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Global regularity results for the 2D Boussinesq equations with partial dissipation

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Abstract

The two-dimensional (2D) incompressible Boussinesq equations model geophysical fluids and play an important role in the study of the Raleigh–Bernard convection. Mathematically this 2D system retains some key features of the 3D Navier–Stokes and Euler equations such as the vortex stretching mechanism. The issue of whether the 2D Boussinesq equations always possess global (in time) classical solutions can be difficult when there is only partial dissipation or no dissipation at all. This paper obtains the global regularity for two partial dissipation cases and proves several global *a priori* bounds for two other prominent partial dissipation cases. These results take us one step closer to a complete resolution of the global regularity issue for all the partial dissipation cases involving the 2D Boussinesq equations.

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1. Introduction

This paper is concerned with the global regularity problem on the two-dimensional (2D) incompressible Boussinesq equations with partial dissipation. This problem has recently attracted considerable attention and progress has been made [1–3,5,6,10–13,16–21,23,25,26,32, 34,38–45]. The aim of this paper is twofold: first, to establish the global existence of classical solutions for two partial dissipation cases; and second, to present results that are useful for the eventual resolution of the global well-posedness problem on two prominent partial dissipation cases.

The Boussinesq equations concerned here model geophysical flows such as atmospheric fronts and oceanic circulation (see, e.g., [15,30,35]). In addition, they play an important role in the study of Raleigh–Bernard convection (see, e.g., [9,37]). The standard 2D Boussinesq equations can be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\,\Delta\mathbf{u} + \theta\mathbf{e}_2, & x\in\mathbb{R}^2,\ t>0,\\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = \eta\,\Delta\theta, & x\in\mathbb{R}^2,\ t>0,\\ \nabla\cdot\mathbf{u} = 0, & x\in\mathbb{R}^2,\ t>0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = \mathbf{u}(x,t)$ denotes the 2D velocity, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, $\mathbf{e}_2$ the unit vector in the vertical direction, and $\nu \geq 0$ and $\eta \geq 0$ are parameters representing the viscosity and the thermal diffusivity, respectively. In order to model anisotropic flows with different diffusion properties in the horizontal and vertical directions, (1.1) is generalized to the form

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \mu_1\partial_{xx}u + \mu_2\partial_{yy}u,\\ \partial_t v + u\partial_x v + v\partial_y v = -\partial_y p + \nu_1\partial_{xx}v + \nu_2\partial_{yy}v + \theta,\\ \partial_t\theta + u\partial_x\theta + v\partial_y\theta = \eta_1\partial_{xx}\theta + \eta_2\partial_{yy}\theta,\\ \partial_x u + \partial_y v = 0, \end{cases} \tag{1.2}$$

where u and v are the horizontal and vertical components of $\mathbf{u}$, respectively. Clearly, when $\mu_1 = \mu_2 = \nu_1 = \nu_2$ and $\eta_1 = \eta_2$, (1.2) reduces to (1.1). What we care about here is whether (1.2) always has a global solution when the initial data

$$(u(x,y,0), v(x,y,0)) = (u_0(x,y), v_0(x,y)) \tag{1.3}$$

is sufficiently smooth. Due to the similarities between the 2D Boussinesq equations and the 3D hydrodynamics equations [31], the study of this problem may shed light on the mysterious global existence and regularity problem on the 3D Navier–Stokes and Euler equations.

Considerable efforts have been devoted to the global regularity problem on (1.2) with various partial dissipation and several cases have been resolved. To position our results in a suitable context, we summarize some existing results in this direction. For the sake of clarity, we divide the description into seven cases:

(I) No dissipation and no thermal diffusion:

$$\mu_1 = \mu_2 = \nu_1 = \nu_2 = \eta_1 = \eta_2 = 0;$$

(II) Both dissipation and thermal diffusion:

$$\mu_1 > 0, \quad \mu_2 > 0, \quad \nu_1 > 0, \quad \nu_2 > 0, \quad \eta_1 > 0, \quad \eta_2 > 0;$$

(III) Velocity dissipation only:

$$\mu_1 > 0, \quad \mu_2 > 0, \quad \nu_1 > 0, \quad \nu_2 > 0, \quad \eta_1 = \eta_2 = 0;$$

(IV) Thermal diffusion only:

$$\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0, \quad \eta_1 > 0, \quad \eta_2 > 0;$$

(V) Horizontal velocity dissipation only:

$$\mu_1 = \nu_1 > 0, \quad \mu_2 = \nu_2 = \eta_1 = \eta_2 = 0;$$

(VI) Horizontal thermal diffusion only:

$$\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0, \quad \eta_1 > 0, \quad \eta_2 = 0;$$

(VII) Vertical velocity and vertical thermal diffusion only:

$$\mu_1 = \nu_1 = 0, \quad \mu_2 = \nu_2 > 0, \quad \eta_1 = 0, \quad \eta_2 > 0.$$

For the case in (I), the global regularity problem remains outstandingly open and it is not clear how to proceed on this difficult problem. The standard idea of proving the global *a priori* bounds in Sobolev spaces fails. Potential finite time singularities have been explored from different perspectives including boundary effects and 1D models [29,8,7,36]. For (II), the global existence of classical solutions can be easily obtained, in a similar fashion as for the 2D Navier–Stokes equations [14,31]. The two cases (III) and (IV) were dealt with in [6] and in [19] and the global regularity was established for both cases.

More recent work further sharpens the global regularity results. [13] and [26] examined the two cases (V) and (VI) and obtained the global existence of suitably regular solutions. In addition, [26] proved the uniqueness for the case (V) under the natural assumption that $\theta_0 \in L^\infty$. We remark that the horizontal dissipation in the case (V) is advantageous in the estimates for global bounds. In fact, a simple energy estimate involving the vorticity ($\omega = \nabla \times \mathbf{u}$) equation

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = \nu_1 \partial_x^2 \omega + \partial_x \theta$$

gives, after integration by parts,

$$\partial_t \|\omega\|_{L^2}^2 + 2\nu_1 \|\partial_x \omega\|_{L^2}^2 = -2 \int \theta \partial_x \omega \, dx \le \nu_1 \|\partial_x \omega\|_{L^2}^2 + C(\nu_1) \|\theta_0\|_{L^2}^2,$$

which immediately yields the global L^2-bound for ω. In contrast, the vertical dissipation case (VII) is different and the global regularity issue appears to be difficult. Nevertheless, Cao and Wu [5] were able to prove the global existence of classical solutions in the case (VII) by overcoming several difficulties. As can be seen from the vorticity equation (writing ν for ν_2)

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = \nu \partial_y^2 \omega + \partial_x \theta, \tag{1.4}$$

the dissipation $\nu \partial_y^2 \omega$ is no longer useful in hiding the "vortex stretching" term $\partial_x \theta$. Therefore, in order to obtain the global H^1-bound for $\mathbf{u}$, or equivalently, the L^2-norm of ω, one is forced to combine (1.4) with the equation of $\nabla^\perp \theta$,

$$\partial_t \nabla^\perp \theta + (\mathbf{u} \cdot \nabla)\nabla^\perp \theta = -\nabla \mathbf{u}\, \nabla^\perp \theta + \eta\, \partial_y^2 \nabla^\perp \theta, \tag{1.5}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$. Through careful energy estimates and applying anisotropic inequalities for triple-product terms, [5] observe that, if

$$\int_0^T \|v(t)\|_{L^\infty}^2\, dt < \infty, \tag{1.6}$$

then the solution is regular on $[0, T]$, namely the H^2-norm of the solution (u, v, θ) is bounded on $[0, T]$. It is difficult to verify (1.6) directly and [5] showed instead

$$\sup_{q \ge 2} \frac{\|v(t)\|_{L^q}}{\sqrt{q \log q}} \le C(T) \tag{1.7}$$

for any $T > 0$ and $t \le T$. [5] then bridged what is needed in (1.6) and what is shown in (1.7) via the following Sobolev interpolation inequality of the logarithmic type

$$\|f\|_{L^\infty} \le C \sup_{q \ge 2} \frac{\|f\|_{L^q}}{\sqrt{q \log q}} \log(e + \|f\|_{H^s}) \log(e + \log(e + \|f\|_{H^s})) + C,$$

where $s > 1$ and $C = C(s)$ are constants. This leads to an Osgood type differential inequality for $\|(u, v, \theta)\|_{H^2}$ and thus a global bound for $\|(u, v, \theta)\|_{H^2}$.

