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STABILIZATION OF A BACKGROUND MAGNETIC FIELD ON A 2 DIMENSIONAL MAGNETOHYDRODYNAMIC FLOW*

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Abstract. This paper rigorously establishes the stabilization effect of a background magnetic field on electrically conducting fluids, a phenomenon that has been widely observed in physical experiments and numerical simulations. This study is based on a 2 dimensional (2D) magnetohydrodynamic (MHD) system in which the velocity equation involves no dissipation and there is only damping in the vertical component equation. Without the magnetic field, the corresponding vorticity equation is a 2D Euler-like equation with an extra Riesz transform type term. The global in time regularity and the stability near the trivial solution are well known open problems. When coupled with the magnetic field through the MHD system, the background magnetic field stabilizes the fluid, and the velocity as well as the vorticity remain small if they are initially so and decay algebraically in time. To overcome the difficulties due to the lack of full dissipation or damping, we construct suitable Lyapunov functionals and reduce the system to wave type equations.

Key words. background magnetic field, magnetohydrodynamic equation, partial dissipation, stability, decay rate

AMS subject classifications. 35A05, 35Q35, 76D03

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1. Introduction.

Well known to the community of mathematical fluid mechanics is the open problem of whether or not the 2 dimensional (2D) Euler-like equations,

$$
\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \mathcal{R}_1\omega, & x\in\mathbb{R}^2,\ t>0,\\ u=\nabla^{\perp}\Delta^{-1}\omega, \end{cases} \tag{1.1}
$$

and

$$
\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \mathcal{R}_1^2\omega, & x\in\mathbb{R}^2,\ t>0,\\ u=\nabla^{\perp}\Delta^{-1}\omega, \end{cases} \tag{1.2}
$$

always possess global (in time) classical solutions. Here $\mathcal{R}_1 = \partial_1(-\Delta)^{-\frac{1}{2}}$ denotes the Riesz transform and the fractional Laplacian operator is defined via the Fourier transform

$$
\widehat{(-\Delta)^{\beta}f}(\xi) = |\xi|^{2\beta}\widehat{f}(\xi).
$$

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$\nabla^{\perp} = (-\partial_2, \partial_1)$ and $u = \nabla^{\perp}\Delta^{-1}\omega$ represents the Biot–Savart law recovering the velocity u from the vorticity $\omega := \nabla \times u$. The velocity formulation of the 2D Euler equation is given by

$$\partial_t u + (u \cdot \nabla)u = -\nabla p, \qquad \nabla \cdot u = 0, \qquad x \in \mathbb{R}^2,\ t > 0,$$

with the corresponding vorticity $\omega = \nabla \times u$ satisfying

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = 0, \qquad x \in \mathbb{R}^2,\ t > 0, \\ u = \nabla^{\perp}\Delta^{-1}\omega. \end{cases} \tag{1.3}$$

The global well-posedness of (1.3) has been well established [34]. Especially, the classical Yudovich theory [61] asserts that any initial datum $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ leads to a unique global weak solution $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. The proof of this result relies crucially on the fact that ω is conserved along the particle trajectory. However, the Yudovich framework fails on (1.1) and (1.2). The terms $\mathcal{R}_1\omega$ and $\mathcal{R}_1^2\omega$ involve the Calderon–Zygmund type singular integral operators and are not bounded on L^∞. As a consequence, the L^∞-norms of the solutions to (1.1) and (1.2) are not known to be bounded. The L^q-norms of ω for $1 < q < \infty$ are bounded, but these L^q-norms could grow exponentially in q, as pointed out by Elgindi [17]. Therefore, the Yudovich approach and its refinements do not work for (1.1) and (1.2). Whether or not solutions of (1.1) and (1.2) can blow up in a finite time remains an outstanding open problem. Some preliminary investigations on (1.1) and (1.2) and two other closely related models have been conducted [9].

The corresponding velocity formulation for (1.2) is given by

$$\begin{cases} \partial_t u_1 + (u \cdot \nabla)u_1 = -\partial_1 P, \\ \partial_t u_2 + (u \cdot \nabla)u_2 + u_2 = -\partial_2 P, \\ \nabla \cdot u = 0 \end{cases} \tag{1.4}$$

with damping only in the second component of the velocity equation. The global regularity problem as well as the stability near the trivial solution of (1.4) remain open. As we shall reveal in this paper, when the velocity is coupled with the magnetic field via the MHD system, the background magnetic field actually stabilizes the fluid, and both the velocity and vorticity remain small if they are initially so. In fact, they actually decay algebraically in time. The magnetic field smooths and stabilizes the velocity through coupling and interaction. The influence of an external magnetic field on the behavior of electrically conducting fluids has been observed in many experiments and numerical simulations (see, e.g., [1, 2, 3, 4, 21, 22]). One goal of this paper is to establish these observations as mathematically rigorous facts.

We give a more precise description of what we achieve in this paper. Attention is focused on the following 2D incompressible MHD equations

$$\begin{cases} \partial_t u_1 + (u \cdot \nabla)u_1 = -\partial_1 P + (B \cdot \nabla)B_1, \quad x \in \mathbb{R}^2,\ t > 0, \\[2mm] \partial_t u_2 + (u \cdot \nabla)u_2 + \gamma u_2 = -\partial_2 P + (B \cdot \nabla)B_2, \quad x \in \mathbb{R}^2,\ t > 0, \\[2mm] \partial_t B + u \cdot \nabla B = \eta\,\Delta B + B \cdot \nabla u, \quad x \in \mathbb{R}^2,\ t > 0, \\[2mm] \nabla \cdot u = \nabla \cdot B = 0, \quad x \in \mathbb{R}^2,\ t > 0, \end{cases} \tag{1.5}$$

where $u = (u_1, u_2)^\top, B = (B_1, B_2)^\top$, and P denote the velocity field of the fluid, the magnetic field, and the scalar pressure, respectively. The parameters $\gamma > 0$ and $\eta > 0$ represent the damping coefficient and the magnetic diffusivity, respectively. Clearly, (1.5) with $B \equiv 0$ reduces to the velocity equation in (1.4). Our main goal here is to understand the stability problem on perturbations near a background magnetic field and give a precise description on the large-time behavior of the perturbations. It is easy to verify that the special steady state given by the background magnetic field, namely,

$$u^{(0)} \equiv 0, \quad B^{(0)} \equiv e_2 := (0, 1),$$

solves (1.5). The perturbation (u, b) with $b = B - B^{(0)}$ near the steady state $(u^{(0)}, B^{(0)})$ solves the MHD equations

$$\begin{cases} \partial_t u_1 + (u \cdot \nabla)u_1 = -\partial_1 P + (b \cdot \nabla)b_1 + \partial_2 b_1, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t u_2 + (u \cdot \nabla)u_2 + \gamma u_2 = -\partial_2 P + (b \cdot \nabla)b_2 + \partial_2 b_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t b + (u \cdot \nabla)b = \eta\,\Delta b + (b \cdot \nabla)u + \partial_2 u, \quad x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \quad x \in \mathbb{R}^2,\ t > 0. \end{cases} \tag{1.6}$$

The system (1.6) supplemented with the initial data

$$u(x, 0) = u_0(x), \qquad b(x, 0) = b_0(x)$$

will be the centerpiece of our study. By taking the curl of (1.6), we find that $\omega = \nabla \times u$ and $j = \nabla \times b$ satisfy

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = \gamma\,\mathcal{R}_1^2 \omega + (b \cdot \nabla)j + \partial_2 j, \\ \partial_t j + (u \cdot \nabla)j = \eta\,\Delta j + (b \cdot \nabla)\omega + \partial_2 \omega + Q, \end{cases} \tag{1.7}$$

where

$$Q = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

The stability problem appears to be impossible on our first glance at the vorticity equation alone. It is the 2D Euler equation with three forcing terms. As is well known, the gradient of the Euler vorticity and more generally its Sobolev norm can grow rapidly (even double exponentially) in time (see, e.g., [12, 28, 62]). The term $\gamma\,\mathcal{R}_1^2\omega$ can only aggravate the situation. Since Riesz transform type singular integral operators $\mathcal{R}_1^2$ are not bounded on L^∞, this term can actually inflate the L^∞-norm of the vorticity, as demonstrated in [17]. The two other terms $b \cdot \nabla j$ and $\partial_2 j$ are related to the magnetic field b and the current density j, and they do not appear to be useful when the vorticity equation is treated alone.

However, it is the smoothing and stabilization effects of the magnetic field via the coupling and interaction that help stabilize the fluid and make this stability problem possible. To reveal these effects, we first eliminate the pressure P by applying the Leray–Helmholtz projection operator $\mathbb{P} := I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation in (1.6). Noticing that

$$\mathbb{P}(0, u_2)^\top = (0, u_2)^\top - \nabla\Delta^{-1}\nabla \cdot (0, u_2)^\top = \Delta^{-1}\partial_1^2 u = -\mathcal{R}_1^2 u,$$

(1.6) is then converted into

$$\begin{cases} \partial_t u = \gamma \mathcal{R}_1^2 u + \partial_2 b + N_1, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t b = \eta\, \Delta b + \partial_2 u + N_2, \quad x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \quad x \in \mathbb{R}^2,\ t > 0, \end{cases} \tag{1.8}$$

where N_1 and N_2 are the nonlinear terms

$$N_1 = \mathbb{P}((b \cdot \nabla) b - (u \cdot \nabla) u),$$
$$N_2 = (b \cdot \nabla) u - (u \cdot \nabla) b.$$

By differentiating (1.8) in t and making several substitutions, we find

$$\begin{cases} \partial_{tt} u - (\eta \Delta + \gamma\, \mathcal{R}_1^2) \partial_t u - (\gamma \eta\, \partial_{11} u + \partial_{22} u) = N_3, \\ \partial_{tt} b - (\eta \Delta + \gamma\, \mathcal{R}_1^2) \partial_t b - (\gamma \eta\, \partial_{11} b + \partial_{22} b) = N_4, \\ \nabla \cdot u = \nabla \cdot b = 0, \end{cases} \tag{1.9}$$

where

$$N_3 = (\partial_t - \eta\, \Delta) N_1 + \partial_2 N_2,$$
$$N_4 = (\partial_t - \gamma \mathcal{R}_1^2) N_2 + \partial_2 N_1.$$

Similarly, we can rewrite (1.7) as

$$\begin{cases} \partial_{tt} \omega - (\eta \Delta + \gamma\, \mathcal{R}_1^2) \partial_t \omega - (\gamma \eta\, \partial_{11} \omega + \partial_{22} \omega) = N_5, \\ \partial_{tt} j - (\eta \Delta + \gamma\, \mathcal{R}_1^2) \partial_t j - (\gamma \eta\, \partial_{11} j + \partial_{22} j) = N_6, \end{cases} \tag{1.10}$$

where

$$N_5 = (\partial_t - \eta\, \Delta)(b \cdot \nabla j - u \cdot \nabla \omega) + \partial_2 (b \cdot \nabla \omega - u \cdot \nabla j + Q),$$
$$N_6 = (\partial_t - \gamma \mathcal{R}_1^2)(b \cdot \nabla \omega - u \cdot \nabla j + Q) + \partial_2 (b \cdot \nabla j - u \cdot \nabla \omega).$$

Amazingly, all physical quantities u, b, ω, and j satisfy exactly the same wave equations with various nonhomogeneous terms. In comparison with the original system of (ω, j) in (1.7), the wave equations (1.10) obeyed by (ω, j) exhibit much more smoothing and stabilization properties, which make the stability and large-time behavior problem plausible. By taking advantage of these dissipative and dispersive effects, we are able to establish the desired global stability and provide sharp decay rates for the solution. The precise statements of our results are given in the following two theorems.

THEOREM 1.1. *Let $(u_0, b_0) \in H^3(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then there exists sufficiently small $\delta = \delta(\gamma, \eta) > 0$ such that, if*

$$\|\nabla u_0\|_{H^2(\mathbb{R}^2)} + \|\nabla b_0\|_{H^2(\mathbb{R}^2)} \le \delta,$$

then (1.6) *possesses a unique global solution* $(u, b) \in C([0,\infty); H^3(\mathbb{R}^2))$ *satisfying*

$$\|(u,b)(t)\|^2_{H^1(\mathbb{R}^2)} + \int_0^t \left(\|u_2(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla b(\tau)\|^2_{H^1(\mathbb{R}^2)}\right) d\tau$$

$$\le C\,\left(\|u_0\|^2_{H^1(\mathbb{R}^2)} + \|b_0\|^2_{H^1(\mathbb{R}^2)}\right), \tag{1.11}$$

$$\|(\nabla u,\nabla b)(t)\|^2_{H^2(\mathbb{R}^2)} + \int_0^t \Big(\|\partial_1 u(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^2 u(\tau)\|^2_{H^1(\mathbb{R}^2)} + \|\nabla^2 b(\tau)\|^2_{H^2(\mathbb{R}^2)}\Big) d\tau$$

$$\le C\,\delta^2 \tag{1.12}$$

for any $t > 0$ *and some constant* $C > 0$. *Furthermore, the following time decay estimate holds:*

$$\|\nabla u(t)\|_{H^2(\mathbb{R}^2)} + \|\nabla b(t)\|_{H^2(\mathbb{R}^2)} \le C\,(\|(u_0,b_0)\|_{L^2(\mathbb{R}^2)} + \delta)\,(1+t)^{-\frac{1}{2}}, \tag{1.13}$$

when δ *is small enough. In particular, for any* $2 < q < \infty$, *as* $t \to \infty$,

$$\|u_2(t)\|_{L^2(\mathbb{R}^2)} \to 0, \quad \|(u,b)(t)\|_{W^{2,q}(\mathbb{R}^2)} \to 0, \quad \textit{and} \quad \|(u,b)(t)\|_{W^{1,\infty}(\mathbb{R}^2)} \to 0. \tag{1.14}$$

As Theorem 1.1 states, the H^1-norm of the solution is uniformly bounded by the initial H^1-norm regardless of the size of the initial H^1-norm. The smallness assumption is not imposed on $\|(u_0, b_0)\|_{L^2(\mathbb{R}^2)}$. In the uniform bounds in (1.11) and (1.12), several time integral bounds are not a direct consequence of the damping or dissipation in the original system. For example,

$$\int_0^t \|\partial_2 u\|^2_{L^2(\mathbb{R}^2)}\,d\tau \le C\,\delta^2, \quad \int_0^t \|\partial_2 \omega(\tau)\|^2_{H^1(\mathbb{R}^2)}\,d\tau \le C\delta^2$$

are consequences of the smoothing effects due to the wave structure in (1.9).

