *Let f be a real-valued function of locally bounded variation defined on* (0, 1) *and suppose the integral*

$$\int_0^1 x^{\frac{1}{2}} f(x)\,dx$$

converges absolutely. Then the series

$$\mathcal{B}_0(x) + \sum_{m=1}^{\infty} b_m J_\mu(\lambda_m x)$$

with

$$\mathcal{B}_0(x) = \begin{cases} 0, & if\ \mu + H > 0, \\ 2(\mu+1)x^\mu \displaystyle\int_0^1 t^{\mu+1} f(t)\,dt, & if\ \mu + H = 0, \end{cases}$$

converges to the sum

$$\frac{1}{2}\{f(x+0) + f(x-0)\}$$

for all points $x \in (0, 1)$. If f is continuous on [0,1), *the convergence is uniform throughout the interval* $[0, 1-\delta)$ *where δ is any small positive number.*

The focus of interest is on the special case where $\mu = 0$ and $H = 0$. In this case, the values (A.2) reduce to $zJ_0'(z)$ and because $J_0'(z) = -J_1(z)$, $\lambda_m = j_m, m = 1, 2, \ldots,$ are the zeros of J_1. The coefficients b_m become

$$b_m = \frac{2}{J_0(j_m)^2} \int_0^1 tf(t)J_0(j_m t)\,dt$$

as in (18). The convergence theorem then states that for continuous *f*,

$$2\int_0^1 tf(t)\,dt + \sum_{m=1}^{\infty} b_m J_0(j_m r)$$

converges to *f(r)* uniformly on compact subsets of [0,1). In particular, if $f(r) = \omega_0(r)$ and obeys

$$\int_0^1 r\omega_0(r)\,dr = 0,$$

then (19) with (18) is valid for any $r \in [0, 1)$. □

Attention is now turned to the expansion (24). Let H_k denote the space of the spherical harmonics of degree k and let

$$\left\{Y_k^1, Y_k^2, \ldots, Y_k^{d_k}\right\}$$

be an orthonormal basis for H_k with the $L^2(D)$-inner product, where d_k is the dimension of H_k. As above, let J_k denote the Bessel function of the first kind of order k.

LEMMA 1. *Let $k > 0$ be an integer. If $j_{k,1} < j_{k,2} \ldots$ are the complete set of positive zeros of J_k' and*

$$C_k^l = \frac{\sqrt{2j_{k,l}}}{[(j_{k,l})^2 - k^2]J_k(j_{k,l})},$$

then

$$\left\{C_k^l J_k(j_{k,l}r) : \quad l = 1, 2, \cdots\right\}$$

is an orthonormal basis for $L^2(0, 1; r\,dr)$.

Remark. When $k = 0$, the constant function $\sqrt{2}$ must also be included.

LEMMA 2. *The triply-indexed family*

$$\left\{F_k^{lm}(x) = C_k^l J_k(j_{k,l}|x|)Y_k^m(x) : \quad k \geq 0, l \geq 0, 0 \leq m \leq d_k\right\}$$

is an orthonormal basis for $L^2(D)$. Moreover,

$$\Delta F_k^{lm} = -(j_{k,l})^2 F_k^{lm} \quad \textit{on } D$$

and

$$F_k^{lm}(x) = 0, \quad \textit{for } x \in S = \partial D.$$

Consider the Neumann problem

$$\Delta\psi = \omega, \quad \text{in } D$$

$$\frac{\partial \omega}{\partial r} = 0, \quad \text{on } S$$

for $\omega \in L^2$. If

$$\omega(x) = \sum a_k^{lm} F_k^{lm}(x),$$

then

$$\psi(x) = \sum a_k^{lm} (j_{k,l})^{-2} F_k^{lm}(x).$$

Because of the condition (23), the component of ω corresponding to $k = 0$ vanishes.

Materials in the Appendix can be found in Folland's book [8, p. 142].

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