The first aim of this paper is the global regularity for the following two cases.

(VIII) Mixed directional dissipation in the velocity equation:

$$\mu_1 = \nu_2 = 0, \quad \mu_2 = \nu_1 > 0, \quad \eta_1 = \eta_2 = 0;$$

(IX) Dissipation in the vertical velocity component:

$$\mu_1 = \mu_2 = 0, \quad \nu_1 = \nu_2 > 0, \quad \eta_1 = \eta_2 = 0.$$

We are able to show that (1.2) in the two cases above always possess global classical solutions. The precise results are stated in the following two theorems.

Theorem 1.1. *Consider (1.2) with vertical dissipation in the horizontal velocity equation and horizontal dissipation in the vertical velocity equation, namely*

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_{yy} u = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \partial_{xx} v = \theta, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \quad \theta(x,y,0) = \theta_0(x,y). \end{cases} \tag{1.8}$$

Assume $(u_0, v_0, \theta_0) \in H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$ *with* $\partial_x u_0 + \partial_y v_0 = 0$. *Then (1.8) has a unique global classical solution* (u, v, θ) *satisfying*

$$(u, v, \theta) \in L^\infty([0,T]; H^3(\mathbb{R}^2)) \times L^\infty([0,T]; H^3(\mathbb{R}^2)) \times L^\infty([0,T]; H^3(\mathbb{R}^2)),$$
$$(u, v) \in L^2([0,T]; H^4(\mathbb{R}^2)) \times L^2([0,T]; H^4(\mathbb{R}^2))$$

for any given $T > 0$.

The global regularity is also established for the case (IX).

Theorem 1.2. *Consider (1.2) with full Laplacian dissipation in the vertical velocity equation, namely*

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x P = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y P - \partial_{xx} v - \partial_{yy} v = \theta, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \quad \theta(x,y,0) = \theta_0(x,y). \end{cases} \tag{1.9}$$

Assume $(u_0, v_0, \theta_0) \in H^{s+1}(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *for some* $s > 2$ *and* $\partial_x u_0 + \partial_y v_0 = 0$. *Then (1.9) admits a unique global classical solution* (u, v, θ) *satisfying*

$$(u, v, \theta) \in L^\infty([0,T]; H^{s+1}(\mathbb{R}^2)) \times L^\infty([0,T]; H^{s+1}(\mathbb{R}^2)) \times L^\infty([0,T]; H^s(\mathbb{R}^2))$$

and

$$v \in L^2([0,T]; H^{s+2}(\mathbb{R}^2))$$

for any given $T > 0$.

This paper also examines the global regularity problem when there is only vertical velocity dissipation or vertical thermal diffusion (not both), namely the following two cases:

(X) Vertical velocity dissipation only:

$$\mu_1 = \nu_1 = 0, \quad \mu_2 = \nu_2 > 0, \quad \eta_1 = \eta_2 = 0;$$

(XI) Vertical thermal diffusion only:

$$\mu_1 = \nu_1 = \mu_2 = \nu_2 = 0, \quad \eta_1 = 0, \quad \eta_2 > 0.$$

We do not have a complete solution for the cases (X) and (XI), but the global bounds presented below may be helpful in the eventual resolution of these two difficult cases.

Theorem 1.3. *Assume that* $(\mathbf{u}_0, \theta_0) \in H^s(\mathbb{R}^2)$ *with* $s > 2$*. Let* $(\mathbf{u}, p, \theta)$ *be a solution of (1.2) with the parameters given in the case (X), namely*

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_{yy} u = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \partial_{yy} v = \theta, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y), \quad \theta(x, y, 0) = \theta_0(x, y). \end{cases} \tag{1.10}$$

Then, $(\mathbf{u}, p, \theta)$ *admits the following global bounds, for any* $T > 0$ *and* $t \le T$*:*

(a) *Global* L^4*-bound for* $\mathbf{u}$*,*

$$\|\mathbf{u}\|_{L^4}^4 + \int_0^t \| |\mathbf{u}(\tau)|^2 \, |\partial_y \mathbf{u}(\tau)|^2 \|_{L^1} \, d\tau + \int_0^t \|\mathbf{u} \cdot \partial_y \mathbf{u}(\tau)\|_{L^2}^2 \, d\tau \le C,$$

where $C = C(T, \mathbf{u}_0, \theta_0)$*;*

(b) *Global bounds for the pressure* p *in* L^2*,* L^4 *and* $L_t^2 H^1$*,*

$$\|p(t)\|_{L^2} \le C, \quad \|p(t)\|_{L^4} \le C, \quad \int_0^t \|\nabla p(\tau)\|_{L^2}^2 \, d\tau \le C, \tag{1.11}$$

where $C = C(T, \mathbf{u}_0, \theta_0)$*;*

(c) *Global* L^q*-bound for the vertical component* v*,*

$$\sup_{2 \le q < \infty} \frac{\|v(t)\|_{L^q}}{\sqrt{q \log q}} \le C,$$

where $C = C(T, \mathbf{u}_0, \theta_0)$*;*

(d) *Global* L^6*-bound for the horizontal component* u*,*

$$\|u(t)\|_{L^6} \le C,$$

where $C = C(T, \mathbf{u}_0, \theta_0)$*;*

(e) *Global* L^q*-bound for the pressure* p*,*

$$\|p(t)\|_{L^q} \le C, \qquad 2 \le q < \infty$$

where $C = C(q, T, \mathbf{u}_0, \theta_0)$*.*

Finally we turn to the 2D Boussinesq equations with only vertical thermal diffusion, namely the case (XI). The global regularity remains open. The following criterion highlights the key quantity that potentially controls the global regularity.

Theorem 1.4. *Consider the 2D Boussinesq equations with only vertical thermal diffusion, namely,*

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p = \theta, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta - \partial_{yy}\theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \quad \theta(x,y,0) = \theta_0(x,y). \end{cases} \tag{1.12}$$

Assume $(\mathbf{u}_0, \theta_0) \in H^3$ *and let* $(\mathbf{u}, \theta)$ *be the corresponding local solution on* $[0, T_0]$. *If, for* $T > T_0$, θ *satisfies*

$$M(T) \equiv \int_0^T \|\partial_x \theta\|_{L^\infty}\, dt < \infty, \tag{1.13}$$

then the local solution $(\mathbf{u}, \theta)$ *can be extended to* $[0, T]$.

It is not clear if (1.13) can be replaced by the condition

$$\int_0^T \sup_{2 \le q < \infty} \frac{\|\partial_x \theta\|_{L^q}}{q}\, dt < \infty,$$

and the difficulty is due to the fact that

$$\|\nabla \mathbf{u}\|_{L^q} \le C(q)\, \|\omega\|_{L^q}$$

with a coefficient depending on q, $C(q) = \widetilde{C}\,\frac{q^2}{q-1}$ for $q \in (1, \infty)$ and a pure constant $\widetilde{C}$. It is also unknown if (1.13) remains a regularity criterion if we drop the vertical dissipative term $\partial_{yy}\theta$ in (1.12).

The rest of the paper is divided into three main sections. Section 2 presents the proofs of Theorems 1.1 and 1.2. Section 3 establishes Theorem 1.3 while Section 4 proves Theorem 1.4.