Efforts are also devoted to establishing sharp decay rates for the solution established in Theorem 1.1. The regularization effect of the wave structure is exploited to achieve this goal. We solve the linearized system in (1.8) or, equivalently, (1.9) explicitly and represent the nonlinear system in an integral form. More precisely, we convert (1.8) into the system

$$\widehat{u}(\xi,t) = \widehat{M_1}(t)\widehat{u}_0 + \widehat{M_2}(t)\widehat{b}_0 + \int_0^t \left(\widehat{M_1}(t-\tau)\widehat{N}_1(\tau) + \widehat{M_2}(t-\tau)\widehat{N}_2(\tau)\right) d\tau, \tag{1.15}$$

$$\widehat{b}(\xi,t) = \widehat{M_2}(t)\widehat{u}_0 + \widehat{M_3}(t)\widehat{b}_0 + \int_0^t \left(\widehat{M_2}(t-\tau)\widehat{N}_1(\tau) + \widehat{M_3}(t-\tau)\widehat{N}_2(\tau)\right) d\tau, \tag{1.16}$$

where the kernel functions are given by

$$\widehat{M_1} = \eta|\xi|^2 G_1 + G_2, \qquad \widehat{M_2} = i\xi_2\, G_1, \qquad \widehat{M_3} = -\eta|\xi|^2 G_1 + G_3$$

with

$$G_1(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \qquad G_2(t) = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_2 t} + \lambda_1 G_1(t),$$

$$G_3(t) = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1(t).$$

Here λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + (\eta|\xi|^2 + \gamma\xi_1^2|\xi|^{-2})\lambda + (\gamma\eta\xi_1^2 + \xi_2^2) = 0$$

or, more explicitly,

$$\lambda_1 = \frac{-(\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2) - \sqrt{\Gamma}}{2}, \ \lambda_2 = \frac{-(\gamma\xi_1^2|\xi|^2 + \eta|\xi|^{-2}) + \sqrt{\Gamma}}{2},$$

where

$$\Gamma = (\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2)^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2).$$

The integral representation in (1.15) and (1.16) does not appear to be simple with the kernel functions being nonhomogeneous and frequency dependent. By appropriately estimating the Sobolev norms of the solutions, we are able to obtain the sharp decay rates stated in the following theorem.

THEOREM 1.2. *Assume* $(u_0, b_0) \in L^1(\mathbb{R}^2) \cap H^3(\mathbb{R}^2)$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$ *satisfying*

$$\|(u_0, b_0)\|_{L^1(\mathbb{R}^2)} + \|(u_0, b_0)\|_{H^3(\mathbb{R}^2)} \le \delta$$

for some δ *small enough. Then for* $m = 0, 1, 2$*, the small global solution* (u, b) *of the system* (1.6) *obeys*

$$\|D^m u(t)\|_{L^2(\mathbb{R}^2)} + \|D^m b(t)\|_{L^2(\mathbb{R}^2)} \le C\delta(1+t)^{-\frac{1+m}{2}},$$

where $C > 0$ *is a constant independent of* δ *and* t.

The decay rates obtained in Theorem 1.2 for the solution of the nonlinear system in (1.6) are the same as those for the 2D heat equation as well as the Navier–Stokes equations (see, e.g., [39, 41]). They are optimal. This reaffirms the smoothing and stabilization effect of the magnetic field on the fluids.

The results presented in Theorems 1.1 and 1.2 not only rigorously confirm the smoothing and stabilization effects of the magnetic field on electrically conducting fluids, they also advance the courses on how to understand the stability problem when the underlying model involves only partial dissipation. The MHD equations have recently attracted extensive interests due to their wide physical applicability and their mathematical significance. The MHD equations model electrically conducting in the presence of a magnetic field and serve as the center piece of the magnetohydrodynamics initiated by the Nobel laureate Alfvén (see, e.g., [3, 10, 30]). The resistive MHD model studied in this paper is applicable when the fluid viscosity can be ignored while the role of resistivity is important such as in magnetic reconnection and magnetic turbulence [36]. Mathematically the MHD equations share many features with the Navier–Stokes and the Euler equations; however, the MHD equations are not a combination of a pair of the Navier–Stokes type equations, but an integrated and interactive system. The MHD equations can model much richer phenomena than the Navier–Stokes equations such as the Alfvén waves.

There are substantial recent developments on fundamental issues concerning the MHD equations such as the global regularity and stability problems. One recent focus is on the MHD equations with only partial or fractional dissipation. Significant progress has been made (see, e.g., [4, 5, 6, 8, 11, 14, 15, 16, 18, 19, 20, 23, 24, 25, 26, 27, 29, 31, 32, 33, 35, 37, 38, 40, 43, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60]). However, many important issues remain outstandingly open. One of them is the stability problem of the MHD equations with only magnetic diffusion (without the viscous dissipation). Theorems 1.1 and 1.2 presented in this paper solve this stability problem when the velocity equation also involves one component damping and obtain precise and sharp large-time behavior on the solutions. These results are completely new and will be useful for future investigations of PDE systems with only partial dissipation.

We briefly explain how we prove Theorems 1.1 and 1.2. The framework in the proof of Theorem 1.1 is the bootstrapping argument (see, e.g., [44, p. 21]). The first step is to construct a suitable energy functional. In addition to the standard H^3-energy terms, we also include the regularization terms suggested by the wave structure in (1.10). We set the energy functional $E(t)$ to be

$$E(t) = E_1(t) + E_2(t) + E_3(t),$$

where

$$\begin{aligned}
E_1(t) &= \sup_{0\le\tau\le t}\Big(\|\omega(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|j(\tau)\|^2_{L^2(\mathbb{R}^2)}\Big)\\
&\quad + \int_0^t \Big(\|\partial_1 u(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla j(\tau)\|^2_{L^2(\mathbb{R}^2)}\Big)\,d\tau,\\
E_2(t) &= \sup_{0\le\tau\le t}\Big(\|\nabla\omega(\tau)\|^2_{H^1(\mathbb{R}^2)} + \|\nabla j(\tau)\|^2_{H^1(\mathbb{R}^2)}\Big)\\
&\quad + \int_0^t \Big(\|\partial_1 \omega(\tau)\|^2_{H^1(\mathbb{R}^2)} + \|\nabla^2 j(\tau)\|^2_{H^1(\mathbb{R}^2)}\Big)\,d\tau,\\
E_3(t) &= \int_0^t \|\partial_2\omega(\tau)\|^2_{H^1(\mathbb{R}^2)}\,d\tau.
\end{aligned}$$

The inclusion of E_3, suggested by (1.10), helps bound the nonlinear term $(u\cdot\nabla)u$ in the process of estimating the H^3-norm of u. Otherwise, we would not be able to close the estimates. An equivalent process is to design a Lyapunov functional given by

$$L(u,b)(t) = \|(\nabla u(t), \nabla b(t))\|^2_{H^2(\mathbb{R}^2)} + \lambda(\nabla u(t), \partial_2\nabla b(t))_{H^1(\mathbb{R}^2)},$$

where $\lambda > 0$ is a small parameter and $(F,G)_{H^1}$ denotes the H^1-inner product. The main efforts are devoted to estimating $E(t)$. This is a long and tedious process involving applications of various anisotropic inequalities. We are able to show that

$$E(t) \le C_1^* E(0) + C_2^* E^{\frac{3}{2}}(t) + C_3^* E^{\frac{5}{4}}(t).$$

An application of the bootstrapping argument would lead to the desired stability.

To obtain the optimal decay rates stated in Theorem 1.2, we make use of the integral representation in (1.15) and (1.16). By dividing the frequency space into suitable subdomains, we pinpoint the exact behavior of the kernel functions $\widehat{M_1}$, $\widehat{M_2}$, and $\widehat{M_3}$ and provide upper bounds for them in each subdomain. Due to the nonlinearity in the system (1.15) and (1.16), we employ the bootstrapping argument, which starts with the ansatz, for any $t \le T$,

$$\begin{aligned}
\|u(t)\|_{L^2(\mathbb{R}^2)} + \|b(t)\|_{L^2(\mathbb{R}^2)} &\le C_0\delta\ (1+t)^{-\frac{1}{2}},\\
\|Du(t)\|_{L^2(\mathbb{R}^2)} + \|Db(t)\|_{L^2(\mathbb{R}^2)} &\le C_1\delta\ (1+t)^{-1},\\
\|D^2u(t)\|_{L^2(\mathbb{R}^2)} + \|D^2b(t)\|_{L^2(\mathbb{R}^2)} &\le C_2\delta\ (1+t)^{-\frac{3}{2}}
\end{aligned}$$

for suitably chosen C_0, C_1, and C_2. Inserting the ansatz bounds in the integral representation and invoking the upper bounds for the kernel functions, we obtain, after carefully estimating the L^2-norms on each subdomain,

$$\|u(t)\|_{L^2(\mathbb{R}^2)} + \|b(t)\|_{L^2(\mathbb{R}^2)} \le \frac{C_0}{2}\ \delta\ (1+t)^{-\frac{1}{2}},$$
$$\|Du(t)\|_{L^2(\mathbb{R}^2)} + \|Db(t)\|_{L^2(\mathbb{R}^2)} \le \frac{C_1}{2}\ \delta\ (1+t)^{-1},$$
$$\|D^2u(t)\|_{L^2(\mathbb{R}^2)} + \|D^2b(t)\|_{L^2(\mathbb{R}^2)} \le \frac{C_2}{2}\ \delta\ (1+t)^{-\frac{3}{2}}. \tag{1.17}$$

The bootstrapping argument then implies that $T = \infty$ and (1.17) holds for all time.

The rest of this paper is naturally divided into two sections. Section 2 proves Theorem 1.1 while section 3 presents the proof of Theorem 1.2.

2. Proof of Theorem 1.1. This section is devoted to proving Theorem 1.1. We start with several tools to be used frequently in this section. The first provides an anisotropic upper bound for integrals involving triple products. It was previously stated and proven in [7].

LEMMA 2.1. *Assume $f, g, h, \partial_1 g, \partial_2 h \in L^2(\mathbb{R}^2)$. Then, for a constant $C > 0$,*

$$\iint fgh\, dx_1\, dx_2 \le C\|f\|_{L^2(\mathbb{R}^2)}\|g\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\partial_1 g\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|h\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\partial_2 h\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}. \tag{2.1}$$

The second tool provides an easily verifiable condition under which a nonnegative and integrable function actually approaches zero at infinity. It is [13, Lemma 3.1].

LEMMA 2.2. *Let $f = f(t)$ with $t \in [0,\infty)$ be a nonnegative and uniform continuous function. Assume f is integrable on $[0,\infty)$,*

$$\int_0^\infty f(t)\, dt < \infty.$$

Then

$$f(t) \to 0 \quad \text{as } t \to \infty.$$

We remark that the uniform continuity condition in Lemma 2.2 can be replaced by a slightly weaker assumption that for any $\delta > 0$, there is $\rho > 0$ such that, for any $0 \le t_1 < t_2$ with $t_2 - t_1 \le \rho$,

$$\text{either } f(t_2) \le f(t_1) \quad \text{or} \quad f(t_2) \ge f(t_1) \text{ and } f(t_2) - f(t_1) \le \delta.$$

The following lemma assesses the precise decay rate for a nonnegative integrable function when it decreases in a generalized sense.

LEMMA 2.3. *Let $f = f(t)$ be a nonnegative continuous function satisfying, for two constants $a_0 > 0$ and $a_1 > 0$,*

$$\int_0^\infty f(\tau)\, d\tau \le a_0 < \infty \quad \text{and} \quad f(t) \le a_1 f(s) \quad \text{for any } 0 \le s < t. \tag{2.2}$$

Then, for $a_2 = \max\{2a_1 f(0), 2a_0 a_1\}$ and for any $t > 0$,

$$f(t) \le a_2 (1+t)^{-1}.$$

Since the proof of Theorem 1.1 is long, for the sake of clarity, we divide it into three main parts. This section is split into three subsections. The first part establishes the global uniform H^1-bound for the solution (u, b) and related time integral bounds. Besides controlling the standard time integral terms, we are also able to bound the time integral of $\|\nabla u(t)\|_{L^2}^2$. This is not a consequence of the original damping and the magnetic diffusion. It is obtained by taking into account the wave structure in (1.9) and by evaluating a mixed term, namely, the inner product $(\partial_2 u, b)$. It is this bound that helps us obtain the decay rate for $\|(\nabla u(t), \nabla b(t))\|_{L^2}$. This part of the proof is provided in the first subsection.

The second main part is to construct the energy function $E(t)$, given by

$$E(t) = E_1(t) + E_2(t) + E_3(t), \tag{2.3}$$

where

$$E_1(t) = \sup_{0\le\tau\le t}\Big(\|\omega(\tau)\|_{L^2}^2 + \|j(\tau)\|_{L^2}^2\Big) + \int_0^t \Big(\|\partial_1 u(\tau)\|_{L^2}^2 + \|\nabla j(\tau)\|_{L^2}^2\Big) d\tau,$$

$$E_2(t) = \sup_{0\le\tau\le t}\Big(\|\nabla\omega(\tau)\|_{H^1}^2 + \|\nabla j(\tau)\|_{H^1}^2\Big) + \int_0^t \Big(\|\partial_1 \omega(\tau)\|_{H^1}^2 + \|\nabla^2 j(\tau)\|_{H^1}^2\Big) d\tau,$$

$$E_3(t) = \int_0^t \|\partial_2\omega(\tau)\|_{H^1}^2 d\tau.$$

The inclusion of E_3, suggested by (1.10), helps bound the nonlinear term $(u\cdot\nabla)u$ in the process of estimating the H^3-norm of u. Otherwise, we would not be able to close the estimates. An equivalent process is to design a Lyapunov functional given by

$$L(u, b)(t) = \|(\nabla u(t), \nabla b(t))\|_{H^2}^2 + \lambda(\nabla u(t), \partial_2\nabla b(t))_{H^1},$$

where $\lambda > 0$ is a small parameter and $(F, G)_{H^1}$ denotes the H^1-inner product. The main efforts are devoted to estimating $E(t)$. This is a long and tedious process involving applications of various anisotropic inequalities such as Lemma 2.1 above. We are able to show that

$$E(t) \le C_1^* E(0) + C_2^* E^{\frac{3}{2}}(t) + C_3^* E^{\frac{5}{4}}(t). \tag{2.4}$$

A bootstrapping argument is then applied to (2.4) to obtain the desired stability. The second subsection provides the details.