2. Proofs of Theorems 1.1 and 1.2

This section proves Theorem 1.1 and Theorem 1.2. These global existence results are usually proven through two steps. The first step asserts the local existence while the second step establishes global *a priori* bounds for the solution in the initial functional setting, which allow us to extend the local solution into a global one. When $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$, the local existence of (1.2) can be proven through a rather standard procedure and is similar to that for the hydrodynamics equations (see, e.g., [14] and [31]). Thus we shall omit the local existence part and focus our attention on the global *a priori* bounds on (u, v, θ).

We use extensively several elementary tools including the Gagliardo–Nirenberg inequality, the bilinear commutator estimate and anisotropic Sobolev inequalities for triple products. For readers' convenience, we recall them here. We start with the well-known Gagliardo–Nirenberg inequality (see [33]).

Lemma 2.1. *Let $1 \le p, q, r \le \infty$. Let $0 \le j < m$ be integers. Assume $u \in C_c^\infty(\mathbb{R}^n)$. Then there exists a constant $C = C(n, m, j, q, r, \sigma)$ such that*

$$\|D^j u\|_{L^p} \le C \|u\|_{L^q}^{1-\sigma} \|D^m u\|_{L^r}^{\sigma},$$

where

$$\frac{1}{p} - \frac{j}{n} = (1-\sigma)\frac{1}{q} + \sigma\left(\frac{1}{r} - \frac{m}{n}\right)$$

and

$$\sigma \in \begin{cases} \left[\frac{j}{m}, 1\right), & \text{if } m - j - \frac{n}{r} \text{ is an nonnegative integer}, \\ \left[\frac{j}{m}, 1\right], & \text{otherwise}. \end{cases}$$

The following bilinear commutator estimate can be found in several references (see, e.g., [22, 24]). Here we write $\Lambda = (-\Delta)^{\frac{1}{2}}$.

Lemma 2.2. *Let $s > 0$ and $p \in (1, \infty)$. Assume f and g satisfy $\nabla f \in L^{p_1}$, $\Lambda^s f \in L^{p_3}$, $\Lambda^{s-1} g \in L^{p_2}$ and $g \in L^{p_4}$. Then*

$$\|[\Lambda^s, f]g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}), \tag{2.1}$$

where $C = C(s, p, p_1, p_2, p_3, p_4)$, and

$$p_2, p_3 \in (1, \infty), \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

The following anisotropic Sobolev inequality bounds a triple-product in terms of the Lebesgue norms of the functions and their directional derivatives. The following lemma is taken from [5]. More general forms can be found in [4].

Lemma 2.3. *Let $q \in [2, \infty)$. Assume that $f, g, g_y, h_x \in L^2(\mathbb{R}^2)$ and $h \in L^{2(q-1)}(\mathbb{R}^2)$. Then*

$$\iint_{\mathbb{R}^2} |f\, g\, h|\, dxdy \le C\, \|f\|_{L^2}\, \|g\|_{L^2}^{1-\frac{1}{q}}\, \|g_y\|_{L^2}^{\frac{1}{q}}\, \|h\|_{L^{2(q-1)}}^{1-\frac{1}{q}}\, \|h_x\|_{L^2}^{\frac{1}{q}}, \tag{2.2}$$

where C is a constant depending on q only. Two special cases of (2.2) are

$$\iint |f\,g\,h|\,dxdy \le C\,\|f\|_{L^2}\,\|g\|_{L^2}^{\frac{1}{2}}\,\|g_y\|_{L^2}^{\frac{1}{2}}\,\|h\|_{L^2}^{\frac{1}{2}}\,\|h_x\|_{L^2}^{\frac{1}{2}}$$

and

$$\iint |f\,g\,h|\,dxdy \le C\,\|f\|_{L^2}\,\|g\|_{L^2}^{\frac{2}{3}}\,\|g_y\|_{L^2}^{\frac{1}{3}}\,\|h\|_{L^4}^{\frac{2}{3}}\,\|h_x\|_{L^2}^{\frac{1}{3}}.$$

We now prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to $\partial_x u + \partial_y v = 0$, we obtain immediately, from the θ-equation,

$$\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}, \quad \forall p \in [1,\infty] \tag{2.3}$$

for any $t \in [0,\infty)$. Taking the L^2 inner product of the velocity equation with $\mathbf{u}$, we find

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_{L^2}^2 + \|\partial_y u\|_{L^2}^2 + \|\partial_x v\|_{L^2}^2 \le \|\theta\|_{L^2}\|v\|_{L^2}.$$

Integrating in time and use (2.3), one has, for any $T > 0$ and $t \le T$,

$$\|\mathbf{u}(t)\|_{L^2}^2 + \int_0^t (\|\partial_y u(\tau)\|_{L^2}^2 + \|\partial_x v(\tau)\|_{L^2}^2)\,d\tau \le C, \tag{2.4}$$

where $C = C(T, u_0, v_0, \theta_0)$. Next we show the global H^1-bound

$$\|\nabla\mathbf{u}(t)\|_{L^2}^2 + \int_0^T \|\Delta\mathbf{u}(\tau)\|_{L^2}^2\,d\tau \le C. \tag{2.5}$$

Taking the inner product of the velocity equation in (1.8) with $\Delta\mathbf{u}$, we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \partial_{yy}u\,\Delta u\,dx - \int_{\mathbb{R}^2} \partial_{xx}v\,\Delta v\,dx - \int_{\mathbb{R}^2} \theta\,\Delta v\,dx \\ &\triangleq K_1 + K_2 + K_3,\end{aligned} \tag{2.6}$$

where we have used the identity (see, e.g., [28])

$$\int_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,dx = 0.$$

Integration by parts yields

$$\begin{aligned}K_1 &= -\int_{\mathbb{R}^2} \partial_{yy}u(\partial_{xx}u + \partial_{yy}u)\,dx = -\int_{\mathbb{R}^2} (\partial_{yy}u)^2\,dx - \int_{\mathbb{R}^2} \partial_{xx}u\partial_{yy}u\,dx \\ &= -\int_{\mathbb{R}^2} (\partial_{yy}u)^2\,dx - \int_{\mathbb{R}^2} \partial_{xy}u\partial_{xy}u\,dx = -\int_{\mathbb{R}^2} (\partial_{yy}u)^2\,dx - \int_{\mathbb{R}^2} \partial_{yy}v\partial_{yy}v\,dx \\ &= -\int_{\mathbb{R}^2} (\partial_{yy}u)^2\,dx - \int_{\mathbb{R}^2} (\partial_{yy}v)^2\,dx. \end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}K_2 &= -\int_{\mathbb{R}^2} \partial_{xx}v(\partial_{xx}v + \partial_{yy}v)\,dx = -\int_{\mathbb{R}^2} (\partial_{xx}v)^2\,dx - \int_{\mathbb{R}^2} \partial_{xx}v\partial_{yy}v\,dx \\ &= -\int_{\mathbb{R}^2} (\partial_{xx}v)^2\,dx - \int_{\mathbb{R}^2} \partial_{xy}v\partial_{xy}v\,dx = -\int_{\mathbb{R}^2} (\partial_{xx}v)^2\,dx - \int_{\mathbb{R}^2} \partial_{xx}u\partial_{xx}u\,dx \\ &= -\int_{\mathbb{R}^2} (\partial_{xx}v)^2\,dx - \int_{\mathbb{R}^2} (\partial_{xx}u)^2\,dx. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) yields

$$\begin{aligned}-K_1 - K_2 &= \int_{\mathbb{R}^2} (\partial_{xx}u)^2\,dx + \int_{\mathbb{R}^2} (\partial_{yy}u)^2\,dx + \int_{\mathbb{R}^2} (\partial_{xx}v)^2\,dx + \int_{\mathbb{R}^2} (\partial_{yy}v)^2\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} (\partial_{xx}u + \partial_{yy}u)^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} (\partial_{xx}v + \partial_{yy}v)^2\,dx \\ &= \frac{1}{2}\|\Delta\mathbf{u}\|_{L^2}^2. \end{aligned} \tag{2.9}$$

Young's inequality entails

$$K_3 \leq \|\theta\|_{L^2}\|\Delta v\|_{L^2} \leq \frac{1}{4}\|\Delta v\|_{L^2}^2 + C\|\theta\|_{L^2}^2.$$

Inserting the estimates above in (2.6), one has

$$\frac{d}{dt}\|\nabla\mathbf{u}(t)\|_{L^2}^2 + \|\Delta\mathbf{u}\|_{L^2}^2 \leq C\|\theta_0\|_{L^2}^2.$$

(2.5) then follows after integrating in time. It appears difficult to obtain a global bound for the H^1-norm of θ at this stage.

To obtain higher global regularity, we adopt an argument of [27]. For any $T > 0$, we assume the solution is regular for $t < T$ and show that it remains regular at $t = T$. More precisely, we define

$$M(t) = \sup_{0\le\tau\le t} (\|\Lambda^3\mathbf{u}(\tau)\|_{L^2}^2 + \|\Lambda^3\theta(\tau)\|_{L^2}^2) < \infty,$$

and assume that $M(t) < \infty$ for $t < T$ and show that

$$M(T) < \infty \tag{2.10}$$

It follows from the equation of $\nabla\theta$ that, for any $0 \le s < t$,

$$\|\nabla\theta(t)\|_{L^\infty} \le C\|\nabla\theta(s)\|_{L^\infty}\exp\Big[\int_s^t \|\nabla\mathbf{u}(\tau)\|_{L^\infty}\, d\tau\Big]. \tag{2.11}$$

Recall the interpolation inequality, for any $s > 2$,

$$\|\nabla f\|_{L^\infty(\mathbb{R}^2)} \le C\Big(1 + \|f\|_{L^2(\mathbb{R}^2)} + \|\Delta f\|_{L^2(\mathbb{R}^2)} \log\big(e + \|\Lambda^s f\|_{L^2(\mathbb{R}^2)}\big)\Big). \tag{2.12}$$

Therefore, for $T_0 < T$ (close to T) to be specified and $T_0 < t < T$, we have

$$\begin{aligned}
\|\nabla\theta(t)\|_{L^\infty} &\le C\|\nabla\theta(T_0)\|_{L^\infty}\exp\Big[C\int_{T_0}^t \Big(1 + \|\mathbf{u}\|_{L^2} + \|\Delta\mathbf{u}\|_{L^2}\log(1 + \|\Lambda^3\mathbf{u}\|_{L^2}\Big)\, d\tau\Big] \\
&\le C\|\nabla\theta(T_0)\|_{L^\infty}\exp\Big[C\int_{T_0}^t \Big(\|\Delta\mathbf{u}(\tau)\|_{L^2}\log(1 + M(t))\Big)\, d\tau\Big] \\
&\le C\|\nabla\theta(T_0)\|_{L^\infty}\exp\Big[C\int_{T_0}^t \|\Delta\mathbf{u}(\tau)\|_{L^2}\, d\tau \log(1 + M(t))\Big].
\end{aligned} \tag{2.13}$$

Due to the bound in (2.5), namely

$$\int_0^T \|\Delta\mathbf{u}(t)\|_{L^2}^2\, dt < \infty,$$

we can choose T_0 close enough to T such that, for small $\epsilon > 0$,

$$C\int_{T_0}^t \|\Delta\mathbf{u}(\tau)\|_{L^2}\, d\tau \le \epsilon.$$

It then follows that, for $T_0 \le t < T$,

$$\|\nabla\theta(t)\|_{L^\infty} \le C(1+M(t))^\epsilon. \tag{2.14}$$

Taking the inner product of the velocity equation in (1.8) with $\Delta^3\mathbf{u}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^3\mathbf{u}(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^2} \partial_{yy}u\cdot\Delta^3 u\,dx - \int_{\mathbb{R}^2} \partial_{xx}v\cdot\Delta^3 v\,dx - \int_{\mathbb{R}^2} \theta\,\Delta^3 v\,dx + \int_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta^3\mathbf{u}\,dx \\
&\triangleq L_1 + L_2 + L_3 + L_4.
\end{aligned} \tag{2.15}$$

Integrating by parts and using $\partial_x u + \partial_y v = 0$, we have

$$\begin{aligned}
L_1 &= -\int_{\mathbb{R}^2} \partial_{yy}\Delta u\,\Delta^2 u\,dx \\
&= -\int_{\mathbb{R}^2} \partial_{yy}\Delta u(\partial_{xx}\Delta u + \partial_{yy}\Delta u)\,dx \\
&= -\int_{\mathbb{R}^2} (\partial_{yy}\Delta u)^2\,dx - \int_{\mathbb{R}^2} \partial_{xx}\Delta u\,\partial_{yy}\Delta u\,dx \\
&= -\int_{\mathbb{R}^2} (\partial_{yy}\Delta u)^2\,dx - \int_{\mathbb{R}^2} \partial_{xy}\Delta u\,\partial_{xy}\Delta u\,dx \\
&= -\int_{\mathbb{R}^2} (\partial_{yy}\Delta u)^2\,dx - \int_{\mathbb{R}^2} \partial_{yy}\Delta v\,\partial_{yy}\Delta v\,dx \\
&= -\int_{\mathbb{R}^2} (\partial_{yy}\Delta u)^2\,dx - \int_{\mathbb{R}^2} (\partial_{yy}\Delta v)^2\,dx.
\end{aligned} \tag{2.16}$$

Similarly,

$$\begin{aligned}
L_2 &= -\int_{\mathbb{R}^2} \partial_{xx}\Delta v(\partial_{xx}\Delta v + \partial_{yy}\Delta v)\,dx \\
&= -\int_{\mathbb{R}^2} (\partial_{xx}\Delta v)^2\,dx - \int_{\mathbb{R}^2} \partial_{xx}\Delta v\,\partial_{yy}\Delta v\,dx \\
&= -\int_{\mathbb{R}^2} (\partial_{xx}\Delta v)^2\,dx - \int_{\mathbb{R}^2} \partial_{xy}\Delta v\,\partial_{xy}\Delta v\,dx
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^2} (\partial_{xx}\Delta v)^2\,dx - \int_{\mathbb{R}^2} \partial_{xx}\Delta u\,\partial_{xx}\Delta u\,dx \\
&= -\int_{\mathbb{R}^2} (\partial_{xx}\Delta v)^2\,dx - \int_{\mathbb{R}^2} (\partial_{xx}\Delta u)^2\,dx. \qquad (2.17)
\end{aligned}
$$

(2.16) and (2.17) together imply

$$
-L_1 - L_2 \geq \frac{1}{2}\|\Delta^2\mathbf{u}\|_{L^2}^2. \qquad (2.18)
$$

By Young's inequality, we have

$$
L_3 \leq \|\Lambda^3\theta\|_{L^2}\|\Lambda^3 v\|_{L^2} \leq C(\|\Lambda^3 v\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2).
$$

Invoking the commutator estimate (2.1), we can bound the last term by

$$
\begin{aligned}
L_4 &= \int_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Lambda^6\mathbf{u}\,dx \\
&= \int_{\mathbb{R}^2} \Lambda^3(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Lambda^3\mathbf{u}\,dx \\
&= \int_{\mathbb{R}^2} [\Lambda^3,\ \mathbf{u}\cdot\nabla]\mathbf{u}\cdot\Lambda^3\mathbf{u}\,dx \\
&\leq C\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^3\mathbf{u}\|_{L^2}^2. \qquad (2.19)
\end{aligned}
$$

Inserting the estimates for L_1-L_4 in (2.15), one obtains

$$
\begin{aligned}
&\frac{d}{dt}\|\Lambda^3\mathbf{u}(t)\|_{L^2}^2 + \|\Delta^2\mathbf{u}\|_{L^2}^2 \\
&\qquad \leq C(\|\Lambda^3 v\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2) + C\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^3\mathbf{u}\|_{L^2}^2. \qquad (2.20)
\end{aligned}
$$