The third main part is to prove the large-time behavior and decay estimates stated in Theorem 1.1. Both Lemmas 2.2 and 2.3 will be used. In order to obtain the decay rate for $\|\nabla u(t)\|_{H^2}$ and $\|\nabla b(t)\|_{H^2}$, according to Lemma 2.3, we need to verify that, for

$$f(t) := \|\nabla u(t)\|_{H^2} + \|\nabla b(t)\|_{H^2}$$

and for any $0 \le t_1 \le t_2$ and a uniform constant $C > 0$,

$$\int_0^\infty f(t)\, dt < \infty \quad \text{and} \quad f(t_2) \le C\, f(t_1).$$

The time integrability part is a consequence of the first part and (2.4) in the second part, but the generalized decreasing property takes some effort. The idea is to use $E(t)$ defined in (2.3) with $\tau \in [t_1, t]$ as a bridge. Since $f(t)$ is part of $E(t)$, we have $f(t_2) \le E(t_2)$. We then show that, for some constant $C > 0$,

$$E(t_2) \le C\, f(t_1) + C\, E^{\frac{3}{2}}(t_2) + C E^{\frac{5}{4}}(t_2). \tag{2.5}$$

According to the second part, when the initial datum or $E(0)$ is sufficiently small, say, $E(0) \le \delta^2$, then $E(t)$ remains uniformly small, $E(t) \le C\,\delta^2$. By taking δ to be small, (2.5) implies that

$$E(t_2) \le C\,f(t_1).$$

As a consequence, we obtain $f(t_2) \le C\,f(t_1)$ and Lemma 2.3 leads to the desired decay rates. This part, together with the completion of the proof for Theorem 1.1, is presented in the third subsection.

2.1. Uniform bounds in H^1. As described above, this subsection proves the uniform H^1 and related time integral bounds stated in the following proposition.

PROPOSITION 2.4. *Assume the initial datum* $(u_0, b_0) \in H^1$ *with* $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. *Then the corresponding solution* (u, b) *of* (1.6) *satisfies*

$$\begin{aligned}&\big(\|u(t)\|_{H^1}^2 + \|b(t)\|_{H^1}^2\big) + \int_0^t \big(\|u_2(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{H^1}^2\big)\,d\tau\\ &\quad\le C\,(\|u_0\|_{H^1}^2 + \|b_0\|_{H^1}^2).\end{aligned}$$

Proof of Proposition 2.4. Taking the L^2-inner product of (1.6) with (u, b), we obtain

$$\|(u,b)(t)\|_{L^2}^2 + 2\int_0^t (\gamma\|u_2(\tau)\|_{L^2}^2 + \eta\|\nabla b(\tau)\|_{L^2}^2)\,d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

To prove the H^1-bound, we resort to the equation of (ω, j) with $\omega = \nabla\times u$, $j = \nabla\times b$,

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \gamma\mathcal{R}_1^2\omega + (b\cdot\nabla)j + \partial_2 j, \\ \partial_t j + (u\cdot\nabla)j = \eta\,\Delta j + (b\cdot\nabla)\omega + \partial_2\omega + Q, \end{cases} \tag{2.6}$$

where

$$Q = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

Multiplying (2.6) by (ω, j), integrating over $\mathbb{R}^2$, and applying Hölder's inequality and Gagaliardo–Nirenberg's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\omega,j)(t)\|_{L^2}^2 + \gamma\|\nabla u_2\|_{L^2}^2 + \eta\|\nabla j\|_{L^2}^2\\ &\quad= \int Qj\;dx\\ &\quad\le C\|\nabla b\|_{L^4}\|\nabla u\|_{L^2}\|j\|_{L^4}\\ &\quad\le C\|\nabla b\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}\|j\|_{L^2}^{\frac12}\|\nabla j\|_{L^2}^{\frac12}\\ &\quad\le C\|\nabla j\|_{L^2}\|j\|_{L^2}\|\omega\|_{L^2} \le \frac{\eta}{2}\|\nabla j\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^2,\end{aligned} \tag{2.7}$$

where we have used the facts

$$\begin{aligned}&\|\mathcal{R}_1\omega\|_{L^2} = \|\partial_1 u\|_{L^2} = \|\nabla u_2\|_{L^2},\ \|\nabla u\|_{L^2} = \|\omega\|_{L^2},\\ &\|\nabla b\|_{L^2} = \|j\|_{L^2},\ \|\nabla^2 b\|_{L^2} = \|\nabla j\|_{L^2}.\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
&\|(\omega,j)(t)\|_{L^2}^2+\int_0^t(2\gamma\|\nabla u_2(\tau)\|_{L^2}^2+\eta\|\nabla j(\tau)\|_{L^2}^2)\,d\tau\\
&\qquad\le C(\|(u_0,b_0)\|_{L^2})\,\|(\omega_0,j_0)\|_{L^2}^2.
\end{aligned}\tag{2.8}$$

Next we bound $\int_0^t\|\partial_2 u(\tau)\|_{L^2}^2 d\tau$ and show that

$$\int_0^t\|\partial_2 u(\tau)\|_{L^2}^2 d\tau\le C\,(\|u_0\|_{H^1}^2+\|b_0\|_{H^1}^2).$$

The idea is to evaluate the L^2-inner product $(\partial_2 u,b)$. It follows from (1.6) that

$$\begin{aligned}
-\frac{d}{dt}(\partial_2 u,b)+\|\partial_2 u\|_{L^2}^2-\|\partial_2 b\|_{L^2}^2&=\int\partial_2(u\cdot\nabla u)\cdot\ b\,dx\\
&\quad+\int\Big(\partial_2 u\cdot(u\cdot\nabla b)-\partial_2(b\cdot\nabla b)\cdot b\Big)\,dx\\
&\quad-\int\partial_2 u\cdot(b\cdot\nabla u)\,dx\\
&\quad+\int(\gamma\partial_2 u_2\ b_2-\eta\partial_2 u\cdot\Delta b)\,dx\\
&:=I_1+I_2+I_3+I_4.
\end{aligned}$$

Further dividing I_1 into two terms, and applying Hölder's inequality and the Sobolev embedding inequality, we have

$$\begin{aligned}
I_1&=-\int(u_1\partial_1 u\cdot\partial_2 b+u_2\partial_2 u\cdot\partial_2 b)\,dx\\
&\le\|u_1\|_{L^4}\|\partial_1 u\|_{L^2}\|\partial_2 b\|_{L^4}+\|u_2\|_{L^4}\|\partial_2 u\|_{L^2}\|\partial_2 b\|_{L^4}\\
&\le C\|u_1\|_{H^1}\|\partial_1 u\|_{L^2}\|\partial_2 b\|_{H^1}+C\|u_2\|_{H^1}\|\partial_2 u\|_{L^2}\|\partial_2 b\|_{H^1}\\
&\le C\|u\|_{H^1}(\|\partial_1 u\|_{L^2}^2+\|\nabla b\|_{H^1}^2)+C\|u\|_{H^1}^2\|\nabla b\|_{H^1}^2+\frac18\|\partial_2 u\|_{L^2}^2.
\end{aligned}$$

Similarly, I_2 and I_3 can be bounded as follows:

$$\begin{aligned}
I_2&=\int\Big(\partial_2 u\cdot(u\cdot\nabla b)+(b\cdot\nabla b)\cdot\partial_2 b\Big)\,dx\\
&\le\|\partial_2 u\|_{L^2}\|u\|_{L^4}\|\nabla b\|_{L^4}+\|b\|_{L^2}\|\nabla b\|_{L^4}^2\\
&\le C(\|u\|_{H^1}^2+\|b\|_{L^2})\|\nabla b\|_{H^1}^2+\frac18\|\partial_2 u\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
I_3&=\int(b_1\partial_1 u\cdot\partial_2 u+b_2\partial_2 u\cdot\partial_2 u)\,dx\\
&\le\|b_1\|_{L^\infty}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}+\|b_2\|_{L^\infty}\|\partial_2 u\|_{L^2}^2\\
&\le C\|b\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\big(\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}+\|\partial_2 u\|_{L^2}^2\big)\\
&\le C\|b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\partial_1 u\|_{L^2}^2+C\|b\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2\|\partial_2 u\|_{L^2}^2+\frac18\|\partial_2 u\|_{L^2}^2\\
&\le C\|b\|_{L^2}\|\nabla u\|_{L^2}(\|\nabla^2 b\|_{L^2}^2+\|\partial_1 u\|_{L^2}^2)+C\|b\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla^2 b\|_{L^2}^2+\frac18\|\partial_2 u\|_{L^2}^2,
\end{aligned}$$

where we have used the Gagaliardo–Nirenberg's inequality $\|b\|_{L^\infty} \le C\|b\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}$ in I_3. By integration by parts and Hölder's inequality,

$$\begin{aligned} I_4 &= -\gamma \int u_2\ \partial_2 b_2\,dx - \eta\int \partial_2 u\cdot\Delta b\,dx \\ &\le \frac{\gamma}{2}\Big(\|u_2\|_{L^2}^2 + \|\partial_2 b_2\|_{L^2}^2\Big) + \left(\frac18\|\partial_2 u\|_{L^2}^2 + 2\eta^2\|\Delta b\|_{L^2}^2\right). \end{aligned}$$

Collecting all the estimates above for I_1 through I_4 leads to

$$\begin{aligned} &-2\frac{d}{dt}(\partial_2 u, b) + \|\partial_2 u\|_{L^2}^2 - \big(\gamma\|u_2\|_{L^2}^2 + (2+\gamma)\|\partial_2 b\|_{L^2}^2 + 4\eta^2\|\Delta b\|_{L^2}^2\big) \\ &\quad\le C\Big(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|b\|_{L^2} + \|b\|_{L^2}\|\nabla u\|_{L^2} + \|b\|_{L^2}^2\|\nabla u\|_{L^2}^2\Big)\big(\|\partial_1 u\|_{L^2}^2 + \|\nabla b\|_{H^1}^2\big) \\ &\quad\le C(\eta,\ \|u_0\|_{H^1}^2,\ \|b_0\|_{H^1}^2)\big(\|\partial_1 u\|_{L^2}^2 + \|\nabla b\|_{H^1}^2\big). \end{aligned} \tag{2.9}$$

Integrating (2.9) over $[0,t]$ yields

$$\begin{aligned} &\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2\,d\tau \\ &\quad\le \int_0^t (\gamma\|u_2(\tau)\|_{L^2}^2 + (2+\gamma)\|\partial_2 b(\tau)\|_{L^2}^2 + 4\eta^2\|\Delta b(\tau)\|_{L^2}^2)\,d\tau \\ &\qquad + (\|\partial_2 u\|_{L^2}^2 + \|b\|_{L^2}^2) + (\|\partial_2 u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2) + C\ (\|u_0\|_{H^1}^2 + \|b_0\|_{H^1}^2) \\ &\quad\le C\,(\|u_0\|_{H^1}^2 + \|b_0\|_{H^1}^2). \end{aligned}$$

This completes the proof of Proposition 2.4. ∎

2.2. Proof of (2.4). This subsection is devoted to the proof of (2.4). As previously mentioned at the beginning of this section, a crucial step in proving the desired stability is to prove (2.4). We state it as a proposition for the purpose of easy reference later on.

PROPOSITION 2.5. *Assume* $(u_0, b_0) \in H^3$ *obeys the conditions stated in Theorem* 1.1. *Let* (u, b) *be the corresponding solution of* (1.6). *Let* $E(t)$ *be defined as in* (2.3). *Then* (2.4) *holds.*

Proof of Proposition 2.5. According to (2.3), $E(t)$ consists of three pieces E_1 and E_2 and E_3. The first piece E_1 contains the homogeneous $\dot H$-norm of (u, b) and has been estimated in (2.8),

$$E_1(t) \le CE_1(0). \tag{2.10}$$

E_2 contains the $\dot H^2$ and $\dot H^3$-norms of (u, b). Its upper bound depends on E_3. It does not appear possible to bound E_2 without E_3. The estimate E_3 is not trivial and it is the wave structure in (1.9) that leads to its boundedness. The rest of this proof establishes the following bounds,

$$E_2(t) \le CE^{\frac32}(t) + E_2(0) \tag{2.11}$$

and

$$E_3(t) - C_0^*(\eta,\gamma)(E_1(t) + E_2(t)) \le C\Big(E^{\frac32}(t) + E^{\frac54}(t)\Big) + E(0), \tag{2.12}$$

where $C > 0$ and C_0^* are constants. (2.10), (2.11), and (2.12) yield the desired global bound in (2.4). We start with the proof of (2.11).

Taking the L^2-inner product of (2.6) with $(\Delta\omega, \Delta j)$ and integrating by parts yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\omega, \nabla j)(t)\|_{L^2}^2 + \gamma\|\partial_1\omega\|_{L^2}^2 + \eta\|\Delta j\|_{L^2}^2 \\
&\quad = -\int (\nabla u\cdot\nabla)\omega\cdot\nabla\omega\,dx + \int(\nabla b\cdot\nabla)j\cdot\nabla\omega\,dx - \int(\nabla u\cdot\nabla)j\cdot\nabla j\,dx \\
&\qquad + \int(\nabla b\cdot\nabla)\omega\cdot\nabla j\,dx + \int\nabla Q\cdot\nabla j\,dx,
\end{aligned}$$

where we have used $\|\mathcal{R}_1\nabla\omega\|_{L^2} = \|\partial_1\omega\|_{L^2}$. By Hölder's and Sobolev's inequalities,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla\omega, \nabla j)(t)\|_{L^2}^2 + \gamma\|\partial_1\omega\|_{L^2}^2 + \eta\|\Delta j\|_{L^2}^2 \\
&\quad \le \|\nabla u\|_{L^\infty}\Big(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\Big) + 2\|\nabla b\|_{L^\infty}\|\nabla j\|_{L^2}\|\nabla\omega\|_{L^2} \\
&\qquad + 2\big(\|\nabla u\|_{L^\infty}\|\nabla^2 b\|_{L^2} + \|\nabla b\|_{L^\infty}\|\nabla^2 u\|_{L^2}\big)\|\nabla j\|_{L^2} \\
&\quad \le C(\|\omega\|_{H^2} + \|j\|_{H^2})(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2),
\end{aligned} \tag{2.13}$$

where we have used the simple facts

$$\|\nabla u\|_{L^2} = \|\omega\|_{L^2}, \quad \|\nabla b\|_{L^2} = \|j\|_{L^2},\ \|\nabla^2 u\|_{L^2} = \|\nabla\omega\|_{L^2}, \quad \|\nabla^2 b\|_{L^2} = \|\nabla j\|_{L^2}.$$

Applying Δ to (2.6) and taking the inner product with $(\Delta\omega, \Delta j)$ leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\Delta\omega, \Delta j)\|_{L^2}^2 + \gamma\|\partial_1\nabla\omega\|_{L^2}^2 + \eta\|\nabla\Delta j\|_{L^2}^2 \\
&\quad = -\int \Delta(u\cdot\nabla)\omega\ \Delta\omega\,dx + \int\Delta(b\cdot\nabla)j\ \Delta\omega\,dx - \int\Delta(u\cdot\nabla)j\ \Delta j\,dx \\
&\qquad + \int\Delta(b\cdot\nabla)\omega\ \Delta j\,dx + \int\Delta Q\ \Delta j\,dx := J_1 + J_2 + \cdots + J_5.
\end{aligned}$$

By integration by parts, Hölder's inequality, and Sobolev's inequality,

$$\begin{aligned}
J_1 &= -\int(\Delta u\cdot\nabla\omega)\ \Delta\omega\,dx - 2\int\nabla u\cdot\nabla(\nabla\omega)\ \Delta\omega\,dx \\
&\le \|\Delta u\|_{L^4}\|\nabla\omega\|_{L^4}\|\Delta\omega\|_{L^2} + 2\|\nabla u\|_{L^\infty}\|\nabla^2\omega\|_{L^2}^2 \\
&\le C\|\Delta u\|_{H^1}\|\nabla\omega\|_{H^1}\|\Delta\omega\|_{L^2} + C\|\nabla u\|_{H^2}\|\nabla^2\omega\|_{L^2}^2 \\
&\le C\|\omega\|_{H^2}\|\nabla\omega\|_{H^1}^2.
\end{aligned}$$

J_1 would not be suitably bounded without E_3. J_3 can be bounded in a similar way:

$$\begin{aligned}
J_3 &= -\int(\Delta u\cdot\nabla j)\ \Delta j\,dx - 2\int\nabla u\cdot\nabla(\nabla j)\ \Delta j\,dx \\
&\le \|\Delta u\|_{L^4}\|\nabla j\|_{L^4}\|\Delta j\|_{L^2} + 2\|\nabla u\|_{L^\infty}\|\nabla^2 j\|_{L^2}^2 \\
&\le C\|\omega\|_{H^2}\|\nabla j\|_{H^1}^2.
\end{aligned}$$

We combine J_2 and J_4. By integration by parts,

$$\begin{aligned} J_2 + J_4 = &\int \Big(\Delta b\cdot\nabla j + 2\nabla b\cdot\nabla(\nabla j)\Big)\ \Delta\omega\, dx \\ &+ \int \Big(\Delta b\cdot\nabla \omega + 2\nabla b\cdot\nabla(\nabla \omega)\Big)\ \Delta j\, dx \\ &\le \|\Delta b\|_{L^4}\|\nabla j\|_{L^4}\|\Delta\omega\|_{L^2} + 4\|\nabla b\|_{L^\infty}\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2} \\ &\quad + \|\Delta b\|_{L^4}\|\nabla\omega\|_{L^4}\|\Delta j\|_{L^2} \\ &\le C\|\Delta b\|_{H^1}\|\nabla j\|_{H^1}\|\Delta\omega\|_{L^2} + C\|\nabla b\|_{H^2}\|\nabla^2 j\|_{L^2}\|\nabla^2\omega\|_{L^2} \\ &\quad + C\|\Delta b\|_{H^1}\|\nabla\omega\|_{H^1}\|\Delta j\|_{L^2} \\ &\le C\|j\|_{H^2}\Big(\|\nabla\omega\|_{H^1}^2 + \|\nabla j\|_{H^1}^2\Big). \end{aligned}$$

By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned} J_5 &\le C\int \big(|\Delta\nabla b||\nabla u| + |\nabla^2 b||\nabla^2 u| + |\nabla b||\Delta\nabla u|\big)\ |\Delta j|\, dx \\ &\le C\Big(\|\Delta\nabla b\|_{L^2}\|\nabla u\|_{L^\infty} + \|\nabla^2 b\|_{L^4}\|\nabla^2 u\|_{L^4} + \|\nabla b\|_{L^\infty}\|\nabla\Delta u\|_{L^2}\Big)\|\Delta j\|_{L^2} \\ &\le C\Big(\|\Delta\nabla b\|_{L^2}\|\nabla u\|_{H^2} + \|\nabla^2 b\|_{H^1}\|\nabla^2 u\|_{H^1} + \|\nabla b\|_{H^2}\|\nabla\Delta u\|_{L^2}\Big)\|\Delta j\|_{L^2} \\ &\le C(\|\omega\|_{H^2} + \|j\|_{H^2})(\|\nabla\omega\|_{H^1}^2 + \|\Delta j\|_{L^2}^2). \end{aligned}$$

We have thus obtained

$$\begin{aligned} &\frac{d}{dt}\|(\Delta\omega,\Delta j)(t)\|_{L^2}^2 + 2\gamma\|\partial_1\nabla\omega\|_{L^2}^2 + 2\eta\|\nabla\Delta j\|_{L^2}^2 \\ &\qquad \le C(\|\omega\|_{H^2} + \|j\|_{H^2})(\|\nabla\omega\|_{H^1}^2 + \|\nabla j\|_{H^1}^2). \end{aligned} \tag{2.14}$$

Combining (2.13) with (2.14), we have

$$\begin{aligned} &\frac{d}{dt}\|(\nabla\omega,\nabla j)(t)\|_{H^1}^2 + 2\gamma\|\partial_1\omega\|_{H^1}^2 + 2\eta\|\Delta j\|_{H^1}^2 \\ &\qquad \le C(\|\omega\|_{H^2} + \|j\|_{H^2})(\|\nabla\omega\|_{H^1}^2 + \|\nabla j\|_{H^1}^2). \end{aligned} \tag{2.15}$$

Integrating in time leads to, for some constant $C>0$,

$$\begin{aligned} E_2(t) &\le C\sup_{0\le\tau\le t}(\|\omega(\tau)\|_{H^2} + \|j(\tau)\|_{H^2})\int_0^t (\|\nabla\omega(\tau)\|_{H^1}^2 + \|\nabla j(\tau)\|_{H^1}^2)\, d\tau \\ &\quad + \|(\nabla\omega_0,\nabla j_0)\|_{H^1}^2 \\ &\le CE^{\frac{3}{2}}(t) + E_2(0), \end{aligned}$$

which is (2.11).

We now turn to the proof of (2.12). Due to the wave structure in (1.9) and (1.10), we realize that the time integral term $\int_0^t \|\partial_2 w(\tau)\|_{H^1}\, d\tau$ in E_3 can be generated as a consequence of the inner products

$$(\partial_2\nabla u, \nabla b) \quad \text{and} \quad (\partial_2\nabla\omega, \nabla j).$$

We focus on the time evolution of these two inner products. Using (1.6), we have

$$
\begin{aligned}
\frac{d}{dt}(\partial_2\nabla u,\nabla b) &= (\partial_2\nabla u_t,\nabla b)+(\partial_2\nabla u,\nabla b_t)\\
&=\int \partial_2\nabla\Big(-(u\cdot\nabla)u-\gamma(0,u_2)^\top+(b\cdot\nabla)b\Big)\cdot\nabla b\;\,dx-\|\partial_2\nabla b\|_{L^2}^2\\
&\quad+\int \partial_2\nabla u\cdot\nabla\Big(-(u\cdot\nabla)b+\eta\,\Delta b+(b\cdot\nabla)u\Big)\,dx+\|\partial_2\nabla u\|_{L^2}^2,
\end{aligned}\tag{2.16}
$$

where $\int\partial_2\nabla(\nabla p)\cdot\nabla b\,dx=0$ due to $\nabla\cdot b=0$. Similarly, by (2.6),

$$
\begin{aligned}
\frac{d}{dt}(\partial_2\nabla\omega,\nabla j) &= \int \partial_2\nabla\Big(-(u\cdot\nabla)\omega+\gamma\mathcal{R}_1^2\omega+(b\cdot\nabla)j\Big)\cdot\nabla j\,dx-\|\partial_2\nabla j\|_{L^2}^2\\
&\quad+\int \partial_2\nabla\omega\cdot\nabla\Big(-(u\cdot\nabla)j+\eta\,\Delta j+(b\cdot\nabla)\omega+Q\Big)\,dx+\|\partial_2\nabla\omega\|_{L^2}^2.
\end{aligned}\tag{2.17}
$$

Summing (2.16) and (2.17) yields

$$
\begin{aligned}
&-\frac{d}{dt}\Big[(\partial_2\nabla u,\nabla b)+(\partial_2\nabla\omega,\nabla j)\Big]+\|\partial_2\omega\|_{H^1}^2-\|\partial_2 j\|_{H^1}^2\\
&=\int\big(\partial_2\nabla(u\cdot\nabla)u-\partial_2\nabla(b\cdot\nabla)b\big)\cdot\nabla b\,dx\\
&\quad+\int\partial_2\nabla u\cdot(\nabla(u\cdot\nabla)b-\nabla(b\cdot\nabla)u)\,dx\\
&\quad+\int\big(\partial_2\nabla(u\cdot\nabla)\omega-\partial_2\nabla(b\cdot\nabla)j\big)\cdot\nabla j\,dx\\
&\quad+\int\partial_2\nabla\omega\cdot\big(\nabla(u\cdot\nabla)j-\nabla(b\cdot\nabla)\omega\big)\,dx\\
&\quad-\int\partial_2\nabla\omega\cdot\nabla Q\,dx\\
&\quad+\int\Big[\gamma(\partial_2\nabla u_2\cdot\nabla b_2-\partial_2\nabla\mathcal{R}_1^2\omega\cdot\nabla j)-\eta(\partial_2\nabla u\cdot\nabla\Delta b+\partial_2\nabla\omega\cdot\nabla\Delta j)\Big]\,dx\\
&:=K_1+\cdots+K_6.
\end{aligned}\tag{2.18}
$$

We bound the terms in (2.18) one by one. By integration by parts, Hölder's inequality, the anisotropic inequality (2.1), and Sobolev's inequality $\|v\|_{L^4}\le C\|v\|_{L^2}^{\frac12}\|\nabla v\|_{L^2}^{\frac12}$,

$$
\begin{aligned}
K_1&=\int(u\cdot\nabla)u\cdot\partial_2\Delta b\,dx+\int\big(\partial_2 b\cdot\nabla b+b\cdot\nabla\partial_2 b\big)\cdot\Delta b\,dx\\
&\le C\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\partial_2\Delta b\|_{L^2}+\|\partial_2 b\|_{L^4}\|\nabla b\|_{L^4}\|\Delta b\|_{L^2}\\
&\quad+\|b\|_{L^4}\|\nabla\partial_2 b\|_{L^4}\|\Delta b\|_{L^2}\\
&\le C\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\omega\|_{L^2}^{\frac12}\|\nabla\omega\|_{L^2}^{\frac12}\|\partial_2\nabla j\|_{L^2}\\
&\quad+\|\partial_2 b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\Delta b\|_{L^2}^{\frac32}+\|b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\nabla\partial_2 b\|_{H^1}\|\Delta b\|_{L^2}\\
&\le C\Big(\|u\|_{L^2}^{\frac12}\|\omega\|_{L^2}^{\frac12}+\|j\|_{L^2}+\|b\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\Big)\Big(\|\partial_1 u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla j\|_{H^1}^2\Big)\\
&\le C\Big(\|\omega\|_{L^2}^{\frac12}+\|j\|_{L^2}^{\frac12}+\|j\|_{L^2}\Big)(\|\partial_1 u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla j\|_{H^1}^2),
\end{aligned}\tag{2.19}
$$

where we have used the uniform bound on $\|(u,b)\|_{L^2}$. Similarly, K_2 can be bounded by

$$\begin{aligned}
K_2 &= -\int (u\cdot\nabla)b\cdot\partial_2\Delta u\,dx - \int \big((\nabla b\cdot\nabla)u + (b\cdot\nabla)\nabla u\big)\cdot\partial_2\nabla u\,dx \\
&\le C\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\partial_2\Delta u\|_{L^2} + \|\nabla b\|_{L^4}\|\nabla u\|_{L^4}\|\partial_2\nabla u\|_{L^2} \\
&\quad + \|b\|_{L^4}\|\nabla^2 u\|_{L^4}\|\partial_2\nabla u\|_{L^2} \\
&\le C\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\nabla j\|_{L^2}^{\frac12}\|\nabla^2\omega\|_{L^2} + \|\nabla b\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac32} \\
&\quad + \|b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\nabla^2 u\|_{H^1}\|\partial_2\nabla u\|_{L^2} \\
&\le C\Big(\|j\|_{L^2}^{\frac12} + \|j\|_{L^2}^{\frac12}\|\omega\|_{L^2}^{\frac12}\Big)(\|\partial_1 u\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \|\nabla\omega\|_{H^1}^2). \qquad (2.20)
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
K_3 &= \int (u\cdot\nabla)\omega\ \partial_2\Delta j\,dx - \int (b\cdot\nabla)j\ \partial_2\Delta j\,dx \\
&\le \Big(\|u\|_{L^4}\|\nabla\omega\|_{L^4} + \|b\|_{L^4}\|\nabla j\|_{L^4}\Big)\|\partial_2\Delta j\|_{L^2} \\
&\le C\Big(\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla\omega\|_{H^1} + \|b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\nabla j\|_{H^1}\Big)\|\partial_2\Delta j\|_{L^2} \\
&\le C\Big(\|\omega\|_{L^2}^{\frac12} + \|j\|_{L^2}^{\frac12}\Big)(\|\nabla\omega\|_{H^1}^2 + \|\nabla j\|_{H^2}^2). \qquad (2.21)
\end{aligned}$$

For K_4, we have

$$\begin{aligned}
K_4 &= \int \Big((\nabla u\cdot\nabla)j + (u\cdot\nabla)\nabla j - \big(\nabla b\cdot\nabla)\omega - (b\cdot\nabla)\nabla\omega\Big)\ \partial_2\nabla\omega\,dx \\
&\le \big(\|\nabla u\|_{L^4}\|\nabla j\|_{L^4} + \|u\|_{L^4}\|\nabla^2 j\|_{L^4} + \|\nabla b\|_{L^4}\|\nabla\omega\|_{L^4} + \|b\|_{L^\infty}\|\nabla^2\omega\|_{L^2}\big)\|\partial_2\nabla\omega\|_{L^2} \\
&\le C\Big(\|\nabla u\|_{H^1}\|\nabla j\|_{H^1} + \|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 j\|_{H^1} + \|\nabla b\|_{H^1}\|\nabla\omega\|_{H^1} \\
&\qquad + \|b\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\|\nabla^2\omega\|_{L^2}\Big)\|\Delta\omega\|_{L^2} \\
&\le C\Big(\|\omega\|_{L^2}^{\frac12} + \|\nabla j\|_{L^2}^{\frac12} + \|\omega\|_{H^1} + \|j\|_{H^1}\Big)(\|\nabla\omega\|_{H^1}^2 + \|\nabla j\|_{H^2}^2). \qquad (2.22)
\end{aligned}$$