Taking the L^2 inner product of the θ equation with $\Lambda^6\theta$, one has

$$
\begin{aligned}
\frac{d}{dt}\|\Lambda^3\theta(t)\|_{L^2}^2 &= -2\int_{\mathbb{R}^2} [\Lambda^3,\ \mathbf{u}\cdot\nabla]\theta\,\Lambda^3\theta\,dx \\
&\leq C(\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^3\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\Lambda^3\mathbf{u}\|_{L^2})\|\Lambda^3\theta\|_{L^2} \\
&\leq C\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}\|\Lambda^3\mathbf{u}\|_{L^2}\|\Lambda^3\theta\|_{L^2} \\
&\leq C\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{3}}\|\Lambda^4\mathbf{u}\|_{L^2}^{\frac{2}{3}}\|\Lambda^3\theta\|_{L^2} \\
&\leq \frac{1}{2}\|\Lambda^4\mathbf{u}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^\infty}^{\frac{3}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\Lambda^3\theta\|_{L^2}^{\frac{3}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2}(\|\Delta^2 u\|_{L^2}^2 + \|\Delta^2 v\|_{L^2}^2) + C\|\nabla \mathbf{u}\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2 \\
&\quad + C\|\nabla\theta\|_{L^\infty}^{\frac{3}{2}}\|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\Lambda^3\theta\|_{L^2}^{\frac{3}{2}},
\end{aligned}
\tag{2.21}
$$

where the following Gagliardo–Nirenberg inequality has been applied in the third inequality

$$
\|\Lambda^3 f\|_{L^2} \le C\|\nabla f\|_{L^2}^{\frac{1}{3}}\|\Lambda^4 f\|_{L^2}^{\frac{2}{3}}.
$$

Putting estimates (2.20) and (2.21) together, we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\Lambda^3\mathbf{u}(t)\|_{L^2}^2 + \|\Lambda^3\theta(t)\|_{L^2}^2) + \|\Delta^2\mathbf{u}\|_{L^2}^2 \\
&\quad \le C(1+\|\nabla\mathbf{u}\|_{L^\infty})(\|\Lambda^3\mathbf{u}\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2) + C\|\nabla\theta\|_{L^\infty}^{\frac{3}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\Lambda^3\theta\|_{L^2}^{\frac{3}{2}}.
\end{aligned}
$$

Now we again use the interpolation inequality (2.12) to obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\Lambda^3\mathbf{u}(t)\|_{L^2}^2 + \|\Lambda^3\theta(t)\|_{L^2}^2) + \|\Delta^2\mathbf{u}\|_{L^2}^2 \\
&\le C(1+\|\nabla\mathbf{u}\|_{L^\infty})M(t) + C(1+M(t))^{\frac{3\epsilon}{2}}M(t)^{\frac{3}{4}} \\
&\le C\Big(1+\|\mathbf{u}\|_{L^2} + \|\Delta\mathbf{u}\|_{L^2}\log\big(1+M(t)\big)\Big)M(t) + C(1+M(t))^{\frac{3\epsilon}{2}}M(t)^{\frac{3}{4}} \\
&\le C\Big(1+\|\mathbf{u}\|_{L^2} + \|\Delta\mathbf{u}\|_{L^2}\log\big(1+M(t)\big)\Big)(1+M(t)).
\end{aligned}
\tag{2.22}
$$

Here we have taken ϵ to be sufficiently small (smaller than $\frac{1}{6}$). Integrating in time over (T_0, t) and observing that $M(t)$ is a monotonically increasing function, we have

$$
1+M(t) \le C + C\int_{T_0}^{t}\Big(1+\|\Delta\mathbf{u}\|_{L^2}\log(1+M(\tau)\Big)\big(1+M(\tau)\big)\,d\tau.
\tag{2.23}
$$

It then follows from Osgood's inequality that

$$
M(T) \le C\exp\exp(C\epsilon) - 1 < \infty,
$$

which is the desired bound (2.10). This completes the proof of Theorem 1.1. $\square$

Next we prove Theorem 1.2.

Proof of Theorem 1.2. As we shall see from the rest of this proof, the vorticity $\omega = \nabla \times \mathbf{u}$ satisfies an equation that is the same as the one for the 2D Boussinesq equations with only horizontal velocity dissipation, namely the case (V). The proof has some similarities to that in [13]. The aim here is to establish a global *a priori* bound for $\|(\mathbf{u},\theta)\|_{H^s}$.

Clearly, the following global bounds hold,

$$
\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q}, \quad q \in [1,\infty]
$$

and

$$\|\mathbf{u}(t)\|_{L^2}^2 + \int_0^t \|\nabla v(\tau)\|_{L^2}^2 \, d\tau \le C(t, \mathbf{u}_0, \theta_0).$$

The vorticity $\omega = \nabla \times \mathbf{u}$, due to $\Delta v = \partial_x \omega$, satisfies

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega - \partial_{xx}\omega = \partial_x \theta. \tag{2.24}$$

Dotting (2.24) by ω and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \|\partial_x \omega\|_{L^2}^2 \le \frac{1}{2}\|\partial_x \omega\|_{L^2}^2 + C\|\theta_0\|_{L^2}^2,$$

which yields the global bound

$$\|\omega(t)\|_{L^2}^2 + \int_0^t \|\partial_x \omega(\tau)\|_{L^2}^2 \, d\tau \le \|\omega_0\|_{L^2}^2 + C\, t \|\theta_0\|_{L^2}^2. \tag{2.25}$$

Performing the L^q-estimate with (2.24), we have, for $q \in [2, \infty)$,

$$\frac{1}{q}\frac{d}{dt}\|\omega(t)\|_{L^q}^q + (q-1)\int |\partial_x \omega|^2 |\omega|^{q-2}\, dx \le (q-1)\int |\theta|\, |\partial_x \omega|\, |\omega|^{q-2}\, dx.$$

Bounding the right-hand side by Young's inequality yields

$$\frac{1}{q}\frac{d}{dt}\|\omega\|_{L^q}^q \le (q-1)\, \|\theta\|_{L^q}^2\, \|\omega\|_{L^q}^{q-2},$$

which can be simplified to

$$\frac{d}{dt}\|\omega(t)\|_{L^q}^2 \le q\, \|\theta_0\|_{L^q}^2.$$

Invoking the notation

$$\|f\|_{\sqrt{L}} \triangleq \sup_{q \ge 2} \frac{\|f\|_{L^q}}{\sqrt{q}},$$

we obtain

$$\|\omega(t)\|_{\sqrt{L}}^2 \le \|\omega_0\|_{\sqrt{L}}^2 + \|\theta_0\|_{L^2 \cap L^\infty}^2 t. \tag{2.26}$$

Due to $\nabla \cdot \mathbf{u} = 0$ and $\omega = \partial_x v - \partial_y u$, we have

$$\|\nabla \mathbf{u}\|_{L^2} = \|\omega\|_{L^2}, \quad \|\nabla \partial_x \mathbf{u}\|_{L^2} = \|\partial_x \omega\|_{L^2}.$$

Therefore (2.24) and (2.25) imply that, for any $T > 0$,

$$\int_0^T \|\partial_x u(t)\|_{H^1}^2 \, dt, \quad \int_0^T \|\partial_x v(t)\|_{H^1}^2 \, dt, \quad \int_0^T \|\partial_y v(t)\|_{H^1}^2 \, dt \le C(T, \mathbf{u}_0, \theta_0). \tag{2.27}$$