The terms in K_5 are similar to the first and third terms in (2.22),

$$\begin{aligned}
K_5 &\le C\big(\|\nabla u\|_{L^4}\|\nabla^2 b\|_{L^4} + \|\nabla b\|_{L^4}\|\nabla^2 u\|_{L^4}\big)\|\partial_2\nabla\omega\|_{L^2} \\
&\le C\Big(\|\omega\|_{H^1} + \|j\|_{H^1}\Big)(\|\nabla\omega\|_{H^1}^2 + \|\nabla j\|_{H^1}^2). \qquad (2.23)
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
K_6 &= \int \Big[-\gamma(\partial_2 u_2\ \Delta b_2 + \partial_2\nabla\mathcal{R}_1^2\omega\cdot\nabla j) - \eta(\partial_2\nabla u\cdot\nabla\Delta b + \partial_2\nabla\omega\cdot\nabla\Delta j)\Big]\,dx \\
&\le \frac12\Big(\|\partial_2 u_2\|_{L^2}^2 + \gamma^2\|\Delta b_2\|_{L^2}^2\Big) + \Big(\frac14\|\partial_2\nabla\mathcal{R}_1^2\omega\|_{L^2}^2 + \gamma^2\|\nabla j\|_{L^2}^2\Big) \\
&\quad + \Big(\frac12\|\partial_2\nabla u\|_{L^2}^2 + \frac{\eta^2}{2}\|\nabla\Delta b\|_{L^2}^2\Big) + \Big(\frac14\|\partial_2\nabla\omega\|_{L^2}^2 + \eta^2\|\nabla^3 j\|_{L^2}^2\Big) \\
&\le \frac12(\|\partial_1 u_1\|_{L^2}^2 + 3\gamma^2\|\nabla j\|_{L^2}^2) + \frac12\|\partial_2\omega\|_{H^1}^2 + \eta^2\|\Delta j\|_{H^1}^2, \qquad (2.24)
\end{aligned}$$

where we have used $\|\partial_2\nabla\mathcal{R}_1^2\omega\|_{L^2} \le \|\partial_2\nabla\omega\|_{L^2}$. Inserting the bounds (2.19)–(2.24) into (2.18), we obtain

$$\begin{aligned}
&\|\partial_2\omega\|_{H^1}^2 - C_0^*(\|\partial_1 u_1\|_{L^2}^2 + \|\nabla j\|_{H^2}^2) \\
&\quad\le 2\frac{d}{dt}\Big[(\partial_2\nabla u, \nabla b) + (\partial_2\nabla\omega, \nabla j)\Big] \\
&\qquad + C\Big(\|(\omega, j)\|_{H^1}^{\frac12} + \|(w, j)\|_{H^1}\Big)\big(\|\partial_1 u\|_{L^2}^2 + \|\nabla\omega\|_{H^1}^2 + \|\nabla j\|_{H^2}^2\big)
\end{aligned} \tag{2.25}$$

for $C_0^* = 2 + 2\eta^2 + 3\gamma^2$. Integrating (2.25) over $[0, t]$ yields

$$\begin{aligned}
&\int_0^t \|\partial_2\omega(\tau)\|_{H^1}^2 d\tau - C_0^*\int_0^t (\|\partial_1 u_1(\tau)\|_{L^2}^2 + \|\nabla j(\tau)\|_{H^2}^2)\, d\tau \\
&\quad\le (\|\partial_2\nabla u\|_{H^1}^2 + \|j\|_{H^1}^2) + (\|\partial_2\nabla u_0\|_{H^1}^2 + \|j_0\|_{H^1}^2) \\
&\qquad + C\sup_{0\le\tau\le t}\Big(\|(\omega, j)(\tau)\|_{H^1}^{\frac12} + \|(\omega, j)(\tau)\|_{H^1}\Big) \\
&\qquad \cdot\int_0^t \big(\|\partial_1 u(\tau)\|_{L^2}^2 + \|\nabla\omega(\tau)\|_{H^1}^2 + \|\nabla j(\tau)\|_{H^2}^2\big)\, d\tau \\
&\quad\le (\|\nabla\omega\|_{H^1}^2 + \|j\|_{H^1}^2) + C(E^{\frac32}(t) + E^{\frac54}(t)) + E(0),
\end{aligned}$$

which implies

$$E_3(t) - C_0^*(E_1(t) + E_2(t)) \le C\Big(E^{\frac32}(t) + E^{\frac54}(t)\Big) + E(0).$$

This completes the proof of (2.12). By taking a small number $\lambda > 0$ and considering the combination

$$(2.10) + (2.11) + \lambda\,(2.12),$$

we find that there exist $C_1^* > 0, C_2^* > 0$, and $C_3^* > 0$ such that

$$E(t) \le C_1^* E(0) + C_2^*\, E^{\frac32}(t) + C_3^* E^{\frac54}(t),$$

which is (2.4). This completes the proof of Proposition 2.5. ∎

2.3. Proof of Theorem 1.1. This subsection completes the proof of Theorem 1.1 using the bounds obtained in the previous two subsections. We first apply the bootstrapping argument to show the stability and then prove the part on the large-time behavior of the solution.

Proof of Theorem 1.1. We now combine the uniform bounds in Propositions 2.4 and 2.5 to establish the global existence and stability of solutions to (1.6). Proposition 2.4 gives us the global uniform H^1-bound regardless of the size of the initial datum (u_0, b_0) in H^1, namely,

$$\|(u(t), b(t))\|_{H^1} \le C\,\|(u_0, b_0)\|_{H^1}.$$

The energy inequality obtained in Proposition 2.5,

$$E(t) \le C_1^* E(0) + C_2^*\, E^{\frac32}(t) + C_3^* E^{\frac54}(t), \tag{2.26}$$

allows us to conclude that, if $\|(\nabla u_0, \nabla b_0)\|_{H^2}$ is sufficiently small, say

$$\|(\nabla u_0, \nabla b_0)\|_{H^2} \le \delta := \sqrt{\frac{M}{4C_1^*}}, \tag{2.27}$$

where

$$M := \min\left\{1, \frac{1}{(4\widetilde{C})^4}\right\} \quad \text{with} \quad \widetilde{C} = \max\{C_2^*,\ C_3^*\},$$

then the solution remains uniformly small,

$$E(t) \le 2C_1^*\delta^2 \quad \text{or} \quad \|(\nabla u(t), \nabla b(t))\|_{H^2} \le \sqrt{2C_1^*}\,\delta.$$

This is shown by applying the bootstrapping argument to (2.26). The argument starts with the ansatz that, for $t \le T$,

$$E(t) \le M.$$

By (2.26) and (2.27),

$$\begin{aligned} E(t) &\le C_1^* E(0) + \widetilde{C}(E^{\frac{1}{2}}(t) + E^{\frac{1}{4}}(t))E(t) \\ &\le C_1^*\delta^2 + 2\widetilde{C}E^{\frac{1}{4}}(t)E(t) \\ &\le C_1^*\delta^2 + \frac{1}{2}E(t). \end{aligned}$$

Then

$$E(t) \le 2C_1^*\delta^2 = 2C_1^*\,\frac{M}{4C_1^*} = \frac{M}{2}.$$

The bootstrapping argument implies that $T = \infty$ and for any $t < \infty$,

$$E(t) \le \frac{1}{2}M.$$

This completes the proof for the global existence and stability of solutions to (1.6).

We now prove the large-time behavior estimates stated in Theorem 1.1. First we show

$$\|(\nabla u(t), \nabla b(t))\|_{L^2} \le C(\|u_0\|_{H^1}, \|b_0\|_{H^1})\,(1+t)^{-\frac{1}{2}} \quad \text{and} \quad \|u_2(t)\|_{L^2} \to 0 \quad \text{as } t \to \infty.$$

The decay estimate is obtained by applying Lemma 2.3. We verify the conditions (2.2) in Lemma 2.3. First of all, Proposition 2.4 implies

$$\int_0^\infty \left(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2\right) d\tau \le C\,\|(u_0, b_0)\|_{H^1}^2 < \infty.$$

In addition, as in the proof of (2.8), for $0 \le t_1 < t_2$,

$$\|\nabla u(t_2)\|_{L^2}^2 + \|\nabla b(t_2)\|_{L^2}^2 \le e^{\frac{1}{2\eta}(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)}\left(\|\nabla u(t_1)\|_{L^2}^2 + \|\nabla b(t_1)\|_{L^2}^2\right).$$

Lemma 2.3 then yields

$$\|(\nabla u(t), \nabla b(t))\|_{L^2} \le C\,\|(u_0, b_0)\|_{H^1}\,(1+t)^{-\frac{1}{2}}.$$

Due to the Gagaliardo–Nirenberg's inequality, for any $2 < q < \infty$,

$$\|v\|_{L^q} \le C\|v\|_{L^2}^{\frac{2}{q}}\|\nabla v\|_{L^2}^{1-\frac{2}{q}}, \tag{2.28}$$

we find that $\|(u(t),b(t))\|_{L^q} \to 0$ as $t\to\infty$. Next we turn to the long-time behavior of $\|u_2(t)\|_{L^2}$. We will use Lemma 2.2 to show that

$$\|u_2(t)\|_{L^2} \to 0 \quad \text{as } t\to\infty.$$

By Proposition 2.4,

$$\int_0^\infty \|u_2(t)\|_{L^2}^2\,dt < \infty.$$

It then suffices to verify the uniform continuity part of Lemma 2.2. Multiplying the equation of u_2 in (1.8) by u_2 and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_2(t)\|_2^2 + \gamma\|\mathcal{R}_1 u_2\|_{L^2}^2 = &-\int (\mathbb{P}(u\cdot\nabla u))_2\; u_2\,dx \\ &+\int (\mathbb{P}(b\cdot\nabla b))_2\; u_2\,dx + \int \partial_2 b_2\; u_2\,dx.\end{aligned}$$

Recalling that $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ and using the fact that the singular integral operator $\Delta^{-1}\nabla\cdot\nabla\cdot$ is bounded on L^2 (see [42]), we have

$$\begin{aligned}\left|\int (\mathbb{P}(u\cdot\nabla u))_2\; u_2\,dx\right| &= \left|-\int \partial_2\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\; u_2\,dx\right| \\ &= \left|\int \Delta^{-1}\nabla\cdot\nabla\cdot(u\otimes u)\;\partial_2 u_2 dx\right| \le \|u\otimes u\|_{L^2}\|\partial_2 u_2\|_{L^2} \\ &\le \|u\|_{L^4}^2\|\partial_2 u_2\|_{L^2} \le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\partial_2 u_2\|_{L^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\int (\mathbb{P}(b\cdot\nabla b))_2\; u_2\,dx\right| &= \left|\int b\cdot\nabla b_2\; u_2\,dx + \int \Delta^{-1}\nabla\cdot(b\cdot\nabla b)\;\partial_2 u_2\,dx\right| \\ &\le \|b\|_{L^4}\|\nabla b\|_{L^2}\|u_2\|_{L^4} + \|b\|_{L^4}^2\|\partial_2 u_2\|_{L^2} \\ &\le C\|b\|_{H^1}^2\|u_2\|_{H^1}.\end{aligned}$$

By Hölder's inequality,

$$\left|\int \partial_2 b_2\; u_2\,dx\right| \le \frac{1}{2}(\|\partial_2 b_2\|_{L^2}^2 + \|u_2\|_{L^2}^2).$$

Invoking the uniform bound of $\|(u,b)\|_{H^1}$ in Proposition 2.4, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_2(t)\|_{L^2}^2 + \gamma\|\mathcal{R}_1 u_2\|_{L^2}^2 \\ &\quad\le C\|u\|_{L^2}\|\nabla u\|_{L^2}^2 + C\|b\|_{H^1}^2\|u_2\|_{H^1} + \frac{1}{2}(\|\partial_2 b_2\|_{L^2}^2 + \|u_2\|_{L^2}^2) \\ &\quad\le C^*(\eta,\ \gamma,\ \|u_0\|_{H^1},\ \|b_0\|_{H^1}), \\ &\frac{1}{2}\frac{d}{dt}\|u_2(t)\|_{L^2}^2 + \gamma\|\mathcal{R}_1 u_2\|_{L^2}^2 \\ &\quad\ge -C\|u\|_{L^2}\|\nabla u\|_{L^2}^2 - C\|b\|_{H^1}^2\|u_2\|_{H^1} - \frac{1}{2}(\|\partial_2 b_2\|_{L^2}^2 + \|u_2\|_{L^2}^2) \\ &\quad\ge -C^*(\eta,\ \gamma,\ \|u_0\|_{H^1},\ \|b_0\|_{H^1}),\end{aligned}$$

which verifies the uniform continuity of Lemma 2.2. As a consequence,

$$\|u_2(t)\|_{L^2} \to 0 \text{ as } \ t \to \infty.$$

Next we prove the decay estimate

$$\|(\omega(t), j(t))\|_{H^2} \le C\,(\|(u_0, b_0)\|_{L^2} + \delta)\,(1+t)^{-\frac{1}{2}}.$$

The tool is Lemma 2.3. We verify that

$$f(t) = \|\omega(t)\|_{H^2}^2 + \|j(t)\|_{H^2}^2$$

satisfies the conditions of Lemma 2.3. First of all, since $E(t) \le C\,\delta^2 < \infty$,

$$\int_0^\infty f(t)dt \le C\,(\|(u_0, b_0)\|_{L^2}^2 + \delta^2) < \infty. \tag{2.29}$$

It then suffices to show the generalized monotonicity that, for any $0 \le t_1 < t_2 < \infty$,

$$f(t_2) \le C\, f(t_1).$$

The idea is to use $E(t)$ as a bridge. With a slight abuse of notation, $E(t)$ here is defined as in (2.3) but with the starting time t_1 instead of 0. Since $f(t)$ is part of $E(t)$, we have $f(t_2) \le E(t_2)$. We then show that, for some constant $C > 0$,

$$E(t_2) \le C\, f(t_1) + C\, E^{\frac{3}{2}}(t_2) + CE^{\frac{5}{4}}(t_2). \tag{2.30}$$

According to the stability shown above, when the initial datum or $E(0)$ is sufficiently small, $E(t)$ remains uniformly small, $E(t) \le C\,\delta^2$. By taking δ to be small, (2.30) implies that

$$E(t_2) \le C\, f(t_1).$$

As a consequence, we obtain $f(t_2) \le C\, f(t_1)$. We now verify (2.30). By (2.7),

$$\frac{d}{dt}\|(\omega, j)(t)\|_{L^2}^2 + 2\gamma\|\nabla u_2\|_{L^2}^2 + \eta\|\nabla j\|_{L^2}^2 \le C\|j\|_{L^2}^2\|\omega\|_{L^2}^2. \tag{2.31}$$

By Gronwall's inequality,

$$\|\omega(t_2)\|_{L^2}^2 + \|j(t_2)\|_{L^2}^2 \le (\|\omega(t_1)\|_{L^2}^2 + \|j(t_1)\|_{L^2}^2)e^{C(\eta)(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)}.$$

If we integrate (2.31) over $[t_1, t_2]$ directly, we have

$$\begin{aligned} &(\|\omega(t_2)\|_{L^2}^2 + \|j(t_2)\|_{L^2}^2) + \int_{t_1}^{t_2} (2\gamma\|\partial_1 u(\tau)\|_{L^2}^2 + \eta\|\nabla j(\tau)\|_{L^2}^2)\, d\tau \\ &\quad \le (\|\omega(t_1)\|_{L^2}^2 + \|j(t_1)\|_{L^2}^2) + \sup_{t_1 \le \tau \le t_2} \|\omega(\tau)\|_{L^2}^2 \int_{t_1}^{t_2} \|j(\tau)\|_{L^2}^2\, d\tau \\ &\quad \le C(\eta)\Big(1 + (\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)e^{C(\eta)(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)}\Big)(\|\omega(t_1)\|_{L^2}^2 + \|j(t_1)\|_{L^2}^2). \end{aligned} \tag{2.32}$$

Next integrating (2.15) over $[t_1, t_2]$ yields

$$\begin{aligned}
&\left(\|\nabla\omega(t_2)\|_{H^1}^2 + \|\nabla j(t_2)\|_{H^1}^2\right) + \int_{t_1}^{t_2} \left(2\gamma\|\partial_1\omega(\tau)\|_{H^1}^2 + 2\eta\|\Delta j(\tau)\|_{H^1}^2\right) d\tau \\
&\le \left(\|\nabla\omega(t_1)\|_{H^1}^2 + \|\nabla j(t_1)\|_{H^1}^2\right) \\
&\quad + C \sup_{t_1\le\tau\le t_2} (\|\omega(\tau)\|_{H^2} + \|j(\tau)\|_{H^2}) \int_{t_1}^{t_2} (\|\nabla\omega(\tau)\|_{H^1}^2 + \|\nabla j(\tau)\|_{H^1}^2)\, d\tau \\
&\le \left(\|\nabla\omega(t_1)\|_{H^1}^2 + \|\nabla j(t_1)\|_{H^1}^2\right) + CE^{\frac{3}{2}}(t_2). \qquad (2.33)
\end{aligned}$$

Similarly, we obtain from (2.25) that

$$\begin{aligned}
&\int_{t_1}^{t_2} \|\partial_2\omega(\tau)\|_{H^1}^2\, d\tau - C_0^* \int_{t_1}^{t_2} (\|\partial_1 u_1(\tau)\|_{L^2}^2 + \|\nabla j(\tau)\|_{H^2}^2)\, d\tau \\
&\quad \le (\|\partial_2\nabla u(t_2)\|_{H^1}^2 + \|j(t_2)\|_{H^1}^2) + (\|\partial_2\nabla u(t_1)\|_{H^1}^2 + \|j(t_1)\|_{H^1}^2) \\
&\qquad + C \sup_{t_1\le\tau\le t_2} \left(\|(\omega, j)(\tau)\|_{H^1}^{\frac{1}{2}} + \|(\omega, j)(\tau)\|_{H^1}\right) \\
&\qquad \cdot \int_{t_1}^{t_2} \left(\|\partial_1 u(\tau)\|_{L^2}^2 + \|\nabla\omega(\tau)\|_{H^1}^2 + \|\nabla j(\tau)\|_{H^2}^2\right) d\tau \\
&\quad \le (\|\nabla\omega(t_2)\|_{H^1}^2 + \|j(t_2)\|_{H^1}^2) + (\|\nabla\omega(t_1)\|_{H^1}^2 + \|j(t_1)\|_{H^1}^2) + C(E^{\frac{3}{2}}(t_2) + E^{\frac{5}{4}}(t_2)). \qquad (2.34)
\end{aligned}$$

(2.32), (2.33), and (2.34) imply that for some $C_4^* > 0, C_5^* > 0$,

$$E(t_2) \le C_4^* f(t_1) + C_5^*(E^{\frac{3}{2}}(t_2) + E^{\frac{5}{4}}(t_2)). \qquad (2.35)$$

As we have shown in the stability part, for a uniform constant C and for all $t \ge 0$,

$$E(t) \le C\,\delta^2$$

if the initial datum is sufficiently small, or $E(0) \le \delta^2$ for small $\delta > 0$. If $\delta > 0$ is sufficiently small, we have

$$C_5^*(E^{\frac{3}{2}}(t_2) + E^{\frac{5}{4}}(t_2)) \le \frac{1}{2}E(t_2).$$

Then (2.35) yields

$$E(t_2) \le C_4^* f(t_1) + \frac{1}{2}E(t_2)$$

or

$$E(t_2) \le C f(t_1).$$

Combining with the simple fact that $f(t_2) \le E(t_2)$, we obtain the generalized monotonicity

$$f(t_2) \le C f(t_1). \qquad (2.36)$$

Therefore, (2.29) and (2.36) verify the conditions of Lemma 2.3, which implies

$$f(t) \le C\,(\|(u_0, b_0)\|_{L^2}^2 + \delta^2)\,(1+t)^{-1}.$$

That is, (1.13) holds:

$$\|\nabla u(t)\|_{H^2} \leq C\left(\|(u_0, b_0)\|_{L^2} + \delta\right)(1+t)^{-\frac{1}{2}}$$
$$\text{and } \|\nabla b(t)\|_{H^2} \leq C\left(\|(u_0, b_0)\|_{L^2} + \delta\right)(1+t)^{-\frac{1}{2}}.$$

The large-time behavior in (1.14) is a consequence of (2.28) and the Gagaliardo–Nirenberg inequality $\|v\|_{L^\infty} \leq C\|v\|_{L^2}^{\frac{1}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}}$. This completes the proof of Theorem 1.1. ∎

3. Proof of Theorem 1.2. This section proves Theorem 1.2, the sharp decay rates for the global solutions obtained in Theorem 1.1. We are assuming that the initial datum (u_0, b_0) satisfies

$$\|(u_0, b_0)\|_{H^3} \leq \delta, \quad \|(u_0, b_0)\|_{L^1} \leq \delta, \tag{3.1}$$

and (u, b) is the corresponding global solution established by Theorem 1.1. We constantly use the following properties of the solution (u, b),

$$\|(u,b)(t)\|_{H^3}^2 + \int_0^t (\|u_2(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{H^2}^2 + \|\nabla b(\tau)\|_{H^3}^2) d\tau \leq C\delta^2 \tag{3.2}$$

and

$$\|\nabla u(t)\|_{H^2} + \|\nabla b(t)\|_{H^2} \leq C\delta(1+t)^{-\frac{1}{2}}, \tag{3.3}$$

where the C's are constants independent of δ.

The sharp decay rates can no longer be shown by energy estimates. We need a more explicit representation of the solution. The idea is to first convert (1.6) into an integral representation. This is achieved by first solving the linearized system of (1.6) or (1.9) and then applying Duhamel's principle. The integral representation involves several kernel functions and the large-time behavior of the solution replies crucially on them. These Fourier multiplier operators are nonhomogeneous and depend crucially on the frequency. Naturally we split the frequency space into subdomains suitably and classify the behavior of these operators on each subdomain. Equivalently we provide upper sharp upper bounds on their symbols. Once this is at our disposal, we then launch the bootstrapping argument on the integral representation to deduce the desired decay rates.

The following two tools will be frequently used in the estimates. The first provides an explicit decay rate for the heat kernel associated with a fractional Laplacian Λ^α ($\alpha \in \mathbb{R}$). Here the fractional Laplacian operator can be defined through the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

The proof of the lemma can be found in many references (see, e.g., [15, 47]).

Lemma 3.1. *Let $\alpha \geq 0$, $\beta > 0$, and $1 \leq q \leq p \leq \infty$. Then there exists a constant C such that, for any $t > 0$,*

$$\|\Lambda^\alpha e^{-\Lambda^\beta t} f\|_{L^p(\mathbb{R}^d)} \leq C\, t^{-\frac{\alpha}{\beta} - \frac{d}{\beta}(\frac{1}{q} - \frac{1}{p})}\, \|f\|_{L^q(\mathbb{R}^d)}. \tag{3.4}$$

The following lemma provides upper bounds for a convolution type integral. Its proof is straightforward.

LEMMA 3.2. *Assume $0 < s_1 \le s_2$. Then, for some constant $C > 0$,*

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2}\ d\tau \le \begin{cases} C(1+t)^{-s_1} & \text{if } \ s_2 > 1, \\ C(1+t)^{-s_1}\ln(1+t) & \text{if } \ s_2 = 1, \\ C(1+t)^{1-s_1-s_2} & \text{if } \ s_2 < 1. \end{cases}$$

We now derive an integral representation satisfied by the solution of (1.8). Taking the Fourier transform of (1.8) yields

$$\partial_t \widehat{V} = A\widehat{V} + \widehat{N},$$

where

$$\widehat{V} = \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix}, \qquad A = \begin{pmatrix} -\gamma\xi_1^2|\xi|^{-2} & i\xi_2 \\ i\xi_2 & -\eta|\xi|^2 \end{pmatrix}, \qquad \widehat{N} = \begin{pmatrix} \widehat{N_1} \\ \widehat{N_2} \end{pmatrix}.$$

The solution of this nonhomogeneous ordinary differential equation can be represented as

$$\widehat{V}(t) = e^{At}\ \widehat{V_0} + \int_0^t e^{A(t-\tau)}\ \widehat{N(\tau)} d\tau.$$

In order to find a more explicit formula of e^{At}, we compute the eigenvalues and eigenvectors of A. The characteristic polynomial associated with A is

$$\lambda^2 + (\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2)\lambda + (\gamma\eta\xi_1^2 + \xi_2^2) = 0.$$

The eigenvalues of the matrix A are given by

$$\lambda_1 = \frac{-(\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2) - \sqrt{\Gamma}}{2}, \qquad \lambda_2 = \frac{-(\gamma\xi_1^2|\xi|^{2} + \eta|\xi|^{-2}) + \sqrt{\Gamma}}{2},$$

where

$$\Gamma = (\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2)^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2).$$

The corresponding eigenvectors are

$$\rho_1 = \begin{pmatrix} \lambda_1 + \eta|\xi|^2 \\ i\xi_2 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} \lambda_2 + \eta|\xi|^2 \\ i\xi_2 \end{pmatrix}.$$

Therefore, the matrix A can be diagonalized as

$$A = (\rho_1, \rho_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\rho_1, \rho_2)^{-1}.$$

Then

$$\begin{aligned} e^{At} &= \frac{1}{(\lambda_1 - \lambda_2)i\xi_2} \begin{pmatrix} \lambda_1 + \eta|\xi|^2 & \lambda_2 + \eta|\xi|^2 \\ i\xi_2 & i\xi_2 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} i\xi_2 & -(\lambda_2 + \eta|\xi|^2) \\ -i\xi_2 & \lambda_1 + \eta|\xi|^2 \end{pmatrix} \\ &= \begin{pmatrix} \eta|\xi|^2 G_1(t) + G_2(t) & G_1(t) i\xi_2 \\ G_1(t) i\xi_2 & -\eta|\xi|^2 G_1(t) + G_3(t) \end{pmatrix}, \end{aligned} \tag{3.5}$$

where

$$G_1(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \qquad G_2(t) = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_2 t} + \lambda_1 G_1(t),$$
$$G_3(t) = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1(t).$$

Therefore, if we write

$$\widehat{M_1}(t) = \eta|\xi|^2 G_1(t) + G_2(t), \quad \widehat{M_2}(t) = i\xi_2\, G_1(t), \quad \widehat{M_3}(t) = -\eta|\xi|^2 G_1(t) + G_3(t), \tag{3.6}$$

then (u, b) can be represented as

$$\widehat{u}(\xi,t) = \widehat{M_1}(t)\widehat{u}_0 + \widehat{M_2}(t)\widehat{b}_0 + \int_0^t \left(\widehat{M_1}(t-\tau)\widehat{N}_1(\tau) + \widehat{M_2}(t-\tau)\widehat{N}_2(\tau)\right) d\tau, \tag{3.7}$$

$$\widehat{b}(\xi,t) = \widehat{M_2}(t)\widehat{u}_0 + \widehat{M_3}(t)\widehat{b}_0 + \int_0^t \left(\widehat{M_2}(t-\tau)\widehat{N}_1(\tau) + \widehat{M_3}(t-\tau)\widehat{N}_2(\tau)\right) d\tau. \tag{3.8}$$

When $\lambda_1 = \lambda_2$, the representation in (3.7) and (3.8) remains valid if we replace G_1 by its limiting form

$$G_1(t) = \lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = t\, e^{\lambda_1 t}.$$

More precisely, when $\lambda_1 = \lambda_2$, we replace $G_1(t)$ by its limit $te^{\lambda_1 t}$, $G_2(t)$ by $e^{\lambda_1 t} + \lambda_1 te^{\lambda_1 t}$, and $G_3(t)$ by $e^{\lambda_1 t} - \lambda_1 te^{\lambda_1 t}$ in (3.5) to get

$$e^{At} = \begin{pmatrix} \eta|\xi|^2 te^{\lambda_1 t} + (1+\lambda_1 t)e^{\lambda_1 t} & i\xi_2 te^{\lambda_1 t} \\ i\xi_2 te^{\lambda_1 t} & -\eta|\xi|^2 te^{\lambda_1 t} + (1-\lambda_1 t)e^{\lambda_1 t} \end{pmatrix}. \tag{3.9}$$

This can also be obtained by a direct calculation. When $\lambda_1 = \lambda_2$, the associated eigenvector of A is

$$\rho = \begin{pmatrix} \lambda_1 + \eta|\xi|^2 \\ i\xi_2 \end{pmatrix},$$

and the general solution of $\partial_t \widehat{V} = A\widehat{V}$ is given by

$$a_3\, \rho\, e^{\lambda_1 t} + a_4\, (\rho\, t + \sigma) e^{\lambda_1 t}, \tag{3.10}$$

where a_3 and a_4 are to be determined by the initial datum, and σ solves

$$(A - \lambda_1 I)\sigma = \rho.$$

After some simple computation, we find

$$\sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We determine a_3 and a_4 by the initial datum $\widehat{u_0}$ and $\widehat{b_0}$. Setting $t = 0$ in (3.10) yields

$$a_3\, \rho + a_4\, \sigma = \begin{pmatrix} \widehat{u_0} \\ \widehat{b_0} \end{pmatrix}. \tag{3.11}$$

Solving (3.11) gives

$$a_3 = \frac{1}{i\xi_2}\widehat{b_0}, \qquad a_4 = \widehat{u_0} - \frac{\lambda_1 + \eta|\xi|^2}{i\xi_2}\widehat{b_0}.$$

Inserting a_3 and a_4 in (3.10) yields

$$\frac{1}{i\xi_2}\widehat{b_0}\begin{pmatrix}\lambda_1+\eta|\xi|^2\\ i\xi_2\end{pmatrix}e^{\lambda_1 t} + \left(\widehat{u_0} - \frac{\lambda_1+\eta|\xi|^2}{i\xi_2}\widehat{b_0}\right)\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\lambda_1+\eta|\xi|^2\\ i\xi_2\end{pmatrix}t\right]e^{\lambda_1 t}$$

$$= \begin{pmatrix} e^{\lambda_1 t} + (\lambda_1+\eta|\xi|^2)te^{\lambda_1 t} & -\frac{1}{i\xi_2}(\lambda_1+\eta|\xi|^2)^2\, te^{\lambda_1 t} \\ i\xi_2 te^{\lambda_1 t} & -(\lambda_1+\eta|\xi|^2)te^{\lambda_1 t} + e^{\lambda_1 t}\end{pmatrix}\begin{pmatrix}\widehat{u_0}\\ \widehat{b_0}\end{pmatrix}.$$

Using the simple fact $\Gamma = 0$ or $-\frac{1}{i\xi_2}(\lambda_1+\eta|\xi|^2)^2 = i\xi_2$, we can see that the coefficient matrix is the same as the one in (3.9).