However, $\int_0^T \|\partial_y u(t)\|_{H^1}^2 \, dt$ cannot be bounded by the above argument. Fortunately, with the aid of (2.26) and $\partial_y u = \partial_x v - w$, we have

$$\int_0^T \|\partial_y u(t)\|_{\sqrt{L}}^2 \, dt \le \int_0^T (\|\partial_x v(t)\|_{\sqrt{L}}^2 + \|w(t)\|_{\sqrt{L}}^2) \, dt \le C. \tag{2.28}$$

Due to the embedding $H^1(\mathbb{R}^2) \hookrightarrow \sqrt{L}(\mathbb{R}^2)$ (see Lemma A.1 in [13]), we combine (2.27) and (2.28) to obtain

$$\int_0^T \|\nabla \mathbf{u}(t)\|_{\sqrt{L}}^2 \, dt \le C. \tag{2.29}$$

Simple energy estimates imply, for any $s > 2$,

$$\begin{aligned} &\frac{d}{dt}(\|\mathbf{u}(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) + \|\nabla v\|_{H^s}^2 \\ &\le C(1 + \|\nabla \mathbf{u}\|_{L^\infty})(\|\mathbf{u}\|_{H^s}^2 + \|\theta\|_{H^s}^2). \end{aligned}$$

Bounding $\|\nabla \mathbf{u}\|_{L^\infty}$ by the logarithmic Sobolev type inequality, for $\sigma > 2$

$$\|\nabla f\|_{L^\infty(\mathbb{R}^2)} \le C\left(1 + \|\nabla f\|_{\sqrt{L}(\mathbb{R}^2)} (\log(e + \|f\|_{H^\sigma(\mathbb{R}^2)}))^{\frac{1}{2}}\right),$$

we obtain

$$\begin{aligned} &\frac{d}{dt}(\|\mathbf{u}(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) \\ &\le C\left(1 + \|\nabla \mathbf{u}\|_{\sqrt{L}} \left(\log(e + \|\mathbf{u}\|_{H^s}^2 + \|\theta\|_{H^s}^2)\right)^{\frac{1}{2}}\right)(\|\mathbf{u}\|_{H^s}^2 + \|\theta\|_{H^s}^2). \end{aligned}$$

Gronwall's inequality and the bound in (2.29) together yield the desired global bound for $\|\mathbf{u}(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2$. This completes the proof of Theorem 1.2. $\square$

3. Proof of Theorem 1.3

This section proves the global *a priori* bounds stated in Theorem 1.3. We believe these bounds will play important roles in the eventual solution of the global regularity problem for the 2D Boussinesq equations with only vertical velocity dissipation.

Proof of Theorem 1.3. Some parts of Theorem 1.3 were previously obtained for (1.2) with both vertical velocity dissipation and vertical thermal diffusion, namely the case (VII) defined in the introduction. We explain that they remain valid even then there is no vertical thermal diffusion. The global bounds in the other parts are new and we provide a complete proof for those parts.

The global L^4-bound for $\mathbf{u}$ was proven for (1.2) with (VII) (see Proposition 3.1 in [5]). The proof of the global bound for $\|\mathbf{u}\|_{L^4}$ makes use of the global L^q bound for θ, $\|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q}$ with $q \in [1, \infty]$. Even when there is no vertical thermal diffusion now, the global L^q-bound for θ is still valid and thus we still have the global L^4-bound for $\mathbf{u}$, that is, (a) is true. Also the global bounds for p in (b) have previously been proven for (1.2) with (VII). For the same reason, they remain valid. Similarly, the global bound for v in (c) still holds even when there is no vertical thermal diffusion.

The global bound for the horizontal velocity u is new. To prove it, we multiply the equation for the velocity component u, namely

$$\partial_t u + (\mathbf{u} \cdot \nabla) u = -\partial_x p + \partial_y^2 u \tag{3.1}$$

by u^5 and integrate in space to obtain

$$\begin{aligned} \frac{1}{6}\frac{d}{dt}\|u\|_{L^6}^6 + 5\int (\partial_y u)^2 u^4\,dx &= 5\int p\,\partial_x u u^4\,dx \\ &= -5\int p\,\partial_y v\,u^4\,dx. \end{aligned}$$

By the anisotropic Sobolev inequality in Lemma 2.3,

$$\begin{aligned} \left|\int p\,\partial_y v\,u^4\,dx\right| &\le C\,\|\partial_y v\,u\|_{L^2}\,\|p\|_{L^4}^{\frac{2}{3}}\,\|\partial_x p\|_{L^2}^{\frac{1}{3}}\,\|u^3\|_{L^2}^{\frac{2}{3}}\,\|\partial_y(u^3)\|_{L^2}^{\frac{1}{3}} \\ &= C\,\|\partial_y v\,u\|_{L^2}\,\|p\|_{L^4}^{\frac{2}{3}}\,\|\partial_x p\|_{L^2}^{\frac{1}{3}}\,\|u\|_{L^6}^{2}\,\|u^2\partial_y u\|_{L^2}^{\frac{1}{3}} \\ &\le \|u^2\partial_y u\|_{L^2}^2 + C\,\|\partial_y v\,u\|_{L^2}^{\frac{6}{5}}\,\|p\|_{L^4}^{\frac{4}{5}}\,\|\partial_x p\|_{L^2}^{\frac{2}{5}}\,\|u\|_{L^6}^{\frac{12}{5}}. \end{aligned}$$

Therefore,

$$\frac{1}{6}\frac{d}{dt}\|u\|_{L^6}^6 + 4\int (\partial_y u)^2 u^4\,dx \le C\,\|\partial_y v\,u\|_{L^2}^{\frac{6}{5}}\,\|p\|_{L^4}^{\frac{4}{5}}\,\|\partial_x p\|_{L^2}^{\frac{2}{5}}\,\|u\|_{L^6}^{\frac{12}{5}}. \tag{3.2}$$

According to (a) and (b), the term $\|\partial_y v\,u\|_{L^2}^{\frac{6}{5}}\,\|p\|_{L^4}^{\frac{4}{5}}\,\|\partial_x p\|_{L^2}^{\frac{2}{5}}$ is time integrable and thus (3.2) implies the desired bound for $\|u\|_{L^6}$ in (d).

Finally we prove (e). It follows from the equation of the horizontal velocity (3.1) that

$$\frac{1}{2}\frac{d}{dt}\|\partial_y u\|_{L^2}^2 + \|\partial_{yy} u\|_{L^2}^2 \le 2\|\partial_x p\|_{L^2}^2.$$

Here we have used the fact

$$\int \partial_y(\mathbf{u}\cdot\nabla u)\,\partial_y u\,dx = 0$$

which can be verified by using the divergence-free condition $\nabla\cdot\mathbf{u}=0$. The global bound for the pressure in (1.11) then implies the global bound

$$\|\partial_y u\|_{L^2} < \infty.$$

Due to $\nabla\cdot\mathbf{u}=0$,

$$-\Delta p = \nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \partial_y\theta = 2\partial_x(v\partial_y u) + \partial_{yy}(v^2) - \partial_y\theta.$$

By the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}
\|p\|_{L^q} &\le C\,\|\Delta^{-1}\partial_x(v\partial_y u)\|_{L^q} + C\,\|\Delta^{-1}\partial_{yy}(v^2)\|_{L^q} + C\,\|\Delta^{-1}\partial_y\theta\|_{L^q}\\
&\le C\,\|v\partial_y u\|_{L^{\frac{2q}{q+2}}} + \|v^2\|_{L^q} + C\,\|\theta\|_{L^{\frac{2q}{q+2}}}\\
&\le C\,(\|v\|_{L^q}\,\|\partial_y u\|_{L^2} + \|v\|_{L^{2q}}^2) + C\,\|\theta_0\|\|_{L^{\frac{2q}{q+2}}}\\
&\le C(q,T,\mathbf{u}_0,\theta_0) < \infty.
\end{aligned}$$

This completes the proof of Theorem 1.3. $\square$

4. Proof of Theorem 1.4

This section proves Theorem 1.4.

Proof of Theorem 1.4. In order to extend (u,θ) to $[0,T]$, it suffices to show that, for $t\le T$,

$$\|(\mathbf{u}(t),\theta(t)\|_{H^s} \le C(T, M(T), u_0, \theta_0).$$

First of all, we have the global L^2-bounds,

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\partial_y\theta(\tau)\|_{L^2}^2\,d\tau = \|\theta_0\|_{L^2}^2, \qquad \|\mathbf{u}(t)\|_{L^2} \le \|\mathbf{u}_0\|_{L^2} + t\|\theta_0\|_{L^2}.$$

It follows from the vorticity equation

$$\partial_t\omega + (\mathbf{u}\cdot\nabla)\omega = \partial_x\theta \tag{4.1}$$

that, for $t \le T$,

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \int_0^t \|\partial_x \theta(\tau)\|_{L^\infty}\, d\tau < \infty.$$

As a consequence, by Sobolev's inequality, we obtain a global bound for the velocity $\mathbf{u}$,

$$\|\mathbf{u}\|_{L^\infty} \le C\, \|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\, \|\omega\|_{L^\infty}^{\frac{1}{2}} < \infty.$$

Now we show that, for any $t \in [0, T]$,

$$\|\omega(t)\|_{L^2} + \|\nabla\theta(t)\|_{L^2} < \infty.$$

A simple L^2-estimate involving (4.1) yields

$$\frac{d}{dt}\|\omega\|_{L^2} \le \|\omega\|_{L^2}\, \|\partial_x\theta\|_{L^2}. \tag{4.2}$$

Taking ∂_x of the equation of θ and then dotting with $\partial_x\theta$ yield

$$\begin{aligned} \frac{d}{dt}\|\partial_x\theta\|_{L^2}^2 + 2\,\|\partial_x\partial_y\theta\|_{L^2}^2 &\le -2\int (\partial_x(\mathbf{u}\cdot\nabla\theta)\,\partial_x\theta \\ &= -2\int \partial_x u(\partial_x\theta)^2 - 2\int \partial_x v\,\partial_y\theta\partial_x\theta. \end{aligned}$$

The two terms on the right can be bounded as follows. By $\nabla\cdot\mathbf{u} = 0$ and integration by parts,

$$\begin{aligned} -\int \partial_x u(\partial_x\theta)^2 = \int \partial_y v\,(\partial_x\theta)^2 &= -2\int v\,\partial_x\theta\,\partial_x\partial_y\theta \\ &\le \frac{1}{16}\|\partial_x\partial_y\theta\|_{L^2}^2 + C\,\|v\|_{L^\infty}^2\,\|\partial_x\theta\|_{L^2}^2. \end{aligned} \tag{4.3}$$

$$\begin{aligned} -\int \partial_x v\,\partial_y\theta\partial_x\theta &\le \|\partial_x\theta\|_{L^\infty}\,\|\partial_x v\|_{L^2}\,\|\partial_y\theta\|_{L^2} \\ &\le \|\partial_x\theta\|_{L^\infty}\,(\|\omega\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2). \end{aligned}$$

Similarly,

$$\frac{d}{dt}\|\partial_y\theta\|_{L^2}^2 + 2\,\|\partial_{yy}\theta\|_{L^2}^2 = -2\int \partial_y u\partial_y\theta\partial_x\theta - 2\int \partial_y v\,(\partial_y\theta)^2. \tag{4.4}$$

By Hölder's inequality,

$$-\int \partial_y u\partial_y\theta\partial_x\theta \le \|\partial_x\theta\|_{L^\infty}\,(\|\omega\|_{L^2}^2 + \|\partial_y\theta\|_{L^2}^2).$$

By Lemma 2.3,

$$
\begin{aligned}
-\int \partial_y v\, (\partial_y \theta)^2 &\le C\, \|\partial_y v\|_{L^2}\, \|\partial_y \theta\|_{L^2} \|\partial_x \partial_y \theta\|_{L^2}^{\frac{1}{2}}\, \|\partial_y^2 \theta\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{1}{16} \|\partial_x \partial_y \theta\|_{L^2}^2 + \frac{1}{16} \|\partial_y^2 \theta\|_{L^2}^2 + C\, \|\omega\|_{L^2}^2 \|\partial_y \theta\|_{L^2}^2.
\end{aligned}
$$

Combining (4.2), (4.3) and (4.4) and the corresponding estimates, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\omega\|_{L^2} + \|\nabla\theta\|_{L^2}^2\right) + \|\partial_y \nabla\theta\|_{L^2}^2 \\
&\le (1 + \|v\|_{L^\infty}^2 + \|\partial_x \theta\|_{L^\infty} + \|\partial_y \theta\|_{L^2}^2)(\|\omega\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}
$$

Applying Gronwall's inequality and using the fact that

$$
\int_0^T (1 + \|v\|_{L^\infty}^2 + \|\partial_x \theta\|_{L^\infty} + \|\partial_y \theta\|_{L^2}^2)\, dt < \infty,
$$

we obtain, for any $t \in [0, T]$,

$$
\|\omega(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2 + \int_0^T \|\partial_y \nabla\theta\|_{L^2}^2\, dt < \infty. \tag{4.5}
$$

Next we show that, for any $t \in [0, T]$,

$$
\|\Delta\omega(t)\|_{L^2} + \|\Lambda^3 \theta(t)\|_{L^2} < \infty.
$$

Taking Δ of (4.1) and dotting with $\Delta\omega$ yield

$$
\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_{L^2}^2 \le C\, \|\nabla \mathbf{u}\|_{L^\infty}\, \|\Delta\omega\|_{L^2}^2 + \|\partial_x \Delta\theta\|_{L^2}\, \|\Delta\omega\|_{L^2}.
$$

Taking the inner product of the equation of θ with $\Delta^3\theta$ and integrating by parts yield

$$
\frac{1}{2}\frac{d}{dt}\|\Lambda^3\theta\|_{L^2}^2 + \|\partial_y \Lambda^3 \theta|\|_{L^2}^2 = J_1 + J_2,
$$

where

$$
\begin{aligned}
J_1 &= \int \partial_x \Delta(\mathbf{u}\cdot\nabla\theta)\, \partial_x \Delta\theta, \\
J_2 &= \int \partial_y \Delta(\mathbf{u}\cdot\nabla\theta)\, \partial_y \Delta\theta.
\end{aligned}
$$

J_1 can be further written as

$$J_1 = \int \partial_x \Delta(u\partial_x\theta + v\partial_y\theta)\,\partial_x\Delta\theta$$

$$= \int (\partial_x\Delta u\,\partial_x\theta + \partial_x\Delta v\,\partial_y\theta)\,\partial_x\Delta\theta \tag{4.6}$$

$$+ \int (\Delta u\,\partial_x^2\theta + \Delta v\,\partial_{xy}\theta)\,\partial_x\Delta\theta \tag{4.7}$$

$$+2\int (\partial_x^2 u\,\partial_x^2\theta + \partial_{xy}u\,\partial_{xy}\theta + \partial_x^2 v\,\partial_{xy}\theta + \partial_{xy}v\,\partial_y^2\theta)\,\partial_x\Delta\theta \tag{4.8}$$

$$+2\int (\partial_x u\,\partial_x^3\theta + \partial_y u\,\partial_x^2\partial_y\theta + \partial_x v\,\partial_x^2\partial_y\theta + \partial_y v\,\partial_x\partial_y^2\theta)\,\partial_x\Delta\theta. \tag{4.9}$$

The two terms in (4.6) can be bounded as follows. By Hölder's inequality,

$$\int \partial_x\Delta u\,\partial_x\theta\,\partial_x\Delta\theta \le C\,\|\partial_x\theta\|_{L^\infty}\,\|\Delta\omega\|_{L^2}\,\|\Lambda^3\theta\|_{L^2}.$$