The kernels $\widehat{M_i}(\xi,t)$ $(i=1,2,3)$ play a crucial role in the decay rates of u and b. Clearly the behavior of $\widehat{M_i}(\xi,t)$ $(i=1,2,3)$ depends on the frequency ξ. We classify their behavior and provide upper bounds by dividing the frequency space into subdomains.

PROPOSITION 3.3. *We divide $\mathbb{R}^2$ into two subdomains, $\mathbb{R}^2 = S_1 \cup S_2$ with*

$$S_1 := \left\{\xi\in\mathbb{R}^2 : \textit{either}\ \Gamma<0 \quad \textit{or} \quad 0\le\Gamma\le\left(\frac{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}{2}\right)^2\right\}$$

$$S_2 := \left\{\xi\in\mathbb{R}^2 : \Gamma>\left(\frac{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}{2}\right)^2 \quad \textit{or} \quad 3(\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2)^2 > 16(\gamma\eta\xi_1^2+\xi_2^2)\right\}.$$

Then we have

(1) *there are two constants $C>0$ and $c_0>0$ such that, for any $\xi\in S_1$,*

$$Re\lambda_1 \le -\frac{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}{2},\ Re\lambda_2 \le -\frac{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}{4},$$
$$|G_1(t)| \le te^{-\frac{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}{4}t}, \qquad |\widehat{M_i}(\xi,t)| \le Ce^{-c_0|\xi|^2 t},\ i=1,2,3;$$

(2) *there is a constant $C>0$ such that, for any $\xi\in S_2$,*

$$\lambda_1 < -\frac{3(\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2)}{4},\ \lambda_2 \le -\frac{\gamma\eta\xi_1^2+\xi_2^2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2},$$
$$|G_1(t)| < \frac{2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}\left(e^{-\frac{3(\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2)}{4}t} + e^{-\frac{C|\xi|^2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}t}\right),$$
$$|\widehat{M_i}(t)| < C(e^{-\frac{3}{4}(\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2)t} + e^{-\frac{C|\xi|^2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}t}),\ i=1,2,3.$$

If we further write $S_2 = S_{21} \cup S_{22}$ *with*

$$S_{21} := \{\xi \in S_2 : \ |\xi| \le 1\},$$
$$S_{22} := \{\xi \in S_2 : \ |\xi| > 1\},$$

then, for $i = 1, 2, 3$ *and some constants* $C > 0, c_1 > 0, c_2 > 0$,

$$|\widehat{M}_i(\xi, t)| < C\ e^{-c_1|\xi|^2 t} \quad \text{if} \quad \xi \in S_{21},$$
$$|\widehat{M}_i(t)| < C\ e^{-c_1|\xi|^2 t} + C\ e^{-c_2 t} \quad \text{if} \quad \xi \in S_{22}.$$

Proof of Proposition 3.3. For notational convenience, we denote $B = \gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2$. Then $\lambda_1,\ \lambda_2,\ \Gamma$ can be rewritten as

$$\lambda_1 = \frac{-B - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-B + \sqrt{\Gamma}}{2}, \quad \Gamma = B^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2).$$

For $\xi \in S_1$, $\Gamma < 0$ or $0 \le \sqrt{\Gamma} \le \frac{B}{2}$. It is then clear that

$$-\frac{3B}{4} \le Re\lambda_1 \le -\frac{B}{2}, \quad Re\lambda_2 \le -\frac{B}{4}, \quad |G_1(T)| \le te^{-\frac{B}{4}t},$$

where we have used the mean-value theorem in bounding $G_1(t)$. If λ_1 is a real number, by the simple fact that $x\ e^{-x} \le C$ for $x \ge 0$, we have

$$|\widehat{M}_1(t)| = \left|\eta|\xi|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\right| \le Bte^{-\frac{B}{4}t} + CBte^{-\frac{B}{4}t} + e^{-\frac{B}{4}t} \le Ce^{-c_0|\xi|^2 t}$$

for some pure constant c_0 dependent on γ and η. If λ_1 is an imaginary number, namely, $\Gamma < 0$ or

$$B^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2) < 0,$$

we further divide the consideration into two subcases: $\sqrt{\gamma\eta\xi_1^2 + \xi_2^2} \le |\sqrt{\Gamma}\ |$ and $\sqrt{\gamma\eta\xi_1^2 + \xi_2^2} \ge |\sqrt{\Gamma}\ |$. In the case when $\sqrt{\gamma\eta\xi_1^2 + \xi_2^2} \le |\sqrt{\Gamma}|$, by the definition of G_1, we have

$$|\lambda_1 G_1(t)| = \frac{\sqrt{\gamma\eta\xi_1^2 + \xi_2^2}}{|\sqrt{\Gamma}|}\,|e^{\lambda_1 t} - e^{\lambda_2 t}| \le Ce^{-\frac{B}{4}t}.$$

In the case when $\sqrt{\gamma\eta\xi_1^2 + \xi_2^2} \ge |\sqrt{\Gamma}\ |$, we have

$$\gamma\eta\xi_1^2 + \xi_2^2 \ge 4(\gamma\eta\xi_1^2 + \xi_2^2) - B^2$$

or

$$3(\gamma\eta\xi_1^2 + \xi_2^2) \le B^2.$$

Then

$$|\lambda_1 G_1(t)| = \sqrt{\gamma\eta\xi_1^2 + \xi_2^2}\,|G_1(t)| \le CBte^{-\frac{B}{4}t} \le Ce^{-\frac{B}{4}t}.$$

As a consequence, if λ_1 is an imaginary number, we obtain

$$|\widehat{M}_1(t)| = \left|\eta|\xi|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\right| \le Bte^{-\frac{B}{4}t} + Ce^{-\frac{B}{4}t} \le Ce^{-c_0|\xi|^2 t}.$$

In summary, for $\xi \in S_1$,

$$|\widehat{M_1}(t)| \le Ce^{-c_0|\xi|^2 t}. \tag{3.12}$$

Similarly,

$$|\widehat{M_3}(t)| = \Big| -\eta|\xi|^2 G_1(t) - \lambda_1 G_1(t) + e^{\lambda_1 t}\Big| \le Ce^{-c_0|\xi|^2 t}.$$

The proof of the bound

$$|\widehat{M_2}(t)| \le Ce^{-c_0|\xi|^2 t} \tag{3.13}$$

is similar to that for $M_1(t)$. By the definition of $\widehat{M_2}$ in (3.6) and the upper bound for G_1 in (3.12), we have

$$|\widehat{M_2}(t)| \le |\xi_2| t\, e^{-\frac{B}{4}t}.$$

To prove (3.13), we consider two cases $|\xi_2| \le |\sqrt{\Gamma}|$ and $|\xi_2| \ge |\sqrt{\Gamma}|$. In the first case $|\xi_2| \le |\sqrt{\Gamma}\,|$, we have

$$\widehat{M_2}(t)| = \left|\frac{\xi_2}{\sqrt{\Gamma}}\right| |e^{\lambda_1 t} - e^{\lambda_2 t}| \le C\, e^{-c_0|\xi|^2 t},$$

where we have used $x\ e^{-x} \le C$ for $x \ge 0$. In the second case, $|\xi_2| \ge |\sqrt{\Gamma}\,|$ or

$$\big|B^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2)\big| \le \xi_2^2,$$

which is equivalent to

$$-\xi_2^2 \le B^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2) \le \xi_2^2.$$

In particular,

$$B^2 \ge 4(\gamma\eta\xi_1^2 + \xi_2^2) - \xi_2^2 \ge \xi_2^2.$$

Therefore,

$$|\widehat{M_2}(t)| \le\ B\ |G_1(t)| \le Bte^{-\frac{B}{4}t} \le C\ e^{-c_0|\xi|^2 t}.$$

Now we assume $\xi \in S_2$. Then $\frac{B}{2} < \sqrt{\Gamma} \le B$ and

$$\begin{gathered}
-B \le \lambda_1 < -\frac{3}{4}B,\\
\lambda_2 = \frac{\Gamma - B^2}{2(B+\sqrt{\Gamma})} \le -\frac{\gamma\eta\xi_1^2 + \xi_2^2}{B} \le -\frac{C|\xi|^2}{B},\\
|G_1(t)| \le \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_1 t} + e^{\lambda_2 t}) < \frac{2}{B}\big(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\big).
\end{gathered}$$

As a consequence,

$$\begin{aligned}
|\widehat{M_1}(t)| &= \Big|\eta|\xi|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\Big| \le 2B|G_1(t)| + e^{\lambda_2 t} < C\Big(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\Big),\\
|\widehat{M_3}(t)| &= \Big| -\eta|\xi|^2 G_1(t) - \lambda_1 G_1(t) + e^{\lambda_1 t}\Big| < C\Big(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\Big).
\end{aligned}$$

Since $\sqrt{\Gamma} > \frac{B}{2}$,

$$\frac{3}{4}B^2 > 4(\gamma\eta\xi_1^2 + \xi_2^2) \ge \xi_2^2.$$

Therefore,

$$|\widehat{M}_2(t)| < CB\ |G_1(t)| < C(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\Big).$$

The upper bound for $|\widehat{M}_i(\xi,t)|$ with $\xi \in S_{21}$ or $\xi \in S_{22}$ is a consequence of the following estimate:

$$\frac{|\xi|^2}{B} = \frac{|\xi|^2}{\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2} \geq \frac{|\xi|^2}{\gamma + \eta|\xi|^2} \geq \begin{cases} C|\xi|^2 & \text{if } \ |\xi| \leq 1, \\ C & \text{if } \ |\xi| > 1. \end{cases}$$

This complete the proof of Proposition 3.3. ◻

With the integral representation in (3.7) and (3.8) and the upper bounds for the kernels in Proposition 3.3 at our disposal, we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By differentiating (3.7) and (3.8), we find, for $k = 1, 2$ and $m = 0, 1, 2$,

$$\begin{aligned} \widehat{\partial_k^m u}(\xi,t) &= \widehat{M}_1(t)\widehat{\partial_k^m u_0} + \widehat{M}_2(t)\widehat{\partial_k^m b_0} \\ &\quad + \int_0^t \left(\widehat{M}_1(t-\tau)\widehat{\partial_k^m N_1}(\tau) + \widehat{M}_2(t-\tau)\widehat{\partial_k^m N_2}(\tau)\right) d\tau, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \widehat{\partial_k^m b}(\xi,t) &= \widehat{M}_2(t)\widehat{\partial_k^m u_0} + \widehat{M}_3(t)\widehat{\partial_k^m b_0} \\ &\quad + \int_0^t \left(\widehat{M}_2(t-\tau)\widehat{\partial_k^m N_1}(\tau) + \widehat{M}_3(t-\tau)\widehat{\partial_k^m N_2}(\tau)\right) d\tau. \end{aligned} \tag{3.15}$$

We apply the bootstrapping argument to (3.14) and (3.15) to establish the sharp decay rates stated in Theorem 1.2. First we recall that the initial datum (u_0, b_0) is assumed to satisfy (3.1), namely,

$$\|(u_0, b_0)\|_{H^3} \leq \delta \quad \text{and} \quad \|(u_0, b_0)\|_{L^1} \leq \delta$$

for sufficiently small $\delta > 0$. The bootstrapping argument makes the ansatz that, for $t \leq T$,

$$\|u(t)\|_{L^2(\mathbb{R}^2)} + \|b(t)\|_{L^2(\mathbb{R}^2)} \leq C_0\delta\ (1+t)^{-\frac{1}{2}}, \tag{3.16}$$

$$\|Du(t)\|_{L^2(\mathbb{R}^2)} + \|Db(t)\|_{L^2(\mathbb{R}^2)} \leq C_1\delta\ (1+t)^{-1}, \tag{3.17}$$

$$\|D^2u(t)\|_{L^2(\mathbb{R}^2)} + \|D^2b(t)\|_{L^2(\mathbb{R}^2)} \leq C_2\delta\ (1+t)^{-\frac{3}{2}}, \tag{3.18}$$

where C_m $(m = 0, 1, 2)$ will be specified later. We then show via (3.14) and (3.15) that $(D^m u(t), D^m b(t))$ admits a smaller upper bound,

$$\|u(t)\|_{L^2(\mathbb{R}^2)} + \|b(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_0}{2}\ \delta\ (1+t)^{-\frac{1}{2}}, \tag{3.19}$$

$$\|Du(t)\|_{L^2(\mathbb{R}^2)} + \|Db(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_1}{2}\ \delta\ (1+t)^{-1}, \tag{3.20}$$

$$\|D^2u(t)\|_{L^2(\mathbb{R}^2)} + \|D^2b(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_2}{2}\ \delta\ (1+t)^{-\frac{3}{2}}. \tag{3.21}$$

The bootstrapping argument then assesses that $T = \infty$ and (3.19), (3.20), and (3.21) actually hold for all time. The rest of the proof focuses on verifying (3.19), (3.20), and (3.21).