By Lemma 2.3,

$$\begin{aligned}\int \partial_x\Delta v\,\partial_y\theta\,\partial_x\Delta\theta &\le C\,\|\Delta\omega\|_{L^2}\,\|\partial_y\theta\|_{L^2}^{\frac12}\,\|\partial_{xy}\theta\|_{L^2}^{\frac12}\,\|\partial_x\Delta\theta\|_{L^2}^{\frac12}\,\|\partial_{xy}\Delta\theta\|_{L^2}^{\frac12}\\
&\le \frac{1}{16}\|\partial_{xy}\Delta\theta\|_{L^2}^2 + C\,\|\partial_y\theta\|_{L^2}^{\frac23}\,\|\partial_{xy}\theta\|_{L^2}^{\frac23}\,\|\Delta\omega\|_{L^2}^{\frac43}\,\|\partial_x\Delta\theta\|_{L^2}^{\frac23}\\
&\le \frac{1}{16}\|\partial_{xy}\Delta\theta\|_{L^2}^2 + C\,\|\partial_y\theta\|_{L^2}^{\frac23}\,\|\partial_{xy}\theta\|_{L^2}^{\frac23}\left(\|\Delta\omega\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2\right).\end{aligned}$$

To bound the terms in (4.7), we use the divergence-free condition $\partial_x u + \partial_y v = 0$ to write $\Delta u = -\partial_y\omega$ and then apply Hölder's inequality to obtain

$$\begin{aligned}\int \Delta u\,\partial_x^2\theta\,\partial_x\Delta\theta &= -\int \partial_y\omega\,\partial_x^2\theta\,\partial_x\Delta\theta\\
&= \int \omega\,\partial_y\partial_x^2\theta\,\partial_x\Delta\theta + \int \omega\,\partial_x^2\theta\,\partial_x\partial_y\Delta\theta\\
&\le \frac{1}{16}\|\partial_x\partial_y\Delta\theta\|_{L^2}^2 + C\,\|\omega\|_{L^\infty}\,\|\Lambda^3\theta\|_{L^2}^2 + C\,\|\omega\|_{L^\infty}^2\,\|\Lambda^2\theta\|_{L^2}^2.\end{aligned}$$

To estimate the second term in (4.7), we integrate by parts and apply Lemma 2.3,

$$\begin{aligned}\int \Delta v\,\partial_{xy}\theta\,\partial_x\Delta\theta &= -\int \partial_x\theta\,\partial_y\Delta v\,\partial_x\Delta\theta - \int \Delta v\,\partial_x\theta\,\partial_{xy}\Delta\theta\\
&\le C\,\|\partial_x\theta\|_{L^\infty}\,\|\Delta\omega\|_{L^2}\,\|\Lambda^3\theta\|_{L^2}\\
&\quad + C\,\|\Delta v\|_{L^2}^{\frac12}\,\|\partial_x\Delta v\|_{L^2}^{\frac12}\,\|\partial_x\theta\|_{L^2}^{\frac12}\,\|\partial_{xy}\theta\|_{L^2}^{\frac12}\,\|\partial_{xy}\Delta\theta\|_{L^2}\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{16}\|\partial_{xy}\Delta\theta\|_{L^2}^2 + C\,\|\partial_x\theta\|_{L^\infty}\left(\|\Delta\omega\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2\right)\\ &\quad + C\,\|\partial_x\theta\|_{L^2}\,\|\partial_{xy}\theta\|_{L^2}\,\|\omega\|_{H^2}^2. \end{aligned}\tag{4.10}$$

We now turn to the terms in (4.8). Integrating by parts yields

$$\begin{aligned}2\int \partial_{xx}u\,\partial_{xx}\theta\,\partial_x\Delta\theta &= \int \partial_{xx}u\,\partial_x((\partial_{xx}\theta)^2) + 2\int \partial_{xx}u\,\partial_{xx}\theta\,\partial_{xyy}\theta\\ &= \int \partial_{xxy}v\,(\partial_{xx}\theta)^2\\ &\quad -2\int \partial_{xxx}u\,\partial_x\theta\,\partial_{xyy}\theta - 2\int \partial_{xx}u\,\partial_x\theta\,\partial_{xxyy}\theta\\ &= -2\int \partial_{xx}v\,\partial_{xx}\theta\,\partial_{xxy}\theta\\ &\quad -2\int \partial_{xxx}u\,\partial_x\theta\,\partial_{xyy}\theta - 2\int \partial_{xx}u\,\partial_x\theta\,\partial_{xxyy}\theta\\ &= 2\int \partial_{xxx}v\,\partial_x\theta\,\partial_{xxy}\theta + 2\int \partial_{xx}v\,\partial_x\theta\,\partial_{xxxy}\theta\\ &\quad -2\int \partial_{xxx}u\,\partial_x\theta\,\partial_{xyy}\theta - 2\int \partial_{xx}u\,\partial_x\theta\,\partial_{xxyy}\theta \end{aligned}\tag{4.11}$$

The terms on the right of (4.11) can be bounded as in (4.10), that is,

$$\begin{aligned}2\int \partial_{xx}u\,\partial_{xx}\theta\,\partial_x\Delta\theta &\le \frac{1}{16}\|\partial_y\Lambda^3\theta\|_{L^2}^2 + C\,\|\partial_x\theta\|_{L^\infty}\left(\|\Delta\omega\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2\right)\\ &\quad + C\,\|\partial_x\theta\|_{L^2}\,\|\partial_{xy}\theta\|_{L^2}\,\|\omega\|_{H^2}^2. \end{aligned}\tag{4.12}$$

The other three terms in (4.8) can be similarly estimated and obey the same bound as in (4.12). The four terms in (4.9) can be in a uniform way and are bounded by

$$\begin{aligned}&2\int (\partial_x u\,\partial_x^3\theta + \partial_y u\,\partial_x^2\partial_y\theta + \partial_x v\,\partial_x^2\partial_y\theta + \partial_y v\,\partial_x\partial_y^2\theta)\,\partial_x\Delta\theta\\ &\quad \le C\,\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2. \end{aligned}$$

This completes the estimates of the terms in J_1. The terms in J_2 can be similarly bounded. In fact, the estimate of J_2 is simpler due to the "favorable" derivative ∂_y. We thus omit the details. Collecting all the estimates above and combining with (4.5), we find

$$\begin{aligned}&\frac{d}{dt}\left(\|\omega\|_{H^2}^2 + \|\theta\|_{H^3}^2\right) + \frac{1}{2}\|\partial_y\Lambda^3\theta\|_{L^2}^2\\ &\quad \le C\,(1 + \|\nabla\mathbf{u}\|_{L^\infty} + \|\partial_x\theta\|_{L^\infty})\left(\|\Delta\omega\|_{L^2}^2 + \|\Lambda^3\theta\|_{L^2}^2\right)\\ &\qquad + C\,(1 + \|\omega\|_{L^\infty}^2 + \|\nabla\theta\|_{L^2}\|\partial_y\nabla\theta\|_{L^2})\left(\|\omega\|_{H^2}^2 + \|\theta\|_{H^3}^2\right). \end{aligned}\tag{4.13}$$

Inserting the interpolation inequality

$$\|\nabla \mathbf{u}\|_{L^\infty} \le C\,(1 + \|\mathbf{u}\|_{L^2} + \|\omega\|_{L^\infty} \log(e + \|\Delta\omega\|_{L^2}))$$

to (4.13) and noticing that the terms

$$\|\omega\|_{L^\infty}^2, \quad \|\partial_x \theta\|_{L^\infty}, \quad \|\nabla\theta\|_{L^2}\,\|\partial_y \nabla\theta\|_{L^2}$$

are all integrable on any finite interval $[0, T]$, we obtain by Gronwall's inequality

$$\|\omega(t)\|_{H^2}^2 + \|\theta(t)\|_{H^3}^2 < \infty$$

for any $T > 0$ and $t \le T$. This completes the proof of Theorem 1.4. □

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