We start with the estimate of $\|\partial_k^m u\|_{L^2(\mathbb{R}^2)}$. Taking the L^2 norm on both side of (3.14), we have

$$\begin{aligned}\|\partial_k^m u\|_{L^2(\mathbb{R}^2)} = \|\widehat{\partial_k^m u}(t)\|_{L^2(\mathbb{R}^2)} &\le \|\widehat{M_1}(t)\widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} + \|\widehat{M_2}(t)\widehat{\partial_k^m b_0}\|_{L^2(\mathbb{R}^2)} \\ &+ \int_0^t \|\widehat{M_1}(t-\tau)\widehat{\partial_k^m N_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &+ \int_0^t \|\widehat{M_2}(t-\tau)\widehat{\partial_k^m N_2}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau. \end{aligned} \tag{3.22}$$

We will estimate only the first term and the third term since the estimates for the other two terms are similar. Without loss of generality, we assume $t > 1$. By Proposition 3.3 and Lemma 3.1, the first term on the right-hand side of (3.22) can be bounded as

$$\begin{aligned}\|\widehat{M_1}(t)\widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} &\le C\|e^{-\widetilde{c}_0|\xi|^2 t}\widehat{\partial_k^m u_0}\ \|_{L^2(\mathbb{R}^2)} + \|e^{-c_2 t}\ \widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} \\ &= \|\ |\xi|^m e^{-\widetilde{c}_0|\xi|^2 t}\widehat{\Lambda^{-m}\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} + e^{-c_2 t}\|\ \widehat{\partial_k^m u_0}\ \|_{L^2(\mathbb{R}^2)} \\ &\le C(1+t)^{-\frac{1+m}{2}}\|u_0\|_{L^1(\mathbb{R}^2)} + C(1+t)^{-\frac{1+m}{2}}\|u_0\|_{L^2(\mathbb{R}^2)} \\ &\le C(1+t)^{-\frac{1+m}{2}}\delta, \end{aligned} \tag{3.23}$$

where $\widetilde{c}_0 = \min\{c_0, c_1\}$ and we have used $e^{-c_2 t}(1+t)^s \le C(c_2, s)$ for any $s \ge 0$. Now we bound the third term in (3.22). Invoking Proposition 3.3 and using the fact that the projection operator $\mathbb{P}$ is bounded in L^2, we have

$$\begin{aligned}\int_0^t \|\widehat{M_1}(t-\tau)\widehat{\partial_k^m N_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau &\le \int_0^t \|\widehat{M_1}(t-\tau)\widehat{\partial_k^m Q_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\le C\int_0^t \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{\partial_k^m Q_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &+ C\int_0^t e^{-c_2(t-\tau)}\|\widehat{\partial_k^m Q_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau, \end{aligned} \tag{3.24}$$

where $Q_1 = u\cdot\nabla u - b\cdot\nabla b$. When $m = 0$, we split the time integral in the first term into two parts,

$$\begin{aligned}\int_0^t \|\ e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q}_1(\tau)\ \|_{L^2}\, d\tau &= \int_0^{\frac{t}{2}} \|\ e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q}_1(\tau)\ \|_{L^2}\, d\tau \\ &+ \int_{\frac{t}{2}}^t \|\ e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q}_1(\tau)\ \|_{L^2}\, d\tau. \end{aligned}$$

By Lemma 3.1, the ansatz (3.16), and (3.2), we get

$$\begin{aligned}&\int_0^{\frac{t}{2}} \|\ e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q}_1(\tau)\ \|_{L^2}\, d\tau \\ &= \int_0^{\frac{t}{2}} \|\ |\xi|\ e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{\Lambda^{-1}Q_1}(\tau)\ \|_{L^2}\, d\tau \\ &\le C\int_0^{\frac{t}{2}} (t-\tau)^{-1}(\|u(\tau)\otimes u(\tau)\|_{L^1} + \|b(\tau)\otimes b(\tau)\|_{L^1})\, d\tau \\ &\le C\Big(\frac{t}{2}\Big)^{-1}\sup_{0\le\tau\le t}(\|u(\tau)\|_{L^2} + \|b(\tau)\|_{L^2})\int_0^{\frac{t}{2}}(\|u(\tau)\|_{L^2} + \|b(\tau)\|_{L^2})\, d\tau \\ &\le CC_0\Big(\frac{t}{2}\Big)^{-1}\delta^2\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{1}{2}}\, d\tau \le CC_0\delta^2(1+t)^{-\frac{1}{2}}, \end{aligned}$$

where we have used $u \cdot \nabla u = \nabla \cdot (u \otimes u)$ and $b \cdot \nabla b = \nabla \cdot (b \otimes b)$. The estimate of the second integral is slightly different:

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} \| \, e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q}_1(\tau) \, \|_{L^2}\, d\tau \\
&\quad \le C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|u\cdot\nabla u - b\cdot\nabla b\|_{L^1}\, d\tau \\
&\quad \le C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (\|u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2} + \|b(\tau)\|_{L^2}\|\nabla b(\tau)\|_{L^2})\, d\tau \\
&\quad \le CC_0\delta^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1}\, d\tau \\
&\quad \le CC_0\delta^2\Big(1+\frac{t}{2}\Big)^{-1}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\, d\tau \le CC_0\delta^2(1+t)^{-\frac{1}{2}},
\end{aligned}$$

where we have used (3.3) for $\|(\nabla u, \nabla b)\|_{L^2}$. Due to the fact $e^{-c_2 t}(1+t)^s \le C(c_2,\ s)$ for any $s>0$, the second term in (3.24) can be estimated as

$$\begin{aligned}
&\int_0^t e^{-c_2(t-\tau)}\|\widehat{Q}_1(\tau)\|_{L^2}\, d\tau \\
&\quad \le C\int_0^t (1+t-\tau)^{-s}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u(\tau)\|_{L^2}\|\nabla^2 u(\tau)\|_{L^2}^{\frac{1}{2}}\, d\tau \\
&\quad \le CC_0^{\frac{1}{2}}\delta^2\int_0^t (1+t-\tau)^{-s}(1+\tau)^{-\frac{1}{2}}\, d\tau \le CC_0^{\frac{1}{2}}\delta^2(1+t)^{-\frac{1}{2}},
\end{aligned}$$

where $s>1$ and we have used (3.16) and (3.3). In summary, when $m=0$, the third term in (3.22) is bounded by

$$\int_0^t \|\widehat{M}_1(t-\tau)\widehat{N}_1(\tau)\|_{L^2}\, d\tau \le C(C_0 + C_0^{\frac{1}{2}})\delta^2(1+t)^{-\frac{1}{2}}.$$

The second term in (3.22) admits the same bound as the first term while the fourth shares the bound with the third term. Therefore, we have shown that there exist $C_3>0$ and $C_4>0$ such that

$$\|u(t)\|_{L^2} \le C_3\delta(1+t)^{-\frac{1}{2}} + C_4(1+C_0)\delta^2(1+t)^{-\frac{1}{2}}.$$

If C_0 and δ satisfy

$$C_3 \le \frac{C_0}{4}, \qquad C_4(1+C_0)\delta \le \frac{C_0}{4},$$

then

$$\|u(t)\|_{L^2} \le \frac{C_0}{2}(1+t)^{-\frac{1}{2}}. \tag{3.25}$$

We now turn to the case when $m=1,2$. We again focus on the third term in (3.22). First of all, we split the first time integral in (3.24) into two terms. The first term is further estimated via (3.4) and the fact that

$$(t-\tau)^{-\frac{m+1}{2}} \le C\,(1+t-\tau)^{-\frac{m+1}{2}} \quad \text{for any } \tau \in [0, t-1]$$

while we use the fact $1+t-\tau \le 2$ for $\tau \in [t-1, t]$ in the second term. We obtain

$$\begin{aligned}
&\int_0^t \|\, e^{-\widetilde{c}_0|\xi|^2(t-\tau)} \widehat{\partial_k^m Q}_1(\tau) \,\|_{L^2}\, d\tau \\
&\quad \le \int_0^{t-1} \|\, |\xi|^{m+1} e^{-\widetilde{c}_0|\xi|^2(t-\tau)} \widehat{\Lambda^{-(m+1)}\partial_k^m} Q_1(\tau) \,\|_{L^2}\, d\tau + \int_{t-1}^t \|\widehat{\partial_k^m Q}_1(\tau) \,\|_{L^2}\, d\tau \\
&\quad \le C \int_0^{t-1} (1+t-\tau)^{-\frac{m+1}{2}} \|\Lambda^{-1} Q_1(\tau)\|_{L^2}\, d\tau \\
&\qquad + C(m) \int_{t-1}^t (1+t-\tau)^{-\frac{m+1}{2}} \|\widehat{\partial_k^m Q}_1(\tau) \,\|_{L^2}\, d\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-\frac{m+1}{2}} \|\Lambda^{-1} Q_1(\tau)\|_{L^2}\, d\tau \\
&\qquad + C(m) \int_0^t (1+t-\tau)^{-\frac{m+1}{2}} \|\widehat{\partial_k^m Q}_1(\tau) \,\|_{L^2}\, d\tau.
\end{aligned}$$

Thanks to the estimates $e^{-c_2 t}(1+t)^s \le C(c_2,\ s)$ for any $t > 0$ and any constant $s > 0$, the second integral term in (3.24) can be bounded by

$$\int_0^t e^{-c_2(t-\tau)} \|\widehat{\partial_k^m Q}_1(\tau)\|_{L^2}\, d\tau \le C(m) \int_0^t (1+t-\tau)^{-\frac{m+1}{2}} \|\widehat{\partial_k^m Q}_1(\tau)\|_{L^2}\, d\tau.$$

Thus,

$$\begin{aligned}
\int_0^t \|\widehat{M}_1(t-\tau) \widehat{\partial_k^m N}_1(\tau)\|_{L^2}\, d\tau &\le C \int_0^t (1+t-\tau)^{-\frac{m+1}{2}} \|\Lambda^{-1} Q_1(\tau)\|_{L^2}\, d\tau \\
&\quad + C(m) \int_0^t (1+t-\tau)^{-\frac{m+1}{2}} \|\widehat{\partial_k^m Q}_1(\tau) \,\|_{L^2}\, d\tau.
\end{aligned} \tag{3.26}$$

For $m = 1$, by Hölder's inequality and Sobolev's inequality, we have

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-1} \|\Lambda^{-1} Q_1(\tau)\|_{L^2}\, d\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-1} (\|u(\tau)\|_{L^4}^2 + \|b(\tau)\|_{L^4}^2)\, d\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-1} (\|u(\tau)\|_{L^2} \|\nabla u(\tau)\|_{L^2} + \|b(\tau)\|_{L^2} \|\nabla b(\tau)\|_{L^2})\, d\tau.
\end{aligned}$$

Then, by (3.25), the ansatz (3.17), and Lemma 3.2,

$$\begin{aligned}
\int_0^t (1+t-\tau)^{-1} \|\Lambda^{-1} Q_1(\tau)\|_{L^2}\, d\tau &\le C C_0 C_1 \delta^2 \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-\frac{3}{2}}\, d\tau \\
&\le C C_1 \delta^2 (1+t)^{-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-1}\|\partial_k Q_1(\tau)\|_{L^2}\,d\tau \\
&\quad \le \int_0^t (1+t-\tau)^{-1}(\|\nabla u(\tau)\|_{L^4}^2 + \|u(\tau)\|_{L^4}\|\nabla^2 u(\tau)\|_{L^4} \\
&\qquad + \|\nabla b(\tau)\|_{L^4}^2 + \|b(\tau)\|_{L^4}\|\nabla^2 b(\tau)\|_{L^4})\,d\tau \\
&\quad \le C\int_0^t (1+t-\tau)^{-1}(\|\nabla u(\tau)\|_{L^2}\|\nabla^2 u(\tau)\|_{L^2} + \|u(\tau)\|_{L^2}^{\frac12}\|\nabla u(\tau)\|_{L^2}^{\frac12}\|\nabla^2 u(\tau)\|_{H^1} \\
&\qquad + \|\nabla b(\tau)\|_{L^2}\|\nabla^2 b(\tau)\|_{L^2} + \|b(\tau)\|_{L^2}^{\frac12}\|\nabla b(\tau)\|_{L^2}^{\frac12}\|\nabla^2 b(\tau)\|_{H^1})\,d\tau \\
&\quad \le C(C_1 + C_0^{\frac12}C_1^{\frac12})\delta^2\int_0^t (1+t-\tau)^{-1}(1+\tau)^{-\frac54}\,d\tau \le C(C_1 + C_1^{\frac12})\delta^2(1+t)^{-1},
\end{aligned}$$

where we have used (3.25), the ansatz (3.17), and the decay estimate

$$\|(\nabla^2 u(t), \nabla^2 b(t))\|_{H^1} \le C\delta(1+t)^{-\frac12}.$$

Therefore, the third term in (3.22) for $m=1$ can be bounded by

$$\int_0^t \|\widehat{M_1}(t-\tau)\widehat{\partial_k N_1}(\tau)\|_{L^2}\,d\tau \le C(1+C_1)\delta^2(1+t)^{-1}. \tag{3.27}$$

Collecting the estimates (3.23) and (3.27) yields

$$\|\nabla u\|_{L^2} \le C_5\delta(1+t)^{-1} + C_6(1+C_1)\delta^2(1+t)^{-1}$$

for some constants $C_5 > 0$ and $C_6 > 0$. Therefore, if C_1 and δ satisfy

$$C_5 \le \frac{C_1}{4}, \quad C_6(1+C_1)\delta \le \frac{C_1}{4},$$

then

$$\|\nabla u\|_{L^2} \le \frac{C_1}{2}\delta(1+t)^{-1} \tag{3.28}$$

and the bootstrapping argument implies $T=\infty$. Thus, the decay rate (3.28) indeed holds for all time. Finally, we bound (3.26) for $m=2$. With a similar argument as for $m=1$, we get

$$\begin{aligned}
\int_0^t (1+t-\tau)^{-\frac32}\|\Lambda^{-1}Q_1(\tau)\|_{L^2}\,d\tau &\le CC_0C_1\delta^2\int_0^t (1+t-\tau)^{-\frac32}(1+\tau)^{-\frac32}\,d\tau \\
&\le C\delta^2(1+t)^{-\frac32}.
\end{aligned}$$

Also, by Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|\partial_k^2 Q_1(\tau)\|_{L^2}\, d\tau \\
&\quad \le C\int_0^t (1+t-\tau)^{-\frac{3}{2}} (\|\nabla u(\tau)\|_{L^4}\|\nabla^2 u(\tau)\|_{L^4} + \|u(\tau)\|_{L^\infty}\|\nabla^3 u(\tau)\|_{L^2} \\
&\qquad + \|\nabla b(\tau)\|_{L^4}\|\nabla^2 b(\tau)\|_{L^4} + \|b(\tau)\|_{L^\infty}\|\nabla^3 b(\tau)\|_{L^2})\, d\tau \\
&\quad \le C\int_0^t (1+t-\tau)^{-\frac{3}{2}} (\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u(\tau)\|_{L^2}\|\nabla^3 u(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\qquad\qquad + \|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^3 u(\tau)\|_{L^2} \\
&\qquad\qquad + \|\nabla b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b(\tau)\|_{L^2}\|\nabla^3 b(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\qquad\qquad + \|b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla^3 b(\tau)\|_{L^2})\, d\tau.
\end{aligned}$$

Then using (3.25), (3.28), the ansatz (3.18), and $\|(\nabla^3 u, \nabla^3 b)\|_{L^2} \le C\delta(1+t)^{-\frac{1}{2}}$, we have

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|\partial_k^2 Q_1(\tau)\|_{L^2}\, d\tau \\
&\quad \le C(C_1^{\frac{1}{2}} C_2 + C_0^{\frac{1}{2}} C_2^{\frac{1}{2}})\delta^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{2}}\, d\tau \\
&\quad \le C(C_2 + C_2^{\frac{1}{2}})\delta^2 (1+t)^{-\frac{3}{2}}.
\end{aligned}$$

Therefore,

$$\int_0^t \|\widehat{M}_1(t-\tau)\widehat{\partial_k^2 N}_1(\tau)\|_{L^2}\, d\tau \le C(1+C_2)\delta^2(1+t)^{-\frac{3}{2}}.$$

As a consequence,

$$\|\nabla^2 u\|_{L^2} \le C_7\delta(1+t)^{-\frac{3}{2}} + C_8(1+C_2)\delta^2(1+t)^{-\frac{3}{2}}$$

for the constant $C_7 > 0$ and $C_8 > 0$. Then the decay rate for $\|\nabla^2 u\|_{L^2}$ follows from a similar argument as the case $m = 0, 1$. This completes the proof of Theorem 1.2. ◻

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